From Quantum to Classical Scattering of Kerr Black Holes

A construction of massive higher-spin scattering amplitudes and their classical limits.

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Abstract

Gravitational scattering processes involving black holes as asymptotic states can provide insight into the classical dynamics of binary black hole systems. The observed gravitational waves emitted during mergers need to be compared to high-precision theoretical predictions. By modelling black holes as massive point particles in an effective quantum field theory, one can take advantage of the advanced computational tools originally designed for collider physics. For Schwarzschild black holes the natural objects to study are scattering amplitudes involving massive scalar fields with interactions mediated by gravitons. The classical physics is extracted by considering limits of the kinematics.

Extending this effective description to rotating Kerr black holes introduces subtleties. To leading order in the post-Minkowskian perturbation scheme, there now exists candidate three-point scattering amplitudes for massive higher-spin particles that in the classical limit reproduce the Kerr metric. For small quantum spins, these are given by familiar theories of interacting massive fields which have a well-behaved massless limit. These theories are sufficient to capture the first few spin-multipole orders for the classical observables; however, to capture more orders one is required to use input from higher-spin theories. The three-point higher-spin amplitudes were originally introduced without reference to an underlying Lagrangian description. Lagrangians for interacting higher-spin fields are notoriously complicated as they necessarily describe composite fields in an effective higher-derivative theory.

This thesis explores the underlying higher-spin effective theories suitable for describing rotating black holes, and proposes a new spin-s family of Compton scattering amplitudes. We present two complementary constructions for consistent interacting higher-spin Lagrangians: the first relies on massive higher-spin gauge symmetry to remove unwanted states, and the second one manifests the correct degrees of freedom using a chiral field framework. A significant portion of the thesis discusses how to extract classical physics from quantum amplitudes, focusing on consistent treatments of the spin degrees of freedom. The resulting quantum and classical Compton amplitudes are built to be consistent with perturbations of the Kerr metric, through a combination of constraints from higher-spin considerations and classical analysis.

In addition to the black-hole amplitudes, we study the scattering of higher-spin fields in a gauge theory referred to as root-Kerr. The three point amplitudes of this gauge theory are closely related to the Kerr ones, such that it provides an instructive model for both higher-spin consistency and classical analysis. Another toy model discussed is the scattering of higher-spin superstring states on the leading Regge trajectory.

Keywords: Higher-spin theory, higher-spin amplitudes, Kerr black holes, EFT, black hole scattering, classical limit

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.


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1. Introduction

General relativity, formulated by Albert Einstein in 1915, revolutionised our understanding of gravity by describing it as the curvature of space-time caused by mass and energy. Among its most intriguing solutions are black holes. A black hole is characterised by an immense gravitational pull so strong that nothing, not even light, can escape its gravitational grip beyond a region called the event horizon. Inside this boundary lies the singularity, a point of infinite density where the laws of physics as we understand them break down. The Kerr black hole solution, proposed by Roy Kerr in 1963, introduced the concept of rotating black holes, which possess not only mass but also angular momentum. The Kerr black hole with angular momentum $J = m|a|$ has a ring-shaped singularity with a radius $|a|$. While the static solution for the Kerr black hole has been known since 1963, studying the dynamics of the Kerr black hole is complicated by the highly non-linear nature of Einstein’s field equations, the equations governing general relativity. Of particular interest is the dynamics of binary black holes systems; these are systems involving two black holes orbiting one-another until, ultimately, merging into a single large black hole. The gravitational waves emitted throughout this process can be detected by the ground-based detectors LIGO/Virgo/KAGRA [1].

These gravitational waves encode information about their astrophysical sources and extraction of this physical data relies on matching to high-precision theoretical models. Due to the highly non-linear nature of general relativity, solving such systems exactly is not tractable. Instead the theoretical waveforms are usually constructed via a combination of numerical and analytical methods that are applicable in binary systems characterised by different physical parameters.

Early in the inspiral, the black holes are still well-separated and the dynamics are governed by weak gravitational interactions. Therefore we can solve the equations of motion using a perturbative approach in the gravitational coupling $G$, the so-called post-Minkowski expansion (PM). To further simplify the analysis, we can study the scattering of two black holes where the gravitational interaction is a transient phenomena, in contrast to the bound system where black-holes are constantly interacting. In many cases, the scattering data can be related to the bound physics; a notable example is the effective potential that governs the conservative interactions of bound systems [42, 15, 16, 49].
In the scattering scenario, we can take advantage of the wealth of high-precision tools originally designed for collider physics by modelling black holes as fundamental massive particles in a quantum field theory (QFT). This is remarkable given we currently do not have a fully consistent quantum description of gravity. The success of the QFT methods relies on considering the Einstein-Hilbert action as an effective theory which is compatible with general-relativity at low energies [32]. We can avoid the usual inconsistencies related to quantising gravity by restricting ourselves to energy scales below the Planck scale, this is compatible with the energy scales of astrophysical black-hole mergers currently detected [1].

The dynamics of non-rotating black holes have been successfully computed from scattering amplitudes involving minimally coupled massive scalar fields, such that the potential and the scattering angle is currently known to $O(G^4)$ [11, 12]. In order to incorporate rotational degrees of freedom, one can study minimally coupled theories for massive spinning fundamental fields. This method has been successful for low spins, such that low orders of the classical spin dependence have also been computed to $O(G^4)$ [43]. Observables for classical spinning objects can be series expanded in the classical spin vector $a^\mu$, and contributions $O(a^n)$ are called the \textit{n-th spin multipole coefficient}. In the case of the Kerr black hole, the spin parameter $a^\mu$ is a four-dimensional repackaging of the spin degrees of freedom of the black hole normalised such that $|a|$ corresponds to the radius of the ring singularity.

There is a subtle relation between the classical spin $a^\mu$, a continuous parameter, and the quantum spin $s$, a discrete quantum number. The guiding principle is that the classical limit of the interactions of a massive spin-$s$ field can generate, at most, the first $2s$ classical spin multipoles [58]. In this framework, computing classical observables to higher orders in the classical spin necessitates studying theories of interacting higher-spins.

