

*Digital Comprehensive Summaries of Uppsala Dissertations  
from the Faculty of Science and Technology 2378*

# Color-Kinematics Duality and Gravitational Waves

MAOR BEN-SHAHAR



ACTA UNIVERSITATIS  
UPSALIENSIS  
2024

ISSN 1651-6214  
ISBN 978-91-513-2078-6  
urn:nbn:se:uu:diva-525574



UPPSALA  
UNIVERSITET

Dissertation presented at Uppsala University to be publicly examined in Högssalen, Ångströmlaboratoriet, Lägerhyddsvägen 1, Uppsala, Thursday, 16 May 2024 at 13:00 for the degree of Doctor of Philosophy. The examination will be conducted in English. Faculty examiner: Professor Jan Plefka.

### **Abstract**

Ben-Shahar, M. 2024. Color-Kinematics Duality and Gravitational Waves. *Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology* 2378. 68 pp. Uppsala: Acta Universitatis Upsaliensis. ISBN 978-91-513-2078-6.

Recent developments in theoretical physics have led to new insights for gauge theory and gravity scattering amplitudes. The color-kinematics duality, in particular, describes an intriguing set of identities obeyed by the kinematic numerators of gauge-theory scattering amplitudes, mirroring the Jacobi identity of the color factors. The kinematic Jacobi identities suggest the existence of some unknown kinematic algebra underlying the gauge-theory Feynman rules. However, as of yet, there is no complete Lagrangian construction of duality-satisfying numerators, nor an off-shell realization of a kinematic algebra even for pure Yang-Mills gauge theory. This thesis presents substantial progress on these open problems, first through a Lagrangian whose Feynman rules compute duality-satisfying numerators in the NMHV sector of Yang-Mills theory. In addition, Chern-Simons gauge theory is shown to obey the color-kinematics duality completely off shell, giving rise to a kinematic algebra of volume preserving diffeomorphisms. Similar structures are also identified in the pure-spinor description of super Yang-Mills theory.

The recent detection of gravitational waves by the LIGO/Virgo/KAGRA collaboration, as well as anticipated improvement in sensitivity of future detectors, call for improved precision of the theoretical predictions for binary merger events. Analytical computations involving gravitating and rotating compact objects require both increased classical loop orders in the gravitational coupling as well as the incorporation of spin effects, which have important contributions to the dynamics. For this purpose, an extension of the worldline quantum field theory is presented, based on the effective worldline action of a classical spinning compact object. The formalism is used to compute tree and one-loop amplitudes up to fourth order in spin, and coefficients in the effective worldline action are fixed such that it reproduces known Kerr observables from black hole perturbation theory.

*Keywords:* Quantum field theory, gravitational waves, scattering amplitudes

*Maor Ben-Shahar, Department of Physics and Astronomy, Theoretical Physics, Box 516, Uppsala University, SE-751 20 Uppsala, Sweden.*

© Maor Ben-Shahar 2024

ISSN 1651-6214

ISBN 978-91-513-2078-6

URN urn:nbn:se:uu:diva-525574 (<http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-525574>)

*To all my friends.*



# List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

- I M. Ben-Shahar and M. Guillen, *10D super-Yang-Mills scattering amplitudes from its pure spinor action*, JHEP 12 (2021) 014
- II M. Ben-Shahar and H. Johansson *Off-shell color-kinematics duality for Chern-Simons*, JHEP 08 (2022) 035
- III M. Ben-Shahar, L. Garozzo, and H. Johansson, *Lagrangians manifesting color-kinematics duality in the NMHV sector of Yang-Mills*, JHEP 08 (2023) 222
- IV M. Ben-Shahar, *Scattering of spinning compact objects from a worldline EFT*, JHEP 03 (2024) 108.

The following papers were not included in this thesis.

- V M. Ben-Shahar and M. Chiodaroli, *One-loop amplitudes for  $\mathcal{N} = 2$  homogeneous supergravities*, JHEP 03 (2019) 153
- VI M. Ben-Shahar and M. Guillen, *Superspace expansions of the 11D linearized superfields in the pure spinor formalism, and the covariant vertex operator*, JHEP 09 (2023) 018

Reprints were made with permission from the publishers.



# Contents

1	Introduction .....	9
1.1	Gauge Theory and Gauge Fixing .....	11
1.2	Color-Kinematics Duality .....	14
1.3	Gravitational Waves and WQFT .....	17
1.4	From Worldlines to Fields .....	22
2	Actions for Color-Kinematics Duality .....	24
2.1	NMHV Lagrangian for Yang-Mills Theory .....	24
2.2	Chern-Simons Theory .....	30
2.3	Pure Spinor Super Yang-Mills Theory .....	37
2.4	Discussion .....	40
3	Spinning Compact Object .....	42
3.1	The Spinning Particle EFT .....	42
3.2	WQFT for the Spinning Particle .....	47
3.3	Fields for the Spinning Particle .....	51
3.4	Future research directions .....	56
	Appendix A: Yang-Mills as a Generalized Chern-Simons .....	62
	References .....	63





# 1. Introduction

This thesis explores the color-kinematics duality and gravitational waves, employing worldline theories and their quantized fields throughout. The color-kinematics duality and associated double copy [1, 2] are a set of intriguing identities obeyed by the kinematic factors of gauge-theory amplitudes. By now, there is a wide web of theories that have been shown to satisfy the duality, including theories with spontaneous symmetry breaking, Yang-Mills-Einstein and Einstein-Maxwell theories, gauged supergravities, and more [3]. But in general, a Lagrangian-based understanding of the duality itself is still missing. Assuming a theory obeys the color-kinematics duality, it can be used in the double-copy formalism, which allows one to obtain observables for theories that typically include gravitation. The benefit of the double copy is that it does not require the use of Feynman rules for gravity, which are cumbersome for practical computations.

A key focus in this thesis is on understanding Feynman rules and Lagrangians that manifest the color-kinematics duality. In particular, for Chern-Simons theory it is shown that in Lorenz gauge the Feynman rules give rise to a completely off-shell realization of the color-kinematics duality. The consequence of this is that the kinematics of Chern-Simons theory are described by a certain kinematic algebra, mirroring the structure of the gauge-group algebraic factors of the theory. For Chern-Simons coupled to scalar and spinor matter it is shown that the constraints from color-kinematics duality imply the action has to have  $\mathcal{N} = 4$  supersymmetry, and its double copy is  $\mathcal{N} = 8$  Dirac-Born-Infeld theory. It is also shown that supersymmetric Yang-Mills Feynman rules in the pure-spinor formulation can be written in a way that closely resembles those of Chern-Simons theory, falling short of full off-shell color-kinematics duality only due to regularization issues of divergences in pure-spinor superspace. In addition, an action is presented for pure Yang-Mills theory which, in a certain kinematic sector, gives rise to on-shell color-kinematics duality.

Actions that emerge from the quantization of worldline theories will prove to be an important ingredient in the study of the color-kinematics duality off-shell. These actions are automatically in the form of the extended Batalin-Vilkovitsky (BV) actions [4], and have a nice Chern-Simons-like structure allowing for straightforward manipulation of the Feynman rules in terms of differential operators. Such actions are not

restricted only to gauge theories, and in chapter 3 the quantization of the spinning point-particle action is presented using these techniques. The main subject of chapter 3, however, is on the application of the worldline quantum field theory (WQFT) formalism [5] to the computation of classical observables for this spinning particle.

The benefit of the action used for the spinning particle is that it allows to extend the WQFT to arbitrary orders in spin. This extension involves a study of the gauge symmetries of the classical action in curved space, and works due to a gauge choice that eliminates Lagrange multipliers from the action. Removing Lagrange multipliers is particularly useful for WQFT since in this formalism the worldline degrees of freedom are integrated out together with the metric fluctuations, employing Feynman rules that account for both worldline and bulk degrees of freedom. WQFT amplitudes without spin have been shown to admit a double-copy formulation [6], but much work still remains to be done in order to extend this double-copy prescription to include spin. Having an off-shell understanding of the double copy would likely find immediate application to such problems.

The thesis is structured as follows. In the remainder of this chapter we give some introductory material for the different topics which are then dealt with in detail in subsequent chapters. We start with gauge theories and the BV quantization, which will be used when gauge fixing later on. Next, the color-kinematics duality is introduced, setting the notation and conventions for chapter 2. To introduce the WQFT, the Hamiltonian of the non-spinning point particle is studied and used to compute classical Compton amplitudes within the WQFT framework. Finally, the quantization of the same worldline Hamiltonian in flat space is presented, illustrating methods used to construct field-theory actions in later sections.

In chapter 2 we will see three different formulations of Lagrangian-based color-kinematics duality, starting with a certain kinematic sector of Yang-Mills theory. Next, Chern-Simons theory is shown to obey the duality fully off-shell, and matter extensions of it are studied, revealing a connection between color-kinematics duality and supersymmetry. Finally, the pure-spinor formulation of supersymmetric Yang-Mills theory is shown to exhibit similar structures to Chern-Simons theory. In chapter 3, the classical spinning point-particle and its gauge symmetries are studied in curved space. An effective field theory for it is constructed, allowing to describe generic spinning compact objects to any order in the spin. Classical scattering amplitudes for these spinning particles are computed using the WQFT, and finally a quantization of the free particle is also presented.

## 1.1 Gauge Theory and Gauge Fixing

We begin with a short introduction to Yang-Mills theory and the BV quantization, setting notation and conventions for the remainder of the thesis.

The fields in Yang-Mills theory are Lie-algebra valued vector fields,  $A_\mu = A_\mu^a T^a$ , where  $T^a$  are the generators of the gauge group. We will take  $T^a$  to be generators of  $SU(N)$ , and assume they are normalized such that  $\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$ , with structure constants defined by  $[T^a, T^b] = i f^{abc} T^c$ . The Jacobi identity  $[[T^a, T^b], T^c] + \text{cyclic} = 0$  can also be written in terms of the structure constants as  $f^{abx} f^{xcd} + f^{cax} f^{xbd} + f^{bcx} f^{xad} = 0$ . In  $d$  dimensions, the action for Yang-Mills theory is,

$$S = -\frac{1}{2g^2} \int d^d x \text{tr}(F_{\mu\nu} F^{\mu\nu}) , \quad (1.1)$$

with coupling constant  $g$  and field strength,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] . \quad (1.2)$$

In this thesis we will study Yang-Mills theory in the perturbative regime, namely we will take  $g$  to be small.

The Yang-Mills action has a local gauge symmetry, with transformations

$$\delta A_\mu = D_\mu \epsilon = \partial_\mu \epsilon + i[A_\mu, \epsilon] , \quad (1.3)$$

where we introduced the covariant derivative  $D_\mu$  in the adjoint representation. Under this transformation, the field strength transforms by  $\delta F_{\mu\nu} = [F_{\mu\nu}, \epsilon]$ . Using the cyclicity of the trace, it is then immediate that the action is left invariant by this operation. Before we can construct the Feynman rules of the theory, the gauge symmetry must be fixed. We will follow the Batalin-Vilkovitsky gauge fixing procedure [4], since later on in the thesis some of our constructions will lead precisely to BV actions. The presentation here is loosely based on ref. [7].

The first step in the BV quantization is to extend the space of fields by adding the Grassmann-odd ghost  $c$  and antifield  $\check{A}_\mu$ , as well as the Grassmann-even antighost  $\check{c}$ . Note that each field is paired with an antifield of opposite statistics. By following the BV procedure the action is extended to,

$$S_{\text{BV}} = \int d^d x \text{tr} \left( -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \check{A}^\mu D_\mu c + \check{c} c c \right) , \quad (1.4)$$

where we suppress the coupling constant. In all the examples presented later in the thesis, the BV extended action will emerge naturally from worldline quantization. As it stands, the action has both a new local gauge symmetry as well as a global Becchi-Rouet-Stora-Tyutin (BRST)

symmetry. The local gauge symmetry has a nice formulation in the Chern-Simons-type actions we will encounter later in the thesis, see equation (2.25) for Chern-Simons and Appendix A for Yang-Mills, and it will still need to be gauge fixed, we will do this shortly. To see the global symmetry we introduce the “antibracket”, defined by,

$$\{X, Y\} = \int d^d x \left( \frac{\overrightarrow{\delta} X}{\delta \Phi^i} \frac{\overleftarrow{\delta} Y}{\delta \check{\Phi}_i} - \frac{\overrightarrow{\delta} X}{\delta \check{\Phi}_i} \frac{\overleftarrow{\delta} Y}{\delta \Phi^i} \right), \quad (1.5)$$

where  $\Phi^i$  stands for the fields  $A_\mu$  and  $c$ ,  $\check{\Phi}_i$  contains the antifields, and the arrow on top of the functional derivative indicates in which direction it acts. The BV action is engineered such that it has a global BRST symmetry, defined by

$$\delta_B \Phi^i = \{\Phi^i, S_{BV}\}, \quad (1.6)$$

$$\delta_B \check{\Phi}_i = \{\check{\Phi}_i, S_{BV}\}, \quad (1.7)$$

immediately implying that the so-called master equation  $\{S_{BV}, S_{BV}\} = 0$  holds. The explicit form of the symmetry transformations on our fields are

$$\delta_B c = cc, \quad (1.8)$$

$$\delta_B A_\mu = D_\mu c, \quad (1.9)$$

$$\delta_B \check{A}^\mu = D_\nu F^{\mu\nu} - i\{\check{A}^\mu, c\}, \quad (1.10)$$

$$\delta_B \check{c} = [c, \check{c}] - D_\mu \check{A}^\mu, \quad (1.11)$$

where non-bold curly braces denote the anticommutator. One of the important properties of this construction is that the BRST transformation of the physical fields is just the gauge transformation where the gauge parameter has been replaced by the  $c$  ghost. This implies that any gauge invariant object in the original theory is also BRST invariant in the extended BV description.

We now seek to gauge fix in such a way that observables are independent of the gauge choice. We will achieve this by introducing a Grassmann-odd functional of the fields  $\mathbf{F}$ , known as the gauge-fixing fermion, and set  $\check{\Phi}_i = \frac{\overrightarrow{\partial} \mathbf{F}}{\partial \Phi^i}$ . It turns out that the BRST symmetry of the action is precisely the condition necessary for classical observables to be independent of the choice of gauge [4]. To facilitate the gauge fixing we will add a trivial boson-fermion pair of fields,  $B$  and  $\bar{c}$  to the path integral, together with their antifields,  $\check{B}$  and  $\check{\bar{c}}$ . We now modify the action in such a way that we still have a solution to the master equation  $\{S_{BV}, S_{BV}\} = 0$ , finding,

$$S_{BV} = \int d^d x \operatorname{tr} \left( -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \check{A}^\mu D_\mu c + \check{c} c c + B \check{\bar{c}} \right). \quad (1.12)$$

Choosing a gauge-fixing fermion

$$\mathbf{F} = \int d^d x \operatorname{tr} \bar{c} (\partial_\mu A^\mu - \xi B/2) , \quad (1.13)$$

with arbitrary parameter  $\xi$ , we can reproduce the standard  $R_\xi$  family of gauge choices for Yang-Mills theory. Indeed, from the requirement  $\check{\Phi}_i = \frac{\partial \mathbf{F}}{\partial \Phi^i}$  we find,

$$\check{c} = 0 , \quad (1.14)$$

$$\check{\bar{c}} = \partial_\mu A^\mu - \xi B/2 , \quad (1.15)$$

$$\check{A}^\mu = \partial^\mu \bar{c} , \quad (1.16)$$

which when substituted back in to the Yang-Mills action gives,

$$S_{\text{BV}} = \int d^d x \operatorname{tr} \left( -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \partial^\mu \bar{c} D_\mu c + B(\partial_\mu A^\mu - \xi B/2) \right) . \quad (1.17)$$

Upon integrating out the auxiliary field  $B$  we find,

$$S_{\text{BV}} = \int d^d x \operatorname{tr} \left( -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \partial^\mu \bar{c} D_\mu c + \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right) . \quad (1.18)$$

The Feynman rules in Feynman gauge, obtained by setting  $\xi = -1/2$ , are

$$\text{wavy line} = \frac{-i\eta_{\mu\nu}}{k^2} \delta^{ab} , \quad (1.19)$$

$$\text{3-gluon vertex} = g f^{a_1 a_2 a_3} (\eta^{\mu_1 \mu_2} (k_1 - k_2)^{\mu_3} + \text{cyclic}) , \quad (1.20)$$

$$\text{4-gluon vertex} = -ig^2 (f^{a_1 a_2 x} f^{x a_4 a_3} (\eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_3} - \eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_4}) + \text{perms}) . \quad (1.21)$$

Feynman diagrams are then computed by contracting tensors along diagram lines with external states given by polarization vectors  $\varepsilon^\mu$ . Additional Feynman rules are needed for the ghost fields, but for tree-level computations the ghosts do not contribute (for Chern-Simons theory the Feynman rules will automatically include all ghost contributions as well as the physical fields). Although we will not explicitly make use of these Feynman rules, it is important to note that the color and kinematic sectors in them are factorized, and amplitudes can be written with only  $1/k^2$ -type poles.

The worldline actions used in sections 2.2, 2.3 and 3.3 will automatically contain the antifields as in the BV action (1.4), therefore the gauge fixing procedure seen here could be repeated in those cases. In addition to the antifields, the worldline constructions will also give rise to a superfield structure which streamlines the analysis of the BRST and gauge symmetries of the actions.

## 1.2 Color-Kinematics Duality

We learn from the structure of the Feynman rules that a general tree-level amplitude with  $m$  external states can be written in the form,

$$\mathcal{A}_m = g^{m-2} \sum_{i \in \Gamma_m} \frac{n_i c_i}{D_i}, \quad (1.22)$$

where  $\Gamma_m$  denotes cubic  $m$ -point trees, and  $c_i$ ,  $n_i$  and  $D_i$  are the color-factor, numerator, and denominator of each Feynman diagram. The color factors and denominators can be read off of the Feynman diagram directly, the latter taking the form products of  $1/k_j^2$  with  $k_j$  denoting internal momenta in the diagrams. The numerators, on the other hand, are complicated functions of the momenta and polarization vectors. The Feynman rules actually involve a quartic vertex as well, but it can be absorbed in to cubic diagrams by multiplying and dividing by propagators, or equivalently by introducing some auxiliary fields in to the action. Importantly, the numerators are not unique, they are affected by different choices of gauge, field redefinitions, and they can even be manipulated in ways that can not obviously be reproduced from an action (for instance if permutation symmetry is broken).

Due to the Jacobi identity of the structure constants, sums of color factors of certain triplets of graphs vanish,

$$c_i + c_j + c_k = 0. \quad (1.23)$$

For example, the color-factor of the diagram

$$c \left( \begin{array}{cc} 2 & 3 \\ \text{---} & \text{---} \\ \text{---} & \text{---} \\ 1 & 4 \end{array} \right) = f^{a_1 a_2 x} f^{x a_3 a_4} \quad (1.24)$$

vanishes under the sum of cyclic permutations of any three legs, irrespective of what external states are contracted into it. The statement of the color-kinematics duality is that there exists some choice of numerators such that they obey the same relations as their associated color

factors [1, 2],

$$c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0 . \quad (1.25)$$

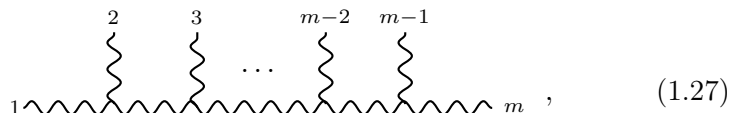
Such numerators are often referred to as color-kinematics duality satisfying numerators, or BCJ numerators, after the names of the authors of ref. [1]. At tree level, the existence of such numerators has been proven or motivated in various contexts (see refs. [8, 9, 10, 11, 12, 13, 14, 15]). For loops in certain cases it is known to hold also, for example in  $\mathcal{N} = 4$  super Yang-Mills, BCJ numerators were found at four loops in for four-point amplitudes [16], and up to five loops for the Sudakov form factor [17]. This intriguing Jacobi-like equation for numerators suggests that the kinematics of Yang-Mills theory could be described by some *kinematic algebra*. Certain algebraic structures do exist in the on-shell numerators [18, 19], however, identifying a Lie algebra whose structure constants, or a generalization thereof, is produced by off-shell cubic Feynman rules remains an open problem.

An interesting application of the color-kinematics duality is the double-copy construction, which gives gravity amplitudes by replacing color factors with numerator factors,

$$\mathcal{M}_m = \left(\frac{\kappa}{2}\right)^{m-2} \sum_{i \in \Gamma_m} \frac{n_i \tilde{n}_i}{D_i} , \quad (1.26)$$

as well as introducing the gravitational coupling constant  $\kappa/2$  instead of  $g$ . The two sets of numerators can in general come from different theories, which enables the double-copy construction of amplitudes for a wide range of (super-)gravities, see ref. [3] for a review. However, in what follows, we will mostly be concerned with trying to understand the color-kinematics duality itself.

