The Ext-Algebra of Standard Modules of Bound Twisted Double Incidence Algebras

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Abstract

Quasi-hereditary algebras are an important class of algebras with many applications in representation theory, most notably the representation theory of semisimple complex Lie-algebras. Such algebras sometimes admit an exact Borel subalgebra, that is a subalgebra satisfying similar formal properties to the Borel subalgebras from Lie theory. This thesis is divided into two parts. In the first part we classify quasi-hereditary algebras with two simple modules over perfect fields up to Morita equivalence, generalizing a similar result by Membrillo-Hernandez for the algebraically closed case. In the second part, we take a poset $X$, a certain set $M$ of constants, and a finite set $\rho$ of paths in the Hasse-diagram of $X$ and construct an algebra $\mathcal{A}(X, M, \rho)$ that generalizes the twisted double incidence algebras originally introduced by Deng and Xi. We provide necessary and sufficient conditions for this algebra to be quasi-hereditary when $X$ is a tree, and we show that $\mathcal{A}(X, M, \rho)$ admits an exact Borel subalgebra when these conditions are satisfied. Following this, we compute the Ext-algebra of the standard modules of $\mathcal{A}(X, M, \rho)$.

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Introduction

Quasi-hereditary algebras first arose from the study of semi-simple complex Lie algebras. It is a known fact that the category $\mathcal{O}$ of a semi-simple complex Lie algebra decomposes into blocks, and quasi-hereditary algebras were designed to capture many of the properties shared by these blocks. Since then, quasi-hereditary algebras have found many applications throughout the field of representation theory, such as the representation theory of Schur algebras and algebraic groups.

Central to the theory of quasi-hereditary algebras is a distinguished class

$$\Delta = \{ \Delta(\lambda) : \lambda \in \Lambda \}$$

of modules called the standard modules obtained as certain quotients of the projective covers of the simple modules. Of particular interest is the category $\mathcal{F}(\Delta)$ of modules that admit filtrations by standard modules, and understanding this category is a problem of great importance. A useful tool for studying this category is the notion of an exact Borel subalgebra $B$ of a quasi-hereditary algebra $A$. Such algebras capture many properties shared by universal enveloping algebras of Borel subalgebras from Lie theory. One of their key features is that the induction functor

$$A \otimes_B - : B-\text{mod} \to A-\text{mod}$$

restricts to an exact functor $B-\text{mod} \to \mathcal{F}(\Delta)$, which allows for the study of $\Delta$-filtered modules in terms of $B$-modules. For this reason, it is an unfortunate fact that not every quasi-hereditary algebra admits an exact Borel subalgebra.

In a paper by Koenig, Külshammer, and Ovsienko [16] a large step was taken towards amending this issue. They showed that for every quasi-hereditary algebra $A$, there is a directed box $\mathfrak{B}$, that is a bimodule with a coassociative comultiplication and a counit, such that $A$ is Morita equivalent to the right Burt-Butler algebra of $\mathfrak{B}$. Then $\mathcal{F}(\Delta)$ is equivalent as an exact category to the category of representations of $\mathfrak{B}$. Moreover, the box $\mathfrak{B}$ can be computed explicitly by using the $A_\infty$-structure on the algebra $\text{Ext}^*_A(\Delta, \Delta)$. An important corollary of this result is that every quasi-hereditary algebra is Morita equivalent to a quasi-hereditary algebra admitting an exact Borel subalgebra. In fact, this subalgebra is given by the underlying algebra of the box $\mathfrak{B}$.

Arguably the most complicated and time-consuming step in this calculation is the computation of the $A_\infty$-structure on $\text{Ext}^*_A(\Delta, \Delta)$. Informally, an $A_\infty$-algebra is a differential graded algebra where the associativity axiom is replaced by a family of higher operations satisfying a set of identities that express associativity up to homotopy. Due to the inherent complexity of such structures, the computation of the $A_\infty$-structure on $\text{Ext}^*_A(\Delta, \Delta)$ has only been done explicitly for a relatively small class of examples.

In an article [21], Thuresson considered this problem for a special class of algebras called dual extension algebras. Given two suitable algebras $A$ and $B$ one can construct the dual extension algebra $\mathcal{A}(B, A^{op})$. This algebra is quasi-hereditary and contains $B$ as an exact Borel subalgebra. Thuresson proved that there is an isomorphism of graded algebras:

$$\text{Ext}^*_\mathcal{A}(B, A^{op})(\Delta, \Delta) \cong \mathcal{A}(\text{Ext}^*_B(E, E), A) \quad (*)$$

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where $E$ denotes the direct sum of the simple $B$-modules. Furthermore, it was shown that the $A_\infty$-structure on $\text{Ext}^*_{A(B,\text{op})}(\Delta, \Delta)$ can be computed explicitly in terms of the $A_\infty$-structure on $\text{Ext}^*_I(E, E)$.

In this thesis, we consider a class of algebras defined in a similar way to dual extension algebras. If $X$ is a partially ordered set, one can construct its incidence algebra $I(X)$ by identifying all paths whose starting vertices and ending vertices coincide. By assigning a set $M$ of constants to certain subdiagrams of $X$, one can construct the twisted double incidence algebra $A(X, M)$. In the original 1995 paper by Deng and Xi [7] where these algebras were introduced, it was shown that $A(X, M)$ is quasi-hereditary whenever $X$ is a tree, and $I(X)$ is an exact Borel subalgebra of $A(X, M)$. The main invention of this thesis is the introduction of a more general class of algebras as follows. If $\rho$ is a finite set of paths in $X$, taking appropriate quotients by ideals generated by $\rho$ yields algebras $I(X, \rho)$ and $A(X, M, \rho)$, called the bound incidence algebra and the bound twisted double incidence algebra respectively. Examples of such algebras appear as the blocks of Schur algebras of finite or tame representation type.

This thesis contains two main results about these algebras. The first is a generalization of the corresponding results by Deng and Xi for twisted double incidence algebras:

**Theorem A.** If $X$ is a tree, then $A(X, M, \rho)$ is quasi-hereditary if and only if $M$ and $\rho$ satisfy certain compatibility criteria (see Definition 2.8). Furthermore, if these criteria are satisfied, then $I(X, \rho)$ is an exact Borel subalgebra of $A(X, M, \rho)$.

The second result concerns the computation of the algebra $\text{Ext}^*_{A(X, M, \rho)}(\Delta, \Delta)$.

**Theorem B.** If $X$ is a tree and $A(X, M, \rho)$ is quasi-hereditary, then $\text{Ext}^*_{A(X, M, \rho)}(\Delta, \Delta)$ is generated by $\text{Ext}^*_I(E, E)$ and $\text{Hom}(\Delta, \Delta) \cong I(X, \rho)$.

In fact, by generalizing the construction of the bound twisted double incidence algebras, we can express $\text{Ext}^*_{A(X, M, \rho)}(\Delta, \Delta)$ explicitly as:

$$\text{Ext}^*_{A(X, M, \rho)}(\Delta, \Delta) \cong A(\text{Ext}^*_I(E, E), I(X, \rho), \widetilde{M})$$

which is a direct analogue of the formula (**) for dual extension algebras.

The structure of this thesis is as follows. In the first chapter, we recall the definitions of quasi-hereditary algebras, exact Borel subalgebras and some of the main properties of these structures. We then consider the problem of classifying quasi-hereditary algebras up to Morita equivalence. Over algebraically closed fields, we show that every quasi-hereditary algebra with two simple modules is Morita equivalent to the path algebra of a certain type of quiver with relations. By considering species instead of quivers, we extend this result to a larger class of quasi-hereditary algebras with two simple modules that satisfy a certain splitting property. Such algebras include all quasi-hereditary algebras over perfect fields, hence also all quasi-hereditary algebras over finite fields or fields of characteristic zero.

In the second chapter, we introduce the algebras $A(X, M, \rho)$ and it is shown in Theorem 2.17 and Theorem 2.22 that these algebras are quasi-hereditary when $X$ is a tree if and only if the compatibility conditions of Definition 2.8 are satisfied. Next, we prove in Theorem 2.25 that $I(X, \rho)$ is an exact Borel subalgebra of $A(X, M, \rho)$ whenever $A(X, M, \rho)$
is quasi-hereditary. Together, these results prove Theorem A. Following this, we pro-
provide a combinatorial description of the algebra $\text{Ext}^*_I(X,\rho)(\mathbb{E},\mathbb{B})$ using the theory of Anick chains. With this computation, we prove Theorem B and give an explicit description of the algebra $\text{Ext}^*_{A(X,M,\rho)}(\Delta,\Delta)$. In the final section, we take a pair of suitable graded directed algebras $B$ and $A$ and a set $M$ of constants, and construct a graded algebra $A(B,A,M)$, generalizing the construction of $A(X,M,\rho)$. Using this construction, we obtain the formula $(\star\star)$.

Preliminaries

In this section we briefly cover some concepts that will be required throughout, in part for the sake of establishing notational conventions and in part to provide a brief introduc-
tion to these topics together with suitable references to texts containing more in depth
expositions.

Throughout this text, we will let $k$ be a (not necessarily algebraically closed) field. All algebras will be assumed to be associative and unital algebras over $k$, and all mod-
ules are assumed to be finite-dimensional left modules unless stated otherwise. We shall let $A$–$\text{Mod}$ and $\text{Mod}$–$A$ denote the categories of left and right $A$-modules respectively, and we let $A$–mod and mod–$A$ denote the corresponding full subcategories of finite-
dimensional modules.

Basic Morita Theory

The general slogan for representation theory is the idea that one studies an algebra by studying its category of modules. The drawback behind this is that non-isomorphic algebras may have equivalent module categories, so this approach does not perfectly capture an algebra up to isomorphism, but only up to some weaker notion of equivalence. This motivates the following definition.

Definition 1. Let $A$ and $B$ be two algebras. We say that $A$ is Morita equivalent to $B$ if there is an equivalence of categories $A$–$\text{Mod} \cong B$–$\text{Mod}$, or equivalently if $\text{Mod}$–$A \cong \text{Mod}$–$B$. Write $A \sim_{\text{Mor}} B$ for this relation.

Example 2. [17, Thm. 17.20] Every algebra $A$ is Morita equivalent to the matrix algebra $M_n(A)$ for every positive integer $n$.

Determining whether the algebras $A$ and $B$ are Morita equivalent can be reduced to determining the existence of a certain module in $A$–$\text{Mod}$ or $B$–$\text{Mod}$. For this we need the following notions:

Definition 3. Let $G$ be an $A$-module.

(i) $G$ is said to be a generator of $A$–$\text{Mod}$ if for all morphisms $f, g : M \to N$ of $A$-modules, there is a morphism $h : G \to M$ such that $fh = gh$.

(ii) If $G$ is a generator that is also projective, then we say that it is a projective generator of $A$–$\text{Mod}$.
Theorem 4. [17] Thm. 17.25] Two algebras $A$ and $B$ are Morita equivalent if and only if there exists a projective generator $P$ in $A-\text{Mod}$ such that $\text{End}_A(P) \cong B$.

If $A$ is a finite-dimensional algebra, then the regular module $AA$ admits a decomposition $AA \cong \bigoplus_{i=1}^n P_i^{e_{ni}}$ where $P_1, \ldots, P_n$ is a complete and irredundant list of the isomorphism classes of indecomposable projective modules of $A$. We say that $A$ is basic if the multiplicities $m_i$ are all equal to one.

Theorem 5. Every finite-dimensional algebra is Morita equivalent to a basic algebra.

This result follows from Theorem 4 by taking the projective generator $P = \bigoplus_{i=1}^n P_i$, so that $A \sim \text{Mor} \text{End}_A(P)$.

For a more comprehensive treatment of Morita theory, we refer the reader to Chapter 7 in [17].

Quivers and Path Algebras

A quiver is a tuple $Q = (Q_0, Q_1)$ where $Q_0$ is a set whose elements are called vertices and where $Q_1$ is a set of arrows between the vertices. Throughout, we will always assume the sets $Q_0$ and $Q_1$ to be finite. If $\alpha$ is an arrow in $Q$, then we let $s(\alpha)$ and $t(\alpha)$ denote the starting and ending vertex of $\alpha$ respectively. Visually, we represent an arrow $\alpha \in Q_1$ by the diagram:

$$
\begin{array}{c}
  s(\alpha) \\
\end{array} \xrightarrow{\alpha} \begin{array}{c}
  t(\alpha)
\end{array}
$$

By drawing all arrows in this manner, we can represent the quiver by a directed (multi)graph. As an example, if $Q_0 = \{1, 2, 3\}$ and $Q_1 = \{\alpha, \beta, \gamma\}$ with $s(\alpha) = 1$, $t(\alpha) = 2$, $s(\beta) = 2$, $t(\beta) = 3$, and $s(\gamma) = t(\gamma) = 2$, then we obtain the diagram:

$$
\begin{array}{ccc}
  1 & \xrightarrow{\alpha} & 2 \\
  & \beta \circlearrowleft & 3
\end{array}
$$

Given a quiver $Q = (Q_0, Q_1)$, we let $Q^{\text{op}}$ denote the quiver with the same vertices as $Q$, but where the directions of the arrows are reversed. We shall refer to $Q^{\text{op}}$ as the opposite quiver of $Q$.

Notation. If $Q = (Q_0, Q_1)$ and $Q' = (Q'_0, Q'_1)$ are two quivers where $Q_0 = Q'_0$, then we write $Q \sqcup Q'$ for the quiver $(Q_0, Q_1 \sqcup Q'_1)$, where $\sqcup$ is the disjoint union.

A path in $Q$ is a word $p_n \ldots p_1$ where each $p_i \in Q_1$ and $s(p_i) = t(p_{i-1})$ for all $i$. Let $Q^*$ denote the set of all paths in $Q$. For a vertex $x \in Q_0$, we denote by $e_x$ the “empty” path from $x$ to itself.

Definition 6. The path algebra $kQ$ is the associative algebra given by the free vector space on $Q^*$ where the product of paths $p_n \ldots p_1$ and $q_m \ldots q_1$ is defined as follows:

$$q_m \ldots q_1 \cdot p_n \ldots p_1 = \begin{cases} q_m \ldots q_1 p_n \ldots p_1 & \text{if } s(q_1) = t(p_n) \\ 0 & \text{otherwise} \end{cases}$$

The identity element of this algebra is the sum $1 := \sum_{x \in Q_0} e_x$. 

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A relation on a quiver $Q$ is a linear combination $\lambda_1p_1 + \ldots + \lambda_np_n$ where $p_1, \ldots, p_n : x \to y$ are paths in $Q$ of length at least two. If $R$ is a set of relations in $Q$, then we say that the pair $(Q,R)$ is a quiver with relations or bound quiver.

Let $\mathbb{k}Q_+$ denote the ideal of $\mathbb{k}Q$ generated by all arrows. An ideal $I$ in $\mathbb{k}Q$ is said to be admissible if there is some integer $m \geq 2$ such that:

$$(\mathbb{k}Q_+)^m \subseteq I \subseteq (\mathbb{k}Q_+)^2$$

**Theorem 7.** [2, Theorem 3.7] If $\mathbb{k}$ is algebraically closed, then for every finite-dimensional basic algebra $A$, there exists some quiver $Q$ and an admissible ideal $I$ such that $A \cong \mathbb{k}Q/I$.

When paired with Theorem 5 this result implies that every finite-dimensional algebra over an algebraically closed field is Morita equivalent to the quotient of the path algebra of a quiver by an admissible ideal.

**Definition 8.** A graded quiver is a quiver $Q = (Q_0, Q_1)$ equipped with a function $\text{deg} : Q_1 \to \mathbb{Z}_{\geq 0}$.

The function $\text{deg}$ extends to a function $\text{deg} : Q^* \to \mathbb{Z}_{\geq 0}$ by taking $\text{deg}(p_n \ldots p_1) = \sum_{i=1}^{n} \text{deg}(p_i)$ for every path $p_n \ldots p_1$ in $Q$ and $\text{deg}(e_x) = 0$ for all $x \in Q_0$. This induces the structure of a graded algebra on $\mathbb{k}Q$.

As an example, every quiver has a natural grading by setting $\text{deg}(p) = 1$ for every $p \in Q_1$. The resulting grading on $\mathbb{k}Q$ is just the usual grading by path length.

### The Ext-algebra of a Module

Let $A$ be an algebra and let $M, L, N$ be $A$-modules. For each $m, n \geq 0$, there is a linear map:

$$* : \text{Ext}_A^m(L, N) \otimes_{\mathbb{k}} \text{Ext}_A^n(M, L) \to \text{Ext}_A^{n+m}(M, N)$$

called the Yoneda product which is constructed as follows. We shall follow the construction given in Section 2.6 in [3]. Consider projective resolutions:

$$P_* : \ldots \longrightarrow P_2 \overset{p_2}{\longrightarrow} P_1 \overset{p_1}{\longrightarrow} P_0 \overset{p_0}{\longrightarrow} M \longrightarrow 0$$

$$Q_* : \ldots \longrightarrow Q_2 \overset{q_2}{\longrightarrow} Q_1 \overset{q_1}{\longrightarrow} Q_0 \overset{q_0}{\longrightarrow} L \longrightarrow 0$$

By definition, the Ext-spaces are given by the cohomology:

- $\text{Ext}_A^n(M, L) = H^n(\text{Hom}_A(P_*, L))$
- $\text{Ext}_A^n(L, N) = H^n(\text{Hom}_A(Q_*, N))$

An element $[f] \in \text{Ext}_A^n(M, L)$ is represented by a morphism $f : P_n \to L$ such that $f \circ p_{n+1} = 0$. We may lift $f$ to a chain map:

$$P_* : \ldots \longrightarrow P_{n+2} \overset{p_{n+2}}{\longrightarrow} P_{n+1} \overset{p_{n+1}}{\longrightarrow} P_n \overset{p_n}{\longrightarrow} \ldots$$

$$Q_*[n] : \ldots \longrightarrow Q_2 \overset{q_2}{\longrightarrow} Q_1 \overset{q_1}{\longrightarrow} Q_0 \overset{q_0}{\longrightarrow} L$$

\[ f \]

\[ f_0 \]

\[ f \]

\[ f_0 \]

\[ f \]

\[ f_0 \]

\[ f \]

\[ f_0 \]
Now, an element \([g] \in \text{Ext}_A^n(L, N)\) is represented by a morphism \(Q_m \rightarrow N\) such that 
\[ g \circ q_{m+1} = 0. \]
Then \(g \circ f_m : P_{n+m} \rightarrow N\) satisfies:
\[ g \circ f_m \circ p_{m+n+1} = g \circ q_{m+1} \circ f_{m+1} = 0 \]
so we define \([g] \ast [f] \in \text{Ext}_A^{n+m}(M, N)\) to be the homology class \([g \circ f_m]\).

To see that it is well-defined, suppose \([g] = 0\). Then \(g\) is contained in the image of 
\(- \circ q_m\), say \(g = \bar{g} \circ q_m\). Then
\[ g \circ f_m = \bar{g} \circ q_m \circ f_m = \bar{g} \circ f_{m-1} \circ p_{n+m} \]
so \(g \circ f_m\) is in the image of \(- \circ p_{n+m}\). Thus \([g \circ f_m] = 0\) so the Yoneda product is well-defined in the first variable. If \([f] = 0\), then the chain map \(f_*\) is null-homotopic. In other words, there are morphisms \(s_k : P_{n+k} \rightarrow Q_{k+1}\) such that
\[ f_k = q_{k+1} \circ s_k + s_{k-1} \circ p_{n+k} \]
Then:
\[ g \circ f_m = g \circ q_{m+1} \circ s_m + g \circ s_{m-1} \circ p_{n+m} = g \circ s_{m-1} \circ p_{n+m} \]
so \(g \circ f_m\) is in the image of \(- \circ p_{n+m}\). Thus \([g \circ f_m] = 0\) so the Yoneda product is well-defined in the second variable.

For the interested reader, we also mention that there is another characterization of \(\text{Ext}_A^n(M, N)\) as the set equivalence classes of all exact sequences of the form:
\[ 0 \rightarrow M \rightarrow L_1 \rightarrow \ldots \rightarrow L_n \rightarrow N \rightarrow 0 \]
Then the Yoneda product can be described by “splicing together” such sequences. A full description of this construction as well as the characterization of \(\text{Ext}\) in terms of exact sequences can also be found in Section 2.6 in [3].

Now, for a fixed \(A\)-module \(M\), consider the space:
\[ \text{Ext}_A^*(M, M) = \bigoplus_{n=0}^{\infty} \text{Ext}_A^n(M, M) \]
The Yoneda product endows this space with the structure of an associative and unital graded \(k\)-algebra. We shall refer to this as the \(\text{Ext}\text{-algebra}\) of \(M\).