In gravity theories, minimally coupled theories of massive spinning tensor fields develop inconsistencies for $s > 2$ due to propagation of unphysical degrees of freedom [30]. In order to cancel the unphysical degrees of freedom one can introduce auxiliary fields [55]. Zinoviev introduced a streamlined approach that builds from a free theory for general spin involving the physical spin-$s$ field and a tower of Stückelberg fields all related by a massive gauge symmetry [61]. Adding interactions consistently requires maintaining the gauge symmetry at each order in the fields constraining non-minimal interactions in the Lagrangian and in the gauge variations [61, 62, 64, 65]. However, working at the Lagrangian level introduces redundancies related to how the fields and parameters in the theory are defined. To avoid this, in paper II we introduce an on-shell approach to enforce massive gauge invariance at the level of the currents in the form of generalised massive Ward identities.
An alternative Lagrangian construction was introduced in ref. [50] using a massive chiral field as a building block. This field automatically propagates the correct $2s + 1$ degrees of freedom so we do not need to consider a tower of auxiliary fields. However, the minimally coupled theories break parity invariance due to the chirality of the fundamental field. One can impose parity order by order in the Lagrangian at the cost of introducing non-minimal interactions [23].

In both constructions the theories of massive higher-spins involve non-minimal interactions. Indeed, due to the no-go theorems for theories of interacting massless higher-spin fields in flat-space [28, 60], theories involving interacting massive higher-spins must be effective field theories (EFTs) that are only valid below a cut-off energy scale.

The focus of this thesis is to study the EFTs that model Kerr dynamics in the classical limit, which corresponds to a low-energy limit. Previous work has suggested that Kerr amplitudes exhibit better high energy behaviour than a generic higher-spin theory would [4, 39, 26, 24]. Notably, refs. [39, 26] showed that the linearised energy-momentum tensor of the Kerr black hole is related to a set of three-point quantum amplitudes $\mathcal{M}_{\text{Kerr}}$ originally introduced in ref. [4] due to their improved high-energy behaviour. In ref. [4] these amplitudes were written without specifying the Lagrangian, although it was known that for $s \leq 2$ they correspond to the interactions of minimally coupled massive fields. In Paper II, we identified the characteristics of the higher-spin EFTs that uniquely determine $\mathcal{M}_{\text{Kerr}}$ at three point: a combination of massive gauge symmetry and strict power counting.

Higher-point amplitudes, such as the four-point Compton amplitude $\mathcal{M}_{\text{Kerr}}(1^s, 2^s, 3, 4)$ are required for higher-order calculations of classical observables. One can use on-shell methods to construct these amplitudes, however the result contains residual freedom in the contact terms [4, 26, 3, 13]. We take an alternative approach, reducing the contact term freedom by constraining the quartic order of the EFT with higher-spin constraints. The reduction in the number of contact terms is boosted by additional constraints coming from the classical limit.

The classical regime of $\mathcal{M}_{\text{Kerr}}$ corresponds to the long-wave length limit of the gravitons, which is equivalent to a multisoft limit of the graviton momenta. In order to model classical macroscopic spin, the classical limit enforces a large quantum spin limit. One formulation of the classical limit centres on a ‘large charge’ approach [39, 26] where the spin quantum number scales as $s \sim \hbar^{-1}$ while the massless momenta scale as $\hbar$. An alternative approach, introduced in ref. [3], involves scattering coherent states of massive spins, as opposed to a single massive spin-$s$ state, and taking a soft graviton limit $\hbar \to 0$. Notably this approach is also sensitive to the $s \to \infty$ behaviour of the amplitude since the coherent state involves an infinite sum over the spin of the massive
states. In both methods, the large spin limit is required to consistently extract the correct spin multipoles even at low orders in the spin multipole expansion. This limit was explored in Paper I when analysing the classical scattering of leading Regge superstring states.

Throughout this thesis we will also consider a gauge theory which is only known as the EFT that gives rise to the family of three-point amplitudes in ref. [4]. The theory is referred to as $\sqrt{\text{Kerr}}$ due to the simple double copy structure that relates both the quantum and classical amplitudes of $\sqrt{\text{Kerr}}$ and Kerr [44, 5]. In paper III we construct the underlying higher-spin EFTs for $\sqrt{\text{Kerr}}$ using similar higher-spin constraints that are relevant for the Kerr EFTs. The structure of the quantum and classical amplitudes in the gravity and gauge theory is closely related such that $\sqrt{\text{Kerr}}$ is a good toy model for Kerr. In this thesis we will only discuss the electromagnetic gauge theory, see paper III for the full non-abelian treatment.

In Part I, we begin by considering a massive scalar field minimally coupled to gravity in order to motivate modelling black holes using interacting fundamental fields. Next we will introduce the on-shell variables relevant for the scattering of massive spin states, taking the minimally coupled massive fermion as an example. We will also introduce our proposed four point quantum Compton amplitudes for $\sqrt{\text{Kerr}}$ and Kerr. Discussion of the origin of these amplitudes is postponed to Parts II and III since we first need to introduce the quantum and classical constraints necessary. The latter half of Part I introduces spin operator variables, a set of quantum variables that are closely related to the classical spin parameter $a^\mu$.

In Part II we discuss the construction of consistent theories for interacting massive spins. We introduce two approaches based on two different representations of the Lorentz group, the $(s, s)$ tensor fields and the $(2s, 0)$ chiral fields. At cubic order we highlight the constraints that uniquely determine the Kerr and $\sqrt{\text{Kerr}}$ amplitudes. We discuss extensions to quartic order and note that our EFTs are no longer uniquely determined.

In Part III we discuss classical amplitudes; what computations they correspond to in classical physics and how they can be generated as limits of the quantum amplitudes. Section 9 discusses the various classical limits that appear in the literature to extract classical spin multipoles from the scattering of massive spin-$s$ particles. Now we have all the tools necessary to present the full set of quantum and classical constraints that fix the $\sqrt{\text{Kerr}}$ and Kerr EFTs and amplitudes. We will present our proposal for both the quantum and classical amplitudes in both theories.
Part I:
Scattering Amplitudes for Massive Spinning Particles
2. Warm-up: modelling Schwarzschild black holes

A priori it may not be obvious how to model interactions between multiple black holes, such as given by the Schwarzschild metric. However, we may use the approach of effective field theory (EFT), and write down the simplest action that comes to mind, and then systematically add corrections when needed. We know that for a distant observer a Schwarzschild black hole is only characterised by its mass, and that its gravitational pull is similar to that of any other massive source in general relativity.