Given the Jacobi identity of the color factors, we can decompose the color factor of any tree-level diagram in to a basis of color factors of  $(n - 2)!$  half-ladder graphs,



$$1 \text{---} \text{wavy} \text{---} 2 \text{---} 3 \text{---} \dots \text{---} m-2 \text{---} m-1 \text{---} m , \quad (1.27)$$

and we label the corresponding color-factor by the list of labels appearing in the graph, in this case as  $c_{(1, \dots, m)}$ , where legs 1 and  $m$  are fixed. The expression for the half-ladder color factors is simply,

$$\begin{aligned} c_{1, \dots, m} &= f^{a_1 a_2 x} f^{x a_3 y} \dots f^{z a_{m-1} w} f^{w a_m a_m} \\ &= 2(-i)^{m-3} \text{tr}([\dots [[T^{a_1}, T^{a_2}], T^{a_3}], \dots, T^{a_{m-1}}], T^{a_m}) . \end{aligned} \quad (1.28)$$

Analogously, assuming we have a set of BCJ numerators, we can decompose any numerator in terms of numerators of half-ladder graphs, and label them by the permutation of indices of the graph, for example for (1.27) we have  $n_{(1,\dots,m)}$ . We can repeatedly use the Jacobi identities for the color factors and numerators to rewrite our amplitude (1.22) as [1, 20],

$$\mathcal{A}_m = \sum_{\sigma,\rho \in S_{m-2}} c_{(1,\sigma_2,\dots,\sigma_{m-1},m)} M(\sigma,\rho) n_{(1,\rho_2,\dots,\rho_{m-1},m)} , \quad (1.29)$$

where the elements of the matrix  $M(\sigma,\rho)$  are made of combinations of various scalar propagators  $1/D_i$ , and  $\sigma$  and  $\rho$  are permutations of the labels  $2, \dots, m-1$ . Since each of the color factors here are independent, their coefficients must be independently gauge invariant (under infinitesimal transformations  $\epsilon_i \rightarrow k_i$ ). The independent coefficients of the half-ladder numerators are called color-ordered amplitudes and, given we have BCJ numerators, are given by,

$$A(1,\sigma,m) = \sum_{\rho \in S_{m-2}} M(\sigma,\rho) n_{(1,\rho_1,\dots,\rho_{m-1},m)} . \quad (1.30)$$

Since the color-ordered amplitudes are gauge invariant they are independent of the way they are computed. This map from numerators to amplitudes is not invertible, meaning despite the fact the color-ordered amplitudes are unique, the numerators are not determined by it. As a consequence of the uniqueness of (color-ordered) amplitudes, equations (1.29) and (1.30) give a convenient way to test if one has found an acceptable basis of numerators. This is particularly useful when constructing numerators using an ansatz procedure, and it will be employed in section 2.1.

An important consequence of the existence of BCJ numerators for a theory is that the color-ordered amplitudes also obey the BCJ relations [1],

$$\sum_{i=2}^{m-1} k_1 \cdot (k_2 + \dots + k_i) A(2, \dots, i, 1, i+1, \dots, m) = 0 , \quad (1.31)$$

which, since the color-ordered amplitudes are uniquely determined for a given theory, offers a powerful way to check if a theory can obey the color-kinematics duality. We will employ this check when studying Chern-Simons matter theories in section (2.2).

Finally, color-ordered amplitudes which obey the BCJ relations can be used to construct the double copy directly, without extracting numerators. This formulation of the double copy is due to Kawai, Lewellen



and Tye (KLT), and takes the form [21],

$$M_m = \sum_{\sigma, \rho \in S_{m-3}} A(1, \sigma, n-1, n) S(\sigma, \rho) \tilde{A}(1, \rho, n, n-1), \quad (1.32)$$

where the KLT kernel  $S$  can be found in ref. [22] for example, and it is assumed that couplings are appropriately redefined. Explicitly, the expressions for the first few double copy relations in KLT form are,

$$\mathcal{M}_3 = A(1, 2, 3) \tilde{A}(1, 2, 3), \quad (1.33)$$

$$\mathcal{M}_4 = k_{12}^2 A(1, 2, 3, 4) \tilde{A}(1, 2, 4, 3), \quad (1.34)$$

$$\begin{aligned} \mathcal{M}_5 = & k_{12}^2 k_{45}^2 A(1, 2, 3, 4, 5) \tilde{A}(1, 3, 5, 4, 2) \\ & + k_{14}^2 k_{25}^2 A(1, 4, 3, 2, 5) \tilde{A}(1, 3, 5, 2, 4), \end{aligned} \quad (1.35)$$

where  $k_{ij} = k_i + k_j$ . This compact formulation of the double copy will be used when studying Chern-Simons matter in section 2.2.

### 1.3 Gravitational Waves and WQFT

Ongoing detection of gravitational waves [23] emitted from the mergers of binary black holes offer a new avenue for tests of general relativity. The evolution of the binary system can be roughly broken down in to three phases, the inspiral, merger, and ringdown [24]. In the initial, inspiral, phase of the binary merger, black holes are well separated compared to their characteristic size, and therefore they can be approximated by weakly interacting point particles [25]. Such point-particle actions have been extended to include spin and tidal effects, see refs. [26, 27] for a review. Although the point-particle approximation breaks down when the black holes come near each other, computations done in the point-particle regime can be used to calibrate models such as the effective one body Hamiltonian [28, 24], which also model the merger and ringdown phases of the dynamics [29].

Several approaches based on effective field theories exist for studying the dynamics of black holes in the weakly coupled regime, including post-Newtonian [25] and post-Minkowski [30] approximations. Here we work in the post-Minkowski setting, where one takes the separation,  $r$ , of the black holes or other compact objects to be much larger than their Schwarzschild radius  $r \gg r_s = 2Gm$ . In this approximation scheme velocities are left arbitrary, so it is most easily employed for studying the scattering scenario. Such scattering results have nonetheless been used in the effective-one-body formalism, which in turn can be employed in the study of black holes in quasi-circular bound orbits [31]. Future detectors [32, 33] are expected to have dramatically better signal-to-noise

ratios than current ones, and thus theoretical predictions for the binary dynamics would likely be required up to as high as  $\mathcal{O}(G^8)$  [34]. For non-extremal Kerr black holes, the ring radius  $a^\mu = S^\mu/m$  is bound by  $|a| < Gm$ , and so each spin correction to the binary dynamics increases the overall order in  $G$  by one. On dimensional grounds, since there will not be inverse powers of  $G$  or  $a$  in this work, the additional length scales introduced by the spin will be compensated for by inverse powers of the black hole separation, appearing as  $a/r \sim r_s/r$ , effectively increasing the order in the post-Minkowski expansion. This motivated the work in paper IV, where the worldline quantum field theory (WQFT) approach was extended to apply to particles of any spin.

The WQFT [5, 35] offers an efficient way to construct classical scattering amplitudes of point particles using simple Feynman rules, while avoiding the need to take classical limits of observables (for classical limits of QFT amplitudes, see for example refs. [36, 37, 38, 39]). The WQFT incorporates spin effects either through a supersymmetric worldline action [35], suitable up to quadratic order in spin, or by incorporating a dynamical spin tensor in the action, see paper IV. Results have been computed using the WQFT up to fourth order in the Post-Minkowski expansion [40, 41, 42], with partial results at fifth order recently appearing also, see ref. [43].

Here, as a warm-up, we will study the action of a non-spinning point particle, and use it to compute the classical Compton amplitude. The spinning particle is treated in detail in section 3. The starting point for the WQFT is the path integral,

$$Z = \int \mathcal{D}[h]\mathcal{D}[x]\mathcal{D}[p]e^{iS} , \quad (1.36)$$

where the total action is,

$$S = S_{\text{p.p.}} + S_{\text{E.H.}} + \int d^4x \left( \partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h^\nu{}_\nu \right)^2 , \quad (1.37)$$

and  $S_{\text{p.p.}}$  is the action of the point particle with position<sup>1</sup>  $x$ , momentum  $p$ , and Lagrange multiplier  $\ell$ ,

$$S_{\text{p.p.}} = - \int d\tau \left( p_\mu \dot{x}^\mu - \frac{\ell}{2} (p^2 - m^2) \right) , \quad (1.38)$$

$S_{\text{E.H.}}$  is the Einstein-Hilbert action,

$$S_{\text{E.H.}} = - \frac{2}{32\pi G} \int d^4x \sqrt{-g} R , \quad (1.39)$$

---

<sup>1</sup>Note that  $x$  is used both for the worldline position and the general spacetime coordinates in a slight abuse of notation.

and the last term was added to gauge fix to the de Donder gauge. We work in the weak-field approximation, so we can expand the metric in a small perturbation  $h$  around flat space,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} , \quad g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\rho} h_{\rho}^{\nu} - \dots , \quad (1.40)$$

where  $\kappa = \sqrt{32\pi G}$ . It is simpler to study the dynamics of the point particle by integrating out  $p$ , but in its current form the action (1.38) is closer to the spinning case that will be used later. This action has a time-reparametrization invariance, which one can use to fix  $\ell = \frac{1}{m}$ , with this gauge choice  $\tau$  is the proper time of the particle. The coupling to the background metric is through the mass-shell constraint,  $p^2 = p_{\mu} p_{\nu} g^{\mu\nu}$ .

Next, for the purpose of computing scattering amplitudes, the straight-line solutions to the equations of motion for  $x$  and  $p$  are substituted in to the action as background fields for the fluctuations  $z^{\mu}$  and  $\pi_{\mu}$ ,

$$x^{\mu} \rightarrow b^{\mu} + v^{\mu} \tau + z^{\mu} , \quad (1.41)$$

$$p_{\mu} \rightarrow m v_{\mu} + \pi_{\mu} , \quad (1.42)$$

where  $b^{\mu}$  is a constant, and due to our choice of parametrization of the worldline time we have  $v^2 = 1$ . Making the substitutions for the worldline fields in to the action and ignoring total derivatives and constant terms, we find,

$$S_{\text{p.p.}} = - \int d\tau \left( \pi_{\mu} \dot{z}^{\mu} - \frac{1}{2m} (\pi^2 - v \cdot h \cdot v - 2v \cdot h \cdot \pi + v \cdot h \cdot h \cdot v + \dots) \right) , \quad (1.43)$$

where the ellipsis includes additional terms with more gravitons or more worldline perturbations which will not be relevant for the examples here, see diagrams in equation (1.54). It is convenient to work in momentum or energy space using the Fourier transforms,

$$h_{\mu\nu}(x) = \int_k e^{ik \cdot x} h_{\mu\nu}(k) , \quad (1.44)$$

$$z_{\mu}(\tau) = \int_{\omega} e^{i\omega\tau} z(\omega) , \quad (1.45)$$

and similarly for  $\pi^{\mu}$  (we use the same normalization conventions as ref. [5] for momentum integrals and delta functions). When inserting the Fourier transforms of the metric perturbation in to the worldline action, the exponential introduces an infinite number of interaction terms with the  $z^{\mu}$  field, since we have,

$$h_{\mu\nu}(x) = \int_k e^{ik \cdot (b + v\tau + z)} h_{\mu\nu} . \quad (1.46)$$

Substituting these in to the worldline action, and keeping terms with at most two perturbations (either two gravitons or one graviton and one worldline fluctuation) we find,

$$\begin{aligned}
S_{\text{p.p.}} = & - \int d\tau \left( \pi_\mu \dot{z}^\mu - \frac{1}{2m} \pi^2 \right) \\
& - \frac{m}{2} \int_k e^{ik \cdot b} (v \cdot h \cdot v) \delta(k \cdot v) \\
& - \frac{1}{2} \int_{k, \omega} e^{ik \cdot b} (mi(k \cdot z)(v \cdot h \cdot v) + 2(v \cdot h \cdot \pi)) \delta(k \cdot v + \omega) \\
& + \frac{m}{2} \int_{k_1 k_2} e^{i(k_1 + k_2) \cdot b} (v \cdot h_1 \cdot h_2 \cdot v) \delta((k_1 + k_2) \cdot v) . \quad (1.47)
\end{aligned}$$

In the top line we isolated the kinetic term for the worldline fields. In the second and third lines we have vertices that involve one graviton and either no worldline fluctuations or one worldline fluctuation. On the last line we have the two-graviton vertex. The energy conserving delta functions come from the integration over the worldline proper time  $\tau$ , and the vertices in the action have an exponential factor  $e^{ik \cdot b}$  that effectively transforms from momentum space to impact parameter space, but since we will be computing amplitudes in momentum space here, we will strip this factor off.

To invert the kinetic term, we schematically write our action in the form  $\frac{i}{2} W \cdot K \cdot W + iJ \cdot W$ , where  $K$  is the kinetic-term matrix, and we introduced a set of source fields  $J$ . (The reason for pulling out a factor of  $\frac{1}{2}$  is that we have a single real particle propagating.) We invert the kinetic term to obtain the solution to the non-interacting theory, which takes the form  $\frac{i}{2} J \cdot \Delta \cdot J$ . Then the nontrivial two-point functions can be read off from the propagator  $\Delta$ ,

$$\langle z^\mu(-\omega) z^\nu(\omega) \rangle = -i \frac{1}{m\omega^2} \eta^{\mu\nu} , \quad (1.48)$$

$$\langle p^\mu(-\omega) z^\nu(\omega) \rangle = -\frac{1}{\omega} \eta^{\mu\nu} . \quad (1.49)$$

Retarded and advanced propagators are given by replacing  $\omega \rightarrow \omega \pm i0$ . For the amplitudes computed here it is sufficient to average the two [5], implying the background fields are the average of the far past and far future background fields.

Finally, the propagator used for the gravitons is obtained by substituting  $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$  in equation (1.39), combining this together with the gauge-fixing term we get the de Donder propagator,

$$\langle h_{\mu\nu} h_{\rho\sigma} \rangle = \frac{i}{2(k^2 + i0)} (\eta_{\mu\rho} \eta_{\sigma\nu} + \eta_{\mu\sigma} \eta_{\rho\nu} - \eta_{\mu\nu} \eta_{\rho\sigma}) . \quad (1.50)$$

We use the Feynman  $i0$  prescription which is suitable for the amplitude computations we do here.

Next, we can read off Feynman vertices from the expanded action (1.47). Ignoring the overall  $e^{ik\cdot b}$ , energy conserving delta functions, and momentum integrals, we find the following vertices,

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = -\frac{i}{2}m(v\cdot h\cdot v), \quad (1.51)$$

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \frac{im}{2}(v\cdot h_1\cdot h_2\cdot v) + (1 \leftrightarrow 2), \quad (1.52)$$

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \frac{m}{2}(v\cdot h\cdot v)(k\cdot z) - i(\pi\cdot h\cdot v). \quad (1.53)$$

We abuse notation slightly and use the worldline fields  $\{z^\mu, \pi_\mu\}$  and gravitons  $h^{\mu\nu}$  in order to soak up free indices in the vertex. In the diagrams we indicate gravitons by snaked lines, and the outgoing superposition of worldline fluctuations with a solid line. The unperturbed background fields are represented by the dotted line.

To compute the Compton amplitude we have three diagrams to take into account,

$$\mathcal{A}^{\text{tree}} = \begin{array}{c} k_1 \quad k_2 \\ | \quad | \\ \text{---} \end{array} \begin{array}{c} \omega \\ \rightarrow \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} k_1 \quad k_2 \\ | \quad | \\ \text{---} \end{array} \begin{array}{c} \omega \\ \rightarrow \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} k_1 \quad k_2 \\ | \quad | \\ \text{---} \end{array} \begin{array}{c} \omega \\ \rightarrow \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}. \quad (1.54)$$

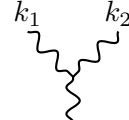
The second diagram is simplest, and is basically given by evaluating the vertex (1.52) with on-shell external gravitons  $h_i^{\mu\nu} = \varepsilon_i^\mu \varepsilon_i^\nu$  satisfying  $\varepsilon_i^2 = 0$ ,  $\varepsilon_i \cdot k_i = 0$ . With these the diagram becomes,

$$\begin{array}{c} k_1 \quad k_2 \\ | \quad | \\ \text{---} \end{array} \begin{array}{c} \omega \\ \rightarrow \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = im(v\cdot \varepsilon_1)(\varepsilon_1\cdot \varepsilon_2)(v\cdot \varepsilon_2). \quad (1.55)$$

The energy-conserving delta functions here simply impose  $v\cdot(k_1 + k_2) = 0$ . The first diagram requires us to use the worldline two-point functions, to compute it we multiply together two copies of the vertex (1.53), keeping track of the fields with outgoing energy or incoming energy by labelling them  $z(-\omega)$  and  $z(\omega)$  respectively (and similarly for  $\pi$ ). We can then contract fields using (1.48), obtaining,

$$\begin{array}{c} k_1 \quad k_2 \\ | \quad | \\ \text{---} \end{array} \begin{array}{c} \omega \\ \rightarrow \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \frac{im}{2\omega}k_2\cdot \varepsilon_1 v\cdot \varepsilon_1 (v\cdot \varepsilon_2)^2 - \frac{im}{2\omega}k_1\cdot \varepsilon_2 v\cdot \varepsilon_2 (v\cdot \varepsilon_1)^2 \\ - \frac{im}{4\omega^2}(v\cdot \varepsilon_1)^2 (v\cdot \varepsilon_2)^2 k_1\cdot k_2. \quad (1.56)$$

Now the energy-conserving delta functions impose  $v \cdot (k_1 + k_2) = 0$  and  $\omega = v \cdot k_1 = -v \cdot k_2$ . Finally, using standard Feynman rules for gravity (see for example ref. [44]), the last remaining diagram is,



$$\begin{aligned}
 &= \frac{im}{4k_1 \cdot k_2} \left( q \cdot \varepsilon_1 v \cdot \varepsilon_1 q \cdot \varepsilon_2 v \cdot \varepsilon_2 + 2\varepsilon_1 \cdot \varepsilon_2 q \cdot \varepsilon_2 v \cdot \varepsilon_1 v \cdot k_1 \right. \\
 &\quad \left. - k_1 \cdot k_2 \varepsilon_1 \cdot \varepsilon_2 v \cdot \varepsilon_1 v \cdot \varepsilon_2 - (v \cdot \varepsilon_1)^2 (q \cdot \varepsilon_2)^2 - \frac{1}{2} (\varepsilon_1 \cdot \varepsilon_2)^2 (v \cdot k_1)^2 \right) \\
 &\quad + (1 \leftrightarrow 2) , \tag{1.57}
 \end{aligned}$$

where  $q = k_1 + k_2$ .

We saw here how the simple point-particle action can be used to compute classical observables in general relativity. In the next section, we will quantize this action in flat space, in order to illustrate some of the ideas that will be used later in the thesis. In chapter 3.2 we will also see how to generalize the point particle to include any classical spin in the context of the WQFT.

## 1.4 From Worldlines to Fields

Many of the actions we see in this thesis will be obtained by quantizing worldline theories. We illustrate this procedure with the relativistic point particle as an example, see also ref. [45], chapters VI and XII. The classical Hamiltonian action for the point particle is exactly the same action we used in the study of the WQFT,

$$S = - \int d\tau \left( p_\mu \dot{x}^\mu - \frac{\ell}{2} (p^2 - m^2) \right) , \tag{1.58}$$

but here we work in flat space. The Poisson brackets are straightforward to work out,  $\{p_\mu, x^\nu\} = \delta_\mu^\nu$ , which corresponds to the quantum commutator  $[p_\mu, x^\nu] = -i\delta_\mu^\nu$ . This algebra is represented by choosing  $p_\mu \rightarrow -i\partial_\mu$ , where we use Cartesian coordinates<sup>2</sup>. Next, our task is to construct a BRST operator. The recipe is to start by multiplying all our constraints by an anticommuting ghost coordinate (worldline field) and add them up. Later we must add corrections to the BRST operator to ensure that it is nilpotent. In our case we only have one such constraint so our BRST operator is,

$$Q = \frac{1}{2} c (\square + m^2) , \tag{1.59}$$

<sup>2</sup>In general we should use  $p_\mu \rightarrow -ig^{-1/2} \partial_\mu g^{1/2}$  where  $g = -\det(g)$  is the absolute value of the determinant of the spacetime metric.

and it is automatically nilpotent,  $\{Q, Q\} = 0$  classically and  $Q^2 = 0$  in the quantized theory also.

The next step in quantization is to introduce wave functions  $\Psi(x, c)$  that depend on all coordinates, including ghost coordinates [46]. The equations of motion of the system are then given by  $Q\Psi = 0$ . This set of equations can also be imposed from an action principle, namely

$$S = \int d^4x dc \Psi Q \Psi , \quad (1.60)$$

where we used the measure appropriate for our choice of coordinates, specializing in particular to 4D. We can now expand the wave functions in the Grassmann-odd coordinate  $c$ ,  $\Psi = \phi + c\check{\phi}$ , and we immediately find that the antifield  $\check{\phi}$  decouples from the action because  $c^2 = 0$ . Therefore, our final action depends only on the physical field  $\phi$ ,

$$S = \int d^4x \frac{1}{2} \phi (\square + m^2) \phi . \quad (1.61)$$

It is also possible to quantize the action in curved space, in which case the result is the scalar field minimally coupled to gravity. We will encounter actions which have very similar forms to the one we saw here, namely  $\Psi Q \Psi$ -type kinetic terms with  $Q$  the worldline BRST operator. In particular, these are the Chern-Simons and pure-spinor actions in equations (2.23) and (2.55), as well as the action for the spinning particle in equation (3.58). In these cases the field-theory action will also have a BRST and gauge symmetries generated by the worldline BRST operator.