In this text, \(M\) will usually come equipped with a decomposition \(M = \bigoplus_{i \in I} M_i\). In this situation, it will be more convenient to compute each Yoneda product
\[ \ast : \text{Ext}_A^n(M_j, M_k) \otimes_k \text{Ext}_A^n(M_i, M_j) \rightarrow \text{Ext}_A^{n+m}(M_i, M_k) \]
individually, rather than the Yoneda product for the whole algebra at once.
1 Quasi-Hereditary Algebras

In this chapter, we discuss quasi-hereditary algebras and their properties. The chapter is further divided into two subsections. In the first we cover the definition of a quasi-hereditary algebra, exact Borel subalgebras and some of their properties that will be used throughout this thesis. In the second, we consider the classification of quasi-hereditary algebras with two simple modules up to Morita equivalence.

1.1 Elementary Definitions and Results

For a finite-dimensional $k$-algebra $A$, choose a partial ordering $(\Lambda, \leq)$ on the set of isomorphism classes of simple $A$-modules and write $E(\lambda)$ for the simple module corresponding to $\lambda \in \Lambda$. Moreover, write $P(\lambda)$ and $Q(\lambda)$ for the projective cover and injective hull of $E(\lambda)$, respectively.

**Definition 1.1.** For each $\lambda \in \Lambda$, define:

(i) The standard module $\Delta(\lambda)$ is the maximal factor module of $P(\lambda)$ whose composition factors are of the form $E(\mu)$ where $\mu \leq \lambda$.

(ii) The costandard module $\nabla(\lambda)$ is the maximal submodule of $Q(\lambda)$ whose composition factors are of the form $E(\mu)$ where $\mu \leq \lambda$.

In the definition above, by “maximal factor module” we mean maximality with respect to length of composition series.

Let $\Delta$ and $\nabla$ denote the sets of standard and costandard modules respectively. We say that an $A$-module $M$ admits a filtration by standard modules, or $\Delta$-filtration, if there is a filtration $0 = M_n \subset M_{n-1} \subset \ldots \subset M_1 \subset M_0 = M$ such that $M_{i-1}/M_i \in \Delta$ for all $1 \leq i \leq n$. We shall write $\mathcal{F}(\Delta)$ for the full subcategory of $A$-mod of all modules with standard filtration. Dually, one may define modules with costandard filtration which gives rise to the corresponding category $\mathcal{F}(\nabla)$.

**Lemma 1.2.** [9, Lemma 1.6] For any $\lambda \in \Lambda$, the following are equivalent:

(i) $\text{End}_A(\Delta(\lambda))$ is a skew field.

(ii) $[\Delta(\lambda) : E(\lambda)] = 1$, where $[M : E]$ denotes the Jordan-Hölder multiplicity of the simple module $E$ in $M$.

For a partial ordering $\Lambda$ of the simple modules, we say that $\Lambda$ is adapted if for every incomparable pair $\lambda_1, \lambda_2 \in \Lambda$ and every $A$-module $M$ such that $\text{top}(M) = E(\lambda_1)$ and $\text{soc}(M) = E(\lambda_2)$, there is some $\mu \in \Lambda$ such that $\mu > \lambda_1, \mu > \lambda_2$ and $[M, E(\mu)] \neq 0$.

For an adapted ordering $\Lambda$, if $\Lambda'$ is a refinement of $\Lambda$ then $\Delta_\Lambda(\lambda) = \Delta_{\Lambda'}(\lambda)$ and $\nabla_\Lambda(\lambda) = \nabla_{\Lambda'}(\lambda)$ for all $\lambda \in \Lambda$. Moreover, $\Lambda'$ is an adapted ordering.

**Lemma 1.3.** [9, Lemma 1.2, 1.3] For any $\lambda, \mu \in \Lambda$, the following holds:

(i) If $\text{Hom}_A(\Delta(\lambda), \Delta(\mu)) \neq 0$ then $\lambda \leq \mu$.

(ii) If $\text{Ext}^1_A(\Delta(\lambda), \Delta(\mu)) \neq 0$ then $\lambda < \mu$ if $\Lambda$ is adapted.
**Definition 1.4.** Fix an adapted partial ordering \((\Lambda, \leq)\) of the simple \(A\)-modules. Then \(A\) is said to be *quasi-hereditary* with respect to \(\Lambda\) if:

(i) \(\text{End}_A(\Delta(\lambda))\) is a skew field for every \(\lambda \in \Lambda\).

(ii) One of the following equivalent conditions is satisfied:

(a) \(\Delta A \in \mathcal{F}(\Delta)\)

(b) \(\mathcal{F}(\Delta) = \{X : \text{Ext}^1(X, \nabla) = 0\}\)

(c) \(\mathcal{F}(\Delta) = \{X : \text{Ext}^i(X, \nabla) = 0 \text{ for all } 1 \leq i\}\)

(d) \(\text{Ext}^2(\Delta, \nabla) = 0\)

In the case where \(k\) is algebraically closed, (i) can be reformulated as the statement \(\text{End}_A(\Delta(\lambda)) \cong k\) for all \(\lambda \in \Lambda\). Moreover, as a consequence of Lemma 1.2, if \(\lambda\) is a minimal element in \(\Lambda\) then the corresponding standard and costandard modules can be computed as \(\Delta(\lambda) = \nabla(\lambda) = E(\lambda)\).

**Remark.** Quasi-hereditary algebras were first defined in [20] from a purely ring theoretic perspective as follows. An idempotent ideal \(I \subset A\) is said to be a *heredity ideal* if \(I\) is projective as a left \(A\)-module and \(\text{Rad}(A)/I = 0\). Then \(A\) is defined to be quasi-hereditary if there exists a chain \(0 = I_n \subseteq \ldots \subseteq I_1 \subseteq I_0 = A\) such that the quotient \(I_i/I_{i-1}\) is a heredity ideal of \(A/I_i\) for all \(1 \leq i \leq n\).

Throughout this article, we shall require a number of useful results about quasi-hereditary algebras and their standard modules. First of all, for adapted orders, there is an alternate characterization of the standard modules given by the following lemma:

**Lemma 1.5.** [9, Lemma 1.1'] Assume \(\Lambda\) is an adapted ordering and let \(\lambda \in \Lambda\). Let \(\eta P(\lambda)\) denote the submodule of \(P(\lambda)\) generated by images of morphisms \(P(\mu) \to P(\lambda)\) where \(\lambda < \mu\). Then \(\Delta(\lambda) \cong P(\lambda)/\eta P(\lambda)\).

**Lemma 1.6.** [9, Lemma 1.4, 1.5] For any finite-dimensional algebra, the category \(\mathcal{F}(\Delta)\) satisfies the following properties:

(i) Assume that \(\Lambda = \{1, \ldots, n\}\) with the usual linear order. For any \(A\)-module \(M\), define \(\eta_i M\) be the submodule of \(M\) generated by images of morphisms \(P(j) \to M\) where \(j > i\). Then \(M\) has a \(\Delta\)-filtration if and only if for every \(1 \leq i \leq n\), there is an isomorphism \(\eta_{i-1} M/\eta_i M \cong \Delta(i)^{\oplus k}\) for some \(k_i\).

(ii) Let \(\Lambda\) be adapted and let \(f\) be an epimorphism in \(\mathcal{F}(\Delta)\). Then \(\text{Ker}(f) \in \mathcal{F}(\Delta)\).

As a consequence of (i), \(\mathcal{F}(\Delta)\) is closed under direct summands, and as a result \(\mathcal{F}(\Delta)\) contains all projective modules if \(A\) is quasi-hereditary. In fact since \(\mathcal{F}(\Delta)\) is also closed under direct sums, the assertion that \(\mathcal{F}(\Delta)\) contains all projective modules is equivalent to the conditions (a)-(d) in Definition 1.4.

**Proposition 1.7.** [9, Lemma 2.2] Every quasi-hereditary algebra is of finite global dimension. More precisely, \(\text{gldim}(A) \leq 2h(\Lambda)\) where \(h(\Lambda)\) denotes the largest integer \(h\) for which there exists a chain \(\lambda_0 < \lambda_1 < \ldots < \lambda_h\) in \(\Lambda\).
Definition 1.8. Let $A$ be a quasi-hereditary algebra with respect to the partial order $(\Lambda, \leq)$ and let $B$ be a subalgebra of $A$ with the same number of simple modules as $A$. Fix an indexing $\{E_B(\lambda)\}_{\lambda \in \Lambda}$ of the simple $B$-modules. Then $B$ is said to be an exact Borel subalgebra provided that:

(i) There is an isomorphism of algebras $\text{End}_B(E_B(\lambda)) \cong \text{End}_A(E_A(\lambda))$ for all $\lambda \in \Lambda$.

(ii) $B$ is quasi-hereditary with respect to $\Lambda$ and all standard $B$-modules are simple.

(iii) The functor $A \otimes_B - : B\text{-mod} \to A\text{-mod}$ is exact.

(iv) There is an isomorphism $A \otimes_B E_B(\lambda) \cong \Delta_A(\lambda)$ for all $\lambda \in \Lambda$.

Note that in the algebraically closed case, the axiom (i) may be omitted.

1.2 Classification of Quasi-Hereditary Algebras

In this section, we shall consider the problem of classifying quasi-hereditary algebras up to Morita equivalence. We begin with the simplest case, that is when the algebra has only one simple module:

Proposition 1.9. Let $A$ be a quasi-hereditary algebra with one simple module. Then $A$ is Morita equivalent to a skew field.

Proof. Let $E$ be the simple module of $A$ and let $P$ be its projective cover. Then $\Delta = P$ so by definition $\text{End}_A(P) = D$ is a skew field. Moreover, $AA \cong P^\oplus n$ for some $n > 0$, so $P$ is a projective generator of $A\text{-mod}$. Thus $A \sim_{\text{Mor}} D$ as desired. \hfill $\square$

In the case where the algebra has two simple modules, there is currently no general classification of quasi-hereditary algebras so extra conditions need to be imposed. We shall present two cases where a classification is possible, the first being when the ground field $k$ is algebraically closed. Under this assumption, quasi-hereditary algebras with two simple modules turn out to be Morita equivalent to path algebras of a certain class of quivers with relations. This has already been proven in [19, Thm. 3.1] using the theory of heredity ideals, but we shall provide an alternative proof from the module-theoretic viewpoint.

Let $r, s \geq 0$ be non-negative integers and define $Q(r, s)$ to be the quiver comprised of two vertices, denoted 1 and 2, together with $r$ arrows $\alpha_i : 1 \to 2$ and $s$ arrows $\beta_j : 2 \to 1$. We shall let $A(r, s)$ denote the path algebra of $Q(r, s)$ over $k$ with relations $\alpha_i \beta_j$ for all $1 \leq i \leq r$ and all $1 \leq j \leq s$. Observe that the dimension of this algebra is $\dim_k A = 2 + r + s + rs$. Note also that $A(r, s)$ has global dimension at most 2, hence it is always quasi-hereditary by [8, Thm. 2].

Proposition 1.10. Assume that $k$ is algebraically closed and let $A$ be a quasi-hereditary algebra over $k$ with two simple modules. Then $A$ is Morita equivalent to $A(r, s)$ for some $r, s \geq 0$.

Proof. Assume that $A$ is a basic algebra. Since $A$ has two simple modules, we may assume that $\Lambda$ is the partial order $1 < 2$. Then $\Delta(2) = P(2)$ since 2 is maximal and $\Delta(1) = E(1)$ by Lemma 1.2 since 1 is minimal. Since $A$ is a basic algebra over an algebraically closed
field, we have \( A \cong \mathbb{k}Q/I \) for some quiver \( Q \) and some admissible ideal \( I \). In particular \( Q \) has exactly two vertices, which we label 1 and 2 and we let \( e_1 \) and \( e_2 \) denote the corresponding idempotents in \( A \).

Then the \( \mathbb{k} \)-vector space of paths from vertex \( i \) to vertex \( j \) is given by \( e_j A e_i \cong \text{Hom}_A(P(j), P(i)) \), i.e a morphism \( P(j) \to P(i) \) is obtained by multiplication on the right by a linear combination of paths from \( i \) to \( j \) in \( Q \) modulo the relations in \( I \). Since \( A \) is quasi-hereditary, we have \( e_2 A e_2 \cong \text{End}_A(P(2)) \cong \mathbb{k} \). Thus \( e_2 A e_2 \) is spanned by the idempotent \( e_2 \) so there are no arrows \( 2 \to 2 \) in \( Q \). Moreover, this means that any path of the form \( \alpha \beta \) is zero where \( \alpha : 1 \to 2 \) and \( \beta : 2 \to 1 \) are arrows in \( Q \).

Next, let \( J \) be the Jacobson radical of \( A \) and recall that \( J \) is the ideal generated by all arrows in \( Q \). Moreover, \( J/J^2 \) is the space spanned by all arrows of \( Q \). Then as a \( \mathbb{k} \)-vector space, \( P(1) \) decomposes as:

\[
P(1) \cong \mathbb{k}e_1 \oplus e_1(J/J^2)e_1 \oplus e_1(J/J^2)e_1(J/J^2)e_1 \oplus ... \]

Applying Lemma 1.5, we compute the standard module:

\[
\Delta(1) \cong \mathbb{k}e_1 \oplus e_1(J/J^2)e_1 \oplus e_1(J/J^2)e_1(J/J^2)e_1 \oplus ... 
\]

But \( \Delta(1) \cong E(1) \) is 1-dimensional, so \( e_1(J/J^2)e_1 = 0 \). Thus there are no arrows of the form \( 1 \to 1 \) in \( Q \), so \( Q = Q(r, s) \) for some \( r, s \geq 0 \). We already know that \( \alpha \beta \in I \) for all \( i, j \), so there is an epimorphism \( A(r, s) \to A \).

Since \( \Delta(1) = E(1) \), Lemma 1.5 states that \( \text{rad} \ P(1) = \eta P(1) = \eta_1 P(1) \), so by Lemma 1.6 we have \( \text{rad} \ P(1) = \Delta(2)^{\oplus m} = P(2)^{\oplus m} \) for some \( m \). Now, since any morphism \( P(2) \to P(1) \) must factor through the inclusion \( \eta P(1) \to P(1) \), this implies that

\[
\text{Hom}_A(P(2), P(1)) \cong \text{Hom}_A(P(2), \eta P(1)) \\
\cong \text{Hom}_A(P(2), P(2)^{\oplus m}) \\
\cong \text{End}_A(P(2))^{\oplus m} \cong \mathbb{k}^m. 
\]

However, since \( \alpha \beta = 0 \), the space \( \text{Hom}_A(P(2), P(1)) \) is spanned by arrows \( 1 \to 2 \). Thus \( \dim_k \text{Hom}_A(P(2), P(1)) = r \) so \( m = r \). This implies that \( P(1) \) has dimension \( 1 + r \cdot \dim_k P(2) = 1 + r(1 + s) \) since \( P(2) \) has basis \( e_2, \beta_1, \ldots, \beta_s \). Thus:

\[
\dim_k A = \dim_k P(1) + \dim_k P(2) \\
= 1 + r + rs + 1 + s \\
= 2 + r + s + rs = \dim_k A(r, s). 
\]

Thus the epimorphism \( A(r, s) \to A \) is an isomorphism. Since every algebra is Morita equivalent to a basic algebra, it follows that every quasi-hereditary algebra over an algebraically closed field with two simple modules is Morita equivalent to some \( A(r, s) \).

The above is far from the most efficient proof of the result, but it outlines the methods by which we shall prove a more general version for quasi-hereditary algebras with two simple modules over fields that are not necessarily algebraically closed. A much more streamlined proof can be obtained by considering the following general results:

**Theorem 1.11.** (No Loops Conjecture) [13, Thm 4.5] Assume that \( A \) is an artinian algebra of finite global dimension over an algebraically closed field. Then \( \text{Ext}_A^1(E, E) = 0 \) for all simple \( A \)-modules \( E \). Consequently, the quiver of \( A \) has no loops.
In view of Proposition \[ \text{Proposition 1.7} \], the above applies to every quasi-hereditary algebra over an algebraically closed field.

**Theorem 1.12.** [5, p.463] Let \( A = \mathbb{k}Q/I \) where \( Q \) is a quiver and \( I \) is a two-sided ideal generated by a set of relations \( R \). Assume further that no proper subset of \( R \) generates \( I \). Then for each pair of vertices \( i, j \) of \( Q \), we have

\[
|R \cap e_j I e_i| = \dim_k \operatorname{Ext}^2_A(E(i), E(j))
\]

where \( E(i) \) denotes the simple module corresponding to the vertex \( i \).

The above result is a consequence of some formulas for \( \operatorname{Tor}^*_A \) originally found by Gavorov [11] in 1973 which garnered little attention at the time. Inspired by private communications with Butler based on earlier work by Gruenberg, Bongartz would later publish these formulae in his 1983 article [5] from which the above theorem was borrowed.

Then an alternate proof of Proposition \[ \text{Proposition 1.10} \] can be obtained as follows:

**Proof.** Assume that \( A \) is basic so that \( A \cong \mathbb{k}Q/I \) for some quiver \( Q \). Then \( Q \) has no loops by Theorem \[ \text{Theorem 1.11} \], so it is of the form \( Q(r, s) \) for some \( r, s \geq 0 \). Like before we assume that \( A \) is quasi-hereditary with respect to \( 1 < 2 \), so we have \( \Delta(1) = E(1) \) and \( \Delta(2) = P(2) \). Moreover, we have \( \nabla(1) = E(1) \) as well. Then \( \operatorname{End}_A(P(2)) \cong k \), so \( \operatorname{End}_A(P(2)) \) is spanned by the morphism corresponding to multiplication by \( e_2 \). This implies that \( \alpha_i \beta_j = 0 \) for any \( \alpha_i \) and \( \beta_j \) in \( Q(r, s) \), so \( \langle \alpha_i \beta_j \rangle \subseteq I \) for all \( i, j \). This gives us an epimorphism \( A(r, s) \to A \).

The only paths of length greater than or equal to two in \( A(r, s) \) are the paths \( \beta_j \alpha_i \), so any additional relations in \( I \) must be linear combinations of such paths. To see that no such relations exist in \( I \), we use the fact that \( \Delta(1) \) trivially admits a \( \Delta \)-filtration. By axiom (d) in the definition of quasi-hereditary algebras, we have \( \operatorname{Ext}^{2}(E(1), E(1)) = \operatorname{Ext}^{2}(\Delta(1), \nabla(1)) = 0 \), so by Theorem \[ \text{Theorem 1.12} \] there are no relations in \( I \) consisting of paths from 1 to 1. Thus \( I = \langle \alpha_i \beta_j \rangle_{i,j} \) so \( A \cong A(r, s) \).

We shall now work towards proving a similar classification for algebras over fields that are not necessarily algebraically closed. First, we require some preliminary notions:

**Definition 1.13.** A modulated quiver or species is a family \( Q = \{ D_i, j M_i \}_{i,j \in I} \) where each \( D_i \) is a skew field and each \( j M_i \) is a \( (D_i, D_j) \)-bimodule, satisfying \( \operatorname{Hom}_{D_i}(j M_i, D_i) \cong \operatorname{Hom}_{D_j}(j M_i, D_j) \) as \( (D_i, D_j) \)-bimodules for all \( i, j \in I \). In addition to the above, all skew fields \( D_i \) are finite-dimensional algebras over some fixed field \( \mathbb{k} \) which acts centrally, all \( j M_i \) are finite-dimensional and \( \lambda x = x \lambda \) for all \( \lambda \in \mathbb{k} \) and all \( x \in j M_i \), then we say that \( Q \) is a \( \mathbb{k} \)-modulated quiver or \( \mathbb{k} \)-species.

Throughout, we shall refer to the skew fields \( D_i \) as the vertices of \( Q \).

**Definition 1.14.** Let \( Q = \{ D_i, j M_i \}_{i,j \in I} \) be a species. Let \( D = \bigoplus_{i \in I} D_i \) and \( M = \bigoplus_{i,j \in I} j M_i \), so that \( M \) is a \( (D, D) \)-bimodule under the obvious actions. Then the tensor algebra of \( Q \) is the graded ring given by:

\[
T(Q) = D \oplus M \oplus M^{\otimes 2} \oplus M^{\otimes 3} \oplus ...\]
with multiplication \( T(Q) \otimes_DT(Q) \to T(Q) \) being given in each degree by the canonical isomorphism:

\[
M^\otimes_Dn \otimes_D M^\otimes_Dm \to M^\otimes_Dm+n
\]

Observe that if \( Q \) is a \( k \)-species, then \( T(Q) \) is an algebra over \( k \).

The bimodule \( M \) generates an ideal \( R \) of \( T(Q) \). We say that an ideal \( I \) of \( T(Q) \) is \textit{admissible} if there exists an integer \( m \geq 2 \) such that \( R^m \subseteq I \subseteq R^2 \).

**Definition 1.15.** Given a species \( Q = \{D_{i,j}M_i\}_{i,j \in I} \) and any two elements \( a, b \in I \), a \textit{relation} is a linear combination \( \rho = r_1 + \ldots + r_n \) where each \( r_k \) is an element of the form:

\[
r_k \in i_k,p_k M_{i_k,p_k-1} \otimes D_{i_k,p_k-1} \ldots \otimes D_{i_k2} i_{k1} M_{i_{k1}}
\]

where \( i_k,p_k = b \) and \( i_{k1} = a \) for all \( 2 \leq k \leq n \).