Hence consider the simplest such effective theory, consisting of two massive scalars minimally coupled to the Einstein-Hilbert action,

\[ S = \int d^4x \sqrt{-g} \left[ \frac{2}{\kappa^2} R + \frac{1}{2} \sum_{i=1}^2 \nabla^\mu \phi_i \nabla_\mu \phi_i - m_i^2 \phi_i^2 \right]. \] (2.1)

Minimal coupling corresponds to covariantising a free matter theory by replacing the derivatives with covariant derivatives \( \partial_\mu \to \nabla_\mu \), the flat metric with the dynamical one \( \eta_{\mu\nu} \to g_{\mu\nu} \) and using the proper volume form \( d^4x \sqrt{-g} \). Of course, for a scalar field the covariant derivative is trivial \( \nabla_\mu \phi = \partial_\mu \phi \). We may then expose the perturbative interactions relevant to scattering amplitudes by expanding the metric around Minkowski space, \( g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \), where \( \kappa = \sqrt{32\pi G} \) is the coupling constant and \( h_{\mu\nu} \) is the dynamical gravitational field.

Massive two-to-two scattering

The classical two-to-two scattering regime corresponds to the case where the two black holes are well separated relative to their size \( 2Gm \), or equivalently, where the deflection \( q_\mu = \Delta p_\mu \) of a black hole’s momentum is small. The dimensionless small parameter is then \( Gm|q| \ll 1 \). The classical interactions are mediated by graviton exchange (neglecting pure-graviton loop diagrams). At post-Minkowskian order \( n\text{-PM} \) one has \( n = L + 1 \) exchanged gravitons that contribute as \( G^n \), where \( L \) is the loop order of the classical Feynman diagram.

Consider the tree-level 1-PM case, consisting of a single Feynman diagram. From the above action one can extract the graviton propagator (in de Donder gauge),

\[ \Delta^{\mu\nu;\rho\sigma}(q) = \frac{1}{q^2 + i0} \left( \frac{1}{2} \eta^{\mu\rho} \eta^{\nu\sigma} + \frac{1}{2} \eta^{\mu\sigma} \eta^{\nu\rho} - \frac{1}{D - 2} \eta^{\mu\nu} \eta^{\rho\sigma} \right), \] (2.2)
where the dimension is $D = 4$, and also extract the scalar-graviton interaction,

$$V^{\mu\nu}(\phi_i, \phi'_i, h) = \frac{\kappa}{2}(p_i^\mu p'_i{}^\nu + p_i^\nu p'_i{}^\mu + \eta^{\mu\nu}(p_i \cdot p'_i - m_i^2)),$$  \hspace{1cm} (2.3)

where the momenta are related as $p'_i{}^\mu = p_i^\mu \pm q^\mu$. Then contracting the vertices with the graviton propagator, gives the following two-to-two scattering amplitude, at leading order in the $|q| \to 0$ limit:

$$\mathcal{M}_{2\text{-to-2}} = \frac{\kappa^2}{2} \frac{m_1^2 m_2^2}{q^2} - \frac{2(p_1 \cdot p_2)^2}{q^2} + O(q^0). \hspace{1cm} (2.4)$$

Following the approach of Iwasaki [42] one can compute the leading order contribution to the relativistic two-body potential by the Fourier transform

$$V_{1PM} = \int \frac{d^3q}{(2\pi)^2} e^{-i\vec{q} \cdot \vec{r}} \mathcal{M}_{2\text{-to-2}} \propto \frac{Gm_1 m_2}{r} (1 - 2\sigma), \hspace{1cm} (2.5)$$

where $\sigma = \frac{(p_1 \cdot p_2)^2}{m_1^2 m_2^2}$ is the Lorentz factor. In the non-relativistic approximation $\sigma = 1 + 1/2v^2 + \ldots$ where $v$ is the relative velocity and we recover Newton’s potential in the static limit $v = 0$.

For a general massive compact object, one would expect corrections to the potential corresponding to the fact it is an extended object with finite size. For a black hole the appropriate size is the Schwarzschild radius $r_S = Gm$, and as such finite-size corrections to the dynamics are characterised by the mass, and possible dimensionless parameters. So called tidal-deformability parameters, or Love numbers, are considered dimensionfull and are expected to vanish for the Schwarzschild case, at least the ones that are important for long-range dynamics. The minimally coupled scalar theory is thus expected to be a good model for interacting black holes, during the inspiral phase of a merger. For neutron stars one should instead construct an effective theory where the tidal effects are added as higher-derivative non-minimal interactions to the above action.

Given the above potential, one can construct an effective Hamiltonian in the centre-of-momentum (COM) frame, using the energies and momenta $p_1 = (E_1, \vec{p})$, $p_2 = (E_2, -\vec{p})$,

$$H = \sqrt{\vec{p}^2 - m_1^2} + \sqrt{\vec{p}^2 - m_2^2} + V(\vec{p}, \vec{r}). \hspace{1cm} (2.6)$$

The potential has the expansion, $V(\vec{p}, \vec{r}) = V_{1PM}(\vec{p}, \vec{r}) + V_{2PM}(\vec{p}, \vec{r}) + \ldots$, which correspond to contributions from classical loop diagrams to the two-to-two scattering amplitude [42, 41, 40]. In the classical regime, the terms that have inverse powers of $|q|$ in the scattering amplitude will
dominate, which corresponds to long-range dynamics. Such non-analytic terms can be fixed by inspecting unitarity cuts of the loop amplitudes, which in turn only require the knowledge of tree-level amplitudes involving a single black hole and an arbitrary number of gravitons [14]. Therefore higher-precision classical calculations depend on knowing the multi-graviton tree-level amplitudes.