## 2. Actions for Color-Kinematics Duality

This chapter deals with the construction of actions that generate BCJ numerators from their Feynman rules. The first part deals with pure Yang-Mills in a certain kinematic sector, but to any number of external states. The subsequent sections develop a superfield and differential-operator approach to color-kinematics duality in Chern-Simons and pure-spinor super Yang-Mills. The latter two actions are constructed from worldline quantization, which automatically gives rise to the full BV action of the theory.

### 2.1 NMHV Lagrangian for Yang-Mills Theory

In this section only, we do not make use of worldline theories, instead jumping straight in to the field theory description of Yang-Mills theory. However, it is interesting to note that certain constructions of BCJ numerators have explicitly relied on worldline theories [47]. In addition, quantization techniques like those used in this thesis have also been employed to construct the BV action of Yang-Mills theory [48, 49], which was double copied off-shell up to quartic order in the fields. Here, however, we will take as our starting point the gauge-fixed action for Yang-Mills theory, and work exclusively on shell.

Previous attempts at constructing an action whose Feynman rules generate BCJ numerators have relied on solving the problem at each order in the number of external states, obtaining results up to six points [50, 51]. We will take a different approach, attempting to solve the problem for any number of particles, but restricting to numerators for the ( $d$ -dimensional generalization of the) NMHV sector, which is defined below.

Recall that local Yang-Mills tree numerators are polynomials of Lorentz scalars  $\{\varepsilon_i \cdot \varepsilon_j, \varepsilon_i \cdot p_j, p_i \cdot p_j\}$ . We can decompose our basis numerators (1.29) in to different *polarization power* sectors [52],

$$n_{(1, \dots, m)} = \sum_{k=1}^{\lfloor m/2 \rfloor} n_{(1, \dots, m)}^{(k)}, \quad (2.1)$$

where each of the subsectors  $n^{(k)}$  contains  $k$  powers of polarization dot products,  $\sim (\varepsilon_i \cdot \varepsilon_k)^k$ . This decomposition is a generalization of the MHV-degree in four dimensions, since there exists a choice of gauge such that



only the numerator sectors up to order  $k$  contribute to the  $N^{k-1}$ MHV amplitudes [52]. More importantly, since the numerator Jacobi relations are linear, they do not mix between different polarization power sectors. In what follows we will focus on obtaining tree-level numerators up to polarization power two, namely  $\sim (\varepsilon_i \cdot \varepsilon_k)^2$ . This will be done in two steps, first we will construct the bi-scalar numerators

$$\bar{n}_{(1,\dots,m)}^{(k)} = \frac{\partial}{\partial \varepsilon_1 \cdot \varepsilon_m} n_{(1,\dots,m)}^{(k)}, \quad (2.2)$$

where  $k = 1, 2$ , as well as a Lagrangian that generates them from its Feynman rules. These can be thought of as a dimensional reduction of the full numerators, where legs 1 and  $m$  are external scalars. The next step will be to use the scalar numerators to obtain full Yang-Mills numerators again using [52, 53],

$$n_{(1,\dots,m)}^{(k)} = \frac{1}{k} \sum_{1 \leq i < j \leq m} \varepsilon_i \cdot \varepsilon_j \bar{n}_{(i, \alpha_i, i+1, \dots, j-1, \beta_j, j)}^{(k)}, \quad (2.3)$$

where  $\alpha_i = [\dots[[1, 2], 3], \dots, i-1]$  and  $\beta_j = [j+1, [\dots, m-2, [m-1, m]] \dots]$  are nested commutators of external labels. The boundary cases of the sum are handled through the identifications  $\alpha_2 = [1] = 1$ ,  $\beta_{n-1} = [n] = n$ , and when either bracket is empty,  $\alpha_1 = \beta_n = [] \rightarrow (-1)$ , then the numerator is multiplied by a minus sign. We will then extend the scalar Lagrangian to also generate all non-scalar numerators as well, including non-half-ladder numerators.

Our starting point is the Lorenz gauge Yang-Mills Lagrangian with the quartic interaction replaced by two auxiliary fields  $B^{\mu\nu}$  and  $\tilde{B}^{\mu\nu}$ ,

$$\begin{aligned} \mathcal{L}_4 = \text{Tr} \left( \frac{1}{2} A_\mu \square A^\mu - \partial_\mu A_\nu [A^\mu, A^\nu] + B_{\mu\nu} \square \tilde{B}^{\mu\nu} \right. \\ \left. + \frac{1}{2} [A_\mu, A_\nu] (B^{\mu\nu} + \square \tilde{B}^{\mu\nu}) \right). \quad (2.4) \end{aligned}$$

The equations of motion for  $B^{\mu\nu}$  and  $\tilde{B}^{\mu\nu}$  are given by

$$B^{\mu\nu} = \square \tilde{B}^{\mu\nu} = -\frac{1}{2} [A^\mu, A^\nu], \quad (2.5)$$

when plugged back in to the action (2.4) they reproduce the standard Yang-Mills action. Note that since the  $B$  fields are quadratic on shell, they do not appear as asymptotic states. The Feynman rules from this Lagrangian give the correct polarization-power one sector of Yang-Mills,  $\mathcal{O}(\varepsilon_i \cdot \varepsilon_j)^1$ , even if we were to drop the  $B$  and  $\tilde{B}$  fields completely from the action. In fact dropping the latter fields reveals a diffeomorphism algebra in the MHV sector, see paper III.

To streamline the discussion of different components of numerators, we introduce a diagrammatic notation for the contractions of Lorentz indices. We use solid lines to track the contraction of pairs of polarization vectors, for example, the scalar-sector numerator at four points can be decomposed in to,

$$\bar{n}^{(1)} = \begin{array}{c} \begin{array}{ccc} & 2 & 3 \\ & \text{wavy} & \text{wavy} \\ 1 & \text{---} & 4 \end{array} \end{array} = u_2 u_3, \quad (2.6)$$

$$\bar{n}^{(2)} = \begin{array}{c} \begin{array}{ccc} & 2 & 3 \\ & | & | \\ 1 & \text{---} & 4 \end{array} \end{array} = -\varepsilon_2 \cdot \varepsilon_3 x_2^2, \quad (2.7)$$

where we introduced the regional momenta  $x_i$  and cubic bi-scalar vertices  $u_i$ ,

$$x_i^\mu \equiv \sum_{j=1}^i p_j^\mu, \quad u_i \equiv 2\varepsilon_i \cdot x_i. \quad (2.8)$$

With these variables, the bi-scalar MHV numerators are simply,

$$\bar{n}_{(1,\dots,m)}^{(1)} = u_2 u_3 \dots u_{m-1}, \quad (2.9)$$

to any multiplicity. We are interested in the next order in polarization power, so our first task is to construct numerators with diagrams such as

$$\bar{n}_{(1,\dots,m)}^{(2)} = \sum_{1 < i < j < m} \begin{array}{c} \begin{array}{ccccccc} & & & i & & j & \\ & & & | & & | & \\ & \text{wavy} & \dots & \text{wavy} & \dots & \text{wavy} & \\ 1 & \text{---} & & \text{---} & & \text{---} & m \end{array} \end{array}, \quad (2.10)$$

to any multiplicity. Starting at five points, we find that BCJ numerators necessarily require terms such as  $\varepsilon_1 \cdot \varepsilon_5 \varepsilon_2 \cdot \varepsilon_4$ , which means that we would need to allow some propagating two-index field in to our Lagrangian description. In other words, diagrams such as (2.10) where the double lines propagate across one or more vertices contribute non-trivially to our numerators. At this stage, however, our Lagrangian (2.4) does not allow such diagrams. To remedy this, we add interactions of the form  $\partial A B \tilde{B}$ , while attempting to ensure that these extra interactions do not contribute to any tree-level amplitudes. We find a solution,

$$\begin{aligned} \mathcal{L}_5 = & \text{Tr} \left( \frac{1}{2} A_\mu \square A^\mu - \partial_\mu A_\nu [A^\mu, A^\nu] + B_{\mu\nu} \square \tilde{B}^{\mu\nu} + 4 \partial_\nu \tilde{B}_{\mu\rho} [A^\mu, B^{\nu\rho}] \right. \\ & \left. + \frac{1}{2} [A_\mu, A_\nu] (B^{\mu\nu} + \square \tilde{B}^{\mu\nu}) - 2 \partial_\rho \tilde{B}_{\mu\nu} [A^\rho, B^{\mu\nu}] \right), \end{aligned} \quad (2.11)$$

which has also been tuned to ensure that the five-point numerators obey the color-kinematics duality. The modification of the Lagrangian now gives us nontrivial contributions from the internal  $B$  fields, for example we find,

$$\begin{array}{c}
 \begin{array}{ccc}
 & 2 & 3 & 4 \\
 & | & | & | \\
 1 & \text{---} & \text{---} & \text{---} & 5 \\
 & | & | & | \\
 & 2 & 3 & 4
 \end{array}
 & = & -\frac{1}{2}\varepsilon_2 \cdot \varepsilon_4 u_3 (x_2^2 + x_3^2).
 \end{array} \quad (2.12)$$

Remarkably, the Lagrangian (2.11) generates the correct bi-scalar numerators  $\bar{n}^{(2)}$  at higher multiplicity as well, this was checked up to 11 points. Since the scalar-sector Lagrangian is fairly simple, it is possible to guess the general form of numerators for diagrams such as (2.10), the result from paper III is summarized below. We introduce several important ingredients, starting with the subdiagrams

$$\mathcal{D}_{ijkn} = \varepsilon_i \cdot \varepsilon_j x_{k-1}^2 U_{2,i-1} U_{i+1,j-1} (x_{j-1} \cdot V_{j+1} \cdots V_{k-1} \cdot \varepsilon_k) U_{k+1,n-1}, \quad (2.13)$$

where the  $(V_i)^{\mu\nu}$  matrices are given by

$$(V_i)^{\mu\nu} = u_i \eta^{\mu\nu} - 2\varepsilon_i^\mu x_{i-1}^\nu, \quad (2.14)$$

and the  $U$ s are products of  $u_i$ ,

$$U_{i,j} = u_i u_{i+1} \cdots u_j. \quad (2.15)$$

In the case  $j = k$  we choose the expression  $(x_{j-1} \cdot V_j \cdots V_{k-1} \cdot \varepsilon_k)$  to be equal to  $(-1/2)$  to avoid double counting the inverse propagator for leg  $j$ . Altogether, using the subdiagrams  $\mathcal{D}_{ijkm}$  the scalar-sector numerator is

$$\bar{n}_{(1,\dots,m)}^{(2)} = \sum_{1 < i < j \leq k}^{n-1} \mathcal{D}_{ijkm} + \text{reflection}, \quad (2.16)$$

where the reflection is obtained by reversing the labels  $\{1, 2, \dots, m\}$  on the momenta  $p_i$  and polarizations  $\varepsilon_i$ , resulting in  $\varepsilon_i \rightarrow \varepsilon_{m-i+1}$ ,  $u_i \rightarrow -u_{m-i+1}$  and  $x_i \rightarrow -x_{m-i+1}$ . For example, we have the scalar-sector numerator at six points,

$$\begin{aligned}
 \bar{n}_{(1,\dots,6)} &= u_2 u_3 u_4 u_5 - 2\varepsilon_2 \cdot \varepsilon_3 (2\varepsilon_4 \cdot x_2 \varepsilon_5 \cdot x_3 x_4^2 - \varepsilon_5 \cdot x_2 u_4 x_4^2) \\
 &\quad - 2\varepsilon_2 \cdot \varepsilon_3 (-\varepsilon_4 \cdot x_2 u_5 x_3^2 + u_4 u_5 x_2^2) - \varepsilon_2 \cdot \varepsilon_5 (x_2^2 + x_4^2) u_3 u_4 \\
 &\quad + \varepsilon_2 \cdot \varepsilon_4 (2\varepsilon_5 \cdot x_3 u_3 x_4^2 - (x_2^2 + x_3^2) u_3 u_5) \\
 &\quad + 2\varepsilon_3 \cdot \varepsilon_4 (\varepsilon_5 \cdot x_3 u_2 x_4^2 + \varepsilon_2 \cdot x_3 u_5 x_2^2 - u_2 u_5 x_3^2) \\
 &\quad + \varepsilon_3 \cdot \varepsilon_5 (2\varepsilon_2 \cdot x_3 u_4 x_2^2 - (x_4^2 + x_3^2) u_2 u_4) \\
 &\quad - 2\varepsilon_4 \cdot \varepsilon_5 (2\varepsilon_2 \cdot x_3 \varepsilon_3 \cdot x_4 x_2^2 - \varepsilon_3 \cdot x_4 u_2 x_3^2 - \varepsilon_2 \cdot x_4 u_3 x_2^2 + u_2 u_3 x_4^2).
 \end{aligned} \quad (2.17)$$

Having the scalar-sector numerators at hand, their non-scalar extension from equation (2.3) is unique. However, the Lagrangian (2.11) does not correctly generate BCJ numerators beyond the scalar sector. Our next goal therefore is to extend our Lagrangian by additional fields such that it generates the full numerators, and we illustrate how this is done using some examples from six points.

To construct corrections to the Lagrangian it is useful to pinpoint terms in the numerator that are not generated correctly, and produce them with additional fields and vertices while not spoiling the scalar-sector numerators or terms at lower multiplicity. Concretely, the term  $u_3 \varepsilon_2 \cdot \varepsilon_4 \varepsilon_5 \cdot \varepsilon_6 \varepsilon_3 \cdot (p_5 - p_6) x_2^2$  is present in the six-point numerator  $n_{1,\dots,6}^{(2)}$ , but it is not generated by the Feynman rules of the Lagrangian in (2.11). This term can be obtained by introducing a pair of vector fields  $Z^\mu$  and  $\tilde{Z}^\mu$ . Choosing new interactions  $A^\mu \tilde{Z}^\nu B_{\mu\nu}$  and  $\partial_\nu A^\mu A_\mu Z^\nu$  one finds a new contribution,

$$u_3 \varepsilon_2 \cdot \varepsilon_4 \varepsilon_5 \cdot \varepsilon_6 \varepsilon_1 \cdot (p_5 - p_6) x_2^2 \quad \longrightarrow \quad \begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \\ | & \text{wavy} & | & | \\ \hline 1 & \text{---} & \text{---} & \text{---} & \partial & \text{---} & 6 \\ \square \tilde{B} & B \tilde{B} & B \tilde{Z} & Z & & & \end{array} \end{array}, \quad (2.18)$$

where the diagram has the relevant interactions indicated, and the derivative in the diagram indicates an interaction contributing  $\varepsilon_1 \cdot (p_5 - p_6)$ . Further adding an interaction  $AA\tilde{Z}$  would spoil the four-point numerator, so it is excluded. To prevent the new fields from contributing to the five-point numerators they can be coupled to  $(B^{\mu\nu} - \square \tilde{B}^{\mu\nu})$ , since this combination of fields is not sourced by the  $A$  field at five points, the  $Z$  fields will only begin to be relevant at six points and higher.

A similar procedure can be applied to other terms that are not generated by the five-point Lagrangian, this reveals new interactions for the existing fields as well as an additional pair of scalars  $X$  and  $\tilde{X}$  that must be added in to the game. Some of the new interactions contain two derivatives, in which case it was assumed that the two derivatives should appear together in a  $\square$  in order to make it manifest that the new interaction only contributes from polarization-power two or higher. To understand why, observe that an  $n$ -point numerator has  $n$ -polarizations and  $n - 2$  momenta, and at polarization-power one all momenta are contracted with polarization tensors. An interaction with  $\square$  has at least one pair of momenta contracted, therefore contributing from polarization power two or higher.

When constructing half-ladder numerators, all interactions contain at least one vector field  $A$ , however the Lagrangians need to also produce non-half-ladder diagrams in which the new fields could interact through, for example,  $[\tilde{Z}_\mu, \tilde{Z}_\nu] B^{\mu\nu}$ . This procedure was repeated at seven points,

and overall it was found in paper III that the Lagrangian,

$$\begin{aligned}
\mathcal{L} = \mathcal{L}_5 + \text{Tr} & \left( Z^\mu \square \tilde{Z}_\mu + X \square \tilde{X} + [\partial_\nu A^\mu, A_\mu] Z^\nu + [A^\mu, \square A_\mu] \tilde{X} \right. \\
& - 2[A_\mu, X] \partial^\mu \tilde{X} + [A_\mu, \tilde{Z}_\nu] (B^{\mu\nu} - \square \tilde{B}^{\mu\nu}) - [A^\mu, X] \tilde{Z}_\mu \\
& + \frac{1}{2} [A^\mu, \square \tilde{X}] \tilde{Z}_\mu - 2[A_\mu, Z_\nu] \partial^\mu \tilde{Z}^\nu - 2[A_\mu, B^{\mu\nu}] \partial_\nu \tilde{X} \\
& + \frac{1}{2} [\tilde{Z}_\mu, \tilde{Z}_\nu] (B^{\mu\nu} + \square \tilde{B}^{\mu\nu}) + 4[B^{\mu\nu}, \partial_\mu \tilde{B}_{\nu\rho}] \tilde{Z}^\rho \\
& \left. - 2[B^{\mu\nu}, \partial_\nu \tilde{X}] \tilde{Z}_\mu \right), \tag{2.19}
\end{aligned}$$

correctly generates NMHV numerators at that order. Interestingly, it also generated correct numerators through ten points, indicating that no further modifications are needed.

The procedure described above does not necessarily generate a unique Lagrangian, meaning there could be alternative constructions with simpler results. This idea was tested in III by making an ansatz for a Lagrangian, and constraining it to generate the same numerators at NMHV level. Some criteria needed to be imposed on the ansatz, such as:

1. The Lagrangian  $\mathcal{L}_5$ , in equation (2.11), is assumed to remain unchanged up to five points.
2. The fields in the extended Lagrangian are assumed to still be the scalar pair  $X, \tilde{X}$ , the vectors  $Z, \tilde{Z}$  and the tensors  $B, \tilde{B}$ .
3. The kinetic terms do not mix between these different fields.
4. The four-point numerators are protected by excluding the  $AA\tilde{Z}$  interaction.
5. Interactions with two derivatives must have a d’Alambertian, thus ensuring that the extra interaction can only contribute at NMHV level.
6. Interactions  $AZZ$  and  $A\tilde{Z}\tilde{Z}$  are excluded, to simplify the ansatz space.

The most general cubic Lagrangian satisfying these constraints has 174 free parameters. These parameters are fixed by non-linear equations obtained from comparing numerators computed from the Lagrangian to those from the uplift of the scalar-sector numerators using equation (2.3). By imposing all possible constraints up to eight points, there were still 129 free parameters left in the Lagrangian. At nine points no further constraints were found on these free parameters, suggesting they are redundant somehow. Therefore, at this stage, solutions were sought that simplify the resulting Lagrangian, with the idea that then the simplified Lagrangians could be tested at higher points. Interestingly, the resulting Lagrangians were all equivalent to the 13 term corrections in (2.19) up to terms proportional to  $\partial \cdot A$ .

The last constraint in the ansatz is of course ad hoc, so it is interesting to replace it with another:

- 6'. The scalars  $X$  and  $\bar{X}$  are excluded, but the interactions  $AZZ$  and  $A\tilde{Z}\tilde{Z}$  are allowed.

In this ansatz space, solutions were found that contain 20 interaction terms in the Lagrangian.

Having Lagrangians at hand, it is interesting to ask if they generate loop-level numerators that still obey the color-kinematics duality, at least up to some order in polarization power. This is not the case however, since in loops, states are sewn together off-shell, and the on-shell conditions turned out to be necessary in order for the kinematic Jacobi identities to work out, at least for numerators computed from this Lagrangian. Nonetheless, it would be interesting to attempt to modify the results here such that the Lagrangians could generate loop level numerators, perhaps by extending  $\mathcal{L}_5$  in other ways. In the next section we will see a Lagrangian that does produce off-shell color-kinematics duality satisfying numerators, and for which a kinematic algebra does exist.

## 2.2 Chern-Simons Theory

In this section we focus on Chern-Simons theory, emphasizing in particular the off-shell color-kinematics duality. We shall see that this holds due to the fact that the propagator numerator, called  $b$  here, is a second-order operator. The connection between the kinematic Jacobi identity and a second-order operator was first observed in ref. [54], where such an operator was constructed for pure Yang-Mills tree-level numerators. In Chern-Simons theory the situation is simpler than in Yang-Mills, and the construction generalizes to all loop orders.