Let \( A \) be an algebra so that the projection \( \pi : A \to A/J \) splits as an algebra homomorphism, i.e., it admits a right inverse \( \iota : A/J \to A \). Then every \( A \)-module can be regarded as an \( A/J \)-module through this morphism. We say that \( A \) is \textit{J-split} if the following exact sequence:

\[
0 \to J^2 \to J \to J/J^2 \to 0
\]

splits as \( A/J-A/J \)-bimodules. Examples of such algebras are abundant, in fact every finite-dimensional \( k \)-algebra is \( J \)-split if \( k \) is a perfect field, hence all finite-dimensional algebras over algebraically closed fields, fields of characteristic zero or finite fields are \( J \)-split. A proof of this can be found in [4, Prop. 3.10].

For any finite-dimensional basic algebra \( A \), the quotient \( A/J \) is a product of skew fields \( D_i \) which we regard as a subalgebra of \( A \) if \( A \) is \( J \)-split. This gives rise to a \( k \)-species \( Q_A = \{D_{i,j}M_i\} \) by taking \( jM_i = D_j(J/J^2)D_i \) so that \( J/J^2 \cong \bigoplus_{i,j} jM_i \).

**Theorem 1.16.** [4 Prop. 3.3] Let \( A \) be a finite-dimensional \( J \)-split basic \( k \)-algebra. Then \( A \cong T(Q_A)/I \) where \( Q_A \) is the species of \( A \) and \( I \) is an admissible ideal.

We are now ready to provide a generalization of the result given in Proposition 1.10 to \( J \)-split algebras.

**Proposition 1.17.** Let \( A \) be a quasi-hereditary \( J \)-split \( k \)-algebra with two simple modules. Then \( A \) is Morita equivalent to the tensor algebra \( T(Q)/R \) of some \( k \)-species \( Q \) where:

(a) \( Q = \{D_{i,j}M_i\} \) has exactly two vertices, \( D_1 \) and \( D_2 \).

(b) The bimodules \( iM_i \) are zero for all \( i = 1, 2 \).

(c) \( R \) is the ideal generated by the relations \( \alpha \otimes \beta \) for all \( \alpha \in 2M_1 \) and \( \beta \in 1M_2 \).

**Proof.** Like before, it suffices to consider basic algebras. Then by Theorem 1.16 \( A \) is isomorphic to the tensor algebra \( T(Q_A)/I \) where \( I \) is an ideal generated by some relations. Since \( A \) has two simple modules, its species has two vertices which we denote by \( D_1 \) and \( D_2 \). Let \( e_1 \) and \( e_2 \) denote the unit element of each of the two skew fields respectively. Then \( e_1, e_2 \) is a complete set of primitive orthogonal idempotents of \( A \). We may assume
without loss of generality that $A$ is quasi-hereditary with respect to the partial ordering $1 < 2$, so $\Delta(1) \cong E(1) \cong D_1$ and $\Delta(2) \cong P(2)$.

For a sequence $i_1, \ldots, i_n$ of elements in $\{1, 2\}$, let $i_n M_{i_{n-1}} M_{i_{n-2}} \cdots i_1 M_{i_2} M_{i_1}$ denote the image of the tensor product:

$$i_n M_{i_{n-1}} \otimes D_{i_{n-1}} \cdots \otimes D_{i_2} i_2 M_{i_1}$$

under the projection map $T(Q_A) \rightarrow T(Q_A)/I$. Then the underlying $D$-$D$-bimodule of the projective module $P(2) = Ae_2$ is:

$$P(2) \cong D_2 \oplus 1 M_2 \oplus 2 M_2 \oplus 1 M_1 M_2 \oplus 1 M_2 M_2 \oplus \cdots$$

and the endomorphism ring $\text{End}_A(P(2)) = e_2 A e_2$ is isomorphic to:

$$\text{End}_A(P(2)) \cong D_2 \oplus 2 M_2 \oplus 2 M_1 M_2 \oplus 2 M_2 M_2 \oplus \cdots$$

as $D_2$-$D_2$-bimodules. Since $\text{End}_A(P(2))$ is a skew field, its radical is zero hence $2 M_2$ and $2 M_1 M_2$ are zero. The latter implies that the ideal $R$ is contained in $I$. It remains to prove that $1 M_1 = 0$ and $R = I$. Note also that the above implies that $\text{End}_A(P(2)) \cong D_2$ as a $D_2$-$D_2$-bimodule.

The projective module $P(1)$ is of the form:

$$P(1) \cong D_1 \oplus 1 M_1 \oplus 2 M_1 \oplus 1 M_1 M_1 \oplus 1 M_2 M_1 \oplus \cdots$$

Then by Lemma 1.5, the standard module $\Delta(1)$ is given by:

$$\Delta(1) \cong D_1 \oplus 1 M_1 \oplus 2 M_1 \oplus 1 M_1 M_1 \oplus 1 M_2 M_1 \oplus \cdots$$

But $\Delta(1) \cong E(1)$, hence $1 M_1 = 0$. Then $Q_A$ is of the desired form, and since $R \subset I$, there exists an epimorphism $T(Q_A)/R \rightarrow A$.

Since $\Delta(1) = E(1)$, Lemma 1.5 states that $\text{rad} P(1) = \eta P(1) = \eta_1 P(1)$, so by Lemma 1.6 we have $\text{rad} P(1) = \Delta(2)^{\oplus m} = P(2)^{\oplus m}$ for some $m$. Now, any morphism $P(2) \rightarrow P(1)$ factors through the inclusion $\eta P(1) \hookrightarrow P(1)$, so we have the following chain of isomorphisms:

$$\text{Hom}_A(P(2), P(1)) \cong \text{Hom}_A(P(2), \eta P(1))$$
$$\cong \text{Hom}_A(P(2), P(2)^{\oplus m})$$
$$\cong \text{End}_A(P(2))^{\oplus m} \cong D_2^{\oplus m}.$$  

However, since $1 M_1$, $2 M_2$, and $2 M_1 M_2$ all vanish, there is an isomorphism $\text{Hom}_A(P(2), P(1)) \cong 2 M_1$, so $m = \text{dim}_D 2 M_1$. This implies that:

$$\text{dim}_k P(1) = \text{dim}_k D_1 + \text{dim}_D 2 M_1 \cdot \text{dim}_k P(2)$$
$$= \text{dim}_k D_1 + \text{dim}_D 2 M_1 \cdot (\text{dim}_k D_2 + \text{dim}_k 1 M_2)$$
$$= \text{dim}_k D_1 + \text{dim}_k 2 M_1 + \text{dim}_D 2 M_1 \cdot \text{dim}_D 1 M_2 \cdot \text{dim}_k D_2$$

So we obtain:

$$\text{dim}_k A = \text{dim}_k P(1) + \text{dim}_k P(2)$$
$$= \text{dim}_k D_1 + \text{dim}_k D_2 + \text{dim}_k 2 M_1 + \text{dim}_k 1 M_2$$
$$+ \text{dim}_D 2 M_1 \cdot \text{dim}_D 2 M_1 \cdot \text{dim}_k D_2$$

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Similarly, we have:
\[
\dim_k T(Q_A)/R = \dim_k D_1 + \dim_k D_2 + \dim_k M_1 + \dim_k M_2 \\
+ \dim_k M_2 \otimes_{D_2} M_1
\]
so it remains to verify that \( \dim_k M_2 \otimes_{D_2} M_1 = \dim D_2 M_2 \cdot \dim D_2 M_1 \cdot \dim D_2 \).

Let \( r = \dim D_2 M_1 \) and let \( s = \dim D_2 M_2 \). Then \( 2M_1 \cong D_2^{\oplus r} \) as a left \( D_2 \)-module and \( 1M_2 \cong D_2^{\oplus s} \) as a right \( D_2 \)-module. Then as a vector space over \( k \), we have:
\[
1M_2 \otimes_{D_2} 2M_1 \cong D_2^{\oplus s} \otimes_{D_2} D_2^{\oplus r} \\
\cong (D_2 \otimes_{D_2} D_2)^{\oplus sr} \\
\cong D_2^{\oplus sr}
\]
so we get:
\[
\dim_k M_2 \otimes_{D_2} M_1 = sr \cdot \dim_k D_2 = \dim D_2 M_2 \cdot \dim D_2 M_1 \cdot \dim D_2
\]
Then the desired dimensional equality follows so \( \dim_k A = \dim_k T(Q)/R \). Thus the projection \( T(Q_A)/R \to A \) is an isomorphism. \( \square \)
2 Bound Twisted Double Incidence Algebras

2.1 Quasi-Hereditity and Existence of Exact Borel Subalgebras

Let \((X, \leq)\) be a finite partially ordered set. Let \(\prec\) denote the covering relation of \(X\), i.e \(x \prec y\) if and only if \(x < y\) and there is no \(x < z < y\) in \(X\). The covering relation gives rise to a quiver \(Q_X\), called the quiver associated to \(X\) which is given by:

(i) The vertices of \(Q_X\) are the elements of \(X\).

(ii) There is exactly one arrow \(x \rightarrow y\) if and only if \(x \prec \cdot y\).

**Definition 2.1.** Let \(\alpha : x \rightarrow y'\) be an arrow in \(Q_X\) and let \(\beta : y \rightarrow x\) be an arrow in \(Q_X^{\text{op}}\). We call such a pair \((\alpha, \beta)\) a twistable pair. A twisting pair of the pair \((\alpha, \beta)\) is a pair \((\beta', \alpha')\) where \(\alpha' : y \rightarrow z\) is an arrow in \(Q_X\) and \(\beta' : z \rightarrow y'\) is an arrow in \(Q_X^{\text{op}}\). Pictorially, this corresponds to the following diagram

\[
\begin{array}{ccc}
  & z & \\
  \alpha' & \downarrow & \beta' \\
y & \downarrow & y' \\
  \beta & \downarrow & \alpha \\
x & \downarrow & y
\end{array}
\]

in the quiver \(Q_X \cup Q_X^{\text{op}}\). We let \(\text{Tw}(\alpha, \beta)\) denote the set of all twisting pairs of \(\alpha\) and \(\beta\).

**Remark.** The set of twisting pairs of \(\alpha : x \rightarrow y'\) and \(\beta : y \rightarrow x\) is in bijection with the set of all \(z \in X\) such that \(y, y' \prec z\).

**Definition 2.2.** A labeling on \(X\) is a set \(M\) comprised of a function \(m_{\alpha\beta} : \text{Tw}(\alpha, \beta) \rightarrow k\) for every pair \((\alpha, \beta)\). We shall refer to the constants \(m_{\alpha\beta}(\beta', \alpha')\) as the twisting constants.

We are now ready to present the main objects of study for this chapter. We begin with a construction due to Deng and Xi in [7]:

**Definition 2.3.** Given a finite poset \((X, \leq)\) and a labeling \(M\) on \(X\), we make the following definitions:

(i) The incidence algebra \(\mathcal{I}(X)\) is the path algebra of the quiver \(Q_X\) modulo the incidence relations \(p - q\) where \(p : x \rightarrow y\) and \(q : x \rightarrow y\) are any two paths in \(Q_X\) of length greater than one.

(ii) The \(M\)-twisted double incidence algebra \(\mathcal{A}(X, M)\) is the path algebra of the quiver \(Q_X \cup Q_X^{\text{op}}\) with the following relations:

(a) The incidence relations in \(Q_X\) and the corresponding relations in \(Q_X^{\text{op}}\).

(b) For any arrow \(\alpha : x \rightarrow y'\) in \(Q_X\) and any arrow \(\beta : y \rightarrow x\) in \(Q_X^{\text{op}}\), take the twisting relations:

\[
\alpha \beta = \sum_{(\beta', \alpha') \in \text{Tw}(\alpha, \beta)} m_{\alpha\beta}(\beta', \alpha') \beta' \alpha'
\]
The original construction of $A(X, M)$ by Deng and Xi is very different in appearance to the definition given above. In their original paper, the labeling was given by a collection of matrices indexed by pairs of elements of $X$, whose entries correspond to the twisting constants as we have defined them. The main motivation behind our change of notation and setup is that the approach using matrices is very unnatural if one wishes to consider generalizations of the algebras $A(X, M)$. In the final subsection of this chapter, we provide a more general definition of the algebras $A(X, M)$ where the quiver $Q_X^\text{op}$ is replaced by some other quiver $Q'$. In this setup, the twisting constants no longer arrange themselves into matrices.

Consider an admissible ideal $I$ of $kQ_X$ that contains all incidence relations. This induces an ideal $\tilde{I}$ of $I(X)$. Since all possible paths between any two vertices of $Q_X$ are identified in the incidence algebra, this is an ideal generated by a finite set $\rho$ of paths of length greater than one in $Q_X$. We will always assume that $\rho$ is a minimal generating set of the ideal. For any such $\rho$, we shall refer to the triple $(X, \leq, \rho)$ as a bound partially ordered set or bound poset. This leads to the following definitions:

**Definition 2.4.** For any bound poset $(X, \leq, \rho)$, we define:

(i) The bound incidence algebra:

$$\mathcal{I}(X, \rho) := \mathcal{I}(X)/\langle \rho \rangle$$

(ii) The bound $M$-twisted double incidence algebra:

$$A(X, M, \rho) := A(X, M)/\langle \rho \cup \rho^\text{op} \rangle$$

The algebra $A(X, M, \rho)$ is finite-dimensional since $A(X, M)$ is finite-dimensional by [7, Prop. 1.6]. Moreover, $A(X, M, \rho)$ contains $\mathcal{I}(X, \rho)$ and $\mathcal{I}(X, \rho)^\text{op} = \mathcal{I}(X^\text{op}, \rho^\text{op})$ as subalgebras.

**Notation.** Let $x \leq y \in X$.

- We shall sometimes let $\alpha_{xy}$ denote the unique path $x \to y$ in $Q_X$. Similarly, we shall let $\beta_{xy} : y \to x$ denote the corresponding path in $Q_X^\text{op}$. By abuse of notation, we take $\alpha_{xx}$ and $\beta_{xx}$ to be the idempotent $e_x$.

- We shall sometimes write $x < y \in \rho$ to indicate that the unique path $\alpha : x \to y$ in $Q_X$ belongs to $\rho$, or equivalently that the unique path $\beta : y \to x$ in $Q_X^\text{op}$ belongs to $\rho^\text{op}$.

If $\alpha : x \to y$ is an arrow in a quiver $Q$, let $\tilde{\alpha}$ denote the corresponding arrow $y \to x$ in $Q^\text{op}$. Then we say that the labeling $M$ is symmetric if:

$$m_{\alpha \beta}(\beta', \alpha') = m_{\tilde{\alpha} \tilde{\beta}}(\tilde{\alpha}', \tilde{\beta}')$$

for all $(\beta', \alpha') \in \text{Tw}(\alpha, \beta)$. If the labeling is symmetric, then $A(X, M, \rho)$ admits an anti-involution that fixes the vertices via the mapping $\alpha \mapsto \tilde{\alpha}$ and $\beta \mapsto \tilde{\beta}$ for all $\alpha \in Q_X$ and $\beta \in Q_X^\text{op}$.

17
Generalized Twisting Relations.

Before we begin discussing whether the algebras $A(X, M, \rho)$ are quasi-hereditary, it will be useful to discuss the effect of the twisting relations on paths of arbitrary length. To achieve this, we extend the definitions of the sets $\text{Tw}(\alpha, \beta)$ and the functions $m_{\alpha \beta}$ to include paths of arbitrary length.

**Definition 2.5.** Let $\alpha : x \to y$ be a path in $Q_X$ and let $\beta : y \to x$ be a path in $Q_X^\text{op}$. Define the sets $\text{Tw}(\alpha, \beta)$ recursively as follows:

1. If $\alpha$ and $\beta$ are arrows, then define $\text{Tw}(\alpha, \beta)$ as before.

2. Assume $\text{Tw}(\alpha, \beta)$ has been defined for all $\alpha$ and $\beta$ with path lengths less than or equal to $n$ and $m$ respectively. Let $\sigma : y' \to u$ be a path in $Q_X$ of length at most $n$ and let $\tau : v \to y$ be a path in $Q_X^\text{op}$ of length at most $m$. Define:
   
   a) $\text{Tw}(\sigma \alpha, \beta)$ is the set of all pairs $(\beta''', \sigma'\alpha')$ where $(\beta', \alpha') \in \text{Tw}(\alpha, \beta)$ and $(\beta''', \sigma') \in \text{Tw}(\sigma, \beta').$
   
   b) $\text{Tw}(\alpha, \beta \tau)$ is the set of all pairs $(\beta', \alpha'')$ where $(\beta', \alpha') \in \text{Tw}(\alpha, \beta)$ and $(\tau', \alpha'') \in \text{Tw}(\alpha', \tau)$.

Pictorially, this recursion step is represented by the following diagrams:

Similarly, we can define functions $m_{\alpha \beta} : \text{Tw}(\alpha, \beta) \to k$ as follows:

1. If $\alpha$ and $\beta$ are arrows, then $m_{\alpha \beta}$ is just the element of the labeling corresponding to $\alpha$ and $\beta$.

2. Assume $m_{\alpha \beta}$ has been defined for all $\alpha$ and $\beta$ with path lengths less than or equal to $m$ and $n$ respectively. Let $\sigma : y' \to u$ be a path in $Q_X$ of length at most $n$ and let $\tau : v \to y$ be a path in $Q_X^\text{op}$ of length at most $m$.

   a) Define $m_{\sigma \alpha, \beta} : \text{Tw}(\sigma \alpha, \beta) \to k$ by:
      
      $$m_{\sigma \alpha, \beta}(\beta''', \sigma'\alpha') := m_{\alpha, \beta}(\beta', \alpha') \cdot m_{\sigma' \beta'}(\beta'', \sigma')$$
      
      where $(\beta', \alpha') \in \text{Tw}(\alpha, \beta)$ and $(\beta'', \sigma') \in \text{Tw}(\sigma, \beta').$
(b) Define \( m_{\alpha,\beta} : \text{Tw}(\alpha, \beta) \rightarrow \mathbb{k} \) by:
\[
m_{\alpha,\beta}(\beta', \alpha'') := m_{\alpha,\beta}(\beta', \alpha') \cdot m_{\alpha'}(\tau', \alpha'')
\]
where \((\beta', \alpha') \in \text{Tw}(\alpha, \beta)\) and \((\tau', \alpha'') \in \text{Tw}(\alpha', \tau)\).

With these definitions, we have the following proposition:

**Proposition 2.6.** Let \( \alpha : x \to y' \) be a path in \( Q_X \) and \( \beta : y \to x \) a path in \( Q_X^{\text{op}} \). Then:
\[
\alpha \beta = \sum_{(\beta', \alpha') \in \text{Tw}(\alpha, \beta)} m_{\alpha,\beta}(\beta', \alpha') \beta \alpha'
\]
in \( \mathcal{A}(X, M, \rho) \). We shall refer to the above formula as the generalized twisting relations.

**Proof.** We shall prove this by double induction on the lengths of \( \alpha \) and \( \beta \). The base case where \( \alpha \) and \( \beta \) are arrows is clear by definition of \( \mathcal{A}(X, M, \rho) \). Assume the formula holds for some \( \alpha : x \to y' \) and \( \beta : y \to x \) and let \( \sigma : y' \to u \) and \( \tau : v \to y \) be arrows in \( Q_X \) and \( Q_X^{\text{op}} \) respectively. Then using the definitions above yields:
\[
\sigma \alpha \beta = \sum_{(\beta', \alpha') \in \text{Tw}(\alpha, \beta)} m_{\alpha,\beta}(\beta', \alpha') \sigma \beta' \alpha'
\]
\[
= \sum_{(\beta', \alpha') \in \text{Tw}(\alpha, \beta)} \sum_{(\beta'', \sigma') \in \text{Tw}(\sigma, \beta')} m_{\alpha,\beta}(\beta', \alpha') \cdot m_{\sigma,\beta}(\beta'', \sigma') \beta'' \sigma' \alpha'
\]
\[
= \sum_{(\beta'', \sigma' \alpha') \in \text{Tw}(\sigma \alpha, \beta)} m_{\sigma \alpha, \beta}(\beta'', \sigma' \alpha') \beta'' \sigma' \alpha'
\]
which is the desired formula. By an identical argument, one obtains the desired formula for the path \( \alpha \beta \tau \).