**Gravitational Compton scattering**

Scattering gravitational waves (GWs) against a Schwarzschild background is the classical process that corresponds to one of the mentioned multi-graviton tree-level amplitudes. The simplest physical case is Compton scattering, which involves two on-shell gravitons, one incoming and one outgoing. In order to do the calculation from first principles in general relativity, one has to study the asymptotic $r \to \infty$ behaviour of solutions to the Regge-Wheeler equation [8], which controls the perturbations to the Schwarzschild metric. Alternatively, in the above point-particle approximation using scalar fields, this corresponds to taking the characteristic graviton wavelength $\lambda$ large enough such that the size of the black hole is not resolved, $\lambda \gg r_S$ [28]. In terms of the graviton angular frequency $\omega \sim 1/\lambda$, the relevant small parameter is then $Gm\omega \ll 1$. Thus the gravitons have soft momenta.

The tree-level Compton amplitude for a Schwarzschild black hole can be worked out from the above action; in the classical limit the graviton momenta scale as $k_4, k_3 \sim \hbar$, giving

$$\mathcal{M}(1, 2, 3, 4) = -4\kappa^2 \frac{(p_1 \cdot F_3 \cdot F_4 \cdot p_1)^2}{q^2(p_1 \cdot q_\perp)^2} + O(\hbar), \quad (2.7)$$

with $F_{ij}^{\mu\nu} = 2k_i^{[\mu} \epsilon_{j]}^{\nu}$ and $q = k_3 + k_4, q_\perp = k_4 - k_3$ [8]. Note that this kinematic regime is consistent with the one relevant for the computation of the classical potential, and thus this amplitude can be used in a unitarity cut where two soft gravitons are being exchanged between two black holes, which is an intermediate step towards obtaining the 2PM potential.

**Extension to Kerr**

Many astrophysical black holes are expected to have non-zero angular momentum, meaning that they are described by the Kerr metric. We would like to repeat the successful exercise of the minimally coupled scalar, except now consider point particles which carry intrinsic angular momentum, also known as spin.

Remarkably, it turns out that one can archive similar success by using three-point amplitudes for minimally coupled massive particles of spins $s = 1/2, 1, 3/2$, corresponding to a fermion, Proca and Rarita-Schwinger fields, respectively. These amplitudes matches the three first
spin-multipole orders of the linearised energy-momentum tensor of the Kerr black hole

\[ M_{\text{min.}}(1, 2, 3) \sim \varepsilon_{\mu\nu}(q)\tilde{T}_{\text{Kerr}}^{\mu\nu}(q), \tag{2.8}\]

where a Kerr energy-momentum tensor was constructed in ref. [59],

\[ \tilde{T}_{\text{Kerr}}^{\mu\nu}(q) = \int d^4x e^{i q \cdot x} T^{\mu\nu}_{\text{Kerr}}(x) = \delta(p_1 \cdot q) p_1^\mu (e^{a \cdot q})^\nu \rho_1 p_1^\rho. \tag{2.9}\]

The spin multipole orders are captured by the exponential of the matrix

\[ (a \ast q)^{\mu\nu} = i \varepsilon_{\mu\nu\rho\sigma} a^\rho q^\sigma, \quad a^\mu = S^\mu/m \]

is the ring radius vector of the Kerr black hole. The magnitude \(|a| := \sqrt{-a^2}\) is the radius of the ring singularity [39, 26]. The details of the classical limit are discussed in Part III.

For \( s \geq 2 \), minimally coupled theories of massive spinning particles are expected to develop inconsistencies due to propagation of unphysical degrees of freedom [30]. Nonetheless, Arkani-Hamed, Huang and Huang, introduced in ref. [4] a natural higher-spin extension of the low-spin minimally coupled amplitudes. It is summarised by the elegant formula

\[ M_{\text{Kerr}}(1^s, 2^s, 3^+, 2^\pm) = \kappa (p_1 \cdot \varepsilon_3^+)^2 \frac{(12)^{2s}}{m^{2s}}, \tag{2.10}\]

where the massive particles carry spin \( s \) and the graviton carry helicity +2. The bracket denotes massive spinor-helicity variables, which we will introduce in detail in the next section. For \( s < 2 \), the amplitudes correspond to minimally coupled spin-\( s \) fields, and beyond this the interpretation is less clear. While these amplitudes were motivated by their simplicity and nice high-energy behaviour, later in refs. [39, 26] they were shown to match the linearised energy-momentum tensor for Kerr up to multipole order \( 2s \), and thus hold for arbitrary classical spin in the limit \( s \to \infty \).

A central result of this thesis is the explicit proposal of a consistent Compton amplitude for Kerr,

\[ M_{\text{Kerr}}(1^s, 2^s, 3^\pm, 4^\pm), \tag{2.11}\]

which we confirm by matching to classical results from general relativity, independently obtained by solving the scattering problem of GWs in a Kerr metric background [9]. In order to get there, we first will need to extensively study and constrain non-minimal interactions contributing to \( M(1^s, 2^s, 3^\pm) \), as well as properly implement a consistent classical limit for spinning particles. We begin by reviewing technical aspects of the variables used for amplitudes with massive spinning particles.
3. On-shell variables for massive spinning particles

3.1 Minimally coupled massive spin-1/2 fermions

We now introduce on-shell variables for general massive spins. Let us first discuss the lowest spin case, the spin-1/2 fermion $\psi$, which transforms as a spinor in the Lorentz group. We choose to work with Dirac as opposed to Marojana fermions since we will couple it to electromagnetism. The on-shell free field satisfies the Dirac equation

\[
(i\gamma^\mu \partial_\mu - m)\psi(x) = 0,
\]

where $\gamma^\mu := \gamma^\mu \partial_\mu$. The two plane-wave solutions are $\psi(x) = u^a e^{-ip \cdot x}$ and $\bar{\psi}(x) = v^a e^{ip \cdot x}$ with $p^2 = m^2$ and correspond to an incoming fermion or anti-fermion respectively. In four dimensions, the massive solutions transform under the little group $SU(2)$, such that $a = 1, 2$. The massless solutions transform under the little group $U(1)$, such that the little group index $a$ corresponds to two helicity choices for the massless fermion, $u^\pm(p)$.