The worldline action for 3D Chern-Simons theory is

$$S = - \int d\tau \left( p_\mu \dot{x}^\mu - \ell_\mu p^\mu \right). \quad (2.20)$$

The theory has no local degrees of freedom, as can be seen from the fact that the constraint plus its associated gauge symmetry can be used to eliminate both  $x^\mu$  and  $p_\mu$ . The worldline BRST operator is just the exterior derivative, written with Grassmann-odd ghosts  $\theta^\mu$ , it is

$$Q = \theta^\mu p_\mu, \quad (2.21)$$

where  $p_\mu = -i\partial_\mu$ . Field-theoretic equations of motion can now be extracted from the BRST operator with  $Q\Psi = 0$ , where  $\Psi$  is taken to be a function of both  $x$  and  $\theta$ . Expanding the wave functions in the ghosts, we find,

$$\Psi = c + \theta^\mu A_\mu + \theta^\mu \theta^\nu \check{A}_{\mu\nu} + \theta^3 \check{c}, \quad (2.22)$$

and from  $Q\Psi = 0$  we recognize the Chern-Simons equations of motion  $F_{\mu\nu} = 0$  for the vector field  $A_\mu$  at ghost number one. The remaining fields are the ghost  $c$  and antighost  $\check{c}$ , as well as the antifield  $\check{A}$ . To obtain non-abelian Chern-Simons theory one could add color degrees of freedom to the worldline (2.20) and quantize it in a similar way to what was done in section 1.4, see ref. [55], the result is the Axelrod-Singer formulation of the action [56],

$$S = \frac{ik}{2\pi} \int d^3x d^3\theta \operatorname{tr} \left( \frac{1}{2} \Psi Q \Psi + \frac{1}{3} \Psi \Psi \Psi \right), \quad (2.23)$$

where the fields are now Lie-algebra valued, e.g.  $A = T^a A_\mu^a dx^\mu$ , and  $k$  is the Chern-Simons level (see ref. [57] for more details). The perturbative regime of the theory is in the limit that  $k$  is large, and it is sometimes convenient to introduce it through the coupling constant  $g = \sqrt{4\pi/k}$ . Although it is possible to describe the BV action for Yang-Mills theory with a similar structure, see Appendix A, the benefit of the worldline construction is that the differential acts on all of the fields in the same way, without the need for projectors, and the three-point vertex is given by simple multiplication of superfields.

This action is the BV action for Chern-Simons theory, namely the analogue of (1.4) which we had for Yang-Mills theory. The antibracket here can conveniently be defined in superspace,

$$\{A, B\} = \int d^3x d^3\theta \frac{\overrightarrow{\delta} A}{\delta \Psi} \frac{\overleftarrow{\delta} B}{\delta \Psi}, \quad (2.24)$$

with which the interacting action satisfies  $\{S, S\} = 0$ .

As it stands, our Chern-Simons action has a gauge symmetry under

$$\Psi \rightarrow Q\Omega + [\Psi, \Omega], \quad (2.25)$$

where  $\Omega$  is some superfield. To gauge fix, we introduce the codifferential  $b = \frac{\partial}{\partial \theta^\mu} p_\mu$ , which obeys the important relation

$$b^2 = 0, \quad \{b, Q\} = p^2, \quad (2.26)$$

and it can be integrated by parts under the  $\int d^3x d^3\theta$  measure. We refer to the codifferential as  $b$  (instead of the usual  $d^\dagger$ ) in order to make a connection with the next sections, where the name  $b$  was chosen since the anticommutator (2.26) resembles the anticommutator of a  $b$ -ghost with a BRST operator for a theory with a mass-shell constraint  $p^2$ .

To proceed with gauge fixing, we use the same extended action we had for Yang-Mills theory, namely we add to our action the non-minimal extension  $+B\check{c}$ . Using the gauge fixing fermion  $\mathbf{F} = \int \star \bar{c} (\xi B + \partial_\mu A^\mu)$ , our gauge-fixed action becomes,

$$S = \frac{ik}{2\pi} \int d^3x d^3\theta \operatorname{tr} \left( \frac{1}{2} \Psi Q \Psi + \frac{1}{3} \Psi \Psi \Psi - i\xi B^2 + iB\partial \cdot A \right), \quad (2.27)$$

subject to the constraints  $\check{A} = \star d\check{c}$  and  $\check{c} = 0$ . There are two ways to proceed, firstly the kinetic term is now invertible, and there is a choice of  $\xi$  for which the inverse kinetic term gives a propagator with  $1/\square$ . The superfields can then be reconstructed after such a gauge fixing. Alternatively we can send  $\xi \rightarrow 0$  and integrate out  $B$ , in this case we would be working on a subspace of fields subject to the constraint  $b\Psi = 0$  (for which  $\check{A} = \star d\check{c}$  and  $\check{c} = 0$  are solutions). On this subspace  $Q$  is invertible so long as  $\square$  is. Altogether, we learn that our  $b\Psi = 0$  gauge is an acceptable gauge fixing condition, see refs. [58] and [59, sec. 5.4] for more details. Proceeding with  $b\Psi = 0$ , we automatically get the superspace Feynman rules,

$$\theta \xrightarrow[k]{\quad} \tilde{\theta} = b_k \frac{1}{k^2} \delta^3(\theta - \tilde{\theta}), \quad \begin{array}{c} | \\ \diagdown \quad \diagup \\ \theta \end{array} = i \int d^3\theta, \quad (2.28)$$

where the arrow indicates the momentum flow, and  $b_k = k^\mu \frac{\partial}{\partial \theta^\mu}$  represents the momentum-space  $b$ -operator acting on a leg with momentum  $k$ . We do not Fourier transform our  $\theta$  coordinates, so vertices are labeled by distinct  $\theta$ s. We have suppressed the coupling constant and momentum conserving delta functions.

We follow II to show that the theory obeys color-kinematics duality. We start by considering the numerator of an off-shell amputated four-point diagram,

$$\begin{array}{c} 2 \\ | \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} \begin{array}{c} 3 \\ | \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ \tilde{\theta} \end{array} = \int d^3\theta \Psi_1 \Psi_2 \int d^3\tilde{\theta} b_{k_{34}} \delta_{\theta, \tilde{\theta}} \Psi_3(\tilde{\theta}) \Psi_4(\tilde{\theta}), \quad (2.29)$$

where  $\delta_{\theta, \tilde{\theta}} = \delta^3(\theta - \tilde{\theta})$  and we assume that  $b\Psi_i = 0$ . This diagram can be simplified by evaluating the  $\tilde{\theta}$  integral and using integration by parts and momentum conservation. We obtain,

$$\begin{array}{c} 2 \\ | \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} \begin{array}{c} 3 \\ | \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ \end{array} = i \int d^3\theta b_{k_{12}} (\Psi_1 \Psi_2) \Psi_3 \Psi_4. \quad (2.30)$$

If this numerator were to obey the same Jacobi identity as the diagram's associated color factor, it would need to vanish under cyclic permutations of legs 1, 2, 3. To see this, note that  $b$  is a second-order operator and so it obeys,

$$\begin{aligned} b(\Psi_1 \Psi_2 \Psi_3) &= b(\Psi_1 \Psi_2) \Psi_3 - \Psi_1 b(\Psi_2 \Psi_3) - b(\Psi_1 \Psi_3) \Psi_2 \\ &\quad - b\Psi_1 \Psi_2 \Psi_3 + \Psi_1 b\Psi_2 \Psi_3 - \Psi_1 \Psi_2 b\Psi_3, \end{aligned} \quad (2.31)$$



which is a simple consequence of the Leibniz rules for  $\frac{\partial}{\partial\theta}$  and  $\partial_\mu$ . In momentum space the same is true, but one has to distribute the momenta in  $b$ , for example,  $b_{k_{12}} = b_{k_1} + b_{k_2}$ . Note, in some diagrams below we abuse notation slightly by using  $b$  without subscript to mean the momentum-space operator when it is clear which states it acts on. In the gauge  $b(\Psi_i) = 0$ , equation (2.31) simplifies to

$$b(\Psi_1\Psi_2\Psi_3) = b(\Psi_1\Psi_2)\Psi_3 + b(\Psi_2\Psi_3)\Psi_1 + b(\Psi_3\Psi_1)\Psi_2 . \quad (2.32)$$

Plugging this in to our diagram (and using the position space representation for  $b$  for readability) we have that it vanishes under cyclic permutations,

$$\int d^3\theta b(\Psi_1\Psi_2)\Psi_3\Psi_4 + \text{cyclic}(1, 2, 3) = \int d^3\theta b(\Psi_1\Psi_2\Psi_3)\Psi_4 = 0 , \quad (2.33)$$

where the last equality follows by integration by parts of  $b$ . The same is true for higher multiplicity diagrams, for example for five points we have the numerator

$$1 \text{ --- } \begin{array}{c} \uparrow 2 \\ \uparrow 3 \\ \uparrow 4 \end{array} \text{ --- } 5 = i \int d^3\theta b(b(\Psi_1\Psi_2)\Psi_3)\Psi_4\Psi_5 . \quad (2.34)$$

In this case external states can be assumed to obey  $b\Psi = 0$ , but we also encounter internal states, such as  $\Psi_{12} \equiv b(\Psi_1\Psi_2)$ . But because of the nilpotency of  $b$ , all internal states (which are dressed by propagators) also obey the gauge condition.

Let us also see a one-loop example, consider the three diagrams below which are related by a Jacobi identity on the propagator between legs 2 and 3. By evaluating the fermionic coordinate integrals we can make all but one of the vertices depend on  $\theta$ , with the remaining vertex depending on  $\tilde{\theta}$ , this vertex is indicated in the diagrams. To simplify the notation we introduce the regional momenta, for example,  $l_{ij} = l + k_i + k_j$ , with which we have the numerators,

$$\begin{array}{c} 2 \text{ --- } \text{---} 3 \\ | \quad | \\ \ell \quad \tilde{\theta} \\ \leftarrow \\ 1 \text{ --- } \text{---} 4 \end{array} = \int d^3\theta d^3\tilde{\theta} \Psi_4 \delta_{\theta, \tilde{\theta}}^3 b_{l_{123}} \left( b_{l_{12}}(X\Psi_2)\Psi_3 \right) , \quad (2.35)$$

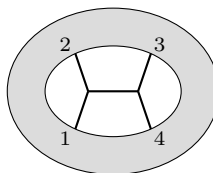
$$\begin{array}{c} 3 \text{ --- } \text{---} 2 \\ | \quad | \\ \ell \quad \tilde{\theta} \\ \leftarrow \\ 1 \text{ --- } \text{---} 4 \end{array} = \int d^3\theta d^3\tilde{\theta} \Psi_4 \delta_{\theta, \tilde{\theta}}^3 b_{l_{123}} \left( b_{l_{13}}(X\Psi_3)\Psi_2 \right) , \quad (2.36)$$

$$\begin{array}{c} 2 \text{ --- } \text{---} 3 \\ \diagdown \quad \diagup \\ \ell \quad \tilde{\theta} \\ \leftarrow \\ 1 \text{ --- } \text{---} 4 \end{array} = \int d^3\theta d^3\tilde{\theta} \Psi_4 \delta_{\theta, \tilde{\theta}}^3 b_{l_{123}} \left( X b_{k_{23}}(\Psi_2\Psi_3) \right) , \quad (2.37)$$

where all three diagrams have the internal state  $X \equiv b_{l_1}(\Psi_1 b_l(\delta_{\theta-\tilde{\theta}}^3))$  in common, and recall that  $b_{l_1} = (l^\mu + k_1^\mu) \frac{\partial}{\partial \theta^\mu}$  and similarly for  $b_l$ . This internal leg satisfies  $bX = 0$  due to  $b^2 = 0$ , therefore, summing up the three numerators and stripping off the integration and the common  $\Psi_4$  factor as well, we find,

$$\begin{aligned} & b(b(X\Psi_2)\Psi_3) - b(b(X\Psi_3)\Psi_2) - b(Xb(\Psi_2\Psi_3)) \\ & = b(b(X\Psi_2)\Psi_3) + b(b(\Psi_3X)\Psi_2) + b(b(\Psi_2\Psi_3)X) = 0. \end{aligned} \quad (2.38)$$

Any diagram with any number of loops can be shown to obey the color kinematics duality too. Considering a generic diagram for which we want to check the Jacobi identity on some internal leg, we can always isolate four-point subdiagram surrounding the leg we want to Jacobi,



$$= \int d^3\theta d^3\tilde{\theta} \left( \theta \Psi_1 \Psi_2 b_{k_{34}} \delta_{\theta, \tilde{\theta}} \Psi_3(\tilde{\theta}) \Psi_4(\tilde{\theta}) \right), \quad (2.39)$$

where now the legs 1, 2, 3 and 4 could refer to either internal or external legs. In either case they obey  $b\Psi_i = 0$ , and it is immediate by the analysis above that the Feynman rules obey the duality.

Having identified off-shell color-kinematics duality, we can also identify an algebra responsible for it. We could write our numerators as nested Poisson brackets,

$$b(\Psi_1\Psi_2) = \frac{\partial}{\partial \theta^\mu} \Psi_1 \partial_\mu \Psi_2 - \partial_\mu \Psi_1 \frac{\partial}{\partial \theta^\mu} \Psi_2 \equiv \{\Psi_1, \Psi_2\}, \quad (2.40)$$

which obey the Jacobi identity. In terms of this Poisson bracket we can formulate our numerators such that the kinematic factors precisely mirror the structure of the color, for example, half-ladder numerators can be written as

$$n_{1, \dots, m} = i \int d^3\theta \{ \dots \{ \{ \Psi_1, \Psi_2 \}, \Psi_3 \}, \dots, \Psi_{m-2} \} \Psi_{m-1} \Psi_m, \quad (2.41)$$

matching the structure of the color factors in (1.28). Instead of the Poisson brackets, one can also introduce diffeomorphism generators labelled by fields  $\Psi$ ,

$$L_\Psi \equiv \frac{\partial}{\partial \theta^\mu} \Psi \partial_\mu - \partial_\mu \Psi \frac{\partial}{\partial \theta^\mu}, \quad (2.42)$$

which satisfy the Lie algebra

$$[L_{\Psi_1}, L_{\Psi_2}] = L_{b(\Psi_1\Psi_2)}, \quad (2.43)$$

and generate infinitesimal diffeomorphisms that preserve the bosonic and fermionic volume forms  $d^3x$  and  $d^3\theta$  separately. Nested commutators of these generators reproduce the Poisson bracket structure above through their structure function, though it is not obvious how to introduce an inner product for the  $L_\Psi$  analogous to the trace for gauge-group generators  $T^a$ . To summarize the off-shell color-kinematics identified here, it follows from the second-order nature of the propagator numerator  $b$ , and the simple relation  $\{b, Q\} = \square$ . We will attempt to extract the same feature from the 10D pure-spinor formulation of super Yang-Mills theory in the next section. But first we add matter to Chern-Simons theory in order to compute non-trivial amplitudes.

The triviality of the Chern-Simons amplitudes can be seen from the fact that the only valid on-shell particle states are  $\Psi = Q\Omega$  for some  $\Omega$ , and also that all amplitudes involving  $Q$ -exact states vanish, as required by gauge invariance. It is possible to obtain non-trivial amplitudes by coupling Chern-Simons to matter, and considering external matter states only. In paper II it was found that a solution exists for a Lagrangian with scalars and spinors in the adjoint representation which is compatible with color-kinematics duality,

$$\begin{aligned} \mathcal{L}_{\mathcal{N}=4} = & \frac{\epsilon_{\mu\nu\rho}}{2} \left( A^{a\mu} \partial^\nu A^{a\rho} - \frac{g}{3} f^{abc} A^{a\mu} A^{b\nu} A^{c\rho} \right) + (D_\mu \bar{\phi})^a (D^\mu \phi)^a \\ & + i \bar{\psi}^a (\not{D} \psi)^a + i g^2 \bar{\psi}^a \psi^b \bar{\phi}^c \phi^d (f^{acx} f^{xbd} + f^{adx} f^{xbc}) \\ & - g^4 \phi^a \bar{\phi}^b \bar{\phi}^c \phi^d \phi^e \bar{\phi}^h f^{abx} f^{xcy} f^{ydz} f^{zeh} , \end{aligned} \quad (2.44)$$

where the barred spinors denote Dirac conjugates. (Note that truncations of this Lagrangian, that is, removing spinors or scalars, are admissible also.) To identify this Lagrangian, an ansatz was made for the interaction terms that could appear in the action, starting at four points, and then the resultant color-ordered amplitudes were plugged into the BCJ relations (1.31). To see an example, consider letting the scalars have either odd or even statistics, then the four-scalar amplitudes are,

$$A_4(\phi_1 \bar{\phi}_2 \phi_3 \bar{\phi}_4) = 2 \epsilon^{k_1 k_2 k_3} \left( \frac{(-1)^{|\phi|}}{k_{23}^2} - \frac{1}{k_{12}^2} \right) , \quad (2.45)$$

$$A_4(\phi_1 \bar{\phi}_2 \bar{\phi}_4 \phi_3) = \frac{2 \epsilon^{k_1 k_2 k_3}}{k_{12}^2} , \quad (2.46)$$

where  $\epsilon^{k_1 k_2 k_3} = \epsilon_{\mu\nu\rho} k_1^\mu k_2^\nu k_3^\rho$ , and  $|\phi|$  indicates the statistics of the scalar field (the explicit Feynman rules are available in paper II). These two amplitudes obey the BCJ relation  $k_{23}^2 A(1234) = k_{13}^2 A(1243)$ , so long as  $|\phi| = 1$ , meaning the scalars are anticommuting. Similar constraints were used to fix the statistics of the spinor fields, as well as the coefficients of all of the non-minimal interactions in the action, finding the unique result presented above, with commuting spinors and anticommuting scalars.

Interestingly, the constraints from the BCJ relations resulted in a Lagrangian that is supersymmetric, with  $\mathcal{N} = 4$  supersymmetry transformations,

$$\begin{aligned}\delta\phi_\alpha &= \bar{\xi}_{\dot{\alpha}}\psi_{\dot{\alpha}} , \\ \delta A_\mu^a &= g\bar{\xi}^{\dot{\alpha}\alpha}\gamma_\mu\psi_{\dot{\alpha}}^b\phi_\alpha^c f^{abc} , \\ \delta\psi_{\dot{\alpha}}^a &= i(\not{D}\phi^\alpha)^a\xi_{\alpha\dot{\alpha}} - \frac{ig^2}{3}f^{abc}\phi_\beta^b(\tau^c)^{\beta\alpha}\xi_{\alpha\dot{\alpha}} ,\end{aligned}\tag{2.47}$$

where  $(\tau^c)_{\alpha\beta} \equiv \phi_\alpha^a\phi_\beta^b f^{abc}$  and an  $SU(2) \times SU(2) \sim SO(4)$  R-symmetry has been made manifest by introducing,

$$\phi_\alpha = (\bar{\phi}, \phi), \quad \psi_{\dot{\alpha}} = (\bar{\psi}, \psi).\tag{2.48}$$

The partial amplitudes from this supersymmetric Chern-Simons theory with opposite matter statistics turn out to be related to more standard formulations of supersymmetric matter Chern-Simons theories where the gauge group is  $SU(N) \times SU(N)$  and the matter transforms in the bi-fundamental representation. In particular, the color-ordered amplitudes computed from the adjoint theory match those of Gaiotto-Witten  $\mathcal{N} = 4$  supersymmetric Chern-Simons-matter theory [60, 61]. This was checked through multiplicity eight by comparing to  $\mathcal{N} = 4$  truncations of  $\mathcal{N} = 6$  Aharony-Bergman-Jafferis-Maldacena (ABJM) amplitudes. The full  $\mathcal{N} = 6$  ABJM amplitudes do not obey the standard BCJ relations, indicating that the  $\mathcal{N} = 4$  Chern-Simons-matter theory has maximal supersymmetry that is compatible with the BCJ relations. This observation is further confirmed by studying the double copy of the supersymmetric theory, which produces amplitudes from maximally supersymmetric  $\mathcal{N} = 8$  Dirac-Born-Infeld (DBI) theory with fields obeying the correct statistics, this was checked through multiplicity six.

The correct statistics after the double-copy can be seen from the KLT formula in equation (1.32). Consider for example the double copy of the anticommuting scalar amplitudes in equation (2.45), which is

$$k_{12}^2 A_4(\phi_1\bar{\phi}_2\phi_3\bar{\phi}_4)A_4(\phi_1\bar{\phi}_2\bar{\phi}_4\phi_3) = 8k_{13}^4.\tag{2.49}$$

Since each of the amplitudes on the left-hand side are antisymmetric in the labels of fields, the right hand side must be symmetric, with bosonic states identified from the double-copy map,

$$\begin{aligned}\phi \otimes \phi &\rightarrow \phi_0 + i\phi_1, & \bar{\phi} \otimes \bar{\phi} &\rightarrow \phi_0 - i\phi_1, \\ \phi \otimes \bar{\phi} &\rightarrow \phi_2 + i\phi_3, & \phi \otimes \bar{\phi} &\rightarrow \phi_2 - i\phi_3,\end{aligned}\tag{2.50}$$

with and  $SO(4)$  symmetry rotating them. But  $\mathcal{N} = 8$  DBI has eight scalars and eight fermions transforming under an  $SO(8)$  R-symmetry

[62, 63], so we expect to find four more scalars. Recall that in 3d the little group is  $\mathbb{Z}_2$ , which acts on on-shell states by multiplication by a minus sign [64]. Physical fields are either odd or even under this transformation, and therefore a double copy involving two odd states produces an even one. We thus find four additional scalars from the double copies of pairs of spinor fields, as well as eight fermions from the double copy of scalars with spinors. In paper II the  $SO(8)$  symmetry was explicitly tested through six points, and amplitudes were matched with an alternative realization of DBI amplitudes through the double copy of  $\mathcal{N} = 8$  super Yang-Mills theory and the non-linear sigma model.

It is interesting that the constraints from the color-kinematics duality landed us on a supersymmetric theory. Similar connections between color-kinematics duality and supersymmetry have been observed before [65], which motivates further exploration of this topic. This partly motivates the next section, where we will study maximally supersymmetric Yang-Mills theory, using the pure-spinor formulation of the action.