Now we shall investigate when the algebra \( \mathcal{A}(X, M, \rho) \) is quasi-hereditary. Deng and Xi proved in [7, Thm. 1.7] that the algebras \( \mathcal{A}(X, M) \) are quasi-hereditary with respect to \((X, \leq)\) if \( X \) is a tree, that is if there is at most one path \( x \to y \) in \( Q_X \) for any \( x, y \in X \). A natural question to ask is if this result generalizes to the algebras \( \mathcal{A}(X, M, \rho) \). Unfortunately, this is not the case for arbitrary \( \rho \) unless extra conditions are imposed on the labeling, as demonstrated by the following example:

**Example 2.7.** Let \( X = \{1, 2, 3, 4\} \) under the usual ordering and let \( \rho \) be the set containing the path \( 1 \to 2 \to 3 \). A labeling on \( X \) amounts to the choice of constants \( m_2, m_3 \), so that \( \mathcal{A}(X, M, \rho) \) is the path algebra of the quiver:

```
1 2 3 4
\alpha_1 \beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3
```

modulo the relations:
\[
\alpha_2 \alpha_1 = 0, \quad \beta_2 \beta_1 = 0,
\]
\[
\alpha_1 \beta_1 = m_2 \beta_2 \alpha_2, \quad \alpha_2 \beta_2 = m_3 \beta_3 \alpha_3, \quad \alpha_3 \beta_3 = 0
\]
A straightforward calculation using Lemma 1.5 shows that the standard modules are
given by $\Delta(i) = \mathbb{k}\{e_i, \beta_i, \beta_i^{-1} \beta_i, \ldots\}$ under the action where the $\alpha_j$ act as zero and $\beta_j$ act
by left multiplication, and it is easily seen that their endomorphism rings are isomorphic
to $\mathbb{k}$. It remains to verify whether the projective modules admit $\Delta$-filtrations.

If the constants $m_2, m_3$ are both non-zero, then $P(3)$ has a basis given by $e_3, \alpha_3, \beta_2, \beta_3 \alpha_3$. Note, that $\beta_3 \beta_3 \alpha_3 = m_2^{-1} \beta_2 \alpha_2 \beta_2 = m_2^{-1} m_3^{-1} \alpha_1 \beta_1 \beta_2 = 0$. By Lemma 1.6, $P(3)$ admits a
$\Delta$-filtration if and only if the filtration:

$$0 = \eta_4 P(3) \subset \eta_3 P(3) \subset \eta_2 P(3) \subset \eta_1 P(3)$$

has factors $\eta_{i-1} P(3)/\eta_i P(3)$ isomorphic to some (possibly zero) power of the standard
module $\Delta(i)$. The above amounts to the following filtration:

$$0 \subset P(4) \alpha_3 \subset P(3)$$

where $P(4) \alpha_3$ denotes the image of $P(4)$ under right multiplication by $\alpha_3$. Then $P(4) \alpha_3$
has to be some power of the standard module $\Delta(4) = \mathbb{k}\{e_4, \beta_3, \beta_2 \beta_3\}$, but $P(4) \alpha_3$ has
basis $\alpha_3, \beta_3 \alpha_3$, so this is impossible.

If at least one of the constants $m_2, m_3$ is zero, then this issue is avoided entirely as
$P(3)$ will now contain an extra basis element $\beta_2 \beta_3 \alpha_3$ so that $P(4) \alpha_3 \cong \Delta(4)$ as desired.

With this example in mind, we shall now work towards giving a full description of
when $A(X, M, \rho)$ is quasi-hereditary if $X$ is a tree.

If $X$ is a tree, and $\alpha : x \to y'$ and $\beta : y \to x$ are paths in $Q_X$ and $Q_X^{\text{op}}$ respectively,
then $\text{Tw}(\alpha, \beta)$ is empty if $y$ and $y'$ are incomparable. If $\alpha$ and $\beta$ are arrows, this means
$\text{Tw}(\alpha, \beta)$ can only be non-empty if $y = y'$. Consequently, a labeling on $X$ amounts to the
choice of a constant $m_{\alpha \beta}(\beta', \alpha')$ for each diagram:

$$x \xrightarrow{\alpha} y \xleftarrow{\beta} z$$

This constant only depends on $x$ and $z$, so to simplify notation we shall sometimes write
$m_{xz}$ for $m_{\alpha \beta}(\beta', \alpha')$.

**Definition 2.8.** Let $(X, \leq, \rho)$ be a bound poset where $X$ is a tree and let $M$ be a labeling
on $X$. We say that $M$ and $\rho$ are compatible if for every $x < y$ in $\rho$ and every $y < z$, at
least one of the following is true:

(i) There is some $x < w < y$ such that $w < z$ belongs to $\rho$.

(ii) There is some $x \leq a < b < c \leq z$ such that $m_{ac} = 0$.

Alternatively, compatibility can be expressed as follows. Let $x < y < z$ be as in
the above definition. Let $\alpha$ denote the unique path $x \to y$ in $Q_X$ and let $\beta$ denote the
unique arrow $x' \to x$ in $Q_X^{\text{op}}$ where $x < x' < y$. Then for $(\beta', \alpha') \in \text{Tw}(\alpha, \beta)$, the first
compatibility axiom is equivalent to demanding that $\alpha' \in \langle \rho \rangle$, and the second compatibly
axiom is equivalent to the statement $m_{\alpha \beta}(\beta', \alpha') = 0$.

**Example 2.9.** If $\rho$ is compatible with every labeling on $X$, then we say that $\rho$ is totally
compatible. Examples of totally compatible sets include the following:
(i) Let $X$ be any tree and let $\ell \geq 2$ be any integer. Write $X^\ell$ for the set of paths of length $\ell$ in $Q_X$. Then $\rho = X^\ell$ is totally compatible.

(ii) Let $X$ be any tree and let $\ell \geq 2$ be any integer. Fix $x \in X$ and write $X^\ell(x)$ for the set of all paths of length $\ell$ in $X$ that start at some vertex $y \geq x$. Then $\rho = X^\ell(x)$ is totally compatible. More generally, if $x_1, \ldots, x_n \in X$, then the set $X^\ell(x_1, \ldots, x_n) := X^\ell(x_1) \cup \ldots \cup X^\ell(x_n)$ is totally compatible.

(iii) Let $X = \{1, \ldots, n\}$ with the usual linear ordering and let $a_0, \ldots, a_k, b \in X$ such that $b + k = n$ and $a_i < a_{i+1} < b + i \leq n$ for all $i$. Then $\rho = \{a_i < b + i : 0 \leq i \leq k\}$ is totally compatible.

There are also several interesting examples when $\rho$ is not assumed to be totally compatible.

Example 2.10. If all functions in $M$ are zero, then we shall refer to $M$ as the zero labeling and we denote it by $0$. Then any $\rho$ is compatible with $0$. In this situation, algebra $A(X, 0, \rho)$ is equal to the dual extension algebra $A(I(X, \rho), I(X, \rho)^{op})$. Dual extension algebras of this form are known to be quasi-hereditary, even when $X$ is not a tree. For instance, the quasi-hereditary algebra $A(r, s)$ introduced at the beginning of Section 1.2 is the dual extension algebra $A(\mathbb{k}Q(r), \mathbb{k}Q(s)^{op})$ where $Q(r)$ is the quiver with two vertices and $r$ arrows $1 \rightarrow 2$. For more details, see [9].

A more exotic but illuminating example is the following:

Example 2.11. Let $X$ be the poset given by the following Hasse-diagram:

```
  4 5
 / \ /
3   2
|   |
1
```

Let $\rho$ be the set $\{1 < 3, 2 < 4\}$. Then $\rho$ is compatible with some labeling $M$ if and only if $m_{13} = 0$ or $m_{25} = 0$. This example highlights the intuitive principle that the first compatibility axiom makes $\rho$ want to “propagate upward” through the Hasse-diagram, while the second axiom can be used to halt this propagation.

We shall now see that when $M$ and $\rho$ are compatible, the algebra $A(X, M, \rho)$ is especially nice.

Lemma 2.12. Let $(X, \leq, \rho)$ be a bound poset where $X$ is a tree and let $M$ be a labeling on $X$ that is compatible with $\rho$. Let $\alpha : x \rightarrow y'$ be a path in $Q_X$ and $\beta : y \rightarrow x$ be a path in $Q_X^{op}$. Let $(\beta', \alpha') \in \text{Tw}(\alpha, \beta)$ such that $m_{\alpha\beta}(\beta', \alpha')$ is non-zero. Then the following holds:

(i) If $\alpha \in \langle \rho \rangle$, then $\alpha' \in \langle \rho \rangle$.

(ii) If $\beta \in \langle \rho^{op} \rangle$, then $\beta' \in \langle \rho^{op} \rangle$. 

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Proof. (i) We proceed by induction on the path length of $\beta$. Suppose $\beta$ is an arrow. Since $\alpha \in \langle \rho \rangle$, there is a decomposition $\alpha = \sigma \gamma \tau$ where $\gamma \in \rho$ and $\sigma, \tau$ are paths in $Q_X$. Then:

$$\alpha \beta = \sigma \gamma \tau \beta = \sum_{(\gamma', \tau') \in \Tw(\tau, \beta)} m_{\tau \beta}(\beta', \tau') \sigma \gamma' \tau'$$

For each $m_{\tau \beta}(\beta', \tau') \sigma \gamma' \tau'$, applying the twisting relations to $\gamma' \tau'$ gives a sum with terms:

$$m_{\tau \beta}(\beta', \tau') \cdot m_{\gamma' \beta'}(\gamma'', \gamma') \sigma \beta'' \gamma' \tau' = m_{\tau \beta, \beta}(\beta'', \gamma' \tau') \sigma \beta'' \gamma' \tau'$$

where $(\beta'', \gamma') \in \Tw(\gamma, \beta')$. If the coefficient is non-zero, then $\gamma' \in \langle \rho \rangle$ by compatibility. Consequently, after applying twisting relations once more, we get a sum:

$$\alpha \beta = \sum_{(\beta''', \gamma' \tau') \in \Tw(\alpha, \beta)} m_{\alpha \beta}(\beta''', \gamma' \tau') \beta''' \gamma' \tau'$$

where $\alpha' := \sigma' \gamma' \tau' \in \langle \rho \rangle$ whenever the coefficients are non-zero.

Next, assume (i) holds for all $\beta$ of length less than or equal to some positive integer $k$. Let $\sigma: u \to y$ be an arrow in $Q_X^{op}$. Then:

$$\alpha \beta \sigma = \sum_{(\beta', \alpha') \in \Tw(\alpha, \beta)} m_{\alpha \beta}(\beta', \alpha') \beta' \alpha' \sigma$$

where $\alpha'$ belongs to $\langle \rho \rangle$ whenever $m_{\alpha \beta}(\beta', \alpha')$ is non-zero by the induction hypothesis. For each non-zero term, applying the twisting relations gives:

$$m_{\alpha \beta}(\beta', \alpha') \beta' \alpha' \sigma = \sum_{(\sigma', \alpha'') \in \Tw(\alpha', \sigma)} m_{\alpha \beta}(\beta', \alpha') \cdot m_{\alpha' \sigma}(\sigma', \alpha'') \beta' \alpha''$$

Now, $\alpha'' \in \langle \rho \rangle$ whenever $m_{\alpha \beta}(\beta', \alpha') \cdot m_{\alpha' \sigma}(\sigma', \alpha'') = m_{\alpha \beta, \sigma}(\beta' \sigma', \alpha'')$ is non-zero by the induction hypothesis. This completes the proof of (i) and (ii) is proved analogously. \qed

Lemma 2.13. Let $(X, \leq, \rho)$ be a bound poset where $X$ is a tree and let $M$ be a labeling on $X$ that is compatible with $\rho$. Let $\alpha: y \to z$ be a path in $Q_X$ that does not belong to $\langle \rho \rangle$ and let $\beta: z \to y$ be a path in $Q_X^{op}$ that does not belong to $\langle \rho^{op} \rangle$. Then $\beta \alpha$ is non-zero in $\mathcal{A}(X, M, \rho)$.

Proof. We shall prove the contrapositive, i.e if $\beta \alpha$ is zero, then either $\alpha \in \langle \rho \rangle$ or $\beta \in \langle \rho^{op} \rangle$. The path $\beta \alpha$ can only be zero in $\mathcal{A}(X, M, \rho)$ if there exists some other path $p = \alpha_n \beta_n \alpha_{n-1} \ldots \beta_2 \alpha_1 \beta_1 : x \to y$ where the $\alpha_i$ are paths in $Q_X$ and the $\beta_j$ are paths in $Q_X^{op}$ such that $\beta \alpha$ appears with non-zero coefficient after applying the twisting relations to this path, and at least one of the $\alpha_i, \beta_i$ belong to $\langle \rho \cup \rho^{op} \rangle$. Suppose that either $\alpha_i \in \langle \rho \rangle$ or $\beta_i \in \langle \rho^{op} \rangle$ for some $i$. Then:

$$p = \sum_{(\beta', \alpha') \in \Tw(\alpha, \beta)} m_{\alpha \beta}(\beta', \alpha') \alpha_n \beta_n \ldots \beta_{i+1} \alpha_i \alpha_{i-1} \ldots \alpha_1 \beta_1$$

By Lemma 2.12, either $\alpha_i' \in \langle \rho \rangle$ or $\beta_i' \in \langle \rho^{op} \rangle$ whenever the coefficient $m_{\alpha \beta}(\beta', \alpha')$ is non-zero. If we repeat this style of reasoning on the non-zero terms, then by induction we eventually get that $\beta \alpha$ is contained in $\langle \rho \cup \rho^{op} \rangle$ as desired. \qed
Corollary 2.14. Let \((X, \leq, \rho)\) be a bound poset where \(X\) is a tree and let \(M\) be a labeling on \(X\) that is compatible with \(\rho\). Then the set of paths of the form \(\beta \alpha\) where \(\alpha \notin \langle \rho \rangle\) and \(\beta \notin \langle \rho^{op} \rangle\) is a basis of \(\mathcal{A}(X, M, \rho)\).

Proof. By the twisting relations, it is clear that such paths span the algebra \(\mathcal{A}(X, M, \rho)\), so it remains to check whether they are linearly independent. Suppose the following is true:

\[
a_1 \beta_1 \alpha_1 + \ldots + a_n \beta_n \alpha_n = 0
\]

for some paths \(\beta_i \alpha_i : y \to y'\) in \(Q_X \cup Q_X^{op}\) and some \(a_i \in \mathbb{k}\). If at least one of the \(a_i\) is non-zero then this can only happen if there exists a linear combination:

\[
b_1 p_1 + \ldots + b_m p_m
\]

where \(b_j \in \mathbb{k}\) and each \(p_j \in \langle \rho \cup \rho^{op} \rangle\), such that the linear combination \((*)\) is the result after applying the twisting relations to \((**)\). However, by Lemma 2.13 this implies that each \(\beta_i \alpha_i\) is contained in \(\langle \rho \cup \rho^{op} \rangle\). Thus the paths \(\beta \alpha\) that do not belong to \(\langle \rho \cup \rho^{op} \rangle\) are linearly independent. \(\square\)

Next, we compute the standard modules of \(\mathcal{A}(X, M, \rho)\) with respect to the ordering on \(X\). First, we note that there is an epimorphism of algebras

\[
\pi : \mathcal{A}(X, M, \rho) \to \mathcal{I}(X, \rho)^{op}
\]

given by mapping each arrow \(\alpha \in Q_X\) to zero and each arrow \(\beta \in Q_X^{op}\) to itself.

Proposition 2.15. For any \(x \in X\), there is an isomorphism:

\[
\Delta(x) \cong P_{\mathcal{I}(X, \rho)^{op}}(x)
\]

where \(P_{\mathcal{I}(X, \rho)^{op}}(x)\) is regarded as an \(\mathcal{A}(X, M, \rho)\)-module via \(\pi\).

Proof. By the twisting relations, any path in \(\mathcal{A}(X, M, \rho)\) can be written as a linear combination of paths \(\beta \alpha\) where \(\beta\) is a path in \(Q_X^{op}\) and \(\alpha\) is a path in \(Q_X\). Consequently, the submodule \(\eta P(x)\) is generated by all paths in \(P(x)\) that factor through some arrow \(x \to y\) in \(Q_X\). This implies that the standard module \(\Delta(x) = P(x)/\eta P(x)\) is spanned by all paths \(\beta\) in \(Q_X^{op}\) starting at \(x\) that do not belong to the ideal \(\langle \rho^{op} \rangle\). By the twisting relations, the paths in \(Q_X\) act on this module by zero and the paths in \(Q_X^{op}\) act by left multiplication. But this is precisely the projective module \(P_{\mathcal{I}(X, \rho)^{op}}(x)\) regarded as an \(\mathcal{A}(X, M, \rho)\)-module via the epimorphism \(\pi\), so we get the desired isomorphism. \(\square\)

Remark. If \(X\) is a tree, then the paths in \(Q_X^{op}\) that start at \(x\) and that do not belong to \(\langle \rho^{op} \rangle\) constitute a basis of \(\Delta(x)\). Indeed, if \(a_1 \beta_1 + \ldots + a_n \beta_n = 0\) for some paths \(\beta_i \in Q_X^{op}\), then we may assume that the \(\beta_i\) all start and end at the same vertex. But this means \(\beta_i = \beta_j\) for all \(i, j\) since \(X\) is a tree.

Lemma 2.16. Let \((X, \leq, \rho)\) be a bound poset where \(X\) is a tree and let \(M\) be a labeling on \(X\) that is compatible with \(\rho\). Then for each path \(\alpha : x \to y\) that does not belong to \(\langle \rho \rangle\), there is an isomorphism:

\[
\Delta(y) \cong P(y)\alpha/\eta P(y)\alpha
\]

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Proof. Since $M$ and $\rho$ are compatible, it follows from Corollary 2.14 that the quotient module $P(y)\alpha/\eta P(y)\alpha$ has a basis given by all paths of the form $\beta\alpha$ where $\beta \notin \langle \rho^{\text{op}} \rangle$. Define a morphism
\[ f : \Delta(y) \to P(y)\alpha/\eta P(y)\alpha \]
by $\beta \mapsto \beta\alpha$. This is bijective on bases, hence an isomorphism. 

We are now ready to state and prove the main theorems of this section.

**Theorem 2.17.** Let $(X, \leq, \rho)$ be a bound poset where $X$ is a tree and let $M$ be a labeling on $X$ that is compatible with $\rho$. Then $A(X, M, \rho)$ is quasi-hereditary with respect to $(X, \leq)$.

**Proof.** Let $x \in X$ and let $f$ be an endomorphism on $\Delta(x)$. Then for each basis element $\beta : x \to y$ in $\Delta(x)$ we have $f(\beta) = \beta f(e_x)$, so $f$ is completely determined by its value on $e_x$. Since $f(e_x) = f(e_x e_x) = e_x f(e_x)$, the element $f(e_x)$ can only be a scalar multiple of $e_x$. Thus $\text{End}_{A(X, M, \rho)}(\Delta(x)) \cong k$ as desired.

Next, we shall prove that the projective module $P(x)$ admits a $\Delta$-filtration. First, we note that the submodule $\eta P(x)$ is given by the following sum:
\[ \eta P(x) = \sum_{x < x_1} P(x_1)\alpha_{xx_1} \]
Using this fact, we shall construct a $\Delta$-filtration of $P(x)$. For each $n \geq 0$, define a submodule $P_n$ of $P(x)$ as follows:
\[ P_n = \sum_{x < x_1 < \ldots < x_n} P(x_n)\alpha_{xx_n} \]
This produces a filtration:
\[ P(x) = P_0 \supset P_1 \supset \ldots \supset P_m = 0 \]
of submodules of $P(x)$. Moreover, we have:
\[ \sum_{x < x_1 < \ldots < x_n} \eta P(x_n)\alpha_{xx_n} = \sum_{x < x_1 < \ldots < x_n, x < x_{n+1}} \sum_{x < x_{n+1}} P(x_{n+1})\alpha_{xx_{n+1}} \alpha_{xx_n} \]
\[ = \sum_{x < x_1 < \ldots < x_{n+1}} P(x_{n+1})\alpha_{xx_{n+1}} = P_{n+1} \]
As a consequence of Corollary 2.14, all of the above sums are direct. Then by Lemma 2.16 we get the following chain of isomorphisms:
\[ P_n/P_{n+1} = \bigoplus_{x < x_1 < \ldots < x_n} P(x_n)\alpha_{xx_n} \bigg/ \bigoplus_{x < x_1 < \ldots < x_n} \eta P(x_n)\alpha_{xx_n} \]
\[ = \bigoplus_{x < x_1 < \ldots < x_n} P(x_n)\alpha_{xx_n} / \eta P(x_n)\alpha_{xx_n} \]
\[ \cong \bigoplus_{x < x_1 < \ldots < x_n} \Delta(x_n) \]
so $P(x)$ admits a filtration by standard modules. \qed
Two important classes of examples are captured by the following corollary:

**Corollary 2.18.** Let $(X, \leq, \rho)$ be a bound poset where $X$ is a tree and let $M$ be a labeling on $X$. Then:

(i) If $\rho$ is totally compatible, then $A(X, M, \rho)$ is quasi-hereditary for any $M$.

(ii) If $M = 0$, then $A(X, M, \rho)$ is quasi-hereditary for any $\rho$. Again, this is the dual extension algebra $A(I(X, \rho), I(X, \rho)^{op})$ which is already known to be quasi-hereditary.

We also consider a few concrete examples of quasi-hereditary $A(X, M, \rho)$.