The free theory is $L = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$, which one can minimally couple to electromagnetism by promoting the derivative into covariant derivatives $\gamma^\mu \rightarrow D^\mu := \gamma^\mu + iQ A^\mu$,

\[
L = \bar{\psi}(iD^\mu - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},
\]

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The resulting three point amplitude between a fermion, anti-fermion and photon, all incoming, is

\[
A(1^a, 2^b, 3) = -\sqrt{2} Q v^b(p_2) \epsilon^\mu_3 u^a(p_1).
\]

Likewise, one can minimally couple the fermion to gravity,

\[
L = \sqrt{-g} \{ \bar{\psi}(i\nabla - m)\psi + R \}.
\]

The slashed covariant derivative is defined as $\nabla = e^\mu \gamma^\nu \nabla_\mu$ where $e^\mu \gamma^\nu$ is the frame field with flat-space index $\gamma$. The covariant derivative is $\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{2} \omega_\mu \gamma^{\nu\rho} M_{\nu\rho} \psi$ with spin connection $\omega_\mu \gamma^{\nu\rho}$ and spin-1/2 Lorentz generators $M_{\nu\rho} = \frac{1}{4} [\gamma^{\nu}, \gamma^{\rho}]$. Note from now on most indices will be flat so we will not use the hatted notation.
The resulting gravitational three-point amplitude is

$$\mathcal{M}(1^a, 2^b, 3) = \frac{1}{2} p_1 \cdot \varepsilon_3 \psi_{2\bar{3}}^b \hat{u}_1^a. \quad (3.5)$$

For spin-1/2, the gauge and gravity amplitudes are compact expressions. However, for general massive spins, the covariant amplitudes can be cumbersome and hide structure imposed by the little group covariance [33].

From now on we will work in the Weyl representation, with the gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^{\mu\alpha} = (1, \sigma^i) = \bar{\sigma}^{\dot{\alpha}\alpha}, \quad (3.6)$$

and using the metric $\eta = \text{diag}(1, -1, -1, -1)$. In this chiral representation, the Dirac equation reduces to two coupled Weyl equations

$$i\bar{\sigma} \cdot \partial \psi_R = m\psi_L, \quad i\sigma \cdot \partial \psi_L = -m\psi_R, \quad (3.7)$$

where the Dirac field, $\psi$, has been projected onto the left- and right-handed chiral fields $\psi_{L,R}$

$$P_- \psi := \frac{1}{2}(1 - \gamma^5)\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \quad P_+ \psi := \frac{1}{2}(1 + \gamma^5)\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \quad (3.8)$$

with $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. We will now introduce spinor-helicity notation formulated using the Weyl spinors [33, 4].

### 3.1.1 Spinor variables for massless fermions

For massless fermions, $k^2 = 0$, we will use the following angle and square bra-ket notation for the Weyl spinors

$$\psi_L = \ket{k}e^{-ik \cdot x}, \quad \psi_R = \ket{k}e^{-ik \cdot x}, \quad (3.9)$$

where $\ket{k}_\alpha$, $\ket{k}_\dot{\alpha}$ are chiral Weyl spinors constructed from the null momentum $k$

$$k_\mu \sigma^{\mu\alpha} = \ket{k}_\alpha \bra{k}_\alpha. \quad (3.10)$$

The two helicities of the corresponding massless Dirac spinors decompose into

$$u^- (k) = \begin{pmatrix} \ket{k} \\ 0 \end{pmatrix}, \quad u^+ (k) = \begin{pmatrix} 0 \\ \ket{k} \end{pmatrix} \quad (3.11)$$

where we can identify the left-handed spinor $\ket{k}$ with negative helicity state and the right-handed spinor $\ket{k}$ with positive helicity state.

---

1The bra spinors are defined by $\bra{k}_\alpha = \epsilon^{\alpha\beta}\ket{k}_\beta$ and $\bra{k}_\dot{\alpha} = \epsilon_{\dot{\alpha}\dot{\beta}}\ket{k}_\dot{\beta}$ where $\epsilon^{12} = 1 = \epsilon_{21}$ is an antisymmetric $SL(2, \mathbb{C})$ Levi-Civita.
Using massless spinors, we can also construct spin-1 polarisations for a massless photon of momentum $k^\mu$ with respect to the null reference momentum $r^\mu$
\[
\varepsilon^\mu_+(k, r) = \frac{\langle r|\sigma^\mu|k\rangle}{\sqrt{2\langle r^2\rangle}} , \quad \varepsilon^\mu_-(k, r) = \frac{\langle k|\sigma^\mu|r\rangle}{\sqrt{2\langle kr\rangle}} . \tag{3.12}
\]
The spinor contractions are defined $\langle r^\alpha\rangle := \epsilon^{\alpha\beta}\langle r^\beta\rangle$ and $\{ r^\alpha\} := \epsilon^{\dot{\alpha}\dot{\beta}}\{ r^\dot{\beta}\}$. The graviton polarisations can be constructed as a tensor product of the spin-1 polarisations $\varepsilon_+^{\mu\nu} = \varepsilon_+^\mu\varepsilon_+^\nu$, $\varepsilon_-^{\mu\nu} = \varepsilon_-^\mu\varepsilon_-^\nu$, which satisfy the required symmetric, traceless properties of the on-shell massless spin-2 field.

We will also make use of the alternative notation $|i\rangle$ where $i$ corresponds to the particle label on momentum $k_i$. In these spinor variables, the three-point amplitude between two massless fermions and a photon can be written compactly
\[
A(1^+_m=0, 2^+_m=0, 3^+) = \sqrt{2}Q v^+(k_2)\tilde{f}_3^+u^-(k_1) = \sqrt{2}Q \frac{[23]^2}{[12]} , \tag{3.13}
\]
Note that we must use complexified momenta otherwise the amplitude vanishes due to the restrictive three point kinematics.