## 2.3 Pure Spinor Super Yang-Mills Theory

In this section we study super Yang-Mills theory using pure spinors [66, 55], summarizing the Feynman rules found in paper I and explaining their connection to color-kinematics duality. On-shell BCJ numerators have successfully been constructed in the pure-spinor description of super Yang-Mills theory [67, 68], and they have been understood through gauge transformations of Lorentz-gauge currents with one off-shell leg [15]. But our aim is to reflect the Chern-Simons construction above as closely as possible, and give some description of color-kinematics duality directly from the Feynman rules of the pure-spinor action.

We use Latin letters to denote 10D Lorentz indices, and Greek letters for 16-component Weil spinor indices. Our  $16 \times 16$  component gamma matrices  $(\gamma^m)_{\alpha\beta}$  and  $(\gamma^m)^{\alpha\beta}$  are symmetric, and we denote their antisymmetrized products as  $\gamma^{mn} = \gamma^{[m}\gamma^{n]} = \frac{1}{2}(\gamma^m\gamma^n - \gamma^n\gamma^m)$ , for example. When spinor indices are contracted in the obvious way we suppress them, as in  $\lambda^\alpha(\gamma_{mn})_\alpha{}^\beta r_\beta = (\lambda\gamma^{mn}r)$ . The worldline action for the pure-spinor superparticle is [55],

$$S = \int d\tau \left( p_m \dot{x}^m + p_\alpha \dot{\theta}^\alpha + \omega_\alpha \dot{\lambda}^\alpha - \frac{1}{2} p_m p^m \right), \quad (2.51)$$

where aside from the position  $x^m$  and momentum  $p_m$  variables, we also have the 16-component fermionic coordinate  $\theta^\alpha$  and its conjugate momentum  $p_\alpha$  (the momenta are distinguished by their index, as are partial derivatives with respect to  $x$  and  $\theta$ , namely  $\partial_m$  and  $\partial_\alpha$ ). The next canonical pair  $\omega_\alpha$  and  $\lambda^\alpha$  are bosonic spinors, and  $\lambda$  obeys the

pure-spinor constraint  $(\lambda\gamma^m\lambda) = 0$ , which induces a gauge symmetry  $\omega_\alpha \sim \omega_\alpha + v_m(\gamma^m\lambda)_\alpha$ . These 10 equations constrain 5 independent components in  $\lambda$ , leaving them with 11 degrees of freedom (see for example ref. [69] for connection to Brink-Schwarz superparticle). Finally, we have a  $p^2$  term in the action from gauge fixing, it will not play a role here as we will switch to the field theory description for the purpose of studying the Feynman rules.

To start constructing the field-theory action we introduce the worldline BRST operator

$$Q = \lambda^\alpha D_\alpha, \quad (2.52)$$

where  $D_\alpha = p_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha p_m$ . Note that  $Q$  is nilpotent due to the pure-spinor condition  $(\lambda\gamma^m\lambda) = 0$ . Following ref. [55] and the Chern-Simons discussion above, we want to introduce some integration measure on the complete space of worldline coordinates, meaning  $x, \theta, \lambda$ . We follow ref. [70] in regularizing the integrals over the bosonic  $\lambda$ . The first step is to introduce an additional worldline bosonic canonical pair  $\bar{\omega}^\alpha$  and  $\bar{\lambda}_\alpha$  satisfying  $(\bar{\lambda}\gamma^m\bar{\lambda}) = 0$  and  $\bar{\omega}^\alpha \sim \bar{\omega}^\alpha + v_m(\gamma^m\bar{\lambda})^\alpha$  as before. In addition we introduce the constraint  $\bar{\omega} = 0$  in order for these extra variables to not contribute to the degrees of freedom (setting  $\bar{\omega} = 0$  allows for arbitrary shifts in the variables  $\bar{\lambda}$ , which means we can always find cohomology representatives that do not depend on the non-minimal sector). This constraint then results in a pair of fermionic spinor ghosts  $s^\alpha$  and  $r_\alpha$  subject to  $(\bar{\lambda}\gamma^m r) = 0$  in order to deal with the gauge symmetry for  $\bar{\omega}$ . Altogether we find a new, non-minimal, BRST operator,

$$\mathbf{Q} = Q + r_\alpha \bar{\omega}^\alpha. \quad (2.53)$$

And finally, with this BRST operator we can define the BRST-exact regulator  $\mathcal{N}$  [71, 70],

$$\mathcal{N} = \exp(-\{\mathbf{Q}, (\bar{\lambda}\theta)\}). \quad (2.54)$$

We can now assemble the superfield action for super Yang-Mills theory [55],

$$S = \frac{i}{g^2} \int [dZ] \mathcal{N} \text{tr} \left( \frac{1}{2} \Psi \mathbf{Q} \Psi + \frac{1}{3} \Psi \Psi \Psi \right), \quad (2.55)$$

which closely resembles the Chern-Simons action. Due to the constraints on our spinor fields, however, we have a much more complicated integration measure. We write it as,

$$[dZ] = d^{10}x d^{16}\theta d^{11}\lambda d^{11}\bar{\lambda} d^{11}r, \quad (2.56)$$

and a more precise definition of the measure can be found in [66, 72, 70]. Since the non-minimal variables are a trivial sector, representatives of the BRST cohomology can be chosen that are independent of

them. The fields are then taken to have a polynomial expansion in the pure-spinor variables, mirroring the ghost-number expansion for Chern-Simons above. In such a case we would have

$$\Psi(x, \theta, \lambda) = \Psi_0(x, \theta) + \lambda^\alpha \Psi_\alpha(x, \theta) + \lambda^\alpha \lambda^\beta \Psi_{\alpha\beta}(x, \theta) + \mathcal{O}(\lambda^3) . \quad (2.57)$$

And again just like in Chern-Simons theory, the terms in this series expansion contain the ghost  $c$  at ghost-number zero, the physical gluon  $A_m$  and gluino  $\psi_\alpha$  at ghost-number one, the antifields  $\bar{A}_m$  and  $\bar{\psi}_\alpha$  at ghost-number two, and finally the antighost  $\bar{c}$  at ghost-number three [55, 73]. Terms at order  $\lambda^4$  decouple from the action, as the integration measure projects on to  $\lambda^3\theta^5$  order only. In the Harnad-Schnider gauge  $\theta^\alpha \Psi_\alpha = 0$  [74], the first few terms at ghost-number one are [75],

$$\begin{aligned} \Psi_\alpha(x, \theta) &= \frac{1}{2}(\gamma^m \theta)_\alpha A_m(x) - \frac{1}{3}(\gamma^m \theta)_\alpha (\theta \gamma_m \psi(x)) \\ &\quad - \frac{1}{16}(\gamma_p \theta)_\alpha (\theta \gamma^{mnp} \theta) p_m A_n(x) + \dots \end{aligned} \quad (2.58)$$

The action (2.55) has a gauge symmetry under  $\Psi \rightarrow \mathbf{Q}\Omega + [\Psi, \Omega]$  for any  $\Omega$ . To gauge fix, we once again use the  $b$ -ghost operator which obeys [70]

$$\mathbf{Q}b + b\mathbf{Q} = p_m p^m , \quad b^2 = 0 . \quad (2.59)$$

The explicit form of  $b$  is

$$b = \frac{(\bar{\lambda} \gamma^m D)}{(\lambda \bar{\lambda})} p_m + \frac{(\bar{\lambda} \gamma^{mnp} r)[-(D \gamma_{mnp} D) + 24 N_{mn} p_p]}{96(\lambda \bar{\lambda})^2} \quad (2.60)$$

$$- \frac{(r \gamma^{mnp} r)(\bar{\lambda} \gamma_m D) N_{np}}{8(\lambda \bar{\lambda})^3} - \frac{(r \gamma^{mnp} r)(\bar{\lambda} \gamma^{pqr} r) N_{mn} N_{qr}}{64(\lambda \bar{\lambda})^4} , \quad (2.61)$$

where we the operator  $N^{mn} = -\frac{1}{2}(\lambda \gamma^{mn} \omega)$  is the generator of rotations for the pure-spinor variables.

The  $b$  ghost can be used to gauge fix by demanding that  $b\Psi = \mathbf{Q}\chi$ , for some  $\chi$ . For the special choice  $\chi = 0$  we will refer to this gauge as the Siegel gauge, which can be imposed by taking the on-shell states to be in the image of  $b$ , resulting in wave functions that depend on the non-minimal variables (the Harnad-Schnider gauge state in equation (2.60) is not compatible with the Siegel gauge, see paper I). We see that when demanding  $b\Psi = \mathbf{Q}\chi$ , the propagator for this theory is simply  $b/\square$ , taking exactly the same form as in Chern-Simons theory. In addition,  $b$  is second order because it is quadratic in the first-order operators  $p_m$ ,  $D_\alpha$ , and  $N_{mn}$ , meaning it obeys equation (2.31) like in Chern-Simons theory ( $N_{mn}$  depends on  $\omega$  which is a  $\lambda$  derivative). The upshot is that the Feynman rules also take the same schematic form as in Chern-Simons

theory, consisting of nested  $b$  operators, and that since  $b$  is a second order operator, it seems that we have off-shell color kinematics duality in the Siegel gauge.

However, divergences from poles in  $(\lambda\bar{\lambda}) \rightarrow 0$  from the  $b$  ghost will pose a problem for the definition of the integration measure. Indeed, the integrals diverge if the poles in the pure-spinor variables exceed  $\lambda^{-8}\bar{\lambda}^{-11}$  [70]. The problem can be understood from the trivialization of the BRST operator; allowing for too big of a divergence in the pure-spinor variables means we could construct the operator  $\xi = (\theta\bar{\lambda})/(\lambda\bar{\lambda} + r\theta)$ , which satisfies  $Q\xi = 1$ . Then  $Q\xi = 1$  together with the fact that  $Q$  is first order implies that the  $Q$  cohomology is trivialized. Interestingly, this trivialized cohomology makes the theory behave much like Chern-Simons theory, in which all scattering amplitudes are zero and color-kinematics duality holds off shell. While there are ways to regularize the  $(\lambda\bar{\lambda}) \rightarrow 0$  divergences in a BRST invariant manner [76], it is not clear if these can be used in such a way that  $b$  remains second order.

Despite the divergence issues, there have been some interesting works building on the construction seen here. The authors of ref. [77] have discussed approaches based on decoupling divergences from the numerators, as well as possible double copies with the pure-spinor actions [78]. It would be interesting to attempt to construct a regularization scheme for  $b$  which preserves its second order property, or in absence of this, to explicitly realize a BRST-covariant way to decouple the divergences from tree and loop numerators such that they still obey the color-kinematics duality.

## 2.4 Discussion

In this section we summarize some of the lessons we learned from Chern-Simons theory and the pure-spinor superparticle. In both cases we found a cubic action of the schematic form  $\langle\psi Q\psi + \psi^3\rangle$ . In addition, we identified a second-order operator  $b$  that obeyed  $bQ + Qb = \square$ . The second-order nature of  $b$ , together with the cubic interaction term and gauge-fixing choice  $b\Psi = 0$ , gives rise to the off-shell identity  $b(b(\psi_1\psi_2)\psi_3) + \text{cyclic} = 0$ . For Chern-Simons theory this implied off-shell color-kinematics duality, yet for super Yang-Mills we ran in to regularization issues. In this context, off-shell color-kinematics duality is understood to mean that not only do numerators obey the duality off-shell, but that it also holds for any set of external states of the theory, and that the Feynman rules are cubic. The earliest example of such off-shell color-kinematics duality is for the self-dual sector of Yang-Mills theory [79], where tree-level amplitudes are zero, and the theory is one-loop exact. Other occurrences of Feynman rules with diffeomorphism algebras in the



kinematic sector have also been identified in two dimensions [80], but for Yang-Mills theory a realization of off-shell color-kinematics duality still has not been found.

Nonetheless, it is interesting to ask if the structure of differential operators we found in Chern-Simons theory can be generalized at all. One way to generalize the construction is to first assume the propagator-numerator is proportional to the codifferential,  $b = id^\dagger$ , in some dimension, and then look for kinetic terms for which this is an appropriate propagator-numerator. By choosing  $Q = -id$  of course, the construction goes through, generalizing the Chern-Simons action to BF-like theory in higher dimensions (see ref. [81] for early constructions of BF theories). But additional deformations of  $Q$  are also permitted, as long as the relation  $bQ + Qb = \square$  is preserved. For example, by introducing an additional Grassmann-odd coordinate  $\xi$ , the superfields are extended by  $\Psi(\theta, \xi) = \psi(\theta) + \xi\phi(\theta)$ , and the kinetic term can be modified to  $Q = d + \frac{\partial}{\partial\xi}$ , producing a large family of topological theories in Alexandrov-Schwarz-Zaboronsky-Kontsevich formulation [82], see for example ref. [83]. These have trivial on-shell scattering amplitudes, just like Chern-Simons theory.

### 3. Spinning Compact Object

This chapter follows paper IV in the construction of the effective field theory (EFT) of a spinning compact object, including an explanation of the degrees of freedom and gauge symmetries of the theory. A convenient gauge-fixing scheme will be presented that allows the EFT to be used together with the worldline quantum field theory (WQFT). The use of the EFT for the spinning compact object enables computations to any order in spin using the WQFT, therefore extending it beyond the spin-squared upper bound from supersymmetric actions [35]. At the end of the chapter, the spinning particle is quantized in flat space, and its ghost-number zero wave functions are presented to any order in spin.

#### 3.1 The Spinning Particle EFT

The minimal action for the spinning particle, without any Wilson coefficients, is [84]

$$S = - \int d\tau \left( p_\mu \dot{x}^\mu + \frac{1}{2} S_{ab} \Omega^{ab} + \frac{D\hat{p}^\mu}{d\tau} S^{\mu\nu} \hat{p}_\nu \right) \quad (3.1)$$

$$- \frac{\ell}{2} (p^2 - m^2) - \ell_a S^{ab} (\hat{p}_b + \Lambda_{0b}) \Big). \quad (3.2)$$

Starting from left to right we see the standard Hamiltonian kinetic term for position and momentum,  $p_\mu \dot{x}^\mu$ , where dot represents time derivative  $\frac{d}{d\tau}$  and Greek indices  $\mu, \nu$  belong to the general spacetime manifold with metric  $g_{\mu\nu}$ . Next we have the kinetic term for the spin sector, where  $S_{ab}$  and  $\Omega^{ab}$  are the spin tensor and angular velocity, the latter being defined by

$$\Omega^{ab} = \Lambda_I^a \frac{D}{d\tau} \Lambda^{bI} = \Lambda_I^a \dot{\Lambda}^{bI} - \dot{x}^\mu \omega_\mu^{ab} \ , \quad (3.3)$$

with spin connection  $\omega_\mu^{ab}$ , and  $\frac{D}{d\tau}$  the covariant time derivative along the worldline. The lowercase Latin indices  $a, b$  belong to a flat local frame with vielbeins  $e_\mu^a$  and metric  $\eta_{ab}$ , these can be used to construct spin tensors, for example, with  $\mu, \nu$  indices. The  $\Lambda_I^a$  are frame fields that rotate from the flat frame to a frame that is “body fixed” or “corotating” with the particle. They obey  $\Lambda_I^a \Lambda_J^b \eta_{ab} = \eta_{IJ}$ , and are defined on the

worldline,  $\Lambda_I^a = \Lambda_I^a(\tau)$ . Next we have a term containing the covariant time derivative of the momentum, sometimes referred to as the acceleration term. Its origin is in a gauge-unfixing procedure [84] starting with a worldline particle subject to the so-called covariant spin supplementary condition (SSC),  $S^{ab}p_b = 0$  and  $\Lambda_0^a = \hat{p}^a$  (here and in the action we used the unit-normalized momentum,  $\hat{p}^\mu = p^\mu / \sqrt{p^2}$ ). Later on we will see how we can return to the covariant SSC by a consistent gauge-fixing procedure.

On the second line we find the mass-shell constraint and the spin constraint

$$\mathcal{C}^a = S^{ab}(\hat{p}_b + \Lambda_{0b}) , \quad (3.4)$$

imposed by Lagrange multipliers  $\ell$  and  $\ell_a$ . The latter constraint is responsible for eliminating three degrees of freedom from the spin sector, and a further three degrees of freedom can be eliminated from the frame fields  $\Lambda_a^I$  using the gauge symmetry,

$$\begin{aligned} \delta S_{\mu\nu} &= 2\hat{p}_{[\mu} S_{\nu]\alpha} \epsilon^\alpha , \\ \delta \Lambda_I^\mu &= 2\epsilon^{[\mu} \hat{p}^{\nu]} \Lambda_{I\nu} + 2\epsilon^{[\mu} \Lambda_0^{\nu]} \Lambda_{I\nu} , \\ \delta \ell^\mu &= -\frac{D\epsilon^\mu}{d\tau} + \dots \end{aligned} \quad (3.5)$$

When checking the gauge invariance of the action, any transformation of the action that results in something proportional to the constraint  $\mathcal{C}^a$  can be absorbed in to a compensating one for the Lagrange multiplier. These transformations are implicitly included in the ellipsis above and are not important to specify in detail.

We have to determine if the transformation (3.5) is a non-trivial gauge symmetry, meaning it actually corresponds to an arbitrariness in the solutions to the equations of motion, which implies that there will be undetermined functions in our solution space (see ref. [85] for examples of trivial transformations). The way to do this is to check if the constraints are conserved in time. To understand why, we write our equations of motion schematically as,

$$\dot{x}^\mu = \dots , \quad (3.6)$$

$$\dot{p}_\mu = \dots , \quad (3.7)$$

$$\dot{S}_{\mu\nu} = \dots , \quad (3.8)$$

$$\dot{\Lambda}_I^\mu = \dots , \quad (3.9)$$

$$p^2 - m^2 = 0 , \quad (3.10)$$

$$S \cdot (\hat{p} + \Lambda_0) = 0 . \quad (3.11)$$

Now the right-hand sides of the first four (dynamical) equations are functions of  $x, p, S, \Lambda$  as well as the two Lagrange multipliers  $\ell$  and  $\ell^a$ .

However, there is no equation of motion for  $\ell$  and  $\ell^a$ . We could try to add equations to our system by demanding that the constraints are preserved in time, e.g.  $\frac{D}{d\tau} S \cdot (\hat{p} + \Lambda_0) = 0$ . If the preservation of the constraints holds, then we would have that any solution initially obeying the constraints would continue to do so throughout its evolution, which we would want to impose in any case. Importantly, the preservation of the constraints could yield algebraic relations for the Lagrange multipliers, because in

$$\frac{D}{d\tau} S \cdot (\hat{p} + \Lambda_0) = 0, \quad \frac{D}{d\tau} (p^2 - m^2) = 0, \quad (3.12)$$

we could replace all time derivatives using the equations of motion in (3.6)-(3.11). If the Lagrange multipliers were determined from the preservation equations (3.12), they would not be arbitrary functions, and therefore we would not be permitted to apply local transformations to them. In such a scenario, the constraints associated to the Lagrange multipliers would be called second class.

In our case, however, the preservation of the constraints holds immediately after eliminating time derivatives in the conservation equations (3.12), irrespective of what values the Lagrange multipliers take (preserved constraints are called first class). This means there is no way to fix the Lagrange multipliers, and they genuinely are arbitrary functions in our solution space. Therefore, our gauge transformations are valid, meaning they relate physically equivalent solutions. Let us show the conservation of the constraints explicitly, following paper IV. We need only use the two equations of motion,

$$\Omega^{\mu\nu} + 2 \frac{D\hat{p}^{[\mu}}{d\tau} \hat{p}^{\nu]} - 2\ell^{[\mu} (\hat{p}^{\nu]} + \Lambda_0^{\nu]}) = 0, \quad (3.13)$$

$$\frac{D}{d\tau} S_{\mu\nu} - 2S_{[\mu|\rho} \Omega^{\rho}_{\nu]} - 2\ell_\rho S^\rho_{[\mu} \Lambda_{\nu]0} = 0. \quad (3.14)$$

It is convenient to introduce “weak equality”, meaning equality up to terms proportional to the constraints, and indicate it by  $\approx$ . The subspace of the coordinate space that obeys the constraints is called “the constraint surface”, thus weak equality means equality on the constraint surface. Using the equation of motion for  $S_{\mu\nu}$  we then find,

$$\begin{aligned} \frac{D}{d\tau} \mathcal{C}^\mu &= (2S_{[\mu|\rho} \Omega^\rho_{\nu]} + 2\ell_\rho S^\rho_{[\mu} \Lambda_{\nu]0}) (\hat{p}^\nu + \Lambda_0^\nu) + S^{\mu\nu} \left( \frac{D\hat{p}_\nu}{d\tau} - \Omega_{\nu\rho} \Lambda_{\rho 0} \right) \\ &\approx (S_{\mu\rho} \Omega^\rho_{\nu]} + \ell_\rho S^\rho_{\mu} \Lambda_{\nu 0}) (\hat{p}^\nu + \Lambda_0^\nu) + S^{\mu\nu} \left( \frac{D\hat{p}_\nu}{d\tau} - \Omega_{\nu\rho} \Lambda_{\rho 0} \right) \\ &= S^{\mu\nu} \frac{D\hat{p}_\nu}{d\tau} + S_{\mu\rho} \Omega^\rho_{\nu} \hat{p}^\nu + \ell_\rho S^{\rho\mu} (\Lambda_0 \cdot \hat{p} + 1). \end{aligned} \quad (3.15)$$

Similarly, substituting  $\Omega_{\mu\nu}$  from (3.13), we find

$$\frac{D}{d\tau} S^{\mu\nu} (\hat{p}_\nu + \Lambda_{\nu 0}) \approx S_{\mu\rho} \hat{p}^\rho \frac{D\hat{p}^\nu}{d\tau} \hat{p}_\nu = 0, \quad (3.16)$$

which is zero due to the fact that  $\hat{p}$  is unit normalized.