**Example 2.19.** Consider the algebra $A(n, M, n^\ell)$ where $n = \{1, \ldots, n\}$ under the usual ordering. This amounts to the path algebra of the quiver:

\[
1 \overset{\alpha_1}{\leftarrow} 2 \overset{\alpha_2}{\leftarrow} 3 \overset{\alpha_3}{\leftarrow} \cdots \overset{\alpha_{n-2}}{\leftarrow} n-1 \overset{\alpha_{n-1}}{\leftarrow} n
\]

with the relations:

\[
\alpha_i \beta_i = m_{i+1} \beta_{i+1} \alpha_{i+1}, \quad \alpha_n \beta_n = 0 \quad \alpha_i \beta_i \cdots \alpha_0 = 0 \quad \beta_i \cdots \beta_{i+\ell} = 0
\]

where $m_{i+1} = m_n \beta_i (\beta_{i+1}, \alpha_{i+1})$. If $M$ and $M'$ are two labelings on $n$ such that $m_i = 0$ if and only if $m_i' = 0$ for all $i$, then it is not hard, albeit slightly tedious, to construct an isomorphism $A(n, M, n^\ell) \cong A(n, M', n^\ell)$. Consequently, one can assume that each $m_i$ is either zero or one.

The set $n^\ell$ is totally compatible, so $A(n, M, n^\ell)$ is quasi-hereditary for any choice of the labeling $M$. The filtration described in the proof of Theorem 2.17 of each projective module $P(i)$ is of the form:

\[
P(i) \supset P(i+1) \alpha_i \supset P(i+2) \alpha_{i+1} \alpha_i \supset \cdots \supset P(i+k) \alpha_{i+k-1} \cdots \alpha_i \supset 0
\]

where $k = \min(n-i, \ell)$. For each $i \leq j < k$, the factor

\[
P(i+j) \alpha_{i+j-1} \cdots \alpha_i / P(i+j+1) \alpha_{i+j} \cdots \alpha_i
\]

is isomorphic to the standard module $\Delta(i+j)$.

**Example 2.20.** Recall the bound poset given in Example 2.11. For some labeling $M$, the corresponding algebra $A(X, M, \rho)$ is given by the quiver:

\[
1 \overset{\alpha_2}{\leftarrow} 2 \overset{\alpha_3}{\leftarrow} 3 \overset{\alpha_4}{\leftarrow} 4 \overset{\beta_4}{\leftarrow} 5
\]

with the relations:

\[
\alpha_3 \alpha_2, \quad \alpha_4 \alpha_3, \quad \beta_2 \beta_3, \quad \beta_3 \beta_4, \quad \alpha_2 \beta_2 = m_{13} \beta_3 \alpha_3, \quad \alpha_3 \beta_3 = m_{24} \beta_4 \alpha_4 + m_{25} \beta_5 \alpha_5
\]

Recall that $\rho$ is compatible with the labeling $M$ if and only if either $m_{13}$ or $m_{25}$ is zero. If we assume $M$ to be of this form, then $A(X, M, \rho)$ is quasi-hereditary. The filtrations described in Theorem 2.17 take the form:
Example 2.21. (Blocks of Schur algebras of finite or tame representation type) Let \( d \) be a positive integer and let \( V \) be a \( n \)-dimensional vector space over \( \mathbb{k} \). There is an action by the symmetric group \( S_d \) on \( V \otimes^d \) given by:

\[
\sigma \cdot (v_1 \otimes \ldots \otimes v_d) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}
\]

for every \( \sigma \in S_d \). We define the Schur algebra to be the algebra:

\[
S(n,d) = \text{End}_{\mathbb{k}S_d}(V \otimes^d)
\]

A finite-dimensional algebra \( A \) is said to be of finite representation type if there are finitely many finite-dimensional indecomposable \( A \)-modules. \( A \) is said to be of tame representation type if, loosely speaking, the indecomposable modules of each dimension can be classified by a finite number of one-parameter families. A more precise definition can be found in [10].

When \( S(n,d) \) is of finite or tame representation type, certain algebras of the form \( \mathcal{A}(X, M, \rho) \) appear as blocks of \( S(n,d) \).

In [10], Erdmann proved that \( S(n,d) \) is of finite representation type if and only if one of the following is true:

(a) \( \text{char}(\mathbb{k}) \geq 2, \, n \geq 3, \text{ and } d < 2 \cdot \text{char}(\mathbb{k}) \).

(b) \( \text{char}(\mathbb{k}) \geq 2, \, n = 2, \text{ and } d < \text{char}(\mathbb{k})^2 \).

(c) \( \text{char}(\mathbb{k}) = 2, \, n = 2, \text{ and } d = 5, 7 \).

Whenever one of the above is true, each block of \( S(n,d) \) of finite representation type is of the form \( \mathcal{A}(m, 1, m^2) \) where \( 1 \) denotes the labeling in which all twisting constants are 1.

Schur algebras of tame representation type were classified by Doty, Erdmann, Martin and Nakano in [13]. The algebra \( S(n,d) \) is of tame representation type if and only if one of the following is true:

(a) \( \text{char}(\mathbb{k}) = 2, \, n = 2, \text{ and } d = 4, 9 \).

(b) \( \text{char}(\mathbb{k}) = 3, \, n = 3, \text{ and } d = 7 \).

(c) \( \text{char}(\mathbb{k}) = 3, \, n = 3, \text{ and } d = 8 \).

(d) \( \text{char}(\mathbb{k}) = 3, \, n = 2, \text{ and } d = 9, 10, 11 \).
In all cases except the first, the basic algebras corresponding to the non-semisimple blocks of \( S(n,d) \) can all be described as bound twisted double incidence algebras.

(b) The basic algebra corresponding to each non-semisimple block of \( S(n,d) \) is of the form \( A(4,1,\{2 < 4\}) \). In other words, it is the path algebra of the quiver:

\[
\begin{array}{cccc}
1 & \xrightarrow{\alpha_1} & 2 & \xleftarrow{\beta_1} \\
2 & \xrightarrow{\alpha_2} & 3 & \xleftarrow{\beta_2} \\
3 & \xrightarrow{\alpha_3} & 4 & \xleftarrow{\beta_3} \\
\end{array}
\]

with relations \( \alpha_3\alpha_2, \beta_2\beta_3, \alpha_1\beta_1 = \beta_2\alpha_2, \alpha_2\beta_2 = \beta_3\alpha_3, \) and \( \alpha_3\beta_3 \).

For case (c) and (d), let \( D_4 \) be the poset whose underlying set is \( \{1, 2, 3, 4\} \) with the ordering \( 1 < 2, 2 < 3 \) and \( 2 < 4 \) so that \( Q_{D_4} \cup Q_{D_4}^\text{op} \) is the quiver:

\[
\begin{array}{cccc}
1 & \xrightarrow{\alpha_1} & 2 & \xleftarrow{\beta_1} \\
2 & \xrightarrow{\alpha_3} & 3 & \xleftarrow{\beta_3} \\
2 & \xrightarrow{\alpha_4} & 4 & \xleftarrow{\beta_4} \\
\end{array}
\]

(c) The basic algebra corresponding to each non-semisimple block of \( S(n,d) \) is of the form \( A(D_4, 1, \{1 < 3\}) \). In other words, it is the path algebra of the above quiver with the relations \( \alpha_3\alpha_1, \beta_1\beta_3, \alpha_1\beta_1 = \beta_3\alpha_3 + \beta_4\alpha_4, \alpha_3\beta_3, \alpha_3\beta_3, \) and \( \alpha_3\beta_3 \).

(d) The basic algebra corresponding to each non-semisimple block of \( S(n,d) \) is of the form \( A(D_4, M, D_4^2) \) where \( M \) is the labeling given by \( m_{\alpha_1\beta_1}(\beta_3, \alpha_3) = 1 \) and \( m_{\alpha_1\beta_1}(\beta_4, \alpha_4) = 0 \). In other words, it is the path algebra of the above quiver with the relations \( \alpha_3\alpha_1, \alpha_4\alpha_1, \beta_1\beta_3, \beta_1\beta_4, \alpha_3\beta_3, \alpha_4\beta_4, \alpha_1\beta_1 = \beta_3\alpha_3 \).

Observe that in each of the above cases, the compatibility conditions are satisfied.

Next, we prove the converse result to Theorem 2.17.

**Theorem 2.22.** Let \( (X, \leq, \rho) \) be a bound poset where \( X \) is a tree and let \( M \) be a labeling on \( X \). If \( \mathcal{A}(X, M, \rho) \) is quasi-hereditary, then \( \rho \) is compatible with \( M \).

**Proof.** We shall prove the contrapositive, that is if \( \rho \) and \( M \) are not compatible, then \( \mathcal{A}(X, M, \rho) \) is not quasi-hereditary. If \( \rho \) and \( M \) are not compatible, then there exists some \( x < y \) in \( \rho \) and some \( y < z \) such that the following is true:

(i) There is no \( x < w < y \) such that \( w < z \) belongs to \( \rho \).

(ii) The constant \( m_{ac} \) is non-zero for all \( a < b < c \leq z \).

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Let $y < z_1, \ldots, z_n$ be all elements with this property for $x < y$. Now, let $(X, \leq)$ be a linear refinement of $(X, \leq)$ so that $X = \{x_1 < x_2 < \ldots < x_N\}$. Choose this refinement so that $y = x_j$ for some $j$ and $z_i = x_{i+j}$. This is possible since the $z_i$ are incomparable in $(X, \leq)$. Since $A(X, M, \rho)$ is quasi-hereditary, $P(y)$ admits a filtration by standard modules. By Lemma 1.6, this filtration takes the form:

$$P(y) = \eta_0 P(y) \supseteq \eta_1 P(y) \supseteq \ldots \supseteq \eta_N P(y) \supseteq 0$$

where $\eta_k P(y)/\eta_{k+1} P(y) \cong \Delta(x_k)^{\oplus a_k}$ for some $a_k \geq 0$. Recall that the submodule $\eta_k P(y)$ was defined as:

$$\eta_k P(y) = \sum_{f: P(x_k) \to P(y)} \text{Im}(f)$$

For $k < j$, this module is equal to $P(y)$. For $k = j$ we get the submodule:

$$\eta_j P(y) = \eta P(y) = \sum_{y < z} P(z) \alpha_{yz}$$

and for $k = j + 1$, we get:

$$\eta_{j+1} P(y) = \sum_{y < z, z \neq z_1} P(z) \alpha_{yz} + \eta P(z_1) \alpha_{yz_1}$$

Then the quotient by these two modules is given by:

$$\eta_j P(y)/\eta_{j+1} P(y) = P(z_1) \alpha_{yz_1}/ (P(z_1) \alpha_{yz_1} \cap \eta_{j+1} P(y))$$

According to the aforementioned lemma, this quotient is isomorphic to $\Delta(z_1)^{\oplus k}$ for some $k$. It is clear that if this is the case, then the multiplicity $k$ is non-zero. Then there exists a monomorphism:

$$\varphi : \Delta(z_1) \hookrightarrow P(z_1) \alpha_{yz_1}/ (P(z_1) \alpha_{yz_1} \cap \eta_{j+1} P(y))$$

Then $\varphi$ maps the idempotent $e_{z_1}$ to a scalar multiple of the equivalence class of the path $\alpha_{yz_1}$, so any $\beta : z_1 \to w$ in $\Delta(z_1)$ is mapped to a scalar multiple of the equivalence class of $\beta \alpha_{yz_1}$. We claim that this cannot be injective.

Since $x < y$ belongs to $\rho$, let $\beta$ denote the corresponding path in $\rho^{\text{op}}$. Let $\alpha : x \to x'$ denote the arrow in $Q_X$ where $x < x' < y$. Then each $z_i$ corresponds to a twisting pair $(\beta_i, \alpha_i) \in \text{Tw}(\alpha, \beta)$, where $\alpha_i = \alpha_{yz_1}$. Then:

$$0 = \alpha \beta = \sum_{i=1}^n m_{\alpha \beta}(\beta_i, \alpha_i) \beta_i \alpha_i$$

since every other twisting pair of $\alpha$ and $\beta$ either factors through $(\rho \cup \rho^{\text{op}})$ or has zero twisting constant. By assumption, the constants $m_{\alpha \beta}(\beta_i, \alpha_i)$ are non-zero. Now there are two cases to consider. If $n = 1$, then the above implies that:

$$m_{\alpha \beta}(\beta_1, \alpha_1) \beta_1 \alpha_1 = 0$$
But this implies that $\varphi$ maps $\beta_1$ in $\Delta(z_1)$ to zero, so it is not injective.

Otherwise if $n > 1$, we get:

$$m_{\alpha\beta}(\beta_1, \alpha_1)\beta_1\alpha_1 = -\sum_{i=2}^{n} m_{\alpha\beta}(\beta_i, \alpha_i)\beta_i\alpha_i$$

This implies that $\beta_1\alpha_1$ is contained in the submodule $P(z_1)\alpha_{y_{z_1}} \cap \eta_{y_{j+1}}P(y)$, so $\varphi(\beta_1)$ is zero. Thus $\varphi$ can never be injective, so the filtration fails. Thus $\mathcal{A}(X, M, \rho)$ is not quasi-hereditary. \qed

To the author’s knowledge, very little is known about when the algebras $\mathcal{A}(X, M, \rho)$ are quasi-hereditary when $X$ is not a tree, even when $\rho$ is assumed to be empty. Below, we consider some of the smallest possible examples of $\mathcal{A}(X, M, \rho)$ when $X$ is not a tree.

**Example 2.23.** The smallest poset which is not a tree is the poset $X$ given by the Hasse diagram:

```
  4
 / \       /
2   3 /\   /\ 
1
```

and let $\rho$ be the set containing the path $1 \rightarrow 2 \rightarrow 4$. A labeling on $X$ amounts to the choice of constants $m_{22}$, $m_{23}$, $m_{32}$, and $m_{33}$ so that $\mathcal{A}(X, M, \rho)$ is the path algebra of the quiver:

```
  4
 / \       /
2   3 /\   /\ 
1
```

with the following relations:

(i) $\alpha_1\beta_1 = m_{22}\beta_2\alpha_2$  (v) $\alpha_2\alpha_1 = \alpha_4\alpha_3$

(ii) $\alpha_1\beta_3 = m_{23}\beta_2\alpha_4$  (vi) $\beta_1\beta_2 = \beta_3\beta_4$

(iii) $\alpha_3\beta_1 = m_{32}\beta_4\alpha_2$  (vii) $\alpha_2\alpha_1 = 0$

(iv) $\alpha_3\beta_3 = m_{33}\beta_4\alpha_4$  (viii) $\beta_1\beta_2 = 0$

A straightforward computation shows that the standard modules all have endomorphism rings isomorphic to $\mathbb{k}$, and the projective modules have $\Delta$-filtrations:

$$P(1) \supset P(2)\alpha_1 \oplus P(3)\alpha_3 \supset P(3)\alpha_3 \supset 0$$

$$P(2) \supset P(4)\alpha_2 \supset 0 \quad P(3) \supset P(4)\alpha_4 \supset 0 \quad P(4) \supset 0$$

So $\mathcal{A}(X, M, \rho)$ is quasi-hereditary for any matrix labeling.
Example 2.24. Consider the algebra \( \mathcal{A}(X, M, \rho) \) given by the quiver:

![Quiver Diagram](image)

with the relations:

(i) \( \alpha_2 \alpha_1 = \alpha_4 \alpha_3 \)  
(ii) \( \beta_1 \beta_2 = \beta_3 \beta_4 \)  
(iii) \( \alpha_1 \beta_3 = 0 \)  
(iv) \( \alpha_3 \beta_1 = 0 \)  
(v) \( \alpha_1 \beta_1 = \beta_2 \alpha_2 \)  
(vi) \( \alpha_3 \beta_3 = \beta_4 \alpha_4 \)  
(vii) \( \alpha_2 \beta_2 = \beta_5 \alpha_5 \)  
(viii) \( \alpha_4 \beta_4 = \beta_5 \alpha_5 \)  
(ix) \( \alpha_5 \alpha_2 = 0 \)  
(x) \( \beta_2 \beta_5 = 0 \)

Here, \( \rho \) is the set consisting of the path \( 2 \to 4 \to 5 \) and \( M \) is the labeling in which all constants are 1, except for the constants \( m_{\alpha_1 \beta_3}(\beta_2, \alpha_4) \) and \( m_{\alpha_3 \beta_1}(\beta_4, \alpha_2) \) which are both taken to be zero.

We claim that this algebra is not quasi-hereditary. To see why, we claim that the projective module \( P(4) \) does not admit a \( \Delta \)-filtration. Indeed, if \( \mathcal{A}(X, M, \rho) \) is quasi-hereditary, then by Lemma 1.6 the module \( P(4) \) admits a \( \Delta \)-filtration of the form:

\[
P(4) \supset P(5) \alpha_5 \supset 0
\]

with factors \( P(4)/P(5)\alpha_5 \cong \Delta(4) \) and \( P(5)\alpha_5 \cong \Delta(5) \). Using Proposition 2.15, we see that \( P(5) \cong \Delta(5) \) has a basis given by \( e_5, \beta_5, \beta_4 \beta_5 \). However, \( P(5)\alpha_5 \) has a basis given by only \( \alpha_5, \beta_5 \alpha_5 \) since:

\[
\beta_4 \beta_5 \alpha_5 = \beta_4 \alpha_4 \beta_4 = \alpha_3 \beta_3 \beta_4 = \alpha_3 \beta_1 \beta_2 = 0
\]

so \( P(5)\alpha_5 \not\cong \Delta(5) \). Note that the failure in the filtration was not due to the added path \( 2 \to 4 \to 5 \), but rather the interplay between the twisting relations and the incidence relations.

We now turn our focus towards determining whether \( \mathcal{A}(X, M, \rho) \) admits an exact Borel subalgebra. In [7], Deng and Xi proved that the incidence algebra \( \mathcal{I}(X) \) is an exact Borel subalgebra of \( \mathcal{A}(X, M) \) whenever \( \mathcal{A}(X, M) \) is quasi-hereditary, not just when \( X \) is a tree. This result generalizes to the algebras \( \mathcal{A}(X, M, \rho) \):

**Theorem 2.25.** Suppose \( \mathcal{A}(X, M, \rho) \) is quasi-hereditary. Then the bound incidence algebra \( \mathcal{I}(X, \rho) \) is an exact Borel subalgebra of \( \mathcal{A}(X, M, \rho) \).

**Proof.** For simplicity, write \( A = \mathcal{A}(X, M, \rho) \) and \( B = \mathcal{I}(X, \rho) \). We must check each of the conditions of Definition 1.8.
(i) It is clear that the endomorphism rings of the simple $A$-modules and the simple $B$-modules are both isomorphic to $k$.

(ii) With respect to the partial order on $X$, the standard modules of $B$ are all simple so it is clear that $B$ admits a $\Delta$-filtration and the endomorphism rings of the standard modules are all skew fields, in fact they are all isomorphic to $k$. Thus $B$ is quasi-hereditary.

(iii) For exactness of the functor $A \otimes_B -$ we shall prove that $A$ is projective as a right $B$-module. For each $x \in X$, let $P^R_A(x)$ and $\Delta^R_A(x)$ and denote the corresponding indecomposable projective and standard right $A$-modules respectively. We claim that for each $x \in X$, there is an isomorphism of right $B$-modules:

$$\Delta^R_A(x) \mid_B \cong P^R_B(x)$$

Indeed, by an analogous computation as in Proposition 2.15 we see that $\eta P^R_A(x)$ is the submodule of $P^R_A(x)$ comprised of all paths in $A$ ending at $x$ that factor through some $\beta \in Q^\text{op}_X$. As a result, $\Delta^R_A(x)$ is the module comprised of precisely the paths in $B$ that end at $x$ where arrows in $Q_X$ act by right multiplication and arrows in $Q^\text{op}_X$ act by zero. Restricting this module to $B$ gives precisely the projective module $P^R_B(x)$. Now, since $A$ is quasi-hereditary, the right regular module $A_A$ admits a filtration by the right standard modules. But this means that $A_B$ admits a filtration by projective right $B$-modules. In other words, there are submodules:

$$A_B = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n = 0$$

such that $P_i := A_{i-1}/A_i$ is a projective right $B$-module for each $1 \leq i \leq n$. Now each short exact sequence

$$0 \longrightarrow A_i \longrightarrow A_{i-1} \longrightarrow P_i \longrightarrow 0$$

splits since $P_i$ is projective, so we get $A_B \cong \bigoplus_{i=1}^n P_i$. Thus $A_B$ is projective so $A \otimes_B -$ is exact.

(iv) For each $x \in X$, consider the module $A \otimes_B E_B(x)$. By the twisting relations, $A$ is generated by paths $\beta \alpha$ where $\alpha$ is a path in $Q_X$ and $\beta$ is a path in $Q^\text{op}_X$, neither of which belong to $\rho$ or $\rho^\text{op}$. Then $A \otimes_B E_B(x)$ is generated by the paths $\beta \alpha \otimes e_x$. If $\alpha$ has positive path length, then this element is zero, so $A \otimes_B E_B(x)$ is generated by the paths $\beta \otimes e_x$. As a left $A$-module, arrows in $Q_X$ act on $A \otimes_B E_B(x)$ by zero, while arrows in $Q^\text{op}_X$ act by right multiplication. Consequently, the assignments $\beta \otimes e_x \mapsto \beta$ and $\beta \mapsto \beta \otimes e_x$ induce mutually inverse $A$-module morphisms between $A \otimes_B E_B(x)$ and $\Delta_A(x)$, so $A \otimes_B E_B(x) \cong \Delta_A(x)$.