### 3.1.2 Spinor variables for massive fermions

One can construct similar plane-wave solutions for the massive fermions, $p^2 = m^2$,
\[
\psi_L = |p^a\rangle e^{-ip\cdot x} , \quad \psi_R = |p^a\rangle e^{-ip\cdot x} . \tag{3.14}
\]
The $SU(2)$ indices run over $a = 1, 2$ such that each variable $|p^a\rangle$, $|p^a\rangle$ is a $2 \times 2$ matrix. Note that the left- and right-handed spinors are not independent, since the Weyl equations in eq. (3.7) imposes the relation
\[
p \cdot \sigma|p^a\rangle = m|p^a\rangle . \tag{3.15}
\]
The parametrisation of the massive spinors is not unique. We use a parameterisation for the massive momenta $p$ that appears in [24],
\[
p = k + \frac{m^2}{2p\cdot r}r , \tag{3.16}
\]
where $r$ is a null reference momentum and null momentum $k$ is defined by solving the on-shell condition $p^2 = m^2$. In this parametrisation the massive spinors are written explicitly as
\[
|p^a\rangle = \left( \frac{m}{\langle kr\rangle} \right) \begin{pmatrix} \langle r\rangle \\ |k\rangle \end{pmatrix} , \quad |p^a\rangle = \left( \frac{m}{\langle kr\rangle} \right) \begin{pmatrix} |k\rangle \\ \langle r\rangle \end{pmatrix} . \tag{3.17}
\]
For labelled massive momenta $p_i$ we will use the notation $|i^a\rangle$, $|i^a\rangle$. In order to avoid working with the free indices, we contract the little group index with a complex wavefunction $z_{i,a}$ and define the bold massive spinors

$$|i\rangle = |i^a\rangle z_{i,a}, \quad |\bar{z}\rangle = |i^a\rangle z_{i,a}. \quad (3.18)$$

Self-contractions of the bold spinors vanish due to symmetrisation imposed by the wavefunctions, for example

$$\langle 11 \rangle = -m \epsilon^{ab} z_{1a} z_{1b} = 0. \quad (3.19)$$

The three point amplitude for the massive fermions coupled to electromagnetism can then be written in the form

$$A(1^{s=1/2}, 2^{s=1/2}, 3^+) = \sqrt{2} Q p_1 \cdot \varepsilon_3^+ \frac{\langle 21 \rangle}{m}. \quad (3.20)$$

This is the lowest spinning case in the $\sqrt{Kerr}$ family of amplitudes introduced in ref. [4],

$$A(1^s, 2^s, 3^+) = \sqrt{2} Q p_1 \cdot \varepsilon_3^+ \frac{\langle 21 \rangle^{2s}}{m^{2s}}. \quad (3.21)$$

We can generate the negative helicity amplitude $A(1^{s=1/2}, 2^{s=1/2}, 3^-)$ by swapping $\varepsilon_3^+ \rightarrow \varepsilon_3^-$ and $|i\rangle \leftrightarrow |\bar{z}\rangle$. We can also compute the four-point Compton amplitudes for the fermion coupled to electromagnetism in the two independent helicity sectors

$$A(1^{s=1/2}, 2^{s=1/2}, 3^+, 4^+) = Q^2 \frac{m[34]^2}{t_{13} t_{14}} \langle 21 \rangle, \quad (3.21)$$

$$A(1^{s=1/2}, 2^{s=1/2}, 3^-, 4^+) = Q^2 \frac{\langle 3[14] \rangle}{t_{13} t_{14}} (\langle 13 \rangle[42] - \langle 23 \rangle[41]), \quad (3.22)$$

where $\langle 3|14 \rangle = p_{1\mu} \langle 3|\sigma^\mu|4 \rangle$ and, as before, the poles $t_{1i} = 2p_1 \cdot k_i$ come from the massive propagator. Note that the three-point and four-point amplitudes can be factored into a scalar amplitude and a spin-dependent coefficient, for example

$$A(1^{s=1/2}, 2^{s=1/2}, 3^+) = A(1^{s=0}, 2^{s=0}, 3^+) \frac{\langle 21 \rangle}{m}, \quad (3.22)$$

Where the scalar amplitudes

$$A(1^{s=0}, 2^{s=0}, 3^+) = \sqrt{2} Q p_1 \cdot \varepsilon_3^+$$

$$A(1^0, 2^0, 3^+, 4^+) = Q^2 \frac{m^2[34]^2}{t_{13} t_{14}}, \quad A(1^0, 2^0, 3^-, 4^+) = Q^2 \frac{\langle 3[14] \rangle}{t_{13} t_{14}}, \quad (3.23)$$

are computed from the Lagrangian $\mathcal{L} = |D_\mu \phi|^2 - m^2|\phi|^2 - \frac{1}{4}(F_{\mu\nu})^2$ for scalars minimally coupled to electromagnetism.
The analogous amplitudes for fermions minimally coupled to gravity are remarkably similar in structure to the gauge theory,

\[
\mathcal{M}(1^{s=1/2}, 2^{s=1/2}, 3^+) = \kappa (p_1 \cdot \varepsilon_3^+) \frac{2 \langle 21 \rangle}{m},
\]

\[
\mathcal{M}(1^{s=1/2}, 2^{s=1/2}, 3^+, 4^+) = \left( \frac{\kappa}{2} \right)^2 m^3 \langle 34 \rangle \frac{t_{13} t_{14}}{s_{12}} \langle 21 \rangle,
\]

\[
\mathcal{M}(1^{s=1/2}, 2^{s=1/2}, 3^+, 4^+) = \left( \frac{\kappa}{2} \right)^2 \langle 3 | 1 | 4 \rangle \frac{t_{13} t_{14}}{s_{12}} \langle 13 | 42 \rangle - \langle 23 | 41 \rangle,
\]

(3.24)

where \( s_{12} = (p_1 + p_2)^2 \). These gravitational amplitudes can be obtained trivially from the gauge theory by a double copy prescription,

\[
\mathcal{M}(1^s, 2^s, 3) = \left( \frac{\kappa}{2} \right)^2 \langle 10 | 20 | 3 \rangle \mathcal{A}(1^s, 2^s, 3) \big|_{Q \to 1},
\]

\[
\mathcal{M}(1^s, 2^s, 3, 4) = \left( \frac{\kappa}{2} \right)^2 \langle 10 | 20 | 3 | 4 \rangle \mathcal{A}(1^s, 2^s, 3, 4) \big|_{Q \to 1},
\]

(3.25)

which holds for \( s \leq 2 \) as discussed in refs. [4, 44].