To see an alternative situation where the constraints are not automatically preserved, let us temporarily return to the covariant SSC, discussed below equation (3.1). We interpret the covariant SSC as a gauge-fixing condition  $\hat{p}^a - \Lambda_0^a = 0$ , imposed through a new Lagrange multiplier. In effect we replace

$$\ell_a S^{ab} (\hat{p}_b + \Lambda_{0b}) \rightarrow \ell_a S^{ab} \hat{p}_b + \bar{\ell}_a (\hat{p}^a - \Lambda_0^a), \quad (3.17)$$

in the action. Since now the gauge symmetry is fixed, the Lagrange multipliers are no longer arbitrary, and should be determined by the preservation of the constraints. See ref. [86], where a similar model was considered and the equations of motion were successfully used to explicitly solve for the Lagrange multipliers. In this thesis however, we will employ another gauge choice, more convenient for the application of the WQFT, see section 3.2.

Finally, we mention that the action is also reparametrization invariant, with all fields transforming as scalars under this transformation (e.g.  $\delta p_\mu = \epsilon \dot{p}_\mu$ ) and both Lagrange multipliers transforming as forms (e.g.  $\delta \ell = \epsilon \dot{\ell} + \dot{\epsilon} \ell$ ). This gauge symmetry will also need to be fixed before we do perturbation theory. When we extend the action to include non-minimal interactions, it would have to be done in such a way that both of the gauge symmetries are respected.

## Non-minimal Interactions

In order to describe generic spinning compact objects, the action (3.1) needs to be extended with additional non-minimal interactions. Some of the additional Wilson coefficients that will be added will turn out to be necessary in order to match the worldline amplitudes to those of Kerr black holes [87, 88]. Note that by only adding interactions to the action (3.1) without adding additional fields, we will be restricted to considering spin-magnitude conserving models (see ref. [89] for a discussion of the connection between the spin constraints and spin conservation).

To facilitate constructing the EFT we will keep track of mass and length scales (we set speed of light to unity), the relevant scaling of operators and parameters in the theory are,

$$\hbar \sim [L][M], \quad S \sim [L][M], \quad G \sim [L][M]^{-1}, \quad (3.18)$$

therefore, as mentioned in section 1.3, we have two length scales, one given by the Schwarzschild radius  $r_s = 2Gm$  and one by the ring radius  $a = |S|/m$ . We will consider the effective action of a single compact object, where by compact it is meant that  $r_s$  and  $a$  are of comparable size, and the object will not have additional intrinsic length scales besides them.

We now seek to classify all operators that could appear in our computations. We must write these operators such that they respect general covariance, the spin gauge symmetry, and worldline reparametrization invariance. We will assume that all additional operators we add to the action will depend on the Riemann curvature tensor  $R_{\mu\nu\rho\sigma}$ , therefore in the flat space limit the non-minimal interactions will vanish (note that terms proportional to the Ricci scalar or tensor can be eliminated using the Einstein field equations). Since we will effectively work perturbatively in  $G$ , at any given level of precision only a finite number of these will appear. This is due to the fact that the curvature has dimensions  $R \sim [L]^{-2}$ , and must be compensated either by factors of  $Gm$  or  $S/m$ . Additional factors of mass that may appear are easily compensated for by multiplying or dividing by  $m$ .

At linear order in curvature we now find an infinite family of possible operators, taking the general form  $(S\partial)^n RS^2$ . To ensure that they respect the spin gauge symmetry we introduce the spin vector,

$$S^\mu = \epsilon^{\mu\nu\rho\sigma} S_{\nu\rho} \hat{p}_\sigma , \quad (3.19)$$

which is clearly gauge invariant on its own. We now also define the electric and magnetic components of the Riemann tensor,

$$E_{\mu\nu} = R_{\mu\rho\nu\sigma} \hat{p}^\rho \hat{p}^\sigma , \quad (3.20)$$

$$B_{\mu\nu} = \frac{1}{2} R_{\alpha\beta\rho\mu} \epsilon^{\alpha\beta\gamma}{}_\nu \hat{p}^\rho \hat{p}^\gamma , \quad (3.21)$$

which are all we need in order to write all independent parity-invariant operators linear in curvature [84],

$$\mathcal{O}(RS^{2n}) \propto D_{\mu_{2n}} \dots D_{\mu_3} E_{\mu_1\mu_2} S^{\mu_1} \dots S^{\mu_{2n}} , \quad (3.22)$$

$$\mathcal{O}(RS^{2n+1}) \propto D_{\mu_{2n+1}} \dots D_{\mu_3} B_{\mu_1\mu_2} S^{\mu_1} \dots S^{\mu_{2n+1}} . \quad (3.23)$$

For such extra operators to also respect the reparametrization invariance we multiply them by  $\ell$ . Instead of multiplying with  $\ell$  one could consider adding operators with a single time derivative, for instance by replacing  $\ell p \rightarrow \dot{x}$ . However, at the cost of redefining some of the Wilson coefficients in the action, time derivatives can be eliminated using the equations of motion, therefore these operators are not independent. In addition, operators where the covariant derivatives contract with each other, such

as  $S^2 D^2 E_{\mu\nu} S^\mu S^\nu$ , vanish after using the Bianchi identities and equations of motion so they are also not independent.

Next, at quadratic order in curvature we see non-trivial local operators starting at fourth order in spin, taking the schematic form  $R^2 S^4$ , where, as before, we ensured the operator does not have an overall length scale. In addition, we have the freedom to replace an  $S^2$  with a  $G^2$ , so we also have operators  $R^2 S^2 G^2$  and  $R^2 G^4$ . We will ignore the latter possibilities though since we are not able to determine their coefficients by comparison with the references [87, 88]. We will not be going to higher orders than  $S^4 R^2$ , but the general pattern is clear, for example we would need  $S^6$  in order to cancel the inverse length scale of  $R^3$  (which would contribute at overall  $G^9$ ), and we could always act on our existing operators with  $S^\mu D_\mu$  in various ways.

Overall, the worldline action we employ is

$$\begin{aligned}
S = - \int d\tau & \left( p_\mu \dot{x}^\mu + \frac{1}{2} S_{ab} \Omega^{ab} + \frac{D\hat{p}_\mu}{d\tau} S^{\mu\nu} \hat{p}_\nu - \frac{\ell}{2} (p^2 - m^2) \right. \\
& - \ell_a S^{ab} (\hat{p}_b + \Lambda_{b0}) + \frac{\ell c_{ES^2}}{2m^2} E_{\mu\nu} S^\mu S^\nu \\
& - \frac{\ell c_{BS^3}}{6m^3} D_\mu B_{\nu\rho} S^\mu S^\nu S^\rho - \frac{\ell c_{ES^4}}{24m^4} D_\mu D_\nu E_{\rho\sigma} S^\mu S^\nu S^\rho S^\sigma \\
& + \frac{\ell c_{E^2 S^4}}{m^6} (E_{\mu\nu} S^\mu S^\nu)^2 + \frac{\ell c_{B^2 S^4}}{m^6} (B_{\mu\nu} S^\mu S^\nu)^2 \\
& + \frac{\ell c'_{E^2 S^4}}{m^6} S^2 S^\mu E_{\mu\nu} E^\nu{}_\rho S^\rho + \frac{\ell c'_{B^2 S^4}}{m^6} S^2 S^\mu B_{\mu\nu} B^\nu{}_\rho S^\rho \\
& \left. + \frac{\ell c''_{E^2 S^4}}{m^6} S^4 E^2 + \frac{\ell c''_{B^2 S^4}}{m^6} S^4 B^2 \right), \tag{3.24}
\end{aligned}$$

where the  $c_{\dots}$  are the dimensionless Wilson coefficients, the mass dimension of the extra operators has been compensated for by factors of  $m^{-1}$ , and each of the new operators has been multiplied by  $\ell$ , thus ensuring reparametrization invariance of the full action. Writing the operators as we did, they can be absorbed in to a so-called dynamical mass, by writing the mass-shell constraint as  $p^2 - \mathcal{M}^2$ , where  $\mathcal{M} = m^2 + \dots$  includes these extra operators. The action with a dynamical mass function written in terms of the spin vector still respects the same conservation of constraints we saw in the previous subsection, as can be expected since it is engineered to respect the gauge symmetry. This is shown explicitly in paper IV.

## 3.2 WQFT for the Spinning Particle

We will now employ the WQFT in order to compute amplitudes from the spinning effective worldline theory, and fix the Wilson coefficients by

matching to Kerr observables [87, 88]. The Feynman rules are extracted from the path integral (1.36), where the full action now has the worldline point-particle action (3.24) instead of the non-spinning Hamiltonian studied in the introduction. Recall that the general idea is to solve the equations of motion in flat space, and use them as background fields for worldline fluctuations. However, the kinetic term is not invertible without first gauge fixing the reparametrization invariance and spin gauge symmetry. While it is possible to gauge fix by imposing some condition on the  $\Lambda_I^a$  fields, see (3.17) for the covariant SSC, such a condition would either have to be solved explicitly (possibly breaking manifest covariance), or it would need to be imposed with additional Lagrange multipliers. Instead, it is more convenient to constrain the Lagrange multipliers themselves. A gauge choice that leads to simple propagators is

$$\ell^a = \frac{1}{\hat{p}} \frac{Dp^a}{d\tau}, \quad \ell = \frac{1}{m}, \quad (3.25)$$

fixing both spin gauge symmetry and reparametrization invariance. The gauge choice for the spin sector matches the covariant SSC up to corrections at higher orders in spin and curvature. With this gauge choice the minimal action simplifies to

$$S = - \int d\tau \left( p_\mu \dot{x}^\mu + \frac{1}{2} S_{ab} \Omega^{ab} - \frac{1}{2m} (p^2 - m^2) - \frac{1}{p} \frac{Dp_\mu}{d\tau} S^{\mu\nu} \Lambda_{\nu 0} \right). \quad (3.26)$$

Next, for the purpose of constructing the WQFT Lagrangian, the flat-space solutions to the equations of motion are constant spin variables,  $\dot{S}_{\mu\nu} = 0$  and  $\dot{\Lambda}_I^\mu = 0$ , and the same position and momentum vectors encountered in the non-spinning case

$$p_\mu = mv_\mu, \quad x^\mu = b^\mu + v^\mu \tau, \quad (3.27)$$

together with  $v^2 = 1$ . These flat-space solutions obey  $\Lambda_0^\mu = v^\mu$  and  $S_{\mu\nu} v^\nu = 0$  due to our gauge choice. Next, we use these flat-space solutions as background fields in the action, with the substitution,

$$p_\mu \rightarrow mv_\mu + \pi_\mu, \quad (3.28)$$

$$x^\mu \rightarrow b^\mu + v^\mu \tau + z^\mu, \quad (3.29)$$

$$S_{ab} \rightarrow S_{ab} + s_{ab}, \quad (3.30)$$

$$\Lambda_I^a \rightarrow \Lambda_I^a + \lambda^{ab} \Lambda_{Ib} + \frac{1}{2} \lambda^{ab} \lambda_{bc} \Lambda_I^c + \dots, \quad (3.31)$$

where now the dynamical fields are  $W = \{z^\mu, \pi_\mu, s^{ab}, \lambda_{ab}\}$ , while  $S_{ab}$  and  $\Lambda_I^a$  as the constant solutions to the equations of motion. Note that





$$\begin{aligned}
\text{---}\overline{\text{---}}\begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} &= \frac{m}{2}(v\cdot h\cdot v)(k\cdot z) - i(\pi\cdot h\cdot v) - \frac{i}{2}(v\cdot h\cdot S\cdot k)(z\cdot k) - \frac{1}{2}(v\cdot h\cdot s\cdot k) \\
&- \frac{i}{2}(z\cdot h\cdot S\cdot k)\omega + \frac{1}{2}(v\cdot h\cdot v)(v\cdot \lambda\cdot S\cdot k) - \frac{1}{2}(v\cdot h\cdot v)(v\cdot s\cdot k) \\
&- \frac{i}{2m}\mathcal{M}^2\Big|_{h,W}. \quad (3.37)
\end{aligned}$$

As in section 1.3, the worldline fields  $W = \{z^\mu, \lambda^{\mu\nu}, s_{\mu\nu}, \pi_\mu\}$  and gravitons  $h^{\mu\nu}$  are used in order to soak up free indices in the vertex. This allows all outgoing perturbations to be considered at once. Finally, non-minimal interactions appear through  $\mathcal{M}^2$ , which is just the Fourier transform of the non-minimal terms in  $\mathcal{M}^2$ . It is important to emphasize that there is no limit to the order in spin which can be studied using this action and Feynman rules, and adding higher-order corrections simply amounts to extending  $\mathcal{M}^2$  to the desired power in spin.

Both the Compton and one-loop scattering amplitudes were computed in paper IV, these results are briefly summarized below. Recall the Compton amplitude is given by the three diagrams,

$$\mathcal{A}^{\text{tree}} = \text{---}\overline{\text{---}}\begin{matrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix} + \text{---}\overline{\text{---}}\begin{matrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix} + \text{---}\overline{\text{---}}\begin{matrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix}. \quad (3.38)$$

This amplitude was computed using the full effective action up to fourth order in spin, and compared to Teukolsky solutions [87, 88], such that the Wilson coefficients could be fixed to produce an effective action for Kerr black holes up to  $\mathcal{O}(S^4 R^2)$ . The results are  $C_{ES^n} = C_{BS^n} = 1$ , while all remaining operators at  $\mathcal{O}(R^2 S^4)$  are zero. These results are in agreement with ref. [90], which constrained three linear combinations of Wilson coefficients  $\mathcal{O}(R^2 S^4)$ .

To compute the one-loop scattering of two compact objects, we add a second worldline action to the path integral, and label the masses  $m_1, m_2$ , spins  $S_1, S_2$  and velocities  $v_1, v_2$  for the two particles. The amplitude is computed from the diagrams,

$$\mathcal{A}^{\text{loop}} = \text{---}\overline{\text{---}}\begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix} \quad (3.39)$$

as well as their “mirror image” obtained by relabeling particles 1 and 2 and changing the sign of the transferred momentum  $q$ . The blob is ex-

panded in essentially the same subdiagrams as for the Compton amplitude, except instead of external on-shell states, the gravitons are sourced by the second worldline,

$$\mathcal{A}^{\text{loop}} = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} \quad (3.40)$$

If both  $q$  and  $k$  are integrated over the result is the Eikonal phase, see e.g. ref. [35], but here we will be interested in the amplitude, so we only integrate over  $k$ . The integrals up to  $S^4$  contain at most six loop momenta in the numerator, and take the general form,

$$I^{\mu_1 \dots \mu_6}(\nu_1, \nu_2, \nu_3) = \int \frac{d^4 k}{(2\pi)^4} \frac{k^{\mu_1} \dots k^{\mu_6}}{(k^2)^{\nu_1} ((k - q)^2)^{\nu_2} (k \cdot v_1)^{\nu_3}} \delta(v_2 \cdot k), \quad (3.41)$$

where we dropped the  $q$  integral and the energy-conserving delta functions impose  $v_2 \cdot k = 0$  as well as  $q \cdot v_2 = 0 = q \cdot v_1$ . Such integrals need to be reduced by making an ansatz for them in terms of tensors which are transverse to  $v_2$ . Those can be constructed from  $q^\mu$ ,  $v_1^\mu - v_2^\mu \gamma$  and  $\eta^{\mu\nu} - v_2^\mu v_2^\nu$ , where  $\gamma = v_1 \cdot v_2$  is the Lorentz factor. Once reduced, the remaining scalar integrals are available in the literature [35]. The resulting one-loop amplitude has been checked against the results in ref. [91], finding that our amplitude with six Wilson coefficients appropriately generalizes the results in ref. [91] with one Wilson coefficient. The full amplitude is also available in paper IV.

### 3.3 Fields for the Spinning Particle

In this section we explore a possible field theory corresponding to the spinning particle action in flat space. This is work in progress which is not published in paper IV. The first step is to construct the Poisson brackets of our worldline coordinates, which in the quantum theory are promoted to commutators. The acceleration term, containing  $\dot{p}^\mu$ , poses a problem for this process since it gives the  $x^\mu$  coordinates a non-trivial Poisson bracket with themselves (see ref. [92] for the Poisson brackets in the presence of electric fields). This non-commutativity is due to the fact that the spin degrees of freedom are entangled with the position. In flat space, however, it is a simple matter to shift the position variables such that the action no longer has the acceleration term, and the constraints remain first class [92]. In addition, we can obtain irreducible constraints by projecting the spin constraint on to the spatial part of the body-fixed frame [93], making it manifest that the constraint eliminates

three degrees of freedom. Applying these changes, the flat-space action becomes

$$S = - \int d\tau \left( p_\mu \dot{x}^\mu + \frac{1}{2} S_{ab} \Omega^{ab} - \frac{\ell}{2} (p^2 - m^2) - \ell_i \Lambda_\mu^i S^{\mu\nu} (p_\nu + m \Lambda_{\nu 0}) \right), \quad (3.42)$$

where the mass-shell constraint has been used in order to make the spin constraint local. The action induces the non-trivial Poisson brackets,

$$\{x^\mu, p_\nu\} = \delta_\nu^\mu, \quad (3.43)$$

$$\{S_{\mu\nu}, S_{\rho\sigma}\} = S_{\mu\rho} \eta_{\nu\sigma} - S_{\mu\sigma} \eta_{\nu\rho} + S_{\nu\sigma} \eta_{\mu\rho} - S_{\nu\rho} \eta_{\mu\sigma}, \quad (3.44)$$

$$\{S_{ab}, \Lambda_I^c\} = \delta_a^c \Lambda_{bI} - \delta_b^c \Lambda_{aI}. \quad (3.45)$$

We will now label the constraints by,

$$H = \frac{1}{2} (p^2 - m^2), \quad \mathcal{C}_i = \Lambda_i^\mu S_{\mu\nu} (p^\nu + m \Lambda_0^\nu), \quad (3.46)$$

and with this the classical algebra of constraints is given by the non-trivial bracket,

$$\{\mathcal{C}_i, \mathcal{C}_j\} = 2H \Lambda_i^\mu \Lambda_j^\nu S^{\mu\nu} = 2H S_{ij}. \quad (3.47)$$

The spin tensor projected onto the spatial part of the body-fixed frame,  $S_{ij}$ , generates global  $SO(3)$  rotations on the  $i, j$  indices, which leaves the action invariant. Therefore, it is a constant of motion, and it is straightforward to see that it indeed commutes with both constraints we have.

We will now add a set of canonical ghost pairs to our action,  $\{b, c\} = 1$  for the Hamiltonian constraint and  $\{b_i, c^j\} = \delta_i^j$  for the spin constraint. With these the classical BRST operator of the theory is

$$Q_{\text{classical}} = cH + c^i \mathcal{C}_i - S_{ij} c^i c^j b. \quad (3.48)$$

Following section (1.4) the next step is to promote the Poisson brackets to operator commutator relations, and find wave functions obeying the physical state conditions  $Q\Psi = 0$  and  $\Psi \sim \Psi + Q\chi$ .

We realize the spin tensor as a composite field  $S_{\mu\nu} = \Lambda_\mu^I \chi_{\nu I} - \Lambda_\nu^I \chi_{\mu I}$ , which is gauge invariant under transformations generated by the constraint  $\Lambda_\mu^I \Lambda_{\nu I} = \eta_{\mu\nu}$ , see paper IV for more details. Now we can effectively promote  $\chi$  to a  $\Lambda$ -derivative, so long as it appears inside the gauge invariant spin tensor. For the position sector we represent  $p$  with a partial derivative as usual,  $p \rightarrow -i\partial$ . Next, we have to check if there are any normal ordering ambiguities in our constraints. It turns out that the only problem comes from the choice between  $\Lambda_i^\mu S_{\mu\nu}$  and  $S_{\mu\nu} \Lambda_i^\mu$  in the spin constraint in (3.46) (commuting  $S_{\mu\nu}$  further to the right

causes no issues). This corresponds to the freedom to add  $\Lambda_{i\mu}p^\mu$  to the spin constraint. We therefore temporarily introduce a constraint  $\tilde{\mathcal{C}}_i = \mathcal{C}_i + \alpha\Lambda_{i\mu}p^\mu$  with a free parameter  $\alpha$ . Computing the commutators of these constraints we find,

$$[\tilde{\mathcal{C}}_i, \tilde{\mathcal{C}}_j] = 2HS_{ij} - 2\Lambda_{i\mu}p^\mu\mathcal{C}_j, \quad (3.49)$$

where on the right-hand side we find  $\mathcal{C}$  and not  $\tilde{\mathcal{C}}$ , and therefore we learn that for  $\alpha \rightarrow 0$  we have a closed algebra. Proceeding with  $\alpha = 0$  our BRST operator becomes

$$Q = cH + c^i\mathcal{C}_i - c^i c^j S_{ij}b + \Lambda_{ip}c^i c^j b_j, \quad (3.50)$$

and since it obeys  $Q^2 = 0$ , no further corrections to it are needed.