In view of Theorem 2.17 we get the following corollary:

**Corollary 2.26.** If $X$ is a tree and $M$ and $\rho$ are compatible, then $I(X, \rho)$ is an exact Borel subalgebra of $A(X, M, \rho)$.

This corollary together with Theorem 2.17 and Theorem 2.22 yields the conclusion of Theorem A.
2.2 The Ext-Algebra of Simple Modules of $\mathcal{I}(X, \rho)$

For the remainder of this chapter, we will always assume that $X$ is a tree and that $\rho$ is compatible with some fixed matrix labeling $M$. Let $A = \mathcal{A}(X, M, \rho)$ and $B = \mathcal{I}(X, \rho)$ and write $\Delta = \bigoplus_{x \in X} \Delta(x)$ and $\mathbb{E} = \bigoplus_{x \in X} \mathbb{E}_B(x)$. The main goal for this section is the computation of the algebra $\text{Ext}^*_B(\mathbb{E}, \mathbb{E})$, which will be used in the next section to compute the Ext-algebra of the standard modules of $\mathcal{A}(X, M, \rho)$.

To determine projective resolutions of the simple modules, we shall first require some combinatorial notions. For any $x \in X$, we make the following definitions:

Definition 2.27. A sequence $x = x_0 < x_1 < \ldots < x_n = y$ is called a $\rho$-chain of length $n$ from $x$ to $y$ provided that for all $2 \leq k \leq n$, the element $x_k$ is a minimal element for which there exists some $x_{k-2} \leq u_k \leq x_{k-1}$ so that $u_k < x_k$ belongs to $\rho$.

We shall often write $x : x \to y$ to indicate that $x$ is a $\rho$-chain from $x$ to $y$.

Example 2.28. For small $n$, $\rho$-chains are given by the following:

(i) $\rho$-chains of length zero are precisely the elements of $X$.

(ii) $\rho$-chains of length one are precisely the pairs of the form $x < y$.

(iii) $\rho$-chains of length two are precisely the triples $x < y < z$ where $x < z$ belongs to $\rho$.

Proposition 2.29. For any $x, y$ in $X$, there is at most one $\rho$-chain from $x$ to $y$.

Proof. Suppose $x = x_0 < x_1 < \ldots < x_n = y$ and $x = x'_0 < x'_1 < \ldots < x'_m = y$ are two $\rho$-chains from $x$ to $y$. Assume without loss of generality that $n \geq m$. Then $x_i$ and $x'_j$ are both contained in the interval $[x, y]$, so they are comparable for all $i, j$. Then, clearly $x_0 = x'_0$ and $x_1 = x'_1$. If now $x_{k-1} = x'_{k-1}$ and $x_k = x'_k$ for some $0 < k < m$, then $x_{k+1} = x'_{k+1}$ by the minimality condition. Then by induction, we obtain $x_m = x'_m$. But the latter is equal to $y$, so we have $y = x_m < x_{m+1} < \ldots < x_n = y$. Thus $n = m$ and $x_k = x'_k$ for all $0 \leq k \leq n$.

We shall let $T(x, n)$ denote the set of all $y$ in $X$ for which there exists a $\rho$-chain of length $n$ from $x$ to $y$. If $x = x_0 < x_1 < \ldots < x_n = y$ is a $\rho$-chain, then we shall write $d_n(y)$, or simply $d(y)$ for the element $x_{n-1}$. This defines a map $d : T(x, n) \to T(x, n-1)$, which is well-defined by the preceding proposition.

With these notions in hand, we may proceed with the computation of the projective resolutions. For any $x \in X$ and any non-negative integer $n$, consider the module:

$$P^B_B(x) = \bigoplus_{x_n \in T(x, n)} P_B(x_n)$$

If $n$ is greater than one, define a morphism $\partial_n : P^B_B(x) \to P^B_B(x_{n-1})$ by taking each component $P_B(x_n) \to P_B(x_{n-1})$ to be right multiplication by $a_{x_{n-1}x_n}$ if $d(x_n) = x_{n-1}$ and zero otherwise. This leads to the following result:

Proposition 2.30. The sequence

$$\ldots \xrightarrow{\partial_2} P^B_B(x) \xrightarrow{\partial_1} P^B_B(x) \xrightarrow{\pi} E_B(x) \to 0$$

is exact, and therefore a projective resolution of $E_B(x)$.
Proof. The kernel of $\pi_x$ is the radical of $P_B(x)$, that is the submodule generated by all arrows $\alpha_{xy}$ where $x < y$. By definition of $\partial_1$, its image is precisely the submodule of all paths that factor through some $\alpha_{xy}$, hence $\text{Im}(\partial_1) = \text{Ker}(\pi_x)$ as desired.

For $n$ greater than one, for each $x_n \in t(x,n)$ and $x_{n-2} \in T(x, n-2)$, the component $P(x_n) \to P(x_{n-2})$ in the composite $\partial_{n-1} \circ \partial_n$ is given by:

$$\partial_{n-1} \circ \partial_n = \begin{cases} 
\alpha_{x_{n-2}x_n} & \text{if } d^2(x_n) = x_{n-2} \\
0 & \text{otherwise}
\end{cases}$$

In the first case, the path $\alpha_{x_{n-2}x_n}$ belongs to $\langle \rho \rangle$, so it is zero. Thus $\partial_{n-1} \circ \partial_n = 0$.

For exactness, the kernel of $\partial_{n-1}$ is the submodule of $P_B^{n-1}(x)$ generated by all paths $\alpha_{x_{n-1}z}$ for which $\alpha_{x_{n-1}z} \alpha_{d(x_{n-1})x_{n-1}} = 0$, where $x_{n-1} \in T(x, n-1)$. But all such paths must factor through some $\alpha_{x_{n-1}x_n}$ where $x_n \in T(x, n)$ and $d_n(x_n) = x_{n-1}$, which is precisely the image of $\partial_n$.

We shall now compute the Ext-algebra of the simple $B$-modules. For any $x, y$ in $X$, consider the spaces $\text{Hom}_B(P_B^n(x), E_B(y))$. Since $\text{Hom}_B(P_B(x), E_B(y))$ is non-zero only if $x = y$, in which case it is one-dimensional, then $\text{Hom}_B(P_B^n(x), E_B(y))$ is non-zero if and only if $y \in T(x,n)$. Then the complex $\text{Hom}_B(P_B^n(x), E_B(y))$ is non-zero in at most one degree, so we obtain:

$$\dim_k \text{Ext}_B^n(E_B(x), E_B(y)) = \dim_k \text{Hom}_B(P_B^n(x), E_B(y)) = \begin{cases} 1 & \text{if } y \in T(x,n) \\
0 & \text{otherwise}
\end{cases}$$

We can reformulate this result as follows: $\text{Ext}_B^*(E_B(x), E_B(y))$ is non-zero if and only if there exists a $\rho$-chain from $x$ to $y$ of length $n$. In this sense, the space $\text{Ext}_B^*(E_B, E_B)$ can be thought of as having a basis comprised of all $\rho$-chains. This motivates the idea that the Yoneda product on $\text{Ext}_B^*(E_B, E_B)$ can be described using some kind of composition operation on $\rho$-chains.

**Definition 2.31.** Let $x : x \to y$ and $y : y \to z$ be $\rho$-chains of length $n$ and $m$ respectively. We define the following notions:

(i) We say that $x$ and $y$ are *composable* if $x$ can be extended to a $\rho$-chain $z$ from $x$ to $z$ of length $n + m$, or equivalently if $z \in T(x, n + m)$. We shall refer to $z$ as the *composite* of $x$ and $y$ whenever it exists, and we write $z = xy$.

(ii) We say that a $\rho$-chain of positive length is *reducible* if it is equal to the composite of two $\rho$-chains of positive length. Otherwise, we say that it is *irreducible*.

(iii) We say that a reducible $\rho$-chain is *completely reducible* if it can be written as the composite of a series of irreducible $\rho$-chains of length one or two.

**Example 2.32.** Below we list a few examples of the above notions.

(i) Every $\rho$-chain of length one is irreducible.

(ii) A $\rho$-chain $x_0 < x_1 < x_2$ is irreducible if and only if $x_1 \notin x_2$. 

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(iii) Any $\rho$-chain $x : x \to y$ is composable with the length zero-chains $x$ and $y$, where $xx = x$ and $yx = x$.

(iv) Let $X = \{1, \ldots, 6\}$ and let $\rho = \{1 < 3, 3 < 6\}$. Then $1 < 2 < 3$ and $3 < 4 < 6$ are not composable.

(v) Let $X = \{1, \ldots, 6\}$ and let $\rho = \{1 < 4, 3 < 6\}$. Then $1 < 2 < 4 < 6$ is an irreducible $\rho$-chain.

(vi) Let $X = \{1, \ldots, 6\}$ and let $\rho = \{1 < 4, 2 < 6\}$. Then $1 < 2 < 4 < 6$ is a reducible $\rho$-chain. Indeed, it is the composite of the $\rho$-chains $1 < 2$ and $2 < 3 < 6$.

For a $\rho$-chain $x : x \to y$ of length $n$, we shall write $\varepsilon^n_x$, or simply $\varepsilon_x$ for the basis element of $\text{Ext}^n_B(E_B(x), E_B(y))$ corresponding to the projection $P^n_B(x) \to E_B(y)$. A chain map representative of this element is a chain map of the form:

\[
P^*_B(x) : \cdots \xrightarrow{\partial_{n+2}} P^{n+1}_B(x) \xrightarrow{\partial_{n+1}} P^n_B(x) \xrightarrow{\partial_n} \cdots
\]

\[
P^*_B(y)[n] : \cdots \xrightarrow{\partial_2} P^1_B(y) \xrightarrow{\partial_1} P^0_B(y) \xrightarrow{\varepsilon_y} E_B(y)
\]

We shall construct such a chain map explicitly. For each $0 \leq k \leq m$, define $\varepsilon_k$ on each component $P(x) \to P(y)$ by $\alpha_{yx}x_{x+k}$ if $y_{x+k} \leq x_{y+k}$ and zero otherwise, where $x_{n+k} \in T(x, n+k)$ and $y_{n+k} \in T(y, k)$. If $y_{x+k} = x_{y+k}$, then we take this component to be the identity. As $y \in T(x, n)$ by assumption, $y$ is incomparable with all other $x_n \in T(x, n)$. Then the map $\varepsilon_0 : P^m_B(x) \to P^m_B(y)$ is non-zero on every component except the component $P_A(y) \to P_A(y)$, where it is the identity. In other words, $\varepsilon_0$ is the projection map of $P^m_B(x)$ onto the summand $P^m_B(y)$. Then the right triangle in the above diagram commutes. It remains to verify that this defines a chain map.

For each $k$, the composite $\varepsilon_{k-1} \circ \partial_{y+k}$ is given in each component $P_B(x_{n+k}) \to P(y_{k-1})$ by $\alpha_{yx_{x+k}}y_{x+k}$ if $y_{x+k} \leq d(x_{y+k})$ and zero otherwise. Likewise, the composite $\partial_k \circ \varepsilon_k$ is given in each component $P_B(x_{n+k}) \to P(y_{k-1})$ by $\alpha_{yx_{x+k}}y_{x+k}$ if there is some $y_k \in T(y, k)$ so that $d(y_k) = y_{k-1}$ and $y_{x+k} \leq y_{x+k}$, and zero otherwise.

For any $x_{n+k} \in T(x, n+k)$, there are three cases to consider. If there is some $y_k \in T(y, k)$ such that $y_k \leq x_{n+k}$, then the restrictions of $\partial_k \circ \varepsilon_k$ and $\varepsilon_{k-1} \circ \partial_{n+k}$ to the summand $P_B(x_{n+k})$ are both equal. If there is no such $y_k$, then either there is some $y_{k-1} \in T(y, k-1)$ so that $y_{k-1} \leq d(x_{n+k})$ or there is not. In the first case, we must verify that $\alpha_{yx_{x+k}}y_{x+k}$ is zero. Let $d^2(x_{n+k}) \leq u_{n+k} < d(x_{n+k})$ be the element for which $u_{n+k} < x_{n+k}$ belongs to $\rho$. Then $y_{k-1} \leq u_{n+k}$, as otherwise $x_{n+k} \in T(y, k)$, which implies that $\alpha_{yx_{x+k}}y_{x+k}$ is zero as desired. In the second case, $\partial_k \circ \varepsilon_k$ and $\varepsilon_{k-1} \circ \partial_{n+k}$ both restrict to the zero morphism on $P_B(x_{n+k})$. Collecting all of this, we conclude that $\partial_k \circ \varepsilon_k = \varepsilon_{k-1} \circ \partial_{n+k}$ so $\varepsilon_\bullet$ is a chain map.

**Proposition 2.33.** Suppose $x$ and $y$ are $\rho$-chains of length $n$ and $m$ respectively. Then:

\[
\varepsilon^m_y \ast \varepsilon^n_x = \begin{cases} 
\varepsilon^{m+n}_{yx} & \text{if } x \text{ and } y \text{ are composable} \\
0 & \text{otherwise}
\end{cases}
\]
Proof. First, if \( x : x \to y \) and \( y : y \to z \) are not composable, then there is no \( \rho \)-chain from \( x \) to \( z \), hence \( \text{Ext}^n_B(E_B(x), E_B(z)) \) is zero so \( \varepsilon^n_y \circ \varepsilon^n_x = 0 \).

If \( x \) and \( y \) are composable, then \( z \in T(x, n + m) \cap T(y, m) \). Let \( \varepsilon_* \) and \( \varphi_* \) denote the chain map representatives of \( \varepsilon^n_x \) and \( \varepsilon^m_y \) respectively, as constructed above. Consider the following diagram:

\[
\begin{array}{cccccc}
P_B^t(x) & \cdots & P_B^{n+m+1}(x) & P_B^{n+m}(x) & \cdots & \\
\varepsilon_* & \downarrow & \varepsilon_{m+1} & \downarrow & \varepsilon_m & \downarrow & \cdots \\
P_B^t(y)[n] & \cdots & P_B^{m+1}(y) & P_B^m(y) & \cdots & \\
\varphi_*[n] & \downarrow & \varphi_1 & \downarrow & \varphi_0 & \cdots \\
P_B^t(z)[n+m] & \cdots & \partial_2 & \cdots & P_B^0(z) & \varepsilon_x & E_B(z)
\end{array}
\]

Since \( z \in T(x, n + m) \), the morphism \( \varepsilon_m \) restricts to the identity on \( P(z) \). Then \( \varphi_0 \circ \varepsilon_m \) is the projection of \( P_B^{n+m}(x) \) onto \( P(z) \), so \( \varphi_*[n] \circ \varepsilon_* \) is a chain map representative of \( \varepsilon^m_{y^{n+m}} \). Thus \( \varepsilon^m_y \circ \varepsilon^n_x = \varepsilon^m_{y^{m+n}} \) as desired.

We have the following immediate corollaries

**Corollary 2.34.** If \( x \) is an irreducible \( \rho \)-chain of length \( n \), then \( \varepsilon^n_x \) cannot be written as a product of some \( \varepsilon^n_{y_1} \) and \( \varepsilon^n_{y_2} \) where \( a, b \) are positive.

**Corollary 2.35.** If \( \{x_i : x_{i-1} \to x_i\}_{i=1}^n \) is a sequence of \( \rho \)-chains for which the composite \( x_n \cdots x_1 \) does not exist, then \( \varepsilon_{x_n} \ast \cdots \ast \varepsilon_{x_1} \) is zero.

Combining the above results, we conclude that the irreducible \( \rho \)-chains form a minimal generating set of the algebra \( \text{Ext}^*_B(\mathbb{E}, \mathbb{E}) \). In fact, it can be described in terms of quivers and relations as follows. Let \( E(X, \rho) \) denote the quiver whose vertices are the elements of \( X \) and where there is one arrow \( b_x : x \to y \) whenever there exists an irreducible \( \rho \)-chain of positive length from \( x \) to \( y \). There is a natural grading on this quiver by setting \( \text{deg}(b_x) = \text{length}(x) \).

**Theorem 2.36.** Let \( I \) be the ideal of \( \mathbb{k}E(X, \rho) \) generated by the following relations:

(i) Let \( b_{x_n} \cdots b_{x_1} \) and \( b_{y_m} \cdots b_{y_1} \) be two paths in \( E(X, \rho) \) from \( x \) to \( y \) of length at least two. If the composite \( \rho \)-chains \( x_n \cdots x_1 \) and \( y_m \cdots y_1 \) both exist, then we take the relation

\[
b_{x_n} \cdots b_{x_1} - b_{y_m} \cdots b_{y_1}
\]

(ii) If \( b_{x_n} \cdots b_{x_1} : x \to y \) is a path in \( E(X, \rho) \) for which the composite \( x_n \cdots x_1 \) does not exist, then we take the relation \( b_{x_n} \cdots b_{x_1} \).

Then there is an isomorphism of graded algebras \( \text{Ext}^*_B(\mathbb{E}, \mathbb{E}) \cong \mathbb{k}E(X, \rho)/I \).

**Proof.** There is an epimorphism of algebras \( \varphi : \mathbb{k}E(X, \rho) \to \text{Ext}^*_B(\mathbb{E}, \mathbb{E}) \) defined by mapping each arrow \( b_x : x \to y \) to \( \varepsilon_x \). Then the desired isomorphism follows provided
that $\text{Ker}(\varphi) = I$. First, if $b_{x_n} \ldots b_{x_1} : x \to y$ is a path in $E(X, \rho)$ for which there is no $\rho$-chain of the appropriate length, then we know from Corollary 2.35 that

$$\varphi(b_{x_n} \ldots b_{x_1}) = \varepsilon_{x_n} \ast \ldots \ast \varepsilon_{x_1} = 0$$

so the relations (ii) belong to $\text{Ker}(\varphi)$. Next, suppose $b_{x_n} \ldots b_{x_1}$ and $b_{y_m} \ldots b_{y_1}$ are paths from $x$ to $y$ in $E(X, \rho)$ for which both composites $x_n \ldots x_1$ and $y_m \ldots y_1$ exist. Then since there can exist at most one $\rho$-chain between any pair of vertices, it follows that $w := x_n \ldots x_1 = y_m \ldots y_1$. Then by Proposition 2.33 we have:

$$\varphi(b_{x_n} \ldots b_{x_1} - b_{y_m} \ldots b_{y_1}) = \varepsilon_{x_n} \ast \ldots \ast \varepsilon_{x_1} - \varepsilon_{y_m} \ast \ldots \ast \varepsilon_{y_1}$$

$$= \varepsilon_{x_n \ldots x_1} - \varepsilon_{y_m \ldots y_1}$$

$$= \varepsilon_w - \varepsilon_w = 0$$

so $b_{x_n} \ldots b_{x_1} - b_{y_m} \ldots b_{y_1} \in \text{Ker}(\varphi)$. Thus $I \subseteq \text{Ker}(\varphi)$. It remains to prove the converse. Suppose we have a family of paths $p_k := b_{x_{kn}} \ldots b_{x_{k1}} : x_k \to y_k$ in $E(X, \rho)$ for $1 \leq k \leq m$. Assume that $p_k \notin \text{Ker}(\varphi)$ and that there are some non-zero constants $\lambda_i$ so that the linear combination $\sum_{k=1}^{m} \lambda_k p_k \in \text{Ker}(\varphi)$. Then we may assume without loss of generality that $x_k = x_l$ and $y_k = y_l$ for all $1 \leq k, l \leq m$. Moreover, since $p_k \notin \text{Ker}(\varphi)$, this implies that the composite $\rho$-chains $x_{kn} \ldots x_{k1}$ exist and are equal to some $\rho$-chain $w$ for all $k$. But then:

$$0 = \varphi \left( \sum_{k=1}^{m} \lambda_k p_k \right) = \sum_{k=1}^{m} \lambda_k \varphi(p_k) = \sum_{k=1}^{m} \lambda_k \varepsilon_{x_{kn} \ldots x_{k1}} = \sum_{k=1}^{m} \lambda_k \varepsilon_w$$

which implies that $\lambda_1 + \ldots + \lambda_m = 0$. But then we can rewrite the sum $\sum_{k=1}^{m} \lambda_k p_k$ as:

$$\sum_{k=1}^{m} \lambda_k p_k = \sum_{k=2}^{m} -\lambda_k p_1 + \lambda_k p_k$$

which is contained in $I$. \hfill \Box

**Remark.** It was unknown to the author at the time of developing the theory of $\rho$-chains, but it turns out that the above construction is a special case of a more general result due to Green and Zacharia [12, Thm. B] that holds for arbitrary monomial algebras. Later work due to Anick and Green [1] focused on the computation of the spaces $\text{Tor}_n^B(E, E)$ using a more general analogue of $\text{Ext}_A^*(\Delta, \Delta)$.