3.2 On-shell variables for general spin

We will now construct higher-spin states from the spin-1/2 building blocks. The massive spin-1 polarisation for particle \( p_i \) can be defined in terms of massive spinors

\[
\varepsilon_i^\mu = \frac{\langle i | \sigma^\mu | i \rangle}{\sqrt{2}m}.
\]

(3.26)

In this parameterisation, the polarisations are explicitly transverse and null given the spinors are explicitly symmetrised by their contraction with \( z^a_{ia} \). Note that one can resolve the three degrees of freedom of massive spin-1 particle by expanding polarisation as a polynomial in \( z^a_{ia} \)

\[
\varepsilon_i^\mu = (z_i^1)^2 \varepsilon_{-,-,i} + \sqrt{2} z_i^1 z_i^2 \varepsilon_{L,-,i} - (z_i^2)^2 \varepsilon_{+,i}.
\]

(3.27)

Where \( \varepsilon_{L,i}^\mu = k_i^\mu / m - m r^\mu / (2p_i \cdot r) \) is the longitudinal degree of freedom of the massive spin-1 particle. In order to describe complex massive vector bosons we take the wavefunctions \( z \) to be complex. In which case, we can introduce the polarisation for the complex conjugated spin-1 field

\[
\bar{\varepsilon}_i^\mu = \frac{(\langle i | \sigma^\mu | i \rangle)^*}{\sqrt{2}m} = -\frac{\langle \bar{i} | \sigma^\mu | i \rangle}{\sqrt{2}m}
\]

(3.28)

where the barred spinors \( |\bar{i}\rangle := |i^a\rangle \bar{z}_{ia} \), \( |\bar{i}\rangle := |i^a\rangle \bar{z}_{ia} \) are functions of \( \bar{z}_{ia} = (z_{ia})^* \), \( \bar{z}_{ia} = -(z_{ia})^* \). The spinor products between barred and unbarred
variables are non-vanishing, $\langle \bar{1}1 \rangle = [1 \bar{1}] = m \bar{z}^a z_a$, and proportional to the normalisation $|z| := \sqrt{\bar{z}^a z_a}$.

We can construct polarisation tensors for higher-spin states as tensor products of the spin-1 and spin-1/2 variables

\begin{align}
\text{integer spin: } & \varepsilon^{\mu_1 \ldots \mu_s} := \varepsilon^a_i \varepsilon^b_i \ldots \varepsilon^s_i, \\
\text{half-integer spin: } & |i\rangle^{\mu_1 \ldots \mu_{\lfloor s \rfloor}} := |i\rangle \varepsilon^a_i \varepsilon^b_i \ldots \varepsilon^\lfloor s \rfloor_i, \tag{3.29}
\end{align}

and similarly for the right-handed polarisation $|\bar{i}\rangle^{\mu_1 \ldots \mu_{\lfloor s \rfloor}}$. These fully-symmetric tensors satisfy the required transversality and tracelessness properties that characterise the irreducible representations of the little group $SO(3)$. Tracelessness and so-called gamma-tracelessness is a consequence of the properties $\varepsilon_i^2 = 0$ and $\varepsilon_i \cdot \bar{\sigma} |i\rangle = 0$ of the spin-1 polarisations, which follow from $\langle ii \rangle = 0$.

**Higher-spin amplitudes from on-shell approaches**

Ref. [4] provides a study of three-point amplitudes involving two states of mass $m$ and spin $s$ with a photon or graviton. At three points, the general amplitudes have the form

\begin{align}
\mathcal{A}(1^s, 2^s, 3^+) &= \sqrt{2} \frac{p_1 \cdot \bar{\varepsilon}_3}{m^{2s}} \sum_{n=0}^{2s} c_n \langle 12 \rangle^{2s-n} [12]^n, \\
\mathcal{M}(1^s, 2^s, 3^+) &= \kappa \frac{(p_1 \cdot \bar{\varepsilon}_3)^2}{m^{2s+1}} \sum_{n=0}^{2s} \tilde{c}_n \langle 12 \rangle^{2s-n} [12]^n, \tag{3.31}
\end{align}

where the coefficients $c_n, \tilde{c}_n$ satisfy the constraints

\begin{align}
\sum_{n=0}^{2s} c_n = Q, & \quad \sum_{n=0}^{2s} \tilde{c}_n = m, \quad \sum_{n=0}^{2s} n\tilde{c}_n = 0. \tag{3.32}
\end{align}

The first two correspond to normalisation of the charge or mass monopole in the respective gauge and gravity theories. The coefficients for the gravity amplitudes satisfy an extra constraint due to the universality of the dipole coupling.

For low spins, $s \leq 1$ in gauge theory and $s \leq 2$ in gravity, the three point amplitudes with coefficients $c_n = Q \delta_{n0}, \tilde{c}_n = m \delta_{n0}$ correspond to the interactions of minimally coupled fundamental massive fields. For these low spin theories, one can compute the four-point Compton amplitudes from the Lagrangians directly. The resulting the all-plus amplitudes have the form

\begin{align}
\mathcal{A}(1^s, 2^s, 3^+, 4^+) &= \frac{m^2 [34]^2}{t_{13} t_{14}} \langle 12 \rangle^{2s}, \\
\mathcal{M}(1^s, 2^s, 3^+, 4^+) &= \frac{m^4 [34]^4}{s_{12} t_{13} t_{14}} \langle 12 \rangle^{2s}. \tag{3.33}
\end{align}
For higher-spins, the Lagrangians that reproduce the three point Kerr and \( \sqrt{K} \) three point amplitudes are no longer theories of interacting fundamental fields. However, one can avoid using the Lagrangian and bootstrap the higher-point amplitudes using factorisation properties as done in [4] or BCFW on-shell recursion [19, 20]. In the same-helicity sector, the resulting amplitudes correspond to those in eq. (3.33).