To describe physical wave functions we start with the ansatz

$$\begin{aligned} \Psi(x, \Lambda, c, c^i) &= \Psi + c^i\Psi_i + c^{ij}\Psi_{ij} + c^3\Psi_{123} \\ &+ c\check{\Psi} + cc^i\check{\Psi}_i + cc^{ij}\check{\Psi}_{ij} + cc^3\check{\Psi}_{123}, \end{aligned} \quad (3.51)$$

where the fields on the second line have opposite statistics to those on the first, and they are not subject to the mass-shell constraint, therefore they behave like antifields.

We will construct ghost-number zero states, namely functions of the coordinates  $x$  and  $\Lambda$  only, and impose that they are BRST closed (they are automatically not exact). We find that the states split into independent spin sectors, each spin sector indicated by the total number of little group indices  $i, j$  present. We present here a few examples of closed states,

$$\Psi_{s=0} = 1, \quad (3.52)$$

$$\Psi_{s=1} = \Lambda_i^a \epsilon_a^i - \Lambda_i^k \Lambda_0^a \epsilon_a^i \frac{1}{\Lambda_0^k + m}, \quad (3.53)$$

$$\begin{aligned} \Psi_{s=2} &= \Lambda_{i_1}^{a_1} \Lambda_{i_2}^{a_2} \epsilon_{a_1 a_2}^{i_1 i_2} \\ &- 2\Lambda_{i_1}^{a_1} \Lambda_{i_2}^k \Lambda_0^{a_2} \epsilon_{a_1 a_2}^{i_1 i_2} \frac{1}{\Lambda_0^k + m} \\ &+ \Lambda_{i_1}^k \Lambda_{i_2}^k \Lambda_0^{a_1} \Lambda_0^{a_2} \epsilon_{a_1 a_2}^{i_1 i_2} \frac{1}{(\Lambda_0^k + m)^2}, \end{aligned} \quad (3.54)$$

$$\begin{aligned} \Psi_{s=3} &= \Lambda_{i_1}^{a_1} \Lambda_{i_2}^{a_2} \Lambda_{i_3}^{a_3} \epsilon_{a_1 a_2 a_3}^{i_1 i_2 i_3} \\ &- 3\Lambda_{i_1}^{a_1} \Lambda_{i_2}^{a_2} \Lambda_{i_3}^k \Lambda_0^{a_3} \epsilon_{a_1 a_2 a_3}^{i_1 i_2 i_3} \frac{1}{\Lambda_0^k + m} \\ &+ 3\Lambda_{i_1}^{a_1} \Lambda_{i_2}^k \Lambda_{i_3}^k \Lambda_0^{a_2} \Lambda_0^{a_3} \epsilon_{a_1 a_2 a_3}^{i_1 i_2 i_3} \frac{1}{(\Lambda_0^k + m)^2} \\ &- \Lambda_{i_1}^k \Lambda_{i_2}^k \Lambda_{i_3}^k \Lambda_0^{a_1} \Lambda_0^{a_2} \Lambda_0^{a_3} \epsilon_{a_1 a_2 a_3}^{i_1 i_2 i_3} \frac{1}{(\Lambda_0^k + m)^3}. \end{aligned} \quad (3.55)$$

These are assumed to be multiplied by a plane-wave factor  $e^{ikx}$ , with  $k^2 = m^2$ . In addition, polarization tensors  $\epsilon_{\mu\nu\dots}^{ij\dots}$  have been introduced, these are traceless, symmetric, and transverse to  $k$ . Note that the  $SO(3)$  little group indices are contracted with the spatial part of the body-fixed frame fields, and therefore  $S_{ij}$  generates little group rotations. At this stage the pattern can be guessed for any spin,

$$\Psi_s = \sum_{n=0}^s \binom{s}{n} (-1)^n T(s, n) \left( \frac{1}{\Lambda_0^p + m} \right)^n, \quad (3.56)$$

where

$$T(s, n) = \Lambda_{i_1}^p \cdots \Lambda_{i_n}^p \Lambda_0^{a_1} \cdots \Lambda_0^{a_n} \Lambda_{i_{n+1}}^{a_{n+1}} \cdots \Lambda_{i_s}^{a_s} \epsilon_{a_1 \dots a_s}^{i_1 \dots i_s}. \quad (3.57)$$

This expression has been explicitly tested up to spin 6.

Beyond ghost number zero, the equations of motion and gauge invariances of the states can be extracted from the  $BF$ -like action,

$$S = \int d^4x d^6\Lambda d^4c \Phi Q \Psi, \quad (3.58)$$

where  $\Phi$  is bosonic and  $\Psi$  fermionic, and both are functions of the ghosts, spacetime coordinates, and frame fields  $\Lambda$ . The measure on the ghosts is just straightforward integration on Grassmann variables, and for the  $\Lambda$  variables the integration should be over all angles and rapidities that define the matrices.

In Minkowski signature the  $\Lambda$  integration is over a non-compact space, and using our wavefunctions from earlier and a naive integration measure would result in divergent expressions. To regularize the integration in a BRST-covariant manner we extend the space of worldline fields by adding a canonical pair of anticommuting variables through the kinetic term  $v_I^\mu \dot{w}_\mu^I$ , as well as the constraint  $v_I^\mu = 0$ . Since the action is completely independent of the  $w$  variable, this constraint is abelian and first class. We can now add it to our BRST operator from before, after introducing a pair of bosonic ghosts with kinetic term  $f_\mu^I \dot{g}_I^\mu$ . The BRST operator is then modified by

$$\mathbf{Q} = Q + g_\mu^I v_I^\mu. \quad (3.59)$$

The new term in the BRST operator allows us to regularize the  $\Lambda$  integrals by multiplying our measure by  $\exp(-\{\mathbf{Q}, w_\mu^I \Lambda_I^\mu g_\nu^J \Lambda_J^\nu\})$ . The new term in the BRST charge will contribute a bosonic piece to the exponential through  $\{g_\mu^I v_I^\mu, w_\mu^I \Lambda_I^\mu g_\nu^J \Lambda_J^\nu\} = (g_\mu^I \Lambda_I^\mu)^2 + \text{ghosts}$ , which will suppress  $\Lambda \rightarrow \infty$  divergences in our integrand. Note that there should be no divergence from  $\Lambda_0^p + m$  appearing in denominators, because the time component of  $\Lambda_0$  and  $p$  (or equivalently  $k$ ) must have the same sign. To see this, consider the classical particle in its rest frame. The spin constraint will be proportional to  $S_{\mu\nu}(\Lambda_0^\nu + \delta_0^\nu)$ . Fixing the covariant SSC

means  $\Lambda_0^\mu = \delta_0^\mu$ , as well as  $S_{\mu 0} = 0$ . However, if one is allowed to introduce time-reversing  $\Lambda$  variables, there is an alternative SSC-like solution to the constraints where  $\Lambda_0^\mu = -\delta_0^\mu$ , which trivializes the constraint on the spin.

For practical purposes it is enough to know that the measure could in principle be constructed, and irrespective of its precise form, whether based on BRST-covariant regularization, Wick rotation, or otherwise, the  $\Lambda$  integrals can be guessed based on symmetries alone, at least for polynomial dependence on  $\Lambda$ . For this purpose we start by normalizing the integration measure such that

$$\int d^6 \Lambda = 1 . \quad (3.60)$$

In four dimensions, integrals over odd numbers of  $\Lambda$  vanish, since there is no invariant tensor we can write down with an odd number of indices. However, we can see that

$$\int d^6 \Lambda (\Lambda_I^a \Lambda_J^b) = \frac{1}{4} \eta^{ab} \eta_{IJ} , \quad (3.61)$$

is consistent with  $\Lambda_I^a \Lambda_J^b \eta_{ab} = \eta_{IJ}$  and the symmetries of the integrand. Similarly,

$$\begin{aligned} \int d^6 \Lambda (\Lambda_{I_1}^{a_1} \Lambda_{I_2}^{a_2} \Lambda_{I_3}^{a_3} \Lambda_{I_4}^{a_4}) &= \frac{5}{18} \eta^{a_1 a_4} \eta_{I_1 I_4} \eta^{a_2 a_3} \eta_{I_2 I_3} - \frac{1}{18} \eta^{a_1 a_3} \eta_{I_1 I_4} \eta^{a_2 a_4} \eta_{I_2 I_3} \\ &\quad - \frac{1}{18} \eta^{a_1 a_4} \eta_{I_1 I_3} \eta^{a_2 a_3} \eta_{I_2 I_4} + \frac{5}{18} \eta^{a_1 a_3} \eta_{I_1 I_3} \eta^{a_2 a_4} \eta_{I_2 I_4} \\ &\quad - \frac{1}{18} \eta^{a_1 a_2} \eta_{I_1 I_4} \eta^{a_3 a_4} \eta_{I_2 I_3} - \frac{1}{18} \eta^{a_1 a_2} \eta_{I_1 I_3} \eta^{a_3 a_4} \eta_{I_2 I_4} \\ &\quad - \frac{1}{18} \eta^{a_1 a_4} \eta_{I_1 I_2} \eta^{a_2 a_3} \eta_{I_3 I_4} - \frac{1}{18} \eta^{a_1 a_3} \eta_{I_1 I_2} \eta^{a_2 a_4} \eta_{I_3 I_4} \\ &\quad + \frac{5}{18} \eta^{a_1 a_2} \eta_{I_1 I_2} \eta^{a_3 a_4} \eta_{I_3 I_4} . \end{aligned} \quad (3.62)$$

However, to actually do computations with the physical fields we constructed earlier, we would need to be able to integrate over an infinite number of  $\Lambda$ s. This, together with coupling to additional fields (gravitons or photons) will be interesting to pursue in future research.

In addition to the integration measure, the gauge symmetry  $\delta \Psi = Q \Omega$  and  $\delta \Phi = Q \xi$  must be fixed. Following our work in the previous chapter, we identify  $b = \frac{\partial}{\partial c}$  which obeys  $bQ + Qb = \frac{1}{2}(p^2 - m^2)$ , so we can impose  $b\Psi = 0$  and similarly for  $\Phi$ , effectively removing all antifields on the second line of equation (3.51). This  $b$ -ghost is non-composite and first order, and so it behaves differently to the  $b$ -operators studied in the context of color-kinematics duality.

It would be particularly interesting to attempt the construction of the spinning particle in curved space, and thus obtain a direct relationship between Wilson coefficients in the worldline theory to those of some higher spin QFT, see ref. [89] for discussions of on-shell comparisons. Unfortunately, although the constraints in (3.24) are first class, the Poisson algebra is dramatically more complicated in curved space due to the difficulty of disentangling the position and spin degrees of freedom.

### 3.4 Future research directions

In this section we explained the gauge symmetries of the EFT of the spinning compact object, and found a convenient gauge choice that allowed the action to be used in the WQFT, extending the latter formalism to arbitrary spin. We computed amplitudes from the EFT action and fixed Wilson coefficients to  $S^4R^2$  order such that the action reproduces Kerr black hole observables. It would be interesting to extend these results to all orders in spin, and attempt to find a symmetry principle that governs the Kerr black hole effective theory.

The Hamiltonian formulation of the spinning particle action we used has four worldline fields,  $x$ ,  $p$ ,  $S$ , and  $\Lambda$ . It can be simplified somewhat by guessing a suitable Routhian for the worldline. The Routhian is a partially Legendre-transformed Hamiltonian, and takes the form

$$S_R = - \int d\tau \left( \frac{1}{\ell} \dot{x}^2 + \frac{1}{2} S_{ab} \Omega^{ab} + \frac{1}{\dot{x}^2} \frac{D\dot{x}_\mu}{d\tau} S^{\mu\nu} \dot{x}_\nu + \frac{\ell}{2} m - \ell_a S^{ab} (\hat{x}_b + \Lambda_{0b}) \right). \quad (3.63)$$

This action has essentially the same spin gauge symmetries as those of the Hamiltonian, after substituting  $p \rightarrow \dot{x}/\ell$ . In addition, non-minimal interactions can be added along similar lines to before, by introducing a dynamical mass and again eliminating  $p \rightarrow \dot{x}/\ell$ . It would be interesting to employ this action in WQFT computations as it may simplify the Feynman rules. For this, a good gauge fixing needs to be identified such that the worldline propagators are not overly complicated. Alternatively, it may be possible to eliminate the acceleration term  $\ddot{x}$  perturbatively in spin in a similar fashion to the elimination of  $\dot{p}$  in paper IV.

As an alternative to an action-based approach, classical scattering amplitudes could potentially be constructed by sewing together other classical amplitudes with fewer legs. Such a procedure will not necessarily have any direct connection to an action principle, but could, at least for scattering scenarios, provide a useful computational tool. In conjunction with on-shell methods, developing a double-copy formulation for the spinning worldline amplitudes would also likely be useful



for computing scattering amplitudes efficiently. Aside from scattering observables, it will also be interesting to generalize the WQFT to bound orbits by including elliptical trajectories as background fields, instead of the straight line used for scattering scenarios.

# Sammanfattning

Yang-Mills gauge-teori är en kvantfältteori som har en unik roll inom modern partikelfysik. Den beskriver bland annat kraftbärarna som förmedlar den elektromagnetiska, svaga och starka kraften. Kraftbärarna är vektorbosoner som i allmänhet kan växelverka med sig själva, vilket ger Yang-Millsteorin en mycket komplicerad dynamik, även i avsaknad av övriga elementarpartiklar, såsom kvarkar och leptoner. Dynamiken kan förenklas med hjälp av störningsteori som approximerar vektorbosonerna som växelverkande planvågor som beskrivs av en samling polarisationsvektorer och rörelsemängder, samt ytterligare frihetsgrader som beskriver färggladdningar hos partiklarna. Denna avhandling studerar de matematiska objekt som kallas spridningsamplituder, vilka beskriver processer där vektorbosonerna kolliderar i spridningsexperiment. Spridningsamplituderna beror på färgfaktorer som matematiskt beskrivs av strukturkonstanter för en ickeabelsk Liealgebra, vilket gör att samband såsom Jacobiidentiteten är avgörande för praktiska beräkningar.

Det har visat sig att de kinematiska faktorerna som bidrar till spridningsamplituderna, nämligen de som beror på polarisationsvektorer och rörelsemängder, också kan omformuleras på ett sätt så att de följer liknande samband som Jacobiidentiteten hos färgfaktorerna. Detta kan tolkas som att den kinematiska variabelerna i Yang-Millsdynamiken kan också ha sitt ursprung i en Liealgebra, även om någon sådan fullständig kinematisk algebra ännu inte har identifierats. I denna avhandling identifieras bland annat en kinematisk Liealgebra för Chern-Simons gauge-teori, som är en tredimensionell kvantfältteori som är besläktad till Yang-Millsteorin. Båda beskriver växelverkande vektorkvantfält med ickeabelska färggladdningar, men dock är Chern-Simonsteorin topologisk vilket innebär att den har triviala spridningsamplituder såvida inte materiepartiklar inkluderas. Den kinematiska Liealgebran som vi finner beskriver en volymbevarande diffeomorfi. Resultatet är ett viktigt steg i rätt riktning för en mer fullständig förståelse, och i denna avhandling presenteras liknande framsteg för den kinematiska algebran för Yang-Millsteorin. Diffeomorfi-algebran förekommer även i en enkel sektor av Yang-Millsteorins Feynmanregler, och för mer komplicerade sektorer används en Lagrangiansk beskrivning med hjälpfält som dramatiskt förbättrar de algebraiska egenskaperna hos de kinematiska faktorerna.

En gauge-teori som uppfyller den ovan beskrivna färg-kinematiska dualiteten kan användas i en matematisk konstruktion som kallas dubbel-

kopian. Den gör det möjligt att erhålla observerbara storheter för allmänna relativitetsteorin samt andra gravitationsteorier, från de kinematiska faktorerna som förekommer i Yang-Millsteorins spridningsamplituder. Fördelen med denna konstruktion är att den kringgår nödvändigheten att beräkna med hjälp av Feynmanreglerna för gravitationsteorin, som är betydligt mer komplicerade än de för en gaugeteori. Det har nyligen visat sig praktiskt användbart att använda spridningsamplituder för att även beskriva klassiska processer i allmän relativitetsteori, såsom fysiken bakom gravitationsvågor.

Gravitationsvågor sänds ut när två kompakta objekt i omloppsbanan, såsom svarta hål eller neutronstjärnor, smälter samman under slutskedet av ett binärt systems liv. Dessa energirika händelser har nyligen upptäckts med gravitationsvågdetektorer, vilket banar väg för en ny era av astrofysik och möjliggör för test av allmän relativitetsteori i extrema situationer. För att tolka signalerna från dessa experiment behövs förutsägelser från allmän relativitetsteori. Även om spridningsamplituder ursprungligen inte utvecklades för att beskriva sådana klassiska händelser har de nyligen framgångsrikt används för beräkningar som innehåller nödvändig information för den tidiga spiralfasen under sammansmältningen. Vid dessa beräkningar är det viktigt att ta hänsyn till de svarta hålets rotation, även kallat rörelsemängdsmoment eller spinn, vilket bidrar till dynamiken i samma storleksordning som de andra parametrarna i approximationen. I denna avhandling visas det att spinnkorrigeringar kan systematiskt beräknas genom att beskriva ett roterande svart hål med en kvantfältteori som lever på en världslinje. Ett roterande svart hål behandlas som en punktpartikel med spinn, och dess världslinje kan krökas genom att kvantmekaniska störningar på världslinjen växelverkar med gravitationsfältet i bakgrunden. Beräkningar med världslinjen resulterar i Feynmandiagram som liknar de som förekommer i vanlig kvantfältteori, men de är mer direkt anpassade till klassisk fysik.

Det är inte uppenbart hur man beskriver växelverkningsför punktpartiklar med spinn så att de matchar roterande svarta hål. De modelleras vanligtvis av en effektiv kvantfältteori, där obestämda Wilsonkoefficienter läggs till så att de beskriver ett allmänt roterande kompakt objekt. För att modellera ett specifikt objekt måste dessa fria koefficienter fixeras genom matchingsberäkningar. Endast ett begränsat antal koefficienter bidrar till en given störningsordning vilket förenklar matchingsberäkningarna. I denna avhandling fixeras Wilsonkoefficienterna för ett roterande svart hål till fjärde ordningen i spinn, vilka kan bidra till praktiska beräkningar upp till den sjätte ordningen i Newtons gravitationskonstant.

# Acknowledgements

First of all I would like to thank my supervisor, Henrik, for all of your kind advice, support, and encouragement during my PhD. I learned from you important lessons about physics, as well as about research and academia. I appreciate our collaborations and discussions and look forward to many more. I am also grateful to Maxim, my co-supervisor, for our recent collaboration and for all the mathematical topics I learned from you. I appreciate your efforts to bridge different “religions”, as you call them, and your contagious motivation. Thank you to Marco also, for supervising my Masters thesis as well as several other projects during my PhD, without your early support I would likely not have embarked on this journey. I am also grateful to Oli, for all of the fun evenings you organized, and for letting me have free reign over a lecture in your course. Aside from top notch research, you also brought a lot of joy to the department!

I am of course grateful to all of the other PhD students in my cycle, in particular my office mate Lucile, for making sure the others never spent too long in our office, for the fun cooking, maker-spacing and driving. Thank you for all the time we spent together, and thank you for your support. Thank you Carlos, bringer of chocolate and chalk (both of which are edible, of course). I am grateful for some of your book recommendations, and for being enthusiastic about contours as you are about having fun with friends, you are always fun and exciting to be around. Thank you to Charel, for hosting the longest board game nights, and for always being willing to share your computer and mathematical expertise, I benefited from many conversations with you. Thank you also to Daniel, for all the years we have known each other and for sharing your advice, in particular, for teaching me which comida is for tontos. Simon, it was great to have you around, and thank you for leaving early, to show us how its done. I enjoyed our sophisticated discussions of Dostoevsky, and our even more sophisticated ultimate games. Thank you also Paolo, for going on small adventures involving cheese, butter, or in general, concerns about lactose.

Importantly, thank you very much Lucia for our collaborations, discussions, and for the mutual support. Thank you to Max, for teaching me so much about pure spinors, quantization, and more. Thank you also to my external collaborators, Gustav, Gustav, Francesco, and Roberto, for sharing your insights about physics and mathematics.

I am especially grateful to Joe, for joining us in our ultimate games, and for joining almost every Journal club. You brought important perspectives that made the experience all the more worthwhile. Thank you to Roman too of course, for the disk golfing and throwing masterclasses, and for catching almost anything that is not a disease. Thank you Magdalena, Dmytro, and Pietro for joining the lunch table, and for bringing some reasonableness to the lunch-time chaos, it was fun to have you around. Thank you Paolo di Vecchia for your insightful discussions, and for giving important perspectives on theoretical physics as a whole.

I am also grateful to my two students, Emil and Abdulmasih, for trusting me with a portion of your education. Working with you I learned both physics and important teaching skills. Thank you for your patience and willingness to learn, I wish you all the best in your future careers. And, of course, I am also grateful to Ulf and Pietro again for supervising my supervision.