### 2.3 The Ext-Algebra of Standard Modules of $\mathcal{A}(X, M, \rho)$

The main goal of this section is the computation of the Ext-algebra $\text{Ext}_A^*(\Delta, \Delta)$. As mentioned previously, in the case where the matrix labeling is zero, then $\mathcal{A}(X, 0, \rho)$ is an example of a similar class of algebras called dual extension algebras. Computation of $\text{Ext}_A^*(\Delta, \Delta)$ for dual extension algebras has already been done by Thuresson in [21], and it was shown that there is a nice description of $\text{Ext}_A^*(\Delta, \Delta)$ in terms of $\text{Ext}_B^*(E, E)$. As such, we shall investigate if there is a similar description for arbitrary labelings using the computation of $\text{Ext}_B^*(E, E)$ from the previous section.

First, we have the following result:
Lemma 2.37. For any $x \in X$, there is an isomorphism $A \otimes_B P_B(x) \cong P_A(x)$. 

Proof. The projective module $P_B(x)$ has a basis comprised of all paths $\alpha : x \to y$ in $Q_X$ that do not belong to $\langle \rho \rangle$. Furthermore, by Corollary 2.14, $A$ has a basis given by the paths $\beta \alpha$, where $\alpha$ is a path in $Q_X$ that does not belong to $\langle \rho \rangle$ and $\beta$ is a path in $Q_X$ that does not belong to $\langle \rho \text{op} \rangle$. Then $A \otimes_B P_B(x)$ is spanned by elements of the form:

$$\beta \alpha \otimes \alpha' = \beta \alpha \alpha' \otimes e_x$$

In other words, $A \otimes_B P_B(x)$ is spanned by elements of the form $\beta \alpha \otimes e_x$ where $\beta \alpha \in P_A(x)$. Since the $\beta \alpha$ constitute a basis of $P_A(x)$, it follows that the $\beta \alpha \otimes e_x$ is a basis of $A \otimes_B P_B(x)$. Then the map:

$$\varphi : A \otimes_B P_B(x) \rightarrow P_A(x)$$

given by $\beta \alpha \otimes e_x \mapsto \beta \alpha$ is an isomorphism. \qed

Since $B$ is an exact Borel subalgebra, the complex $A \otimes_B P_B^n(x)$ is a projective resolution of the module $A \otimes_B E_B(x) \cong \Delta(x)$. By Proposition 2.37, each $A \otimes_B P_B^n(x)$ is isomorphic to the projective $A$-module:

$$P_A^n(x) := \bigoplus_{x_n \in T(x,n)} P_A(x_n)$$

Under this isomorphism, the differential $\text{id} \otimes \partial_n^B$ can be expressed analogously to the differential $\partial_n^B$, i.e. it is given in each component by right multiplication by paths in $Q_X$. In other words, we can write the projective resolution of $\Delta(x)$ as follows:

$$\ldots \longrightarrow \partial_2 P_A^1(x) \longrightarrow \partial_1 P_A^0(x) \longrightarrow \pi_x \Delta(x) \longrightarrow 0$$

where $\partial_n$ denotes the map $\text{id} \otimes \partial_n^B$ under the aforementioned isomorphisms.

Lemma 2.38. For each $n$ and each $x, y \in X$, we have the following formula:

$$\dim_k \text{Hom}_A(P_A^n(x), \Delta(y)) = \begin{cases} 
1 & \exists x_n \in T(x,n) \text{ such that } x_n \leq y \text{ and } x_n \leq y \not\in \langle \rho \rangle, \\
0 & \text{otherwise}
\end{cases}$$

Proof. By definition, we have:

$$\text{Hom}_A(P_A^n(x), \Delta(y)) = \bigoplus_{x_n \in T(x,n)} \text{Hom}_A(P_A(x_n), \Delta(y))$$

For each $x_n \in T(x,n)$, any map $f : P_A(x_n) \rightarrow \Delta(y)$ factors as $\pi_y \circ f_0$ where $f_0 : P_A(x_n) \rightarrow P_A(y)$ is right multiplication by a linear combination of paths from $y$ to $x_n$. Since $\pi_y$ annihilates all paths that factor through some arrow in $Q_X$, we may assume that the paths in $f_0$ are paths in $Q_X^{\text{op}}$. But such a path can only exist if $x_n \leq y$, so $f_0$ is right multiplication by a scalar multiple of the path $\beta_{x_n y}$. Thus:

$$\dim_k \text{Hom}_A(P_A(x_n), \Delta(y)) = \begin{cases} 
1 & \text{if } x_n \leq y \not\in \langle \rho \rangle, \\
0 & \text{otherwise}
\end{cases}$$

Next, we claim that there is at most one $x_n \in T(x,n)$ for which $\text{Hom}_A(P_A(x_n), \Delta(y))$ is non-zero. Indeed, suppose there are $x_{n_1}, x_{n_2} \in T(x,n)$ such that $x_{n_1} \leq y$ and $x_{n_2} \leq y$. Then $x_{n_1} = x_{n_2}$ as otherwise there are two distinct paths between $x$ and $y$ in $Q_X$, which cannot happen since $X$ is a tree. Collecting all of this, we obtain the desired formula. \qed
Proposition 2.39. The morphism:
\[ \partial_n^*: \text{Hom}_A(P_A^{-1}(x), \Delta(y)) \to \text{Hom}_A(P_A^n(x), \Delta(y)) \]
is zero.

**Proof.** We shall assume that there exists some \( x_n \in T(x, n) \) such that \( x_n \leq y \) and neither \( x_n \leq y \) nor \( x_n-1 = d(x_n) < y \) belong to \( \langle \rho \rangle \), as otherwise at least one of the above spaces is zero. Then \( \partial_n^* \) restricts to a morphism:

\[ \text{Hom}_A(P_A(x_{n-1}), \Delta(y)) \to \text{Hom}_A(P_A(x_n), \Delta(y)) \]

If \( p_{n-1} \) is the map \( \pi_y \circ \beta_{x_{n-1}y} \), then:

\[ \partial_n^*(p_{n-1}) = p_{n-1} \circ \partial_n = \pi_y \circ \alpha_{x_{n-1}x_n} \beta_{x_{n-1}y} \]

As \( \pi_y \) annihilates all paths that factor through some arrow in \( Q_X \), it follows that the image of \( \alpha_{x_{n-1}x_n} \beta_{x_{n-1}y} \) is contained in the kernel of \( \pi_y \). Thus \( \partial_n^*(p_{n-1}) \) is zero.

An immediate consequence of the above lemma is the following:

**Corollary 2.40.** For all \( x, y \in X \) and every \( n \geq 0 \), we have:

\[ \text{Ext}_A^n(\Delta(x), \Delta(y)) = \text{Hom}_A(P_A^n(x), \Delta(y)) \]

Whenever \( \text{Ext}_A^n(\Delta(x), \Delta(y)) \) is non-zero, write \( \delta_{xy}^n \) for the basis element corresponding to the composite of the projection \( P_A^n(x) \to P_A(x_n) \) and the map \( \pi_y \circ \beta_{xy} : P_A(x_n) \to \Delta(y) \) where \( x_n \in T(x, n) \) is the element for which \( x_n \leq y \) does not belong to \( \langle \rho \rangle \).

Proposition 2.41. There is a monomorphism of algebras:

\[ \text{Ext}_B^*(E, E) \to \text{Ext}_A^*(\Delta, \Delta). \]

**Proof.** Since the functor \( A \otimes_B - \) is exact, there is an induced algebra homomorphism:

\[ \text{Ext}_B^*(E, E) \to \text{Ext}_A^*(A \otimes_B E, A \otimes_B E) \cong \text{Ext}_A^*(\Delta, \Delta). \]

given by the assignment \( \varepsilon_{xy}^n \mapsto \text{id}_A \otimes \varepsilon_{xy}^n \). It suffices to verify that each map:

\[ \text{Ext}_B^*(E_B(x), E_B(y)) \to \text{Ext}_A^*(\Delta(x), \Delta(y)) \]
is injective for any \( x, y \in X \). If \( \text{Ext}_B^*(E_B(x), E_B(y)) \) is zero, then injectivity is clear. Otherwise, recall that \( \text{Ext}_B^*(E_B(x), E_B(y)) \) is non-zero if and only if \( y \in T(x, n) \). When this is true, then \( \text{Ext}_A^*(\Delta(x), \Delta(y)) \) is also non-zero by the formula of Lemma 2.38. Moreover, both spaces are one-dimensional so the above map is injective provided that \( \text{id}_A \otimes \varepsilon_{xy}^n \) is non-zero. Indeed, under the identifications \( A \otimes_B P_B^n(x) \cong P_A^n(x) \) and \( A \otimes_B E_B(x) \cong \Delta(x) \), the map \( \text{id}_A \otimes \varepsilon_{xy}^n : A \otimes_B P_B^n(x) \to A \otimes_B E_B(y) \) corresponds precisely to \( \delta_{xy}^n : P_A^n(x) \to \Delta(y) \) which is non-zero. Thus \( \text{Ext}_B^*(E, E) \to \text{Ext}_A^*(\Delta, \Delta) \) is injective.

**Proposition 2.42.** Let \( x, z \in X \) be such that \( \text{Ext}_A^n(\Delta(x), \Delta(z)) \) is non-zero. Then \( \delta_{xz}^n \) factors as \( \delta_{yz}^n \ast \delta_{xy}^n \) for some \( y \in T(x, n) \).
Proof. Since \( \text{Ext}_{\mathcal{A}}^n(\Delta(x), \Delta(z)) \) is non-zero there exists some \( y \in T(x,n) \) so that \( y \leq z \not\in \langle \rho \rangle \) by Lemma 2.38. Then the spaces \( \text{Ext}_{\mathcal{A}}^n(\Delta(x), \Delta(y)) \) and \( \text{Ext}_{\mathcal{A}}^0(\Delta(y), \Delta(z)) \) are non-zero, so it remains to verify that \( \delta_{yz}^{y} \circ \delta_{xy}^{n} = \delta_{xz}^{n} \).

As \( y \in T(x,n) \), the map \( \delta_{xy}^{n} \) is the composite of the projection map \( \delta_0 : P_{\mathcal{A}}^n(x) \to P_{\mathcal{A}}(y) \) with the map \( \pi_y : P_{\mathcal{A}}(y) \to \Delta(y) \). Then \( \delta_{xy}^{n} \) lifts to a chain map:

\[
\cdots \xrightarrow{\partial_{n+2}} P_{\mathcal{A}}^{n+1}(x) \xrightarrow{\partial_{n+1}} P_{\mathcal{A}}^n(x) \xrightarrow{\partial_n} \cdots \\
\downarrow \delta_1 \downarrow \delta_0 \downarrow \delta_{xy}^n \\
\cdots \xrightarrow{\partial_2} P_{\mathcal{A}}^1(y) \xrightarrow{\partial_1} P_{\mathcal{A}}^0(y) \xrightarrow{\pi_y} \Delta(y)
\]

so \( \delta_{yz}^{y} \circ \delta_{xy}^{n} \) is given by the composite \( \delta_{yz}^{y} \circ \delta_{xy}^{n} \). By definition, \( \delta_{yz}^{y} \circ \delta_{xy}^{n} \) is the composite \( \pi_{yz} \circ \beta_{yz} \circ \delta_0 \), which is precisely the map \( \delta_{xz}^{n} \).

As a consequence of the above, \( \text{Ext}_{\mathcal{A}}^n(\Delta, \Delta) \) is generated by the subalgebras \( \text{Ext}_{\mathcal{B}}^n(\mathbb{E}, \mathbb{E}) \) and \( \text{Ext}_{\mathcal{A}}^0(\Delta, \Delta) \cong \text{Hom}_{\mathcal{A}}(\Delta, \Delta) \). This proves Theorem [B]. In particular, it is easily seen that \( \text{Hom}_{\mathcal{A}}(\Delta, \Delta) \cong B \) via the map \( \delta_{xy}^{n} \to \alpha_{xy} \), so it follows that \( \text{Ext}_{\mathcal{A}}^n(\Delta, \Delta) \) is generated by the irreducible \( \rho \)-chains and the elements \( \delta_{yz}^{y} \) where \( x < y \).

Before we determine the remaining relations in \( \text{Ext}_{\mathcal{A}}^n(\Delta, \Delta) \), we require some auxiliary results about \( \rho \)-chains.

**Lemma 2.43.** Consider two \( \rho \)-chains:

\[
\mathbf{x} = \{x_0 < x_1 < x_2 < \ldots x_n\} \quad \text{and} \quad \mathbf{y} = \{y_0 < y_1 < y_2 < \ldots y_n\}
\]

where \( x_i < y_i \) for all \( i \). Then \( y_{2k} = x_{2k+1} \) for all \( 0 \leq 2k < n \).

**Proof.** We proceed by induction on \( k \).

For \( k = 0 \), we have \( x_0 < x_1 < y_n \) and \( x_0 < y_0 < y_n \) so \( y_0 = x_1 \).

For \( k \geq 0 \), assume that \( y_{2k} = x_{2k+1} \). Then \( y_{2k} < y_{2k+1} \). By definition of \( \rho \)-chains, \( y_{2k+2} \) is a minimal element for which there is some \( y_{2k} < v < y_{2k+1} \) with \( v < y_{2k+2} \in \rho \). As \( y_{2k} < y_{2k+1} \), we have \( v = y_{2k} \) so \( y_{2k} < y_{2k+2} \in \rho \). Next, we claim that \( y_{2k+2} \in T(x, 2k+3) \). We must find some \( x_{2k+1} \leq u < x_{2k+2} \) such that \( u < y_{2k} \in \rho \) and such that \( y_{2k+2} \) is a minimal element with this property. If we set \( u = y_{2k} \) then we already know that \( y_{2k} < y_{2k+2} \) so it remains to check the minimality of \( y_{2k+2} \). If there is some \( x_{2k+1} < u' < x_{2k+2} < u < y_{2k+2} \) such that \( u' < w < \rho \), then \( y_{2k} \leq u' < w < y_{2k+2} \in \rho \), which contradicts the minimality of \( \rho \). Thus \( y_{2k+2} \) is minimal so \( y_{2k+2} \in T(x, 2k+3) \). But then \( y_{2k+2} = x_{2k+3} \) as desired. \( \square \)

**Lemma 2.44.** Let \( \mathbf{x} \) and \( \mathbf{y} \) be as in the previous lemma. Then \( \mathbf{x} \) and \( \mathbf{y} \) are completely reducible.

**Proof.** By Lemma 2.43 we have \( x_{2k} < y_{2k} = x_{2k+1} \), so \( \mathbf{x} \) is of the form:

\[
x_0 < x_1 < x_2 < x_3 < x_4 < x_5 < \ldots < x_n
\]
If \( n \) is even, then \( x \) is the composite \( x_n x_{n-2} \ldots x_4 x_2 \), where each \( x_{2k} \) is the \( \rho \)-chain:

\[
x_{2k} < x_{2k+1} < x_{2k+2}
\]

If \( n \) is odd, then \( x \) is the composite \( x_n x_{n-1} x_{n-3} \ldots x_4 x_2 \) where each \( x_{2k} \) is the \( \rho \)-chain of length two given above, and where \( x_n \) is the \( \rho \)-chain \( x_{n-1} < x_n \). Thus \( x \) is completely reducible.

Similarly, we have \( y_{2k} = x_{2k+1} < y_{2k+1} \), so \( y \) is completely reducible by an analogous argument. \( \square \)

Given \( x < y \), a chain map representative of \( \delta_{xy}^0 \) is given by a commutative diagram of the form:

\[
\ldots \xrightarrow{\partial_3} P^2_4(x) \xrightarrow{\partial_2} P^1_4(x) \xrightarrow{\partial_1} P^0_4(x) \xrightarrow{\pi_x} \Delta(x) \\
\xrightarrow{\gamma_2} \ldots \xrightarrow{\gamma_1} \ldots \xrightarrow{\gamma_0} \Delta(x)
\]

\[
\ldots \xrightarrow{\partial_3} P^2_4(y) \xrightarrow{\partial_2} P^1_4(y) \xrightarrow{\partial_1} P^0_4(y) \xrightarrow{\pi_y} \Delta(y)
\]

To construct this chain map, first let \( \beta \) denote the unique arrow in \( Q_X^{op} \) from \( y \) to \( x \).

We set \( \gamma_0 : P_A(x) \to P_A(y) \) to be right multiplication by \( \beta \).

For \( n \geq 1 \), if \( x_n \in T(x, n) \) let \( \alpha_n \) denote the unique path in \( Q_X \) from \( x \) to \( x_n \). For each \( y_k \in T(y, n) \), define \( \gamma_n : P^n_A(x) \to P^n_A(y) \) on the component \( P_A(x_n) \to P_A(y_n) \) by:

\[
\begin{cases}
  m_{\alpha_n, \beta}(\beta', \alpha'_n) \beta_n & \text{if } (\beta', \alpha'_n) \in \text{Tw}(\alpha_n, \beta') \text{ and } s(\beta') = t(\alpha'_n) = y_n \\
  0 & \text{otherwise}
\end{cases}
\]

**Lemma 2.45.** The collection \( \gamma_\bullet \) is a chain map \( P^*_A(x) \to P^*_A(y) \).

**Proof.** We must prove that \( \gamma_{n-1} \circ \partial_n = \partial_n \circ \gamma_n \) for all \( n \geq 1 \).

If \( n = 1 \), then the composite \( \gamma_1 = \partial_n \circ \gamma_n \) is given on each component \( P_A(x_1) \to P_A(y) \) by \( \alpha_1 \beta \). Applying the twisting relations, we get:

\[
\alpha_1 \beta = \sum_{(\beta', \alpha'_1) \in \text{Tw}(\alpha_1, \beta)} m_{\alpha_1, \beta}(\beta', \alpha'_1) \beta' \alpha'_1
\]

but this is precisely the component \( P_A(x_1) \to P_A(y) \) of \( \partial_1 \circ \gamma_1 \).

Next, let \( n > 1 \) and assume that \( \gamma_{k-1} \circ \partial_k = \partial_k \circ \gamma_k \) for all \( k < n \). Take any \( x_n \in T(x, n) \) and \( y_{n-1} \in T(y, n-1) \) and let

\[
x = x_0 < x_1 < \ldots < x_n \quad \text{and} \quad y = y_0 < y_1 < \ldots < y_{n-1}
\]

be the corresponding \( \rho \)-chains. The composite \( \gamma_{n-1} \circ \partial_n : P^n_A(x) \to P^n_A(y) \) is given on the component \( P_A(x_n) \to P_A(y_{n-1}) \) by:

\[
\begin{cases}
  m_{\alpha_{n-1}, \beta}(\beta', \alpha'_{n-1}) \alpha \beta' & \text{if } (\beta', \alpha'_{n-1}) \in \text{Tw}(\alpha_{n-1}, \beta) \text{ and } s(\beta') = t(\alpha'_{n-1}) = y_{n-1} \\
  0 & \text{otherwise}
\end{cases}
\]

where \( \alpha_{n-1} : x \to x_{n-1} \) and \( \alpha : x_{n-1} \to x_n \) are the unique paths in \( Q_X \) between their respective vertices. By the twisting relations, we get the sum:

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This implies that \( y \) is an irreducible \( \rho \)-chain of length at least three, then:

\[
\delta^n_y \ast \delta^0_{xy} = 0
\]

if there is some irreducible \( \rho \)-chain \( x : x \to y' \) so that \( (\beta', \alpha_y) \in \mathrm{Tw}(\alpha_x, \beta) \). Otherwise, \( \delta^n_y \ast \delta^0_{xy} = 0 \).

(ii) If \( y : y \to z \) is an irreducible \( \rho \)-chain of length at least three, then:

\[
\delta^n_y \ast \delta^0_{xy} = 0
\]
Proof. The Yoneda product $\delta^n_y \ast \delta^0_{xy}$ is given by the composite $\delta^n_y \circ \gamma_n : P^n_A(x) \to \Delta(n)$, where $\gamma_n$ is the chain map representative of $\delta^n_y$ constructed in Lemma 2.45. Recall that $\delta^n_y$ is given by the composite of the projections $\pi_z : P_A(z) \to \Delta(z)$ and the projection map $P^n_A(y) \to P_A(z)$. Then for each $x_n \in T(x, n)$, the composite $\delta^n_y \circ \gamma_n$ is given on each component $P_A(x_n) \to \Delta(z)$ by:

$$
\begin{cases}
m_{\alpha, \beta}(\beta', \alpha'_n) \cdot \pi_z \circ \beta' & \text{if } (\beta', \alpha'_n) \in T_w(\alpha_n, \beta) \text{ and } t(\alpha'_n) = s(\beta') = z \\
0 & \text{otherwise}
\end{cases}
$$

where $\alpha_n : x \to x_n$ is the corresponding path in $Q_X$. Note that if there is some $x_n$ such that the first case is true, then it is unique as otherwise there would exist some incomparable $x_n, x'_n \in T(x, n)$ such that $y < x_n, x'_n < z$ and $x_n$, which cannot happen since $X$ is a tree. If no such $x_n$ exists, then $\delta^n_y \circ \gamma_n$ is zero so $\delta^n_y \ast \delta^0_{xy} = 0$.