The physics department would not be itself without the lively PhD students and postdocs, thank you to Yoann for your hard organizational work, the proof of which is in the pudding. Thank you Moritz, Kunal, and Yacoub for the lunch-time discussions, and thank you Daniele also for the punctual lunch reminders, the discussions, and for trying one very unique piece of cheese. Thank you also to Sourav, Kaiwen, Sam, Ingrid, Kays, and Jacopo, for sharing your experiences in your transition from PhD to postdoc, and always being ready to give advice. A special thanks goes to Vincent, master of journal clubs and sliced bread. Thank you Martijn and Alex for sharing your insights on integration and programming, and thank you Filippo for visiting us in Sweden and trying local food, you survived.

Thank you Edu, for great discussions about anything, from the foundations of quantum mechanics to economics. Thank you for your hospitality in Spain, and for the great trips we had in Lisbon and Asturias, or just skiing around Uppsala. Despite what some companies think, together with Lucile, we are a financial dream team. Thank you to Emma for joining our games, and Hannes and Sia for the hot pots, enthusiasm, and for teaching us some rules, sometimes. Thank you to Nacho also for organizing the best improv team, at least when I wasn't there. Thank you Axel for the time we spent together in greenrooms all around stockholm, and for demonstrating that comedy can be both a precision science and an artistic craft.

Last but not least, thank you to all my family members, near and far, for your continued support and understanding, and all the delicious food. Thank you in particular to my parents for checking, from time to time, if I was alive.

# Appendix A.

## Yang-Mills as a Generalized Chern-Simons

In order to write down the gauge symmetries for Yang-Mills theory in its Batalin-Vilkovitsky formulation, it is convenient to mimic Chern-Simons theory as closely as possible, and use (2.25). The first step is to turn the Yang-Mills action cubic by using the first-order Lagrangian, schematically taking the form  $\mathcal{L} = BF - B \star B/2$ , with  $B$  an auxiliary  $d-2$  form field. Adding this extra field requires also adding its antifield, which is a 2-form. The fields and antifields of Yang-Mills theory are collected in the differential graded vector space with differential  $D$ ,

$$\begin{array}{ccccc}
 \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 \\
 & & & \nearrow \star \frac{\partial}{\partial \xi} & \\
 \xi \Omega^{d-2} & \xrightarrow{d} & \xi \Omega^{d-1} & \xrightarrow{d} & \xi \Omega^d
 \end{array} \tag{3.64}$$

where the first column contains the ghost  $c$ , the second contains the physical fields  $A$  and  $B$ , the third contains the antifields  $\check{B}$  and  $\check{A}$ , and finally the last column has the antighost  $\check{c}$ . The parameter  $\xi$  is assumed to obey  $\xi^2 = 0$ , and fields are paired by the integration measure, which also projects on to  $\xi^1$  (see refs. [83, 59] for similar constructions in 4d). Altogether, the action for Yang-Mills is,

$$S = ig \int \text{tr} \left( \frac{1}{2} \mathcal{A} Q \mathcal{A} + \frac{1}{3} \mathcal{A}^3 \right), \tag{3.65}$$

where  $\mathcal{A}$  is the sum of all fields and antifields and  $Q = -iD$ . This action has the expected gauge symmetry  $\delta \mathcal{A} = Q\Omega + [\mathcal{A}, \Omega]$ , and analogous BRST transformation. Note that simply having a Chern-Simons-like action does not have any immediate implications for the color-kinematics duality, since the codifferential does not need to be second order in general.

# References

- [1] Z. Bern, J.J.M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [0805.3993].
- [2] Z. Bern, J.J.M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [1004.0476].
- [3] Z. Bern, J.J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, *The Duality Between Color and Kinematics and its Applications*, 1909.01358.
- [4] I.A. Batalin and G.A. Vilkovisky, *Gauge Algebra and Quantization*, *Phys. Lett. B* **102** (1981) 27.
- [5] G. Mogull, J. Plefka and J. Steinhoff, *Classical black hole scattering from a worldline quantum field theory*, *JHEP* **02** (2021) 048 [2010.02865].
- [6] C. Shi and J. Plefka, *Classical double copy of worldline quantum field theory*, *Phys. Rev. D* **105** (2022) 026007 [2109.10345].
- [7] J. Gomis, J. Paris and S. Samuel, *Antibracket, antifields and gauge theory quantization*, *Phys. Rept.* **259** (1995) 1 [hep-th/9412228].
- [8] N.E.J. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard and P. Vanhove, *The Momentum Kernel of Gauge and Gravity Theories*, *JHEP* **01** (2011) 001 [1010.3933].
- [9] N.E.J. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard and B. Feng, *Manifesting Color-Kinematics Duality in the Scattering Equation Formalism*, *JHEP* **09** (2016) 094 [1608.00006].
- [10] Y.-J. Du and F. Teng, *BCJ numerators from reduced Pfaffian*, *JHEP* **04** (2017) 033 [1703.05717].
- [11] G. Chen and T. Wang, *BCJ Numerators from Differential Operator of Multidimensional Residue*, *Eur. Phys. J. C* **80** (2020) 37 [1709.08503].
- [12] A. Edison and F. Teng, *Efficient Calculation of Crossing Symmetric BCJ Tree Numerators*, *JHEP* **12** (2020) 138 [2005.03638].
- [13] S. He, L. Hou, J. Tian and Y. Zhang, *Kinematic numerators from the worldsheet: cubic trees from labelled trees*, 2103.15810.
- [14] C. Cheung and J. Mangan, *Covariant color-kinematics duality*, *JHEP* **11** (2021) 069 [2108.02276].
- [15] E. Bridges and C.R. Mafra, *Algorithmic construction of SYM multiparticle superfields in the BCJ gauge*, *JHEP* **10** (2019) 022 [1906.12252].
- [16] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes*, *Phys. Rev. D* **85** (2012) 105014 [1201.5366].
- [17] G. Yang, *Color-kinematics duality and Sudakov form factor at five loops for  $N=4$  supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **117** (2016) 271602 [1610.02394].

- [18] A. Brandhuber, G. Chen, H. Johansson, G. Travaglini and C. Wen, *Kinematic Hopf Algebra for BCJ Numerators in Heavy-Mass Effective Field Theory and Yang-Mills Theory*, 2111.15649.
- [19] A. Brandhuber, G.R. Brown, G. Chen, J. Gowdy, G. Travaglini and C. Wen, *Amplitudes, Hopf algebras and the colour-kinematics duality*, *JHEP* **12** (2022) 101 [2208.05886].
- [20] D. Vaman and Y.-P. Yao, *Constraints and Generalized Gauge Transformations on Tree-Level Gluon and Graviton Amplitudes*, *JHEP* **11** (2010) 028 [1007.3475].
- [21] H. Kawai, D.C. Lewellen and S.H.H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys. B* **269** (1986) 1.
- [22] N.E.J. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard and P. Vanhove, *The Momentum Kernel of Gauge and Gravity Theories*, *JHEP* **01** (2011) 001 [1010.3933].
- [23] KAGRA, VIRGO, LIGO SCIENTIFIC collaboration, *GWTC-3: Compact Binary Coalescences Observed by LIGO and Virgo during the Second Part of the Third Observing Run*, *Phys. Rev. X* **13** (2023) 041039 [2111.03606].
- [24] A. Buonanno and T. Damour, *Transition from inspiral to plunge in binary black hole coalescences*, *Phys. Rev. D* **62** (2000) 064015 [gr-qc/0001013].
- [25] W.D. Goldberger and I.Z. Rothstein, *An Effective field theory of gravity for extended objects*, *Phys. Rev. D* **73** (2006) 104029 [hep-th/0409156].
- [26] R.A. Porto, *The effective field theorist's approach to gravitational dynamics*, *Phys. Rept.* **633** (2016) 1 [1601.04914].
- [27] M. Levi, *Effective Field Theories of Post-Newtonian Gravity: A comprehensive review*, *Rept. Prog. Phys.* **83** (2020) 075901 [1807.01699].
- [28] A. Buonanno and T. Damour, *Effective one-body approach to general relativistic two-body dynamics*, *Phys. Rev. D* **59** (1999) 084006 [gr-qc/9811091].
- [29] Y. Pan, A. Buonanno, A. Taracchini, L.E. Kidder, A.H. Mroué, H.P. Pfeiffer et al., *Inspiral-merger-ringdown waveforms of spinning, precessing black-hole binaries in the effective-one-body formalism*, *Phys. Rev. D* **89** (2014) 084006 [1307.6232].
- [30] D. Bini and T. Damour, *Gravitational spin-orbit coupling in binary systems, post-Minkowskian approximation and effective one-body theory*, *Phys. Rev. D* **96** (2017) 104038 [1709.00590].
- [31] A. Buonanno, G.U. Jakobsen and G. Mogull, *Post-Minkowskian Theory Meets the Spinning Effective-One-Body Approach for Two-Body Scattering*, 2402.12342.
- [32] LISA CONSORTIUM WAVEFORM WORKING GROUP collaboration, *Waveform Modelling for the Laser Interferometer Space Antenna*, 2311.01300.
- [33] N. Kunert, P.T.H. Pang, I. Tews, M.W. Coughlin and T. Dietrich, *Quantifying modeling uncertainties when combining multiple gravitational-wave detections from binary neutron star sources*, *Phys. Rev. D* **105** (2022) L061301 [2110.11835].



- [34] C.B. Owen, C.-J. Haster, S. Perkins, N.J. Cornish and N. Yunes, *Waveform accuracy and systematic uncertainties in current gravitational wave observations*, *Phys. Rev. D* **108** (2023) 044018 [2301.11941].
- [35] G.U. Jakobsen, G. Mogull, J. Plefka and J. Steinhoff, *SUSY in the sky with gravitons*, *JHEP* **01** (2022) 027 [2109.04465].
- [36] M.-Z. Chung, Y.-T. Huang, J.-W. Kim and S. Lee, *The simplest massive S-matrix: from minimal coupling to Black Holes*, *JHEP* **04** (2019) 156 [1812.08752].
- [37] D.A. Kosower, B. Maybee and D. O’Connell, *Amplitudes, Observables, and Classical Scattering*, *JHEP* **02** (2019) 137 [1811.10950].
- [38] R. Aoude and A. Ochirov, *Classical observables from coherent-spin amplitudes*, *JHEP* **10** (2021) 008 [2108.01649].
- [39] L. Cangemi, M. Chiodaroli, H. Johansson, A. Ochirov, P. Pichini and E. Skvortsov, *From higher-spin gauge interactions to Compton amplitudes for root-Kerr*, 2311.14668.
- [40] G.U. Jakobsen, G. Mogull, J. Plefka, B. Sauer and Y. Xu, *Conservative Scattering of Spinning Black Holes at Fourth Post-Minkowskian Order*, *Phys. Rev. Lett.* **131** (2023) 151401 [2306.01714].
- [41] G.U. Jakobsen, G. Mogull, J. Plefka and B. Sauer, *Dissipative Scattering of Spinning Black Holes at Fourth Post-Minkowskian Order*, *Phys. Rev. Lett.* **131** (2023) 241402 [2308.11514].
- [42] G.U. Jakobsen, G. Mogull, J. Plefka and B. Sauer, *Tidal effects and renormalization at fourth post-Minkowskian order*, *Phys. Rev. D* **109** (2024) L041504 [2312.00719].
- [43] M. Driesse, G.U. Jakobsen, G. Mogull, J. Plefka, B. Sauer and J. Usovitsch, *Conservative Black Hole Scattering at Fifth Post-Minkowskian and First Self-Force Order*, 2403.07781.
- [44] G. Kälin, Z. Liu and R.A. Porto, *Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach*, *Phys. Rev. Lett.* **125** (2020) 261103 [2007.04977].
- [45] W. Siegel, *Fields*, hep-th/9912205.
- [46] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton University Press (1992).
- [47] N. Ahmadinia, F.M. Balli, O. Corradini, C. Lopez-Arcos, A.Q. Velez and C. Schubert, *Manifest colour-kinematics duality and double-copy in the string-based formalism*, *Nucl. Phys. B* **975** (2022) 115690 [2110.04853].
- [48] R. Bonezzi, C. Chiaffrino, F. Diaz-Jaramillo and O. Hohm, *Gauge invariant double copy of Yang-Mills theory: The quartic theory*, *Phys. Rev. D* **107** (2023) 126015 [2212.04513].
- [49] R. Bonezzi, F. Diaz-Jaramillo and O. Hohm, *The gauge structure of double field theory follows from Yang-Mills theory*, *Phys. Rev. D* **106** (2022) 026004 [2203.07397].
- [50] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *Gravity as the Square of Gauge Theory*, *Phys. Rev. D* **82** (2010) 065003 [1004.0693].
- [51] M. Tolotti and S. Weinzierl, *Construction of an effective Yang-Mills Lagrangian with manifest BCJ duality*, *JHEP* **07** (2013) 111 [1306.2975].
- [52] G. Chen, H. Johansson, F. Teng and T. Wang, *On the kinematic algebra*

- for BCJ numerators beyond the MHV sector, *JHEP* **11** (2019) 055 [1906.10683].
- [53] M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, *Explicit Formulae for Yang-Mills-Einstein Amplitudes from the Double Copy*, *JHEP* **07** (2017) 002 [1703.00421].
- [54] M. Reiterer, *A homotopy BV algebra for Yang-Mills and color-kinematics*, 1912.03110.
- [55] N. Berkovits, *Covariant quantization of the superparticle using pure spinors*, *JHEP* **09** (2001) 016 [hep-th/0105050].
- [56] S. Axelrod and I.M. Singer, *Chern-Simons perturbation theory*, in *International Conference on Differential Geometric Methods in Theoretical Physics*, pp. 3–45, 1991 [hep-th/9110056].
- [57] G.V. Dunne, *Aspects of Chern-Simons theory*, in *Les Houches Summer School in Theoretical Physics, Session 69: Topological Aspects of Low-dimensional Systems*, 7, 1998 [hep-th/9902115].
- [58] A.S. Schwarz, *Geometry of Batalin-Vilkovisky quantization*, *Commun. Math. Phys.* **155** (1993) 249 [hep-th/9205088].
- [59] K.J. Costello, *Renormalization and effective field theory*, 2011, <https://api.semanticscholar.org/CorpusID:118477142>.
- [60] D. Gaiotto and E. Witten, *Janus Configurations, Chern-Simons Couplings, And The theta-Angle in N=4 Super Yang-Mills Theory*, *JHEP* **06** (2010) 097 [0804.2907].
- [61] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, *N=4 Superconformal Chern-Simons Theories with Hyper and Twisted Hyper Multiplets*, *JHEP* **07** (2008) 091 [0805.3662].
- [62] P.K. Townsend, *D-branes from M-branes*, *Phys. Lett. B* **373** (1996) 68 [hep-th/9512062].
- [63] H. Elvang, M. Hadjiantonis, C.R.T. Jones and S. Paranjape, *Electromagnetic Duality and D3-Brane Scattering Amplitudes Beyond Leading Order*, *JHEP* **04** (2021) 173 [2006.08928].
- [64] H. Elvang and Y.-t. Huang, *Scattering Amplitudes*, 1308.1697.
- [65] M. Chiodaroli, Q. Jin and R. Roiban, *Color/kinematics duality for general abelian orbifolds of N=4 super Yang-Mills theory*, *JHEP* **01** (2014) 152 [1311.3600].
- [66] N. Berkovits, *Super Poincare covariant quantization of the superstring*, *JHEP* **04** (2000) 018 [hep-th/0001035].
- [67] C.R. Mafra, O. Schlotterer and S. Stieberger, *Explicit BCJ Numerators from Pure Spinors*, *JHEP* **07** (2011) 092 [1104.5224].
- [68] C.R. Mafra and O. Schlotterer, *Berends-Giele recursions and the BCJ duality in superspace and components*, *JHEP* **03** (2016) 097 [1510.08846].
- [69] N. Berkovits and H. Gomez, *An Introduction to Pure Spinor Superstring Theory*, in *9th Summer School on Geometric, Algebraic and Topological Methods for Quantum Field Theory*, Mathematical Physics Studies, pp. 221–246, 2017, DOI [1711.09966].
- [70] N. Berkovits, *Pure spinor formalism as an N=2 topological string*, *JHEP* **10** (2005) 089 [hep-th/0509120].

- [71] R. Marnelius and M. Ogren, *Symmetric inner products for physical states in BRST quantization*, *Nucl. Phys. B* **351** (1991) 474.
- [72] N. Berkovits, *Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring*, *JHEP* **09** (2004) 047 [[hep-th/0406055](#)].
- [73] N. Berkovits, *Cohomology in the pure spinor formalism for the superstring*, *JHEP* **09** (2000) 046 [[hep-th/0006003](#)].
- [74] J.P. Harnad and S. Shnider, *CONSTRAINTS AND FIELD EQUATIONS FOR TEN-DIMENSIONAL SUPERYANG-MILLS THEORY*, *Commun. Math. Phys.* **106** (1986) 183.
- [75] G. Policastro and D. Tsimpis,  *$R^{**4}$ , purified*, *Class. Quant. Grav.* **23** (2006) 4753 [[hep-th/0603165](#)].
- [76] N. Berkovits and N. Nekrasov, *Multiloop superstring amplitudes from non-minimal pure spinor formalism*, *JHEP* **12** (2006) 029 [[hep-th/0609012](#)].
- [77] L. Borsten, B. Jurco, H. Kim, T. Macrelli, C. Saemann and M. Wolf, *Tree-level color-kinematics duality from pure spinor actions*, *Phys. Rev. D* **108** (2023) 126012 [[2303.13596](#)].
- [78] L. Borsten, B. Jurco, H. Kim, T. Macrelli, C. Saemann and M. Wolf, *Double Copy from Tensor Products of Metric  $BV^\square$ -algebras*, [2307.02563](#).
- [79] R. Monteiro and D. O’Connell, *The Kinematic Algebra From the Self-Dual Sector*, *JHEP* **07** (2011) 007 [[1105.2565](#)].
- [80] C. Cheung, J. Mangan, J. Parra-Martinez and N. Shah, *Non-perturbative Double Copy in Flatland*, *Phys. Rev. Lett.* **129** (2022) 221602 [[2204.07130](#)].
- [81] M. Blau and G. Thompson, *Topological Gauge Theories of Antisymmetric Tensor Fields*, *Annals Phys.* **205** (1991) 130.
- [82] M. Alexandrov, A. Schwarz, O. Zaboronsky and M. Kontsevich, *The Geometry of the master equation and topological quantum field theory*, *Int. J. Mod. Phys. A* **12** (1997) 1405 [[hep-th/9502010](#)].
- [83] F. Bonechi, A.S. Cattaneo and M. Zabzine, *Towards equivariant Yang-Mills theory*, *J. Geom. Phys.* **189** (2023) 104836 [[2210.00372](#)].
- [84] M. Levi and J. Steinhoff, *Spinning gravitating objects in the effective field theory in the post-Newtonian scheme*, *JHEP* **09** (2015) 219 [[1501.04956](#)].
- [85] M. Henneaux, A. Kleinschmidt and G. Lucena Gómez, *Remarks on Gauge Invariance and First-Class Constraints*, *Proc. Steklov Inst. Math.* **272** (2011) 141 [[1004.3769](#)].
- [86] T. Scheopner and J. Vines, *Dynamical Implications of the Kerr Multipole Moments for Spinning Black Holes*, [2311.18421](#).
- [87] Y.F. Bautista, A. Guevara, C. Kavanagh and J. Vines, *Scattering in black hole backgrounds and higher-spin amplitudes. Part I*, *JHEP* **03** (2023) 136 [[2107.10179](#)].
- [88] Y.F. Bautista, A. Guevara, C. Kavanagh and J. Vines, *Scattering in black hole backgrounds and higher-spin amplitudes. Part II*, *JHEP* **05** (2023) 211 [[2212.07965](#)].
- [89] Z. Bern, D. Kosmopoulos, A. Luna, R. Roiban, T. Scheopner, F. Teng et al., *Quantum Field Theory, Worldline Theory, and Spin Magnitude*

- Change in Orbital Evolution*, 2308.14176.
- [90] N. Siemonsen and J. Vines, *Test black holes, scattering amplitudes and perturbations of Kerr spacetime*, *Phys. Rev. D* **101** (2020) 064066 [1909.07361].
- [91] R. Aoude, K. Haddad and A. Helset, *Searching for Kerr in the 2PM amplitude*, *JHEP* **07** (2022) 072 [2203.06197].
- [92] J. Steinhoff, *Spin gauge symmetry in the action principle for classical relativistic particles*, 1501.04951.
- [93] J.-H. Kim, J.-W. Kim and S. Lee, *The relativistic spherical top as a massive twistor*, *J. Phys. A* **54** (2021) 335203 [2102.07063].



# Acta Universitatis Upsaliensis

*Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology 2378*

Editor: The Dean of the Faculty of Science and Technology

A doctoral dissertation from the Faculty of Science and Technology, Uppsala University, is usually a summary of a number of papers. A few copies of the complete dissertation are kept at major Swedish research libraries, while the summary alone is distributed internationally through the series Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology. (Prior to January, 2005, the series was published under the title “Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology”.)

Distribution: [publications.uu.se](http://publications.uu.se)  
urn:nbn:se:uu:diva-525574



ACTA UNIVERSITATIS  
UPSALIENSIS  
2024