Suppose such an $x_n$ exists. If $x = \{x = x_0 < x_1 < \ldots < x_n\}$ is the corresponding $\rho$-chain and $y = \{y = y_0 < y_1 < \ldots < y_n\}$, then $x_i < y_i$ by construction of the chain map $\gamma_n$. Then both $x$ and $y$ are completely reducible by Lemma 2.44 if $n \geq 3$ which contradicts irreducibility of $y$. This gives us the relations (ii).

Otherwise, if $n \leq 2$ and such an $x_n$ exists, then

$$
\delta^n_y \ast \delta^0_{xy} = m_{\alpha_n, \beta}(\beta', \alpha'_n) \cdot \pi_z \circ \beta' = m_{\alpha_n, \beta}(\beta', \alpha'_n) \cdot \delta^n_{xx}
$$

Note that $\alpha'_n = \alpha_y$. By Proposition 2.42, $\delta^n \ast \delta^0_{xx}$ factors as $\delta^n_{x_n} \ast \delta^n_{xx}$. If we let $x : x \to x_n$ be the corresponding $\rho$-chain so that $\alpha_x = \alpha_n$, then we get the relations (i). \(\square\)

We are now ready to present the algebra $\text{Ext}_A^*(\Delta, \Delta)$ in terms of quivers and relations. Let $S(X, \rho)$ denote the quiver $E(X, \rho) \cup Q_X$. There is a canonical grading on this quiver by setting $\deg(\alpha) = 0$ for all arrows $\alpha \in Q_X$ and $\deg(b_x) = \text{length}(x)$ for all arrows $b_x \in E(X, \rho)$. We make the following definitions:

Definition 2.47. Consider arrows $\alpha : x \to y$ in $Q_X$ and $b_y : y \to z$ in $E(X, \rho)$.

(i) A twisting pair of $(b_y, \alpha)$ is a pair $(\alpha', b_x)$ where $\alpha' : y' \to z$ is an arrow in $Q_X$ and $b_x : x \to y'$ is an arrow in $E(X, \rho)$ with $\deg(b_x) = \deg(b_y)$.

(ii) Let $m_{b_y \alpha}(\alpha', b_x)$ denote the twisting constant $m_{\alpha \beta}(\beta', \alpha_y)$. Here $\beta : y \to x$ and $\beta' : z \to y'$ denote the arrows in $Q_X$ corresponding to $\alpha$ and $\alpha'$ respectively.

Note that such a twisting pair can only exist if $\deg(b_y) \leq 2$ by Proposition 2.46, and if it does exist then it is unique. With this we can finally state and prove the main theorem of this section:

Theorem 2.48. There is an isomorphism of graded algebras $\text{Ext}_A^*(\Delta, \Delta) \cong kS(X, \rho)/R_M$ where $R_M$ is the ideal generated by the following relations:

(i) All the relations in $\rho$.

(ii) The relations belonging to the ideal $I$ of $kE(X, \rho)$ defined in Theorem 2.36.

(iii) For every arrow $\alpha : x \to y$ in $Q_X$ and every arrow $b_y : y \to z$ in $E(X, \rho)$ of degree $n > 0$, take the relations:
(a) If \( n \) is at most two, take the relation:

\[
b_y \alpha - \tilde{m}_{b_y \alpha}(\alpha', b_x)\alpha'b_x
\]

if there is a twisting pair \((\alpha', b_x)\) of \((b_y, \alpha)\). Otherwise take the relation \(b_y \alpha\).

(b) If \( n \geq 3 \), take the relation \(b_y \alpha\).

**Proof.** Clearly, we have an epimorphism of algebras \( \varphi : \mathbb{k}S(X, \rho) \to \text{Ext}^*_A(\Delta, \Delta) \) defined by mapping each \( \alpha_{xy} \) to \( \delta^0_{xy} \) and each \( b_x \) to \( \delta^0_{x} \). Then the desired isomorphism holds provided that \( \text{Ker}(\varphi) = R_M \). The restriction of \( \varphi \) to the subalgebra \( \mathbb{k}E(X, \rho) \) is the epimorphism onto the subalgebra \( \text{Ext}_B(\mathbb{E}, \mathbb{E}) \) that was described in Theorem 2.36. It was shown that the kernel of this morphism is precisely the ideal \( I \), so \( I \subseteq R_M \). Next, the restriction of \( \varphi \) to \( \mathbb{k}Q_X \) is an epimorphism onto the subalgebra \( B \), and the kernel of this restriction is precisely the paths in \( Q_X \) that belong to \( \rho \) so \( \langle \rho \rangle \subseteq \text{Ker}(\varphi) \). Next, take any path \( b_y \alpha \) in \( S(X, \rho) \), where \( \alpha : x \to y \) is an arrow in \( Q_X \) and \( b_y : y \to z \) is an arrow in \( E(X, \rho) \). Then by Proposition 2.46 we have:

(a) If \( n \) is equal to one or two, then

\[
\varphi(b_y \alpha) = \delta^0_y \delta^0_{xy} = m_{\alpha_{xy}}(\beta', \alpha_y)\delta^0_{y'z} \delta^0_y = \tilde{m}_{b_y \alpha}(\alpha', b_x)\varphi(\alpha'b_x)
\]

if there is a twisting pair \((\alpha', b_x)\) of \((b_y, \alpha)\). Otherwise \( \varphi(b_y a_{xy}) = \delta^0_y \delta^0_{xy} = 0 \).

(b) If \( n \geq 3 \), then \( \varphi(b_y \alpha) = \delta^0_y \delta^0_{xy} = 0 \).

Collecting all of this, we conclude that \( R_M \subseteq \text{Ker}(\varphi) \). To prove the converse take any \( p \in \text{Ker}(\varphi) \). Such an element is a linear combination of the form:

\[
p = \sum_{k=1}^{n} \lambda_k b^{i_{kmk}} \alpha^{i_{km_k-1}} \cdots b^{i_{k2}} \alpha^{i_{k1}}
\]

where \( i_{kl} \geq 0 \) for all \( k, l \). Here we have omitted the lower indices for the sake of simplicity. We may further assume that none of paths \( p_k = b^{i_{kmk}} \alpha^{i_{km_k-1}} \cdots b^{i_{k2}} \alpha^{i_{k1}} \) is contained in \( R_M \) and that every summand starts at some \( x \) and ends at some \( y \). Applying the formul\ae of Proposition 2.46 to \( \varphi(p) \), we obtain:

\[
\varphi(p) = \sum_{k=1}^{n} \lambda_k M_k \varphi(\alpha^{s_k})\varphi(b^t)
\]

where \( M_k \) is the product of all relevant twisting constants and \( s_k, t_k \) are the sums of the respective exponents. By assumption, \( M_k \) is non-zero. We may assume without loss of generality that the homological degrees of all \( \varphi(\alpha^{s_k}b^t) \) all coincide. Then \( \varphi(\alpha^{s_k}b^t) \) are all equal to some \( \delta^0_{xy} \) for all \( k \), so \( \varphi(p) = 0 \) implies that the sum \( \sum_{k=1}^{n} \lambda_k M_k \) is zero. Using this, we see that:

\[
\lambda_1 = -\sum_{k=2}^{n} \lambda_k M_k M_1^{-1}
\]

This in turn implies that:

\[
p = \sum_{k=2}^{n} \lambda_k p_k - \lambda_k M_k M_1^{-1} p_1
\]

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We claim that \( p_k - M_kM_1^{-1}p_1 \in R_M \) for all \( k \). We already know that \( p_k - M_k\alpha^{s_k}b^{s_k} \) and \( p_1 - M_1\alpha^{s_1}b^{s_1} \) both belong to \( R_M \) since they are obtained from repeated application of the relations (iii). Then:

\[
 p_k - M_kM_1^{-1}p_1 - M_k\alpha^{s_k}b^{s_k} + M_k\alpha^{s_1}b^{s_1} \in R_M
\]

Since the paths \( \alpha^{s_k}b^{s_k} \) and \( \alpha^{s_1}b^{s_1} \) both map to the same element \( \delta_{xy}^{N} \), it is easily seen that \( s := s_1 = s_k \) so \( \alpha^{s_1} = \alpha^{s_k} \). Moreover, \( b^{s_1} \) and \( b^{s_k} \) both correspond to composing two sequences of \( \rho \)-chains between the same pair of vertices. Then \( b^{s_1} - b^{s_k} \) belongs to \( R_M \). Using this fact, we can subtract \( M_k\alpha^{s}(b^{s_1} - b^{s_k}) \) from the element above, which proves that \( p_k - M_kM_1^{-1}p_1 \in R_M \). Thus \( p \in R_M \) so \( \text{Ker}(\varphi) \subseteq R_M \). This completes the proof. \( \square \)

We finish this section by listing a few examples of this construction. In each of the examples below we denote the arrows corresponding to \( Q \) in \( S(X, \rho) \) by \( a \) instead of \( \alpha \). Arrows in \( S(X, \rho) \) corresponding to irreducible \( \rho \)-chains will be denoted by \( b, c, d, \ldots \) depending on length. We begin by computing the Ext-algebra of standard modules for the twisted double incidence algebras originally considered by Deng and Xi:

**Example 2.49.** Let \( X \) be any tree and assume that \( \rho = \emptyset \). Then \( A(X, M, \emptyset) = A(X, M) \), so it is quasi-hereditary for any labeling \( M \). Then the only possible \( \rho \)-chains are the chains of length one, i.e. chains of the form \( x < y \). Moreover, any two such chains are non-composable as there are no \( \rho \)-chains of length two. Consequently \( E(X, \emptyset) \) is just the quiver \( Q_X \), so \( \text{Ext}^{*}_{A(X,M,\emptyset)}(\Delta, \Delta) \) is the path algebra of the quiver \( S(X, \emptyset) = Q_X \cup Q_X \) with relations:

(i) \( b_{yz}b_{xy} \) for all arrows \( b_{xy} : x \to y \) and \( b_{yz} : y \to z \) in the first copy of \( Q_X \).

(ii) \( b_{yz}a_{xy} = m_{xz}a_{yz}b_{xy} \) for all arrows \( a_{xy} : x \to y \) in the second copy of \( Q_X \) and all arrows \( b_{yz} : y \to z \) in the first copy of \( Q_X \).

**Example 2.50.** Recall the quasi-hereditary algebra \( A(n, M, \mu^\ell) \) from Example 2.19 where \( \ell \geq 2 \) and \( M \) is any labeling. There are two special cases to consider.

If \( \ell = 2 \), then the irreducible \( n^2 \)-chains are precisely the chains of length 1, i.e the chains \( i < i + 1 \), so \( \text{Ext}^{*}_{A(n, M, \mu^2)}(\Delta, \Delta) \) is given by the path algebra of the quiver:

\[
1 \overset{a_1}{\longrightarrow} 2 \overset{a_2}{\longrightarrow} 3 \overset{a_3}{\longrightarrow} \ldots \overset{a_{n-2}}{\longrightarrow} n-1 \overset{a_{n-1}}{\longrightarrow} n
\]

with the relations:

(i) \( a_{i+1}a_i \)

(ii) \( b_{i+1}a_i = m_{i+1}a_{i+1}b_i \)

Recall from Example 2.21 that the representation finite blocks of Schur algebras are of the form \( A(n, 1, n^2) \). The computation of \( \text{Ext}^{*}(\Delta, \Delta) \) has already been done for this algebra by Klampt and Stroppel in [14].

If \( \ell \geq 3 \), the irreducible \( n^\ell \)-chains are the chains \( i < i+1 \) and the chains \( i < i+1 < i+\ell \), so \( \text{Ext}^{*}_{A(n, M, \mu^\ell)}(\Delta, \Delta) \) is given by the path algebra of the quiver:

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with the relations:

(i) $a_{i+\ell} \ldots a_i$

(ii) $b_{i+1}b_i$

(iii) $c_{i+1}b_i = b_{i+\ell}c_i$

When $M$ is the zero labeling, the above construction is precisely the Ext-algebra computed by Thuresson in [21] for the dual extension algebra $A(I(n, n\ell), I(n, n\ell))$

**Example 2.51.** Recall the quasi-hereditary algebra $A(X, M, \rho)$ from Example 2.20 given by the quiver:

```
  1 ---- a_2 ---- 2 ---- a_3 ---- 3 ---- b_3 ---- 4 ---- a_4 ---- 5
    \       \       \       \       \       \       \       \       \\
     \       \       \       \       \       \       \       \       \\
    b_2 \downarrow \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \beta_5

\text{with the relations:}

(i) $\alpha_3 \alpha_2$

(ii) $\alpha_4 \alpha_3$

(iii) $\beta_2 \beta_3$

(iv) $\beta_3 \beta_1$

(v) $\alpha_2 \beta_2 = m_{13} \beta_3 \alpha_3$

(vi) $\alpha_3 \beta_3 = m_{24} \beta_4 \alpha_4$

The irreducible chains are precisely the chains of length one. Then $\text{Ext}^*_{A(X, M, \rho)}(\Delta, \Delta)$ is given by the quiver:

```
  1 ---- a_2 ---- 2 ---- a_3 ---- 3 ---- b_3 ---- 4 ---- b_4 ---- 5 ---- b_5
```

with the relations:

(i) $a_3 a_2$

(ii) $a_4 a_3$

(iii) $b_5 b_3$
(iv) \( b_3a_2 = m_{13}a_3b_2 \)  
(v) \( b_4a_3 = m_{24}a_4b_3 \)

**Example 2.52.** Recall the blocks of Schur algebras of tame representation type introduced in Example 2.21. The Ext-algebras of their standard modules have already been computed by Kühlshammer in [15] using the theory of bocses, but they can also be computed using Theorem 2.48:

(b) For the block \( \mathcal{A}(4, 1, \{2 < 4\}) \), the corresponding Ext-algebra is given by the quiver:

\[
\begin{array}{cccc}
1 & \overset{a_1}{\rightarrow} & 2 & \overset{a_2}{\rightarrow} & 3 & \overset{a_3}{\rightarrow} & 4 \\
\underset{b_1}{\leftarrow} & \underset{b_2}{\leftarrow} & \underset{b_3}{\leftarrow}
\end{array}
\]

with the relations:

(i) \( a_3a_2 \)  
(ii) \( b_2b_1 \)  
(iii) \( b_2a_1 = a_2b_1 \)  
(iv) \( b_3a_2 = a_3b_2 \)

(c) For the block \( \mathcal{A}(D_4, 1, \{1 < 3\}) \), the corresponding Ext-algebra is given by the quiver:

\[
\begin{array}{cccc}
1 & \overset{a_1}{\rightarrow} & 2 & \overset{a_3}{\rightarrow} & 3 \\
\underset{b_1}{\leftarrow} & \underset{b_3}{\leftarrow} & \underset{b_4}{\leftarrow}
\end{array}
\]

with the relations:

(i) \( a_3a_1 \)  
(ii) \( b_4b_1 \)  
(iii) \( b_3a_1 = a_3b_1 \)  
(iv) \( b_4a_1 = a_4b_1 \)

(d) For the block \( \mathcal{A}(D_4, M, D_4^2) \), the corresponding Ext-algebra is given by the same quiver as in (c), with the relations:

(i) \( a_3a_1 \)  
(ii) \( a_4a_1 \)  
(iii) \( b_3a_1 = a_3b_1 \)  
(iv) \( b_4a_1 \)

### 2.4 Graded Twisted Double Incidence Algebras

Upon closer inspection, the algebra computed in Theorem 2.48 is strikingly similar to the twisted double incidence algebra we began with. This suggests that we can extend the definition of twisted double incidence algebra to include this algebra, which will be the main goal for the remainder of this thesis. There are two key ways we will approach this:

- Replace posets by “poset-like” graded quivers.
Allowing arbitrary quivers of the form $Q \cup Q'$ instead of only $Q \cup Q^{\text{op}}$.

A graded quiver $Q$ is said to be \textit{thin} if for each $n$ there exists at most one arrow between any pair of vertices of degree $n$. If in addition, $Q$ has no oriented cycles then we say that $Q$ is \textit{poset-like}. Examples of such quivers include the quiver $Q_X$ with the zero grading, and the quiver $E(X, \rho)$ graded by length of $\rho$-chains.

With these notions, we can extend the definitions of twisting pairs and labelings of posets to poset-like graded quivers.

**Definition 2.53.** Let $Q$ and $Q'$ be two poset-like graded quivers with the same vertex sets.

(i) A \textit{twistable pair} of arrows in $Q$ and $Q'$ is a pair $(\alpha, \beta)$ where $\alpha : x \to y'$ is an arrow in $Q$ and $\beta : y \to x$ is an arrow in $Q'$.

(ii) A \textit{twisting pair} of $\alpha$ and $\beta$ is a pair $(\beta', \alpha')$ where $\alpha' : y \to z$ is an arrow in $Q$ and $\beta' : z \to y'$ is an arrow in $Q'$ such that $\deg(\alpha) = \deg(\alpha')$ and $\deg(\beta) = \deg(\beta')$. We shall let $\text{Tw}(\alpha, \beta)$ denote the set of all twisting pairs of $(\alpha, \beta)$.

The key difference between the above and the definitions at the beginning of this chapter is the fact that we are required to keep track of the grading. In the event that $Q = Q_X$ and $Q' = Q_X^{\text{op}}$ with the zero gradings, this reduces to the old definition. Moreover, the twisting pairs defined in Definition 2.47 for the quivers $E(X, \rho)$ and $Q_X$ coincide with the above definition.

**Definition 2.54.** Let $Q$ and $Q'$ be two poset-like graded quivers with the same vertex sets. A \textit{labeling} on the pair $(Q, Q')$ is a set $M$ comprised of a function:

$$m_{\alpha \beta} : \text{Tw}(\alpha, \beta) \to k$$

for each twistable pair of arrows in $Q$ and $Q'$.

We are now ready to define the incidence algebra and twisted double incidence algebra of poset-like graded quivers.

**Definition 2.55.** Let $Q$ be a poset-like graded quiver.

(i) Define the \textit{incidence algebra} $\mathcal{I}(Q)$ to be the quotient $kQ/I$ where $I$ is the ideal generated by the homogenous relations $p - q$ where $p, q : x \to y$ are paths in $Q$ of length at least two, such that $\deg(p) = \deg(q)$.

(ii) Any ideal of $\mathcal{I}(Q)$ contained in $\text{rad} \mathcal{I}(Q)^2$ is generated by a finite set $\rho$ of paths in $Q$.

We shall refer to the quotient $\mathcal{I}(Q, \rho) := \mathcal{I}(Q)/\langle \rho \rangle$ as the \textit{bound incidence algebra} of the pair $(Q, \rho)$.

In this setting, the algebra $B = \mathcal{I}(X, \rho)$ is simply $\mathcal{I}(Q_X, \rho)$. Moreover, the algebra $\text{Ext}^*_B(E, E)$ is of the form $\mathcal{I}(E(X, \rho), \bar{\rho})$ where $\bar{\rho}$ is the set of those paths in $E(X, \rho)$ corresponding to sequences of non-composable $\rho$-chains.

**Definition 2.56.** Let $B = \mathcal{I}(Q, \rho)$ and $A = \mathcal{I}(Q', \rho')$ be two bound incidence algebras where the vertex sets of $Q$ and $Q'$ coincide. Let $M$ be a labeling on $(Q, Q')$. Define the \textit{bound twisted double incidence algebra} $\mathcal{A}(B, A, M)$ to be the quotient $k(Q \cup Q')/R$ where $R$ is the ideal generated by the relations:
(i) All the defining relations of $B$ and $A$.

(ii) For each twistable pair $(\alpha, \beta)$ of $Q$ and $Q'$, take the relation:

$$\alpha \beta = \sum_{(\beta', \alpha') \in \text{Tw}(\alpha, \beta)} m_{\alpha \beta}(\beta', \alpha') \beta' \alpha'$$

Note that all the relations are homogenous, so $\mathcal{A}(B, A, M)$ is a graded algebra.

Unlike the algebras $\mathcal{A}(X, M, \rho)$, the notion of symmetric labeling does not make sense for $\mathcal{A}(B, A, M)$ unless $A = B^{\text{op}}$. In other words, the algebras $\mathcal{A}(B, A, M)$ rarely admit anti-involutions that fixes the vertices.

In this setting, the algebra $\mathcal{A}(X, M, \rho)$ is given by $\mathcal{A}(\mathcal{I}(X, \rho), \mathcal{I}(X, \rho)^{\text{op}}, M)$.

Turning back to the Ext-algebra of standard modules of $\mathcal{A}(X, M, \rho)$, we note that Definition 2.47 defines precisely the data of twisting pairs in $E(X, \rho)$ and $Q_X$ and a labeling $M$ on $(E(X, \rho), Q_X)$. Consequently, the following result is immediate from Theorem 2.48:

**Theorem 2.57.** Let $A = \mathcal{A}(X, M, \rho)$ and $B = \mathcal{I}(X, \rho)$ where $X$ is a tree and $M$ and $\rho$ are compatible. Then there is an isomorphism of graded algebras:

$$\text{Ext}^*_A(\Delta, \Delta) \cong \mathcal{A}(\text{Ext}^*_B(E, E), B, \tilde{M})$$

This formula is a direct analogue to the formula computed by Thuresson in [21] for dual extension algebras.

**References**