However the opposite-helicity amplitudes proposed in ref. [4] have the form

\[
A_{\text{AHH}}(s^1, 2s^2, 3^-, 4^+) = Q^2 \frac{\langle 3|1|4 \rangle^{2-2s}}{t_{13}t_{14}} (\langle 13 |42 \rangle | - \langle 23 |41 \rangle )^{2s}, \\
M_{\text{AHH}}(s^1, 2s^2, 3^-, 4^+) = \left( \frac{\kappa}{2} \right)^2 \frac{\langle 3|1|4 \rangle^{4-2s}}{s_{12}t_{13}t_{14}} (\langle 13 |42 \rangle | - \langle 23 |41 \rangle )^{2s},
\]

(3.34)
such that they develop unphysical poles in the kinematic factor \( \langle 3|1|4 \rangle \) for spins \( s > 1 \) and \( s > 2 \) in the gauge and gravity theories. Although one can add contact terms to remove the spurious poles, the resulting amplitude is not unique [26].

### 3.2.1 Consistent spin-\( s \) candidate amplitudes

We take a different approach in this thesis; we construct the effective theories that generate \( A_{\sqrt{K}}(s^1, 2s^2, 3) \) and \( M_{\sqrt{K}}(s^1, 2s^2, 3) \) for higher spins. At quartic order, we use a combination of higher-spin and classical constraints to fix the contact terms in the four-point Compton amplitudes. We will postpone the construction of the effective theories to Part II and only present the resulting quantum general-spin amplitudes in this section.

To avoid writing the couplings, we introduce the notation,

\[
A(s^1, 2s^2, 3, 4) = Q^2 A(s^1, 2s^2, 3, 4) \\
M(s^1, 2s^2, 3, 4) = \left( \frac{\kappa}{2} \right)^2 M(s^1, 2s^2, 3, 4).
\]

(3.35)

**Compton proposal for \( \sqrt{K} \)**

Our proposal for the \( \sqrt{K} \) Compton amplitude in electromagnetism\(^2\) is

\[
A_{\sqrt{K}}(s^1, 2s^2, 3^-, 4^+) = \frac{\langle 3|1|4 \rangle^2}{t_{13}t_{14}} P_1^{(2s)} - \frac{\langle 13 \rangle |3|1|4 \rangle |42 \rangle}{m^2 t_{13}} P_2^{(2s)} \\
+ \frac{\langle 13 \rangle |32 \rangle |41 \rangle |42 \rangle}{m^4} \left[ P_2^{(2s-1)} - \varsigma_3 \varsigma_4 P_4^{(2s-1)} + \frac{\varsigma_3 + \varsigma_4}{2} \left( P_4^{(2s)} - P_2^{(2s-2)} \right) \right].
\]

(3.36)

\(^2\)see Paper III for the full non-abelian amplitude.
We use the helicity-independent variables\(^3\)
\[
\varsigma_1 := \langle 1|4|2]/m^2 + [21]/m, \quad \varsigma_3 := \langle 21]/m, \\
\varsigma_2 := -\langle 2|4|1]/m^2 + [21]/m, \quad \varsigma_4 := [21]/m,
\]
as they form a basis for the dependence on the massive spinors of particles 1 and 2. In these variables, the dependence on the quantum spin \(s\) of the particles 1 and 2 is encoded in the family of totally symmetric homogeneous polynomials \(P_n^{(k)}\). The relevant polynomials in the \(\sqrt{Kerr}\) amplitude are
\[
P_1^{(k)} = \varsigma_1, \quad P_2^{(k)} = \sum_{i=0}^{k-1} \varsigma_1^i \varsigma_2^{k-1-i}, \quad P_4^{(k)} = \sum_{i+j+l+r=k-3} \varsigma_1^i \varsigma_2^j \varsigma_3^l \varsigma_4^r. \tag{3.38}
\]

The degree of the polynomial \(P_n^{(k)}\) is \(k+1-n\), so clearly the polynomials satisfy the identities \(P_n^{(k)} = 0\) for \(k < n-1\) and \(P_n^{(n-1)} = 1\), which can be used to identify at what spin each term in eq. (3.36) starts to contribute. For \(s \leq 1\) only the first three terms contribute, and the amplitudes agree with \(A_{AHH}\) in eq. (3.34) which are spurious pole free for these spins. At \(s = 3/2\) the last two polynomials contribute independently but there is a cancellation such that \(P_4^3 - P_2^1 = 0\) and the amplitude agrees with the \(s = 3/2\) amplitude proposed in ref. [24]. The amplitude does not develop any spurious poles and factorises correctly on the massive poles.

**Compton proposal for Kerr**

One of the central results in this thesis is the spin-\(s\) family of Kerr Compton amplitudes,
\[
M(1^s, 2^s, 3^-, 4^+) = \frac{(3|1|4]^4}{s_{12}t_{13}t_{14}} P_{1}^{(2s)} - \frac{(13)[42](3|1|4]^3}{s_{12}t_{13}m^2} P_{2}^{(2s)}
+ \frac{(13)[32][42]}{m^4 s_{12}} \left( (3|1|4]^2 P_{2}^{(2s-1)} + (3|\rho|4]^2 P_{4}^{(2s-1)} \right)
+ \frac{(13)[32][14][42]}{m^4 s_{12}} (3|1|4]^3 (3|\rho|4](P_{2}^{(2s-2)} - s_{3}s_{4} P_{4}^{(2s-2)})
+ \frac{(13)^2[32]^2[14]^2[42]^2}{2m^6 s_{3}s_{4}} (1+\eta)P_{5}^{(2s-2)} + (1-\eta)P_{5}^{(2s-2)} \right]. \tag{3.39}
\]

The amplitude is expressed in the same spinor variables \(\varsigma_i\) and we have introduced the variable \(\rho^\mu = \frac{1}{2}(\langle 1|\sigma^\mu|2] + [2|\sigma^\mu|1])\). In the last line we also introduce the limits of the \(n = 5\) polynomial which can also be

\(^3\)Note that variables \(\varsigma_i\) are defined with an extra \(m^{-2}\) factor compare to the definitions papers I, III and IV such that in this thesis they dimensionless.
interpreted as a derivative on $P_{4}^{(k)}$,

$$P_{5|\varsigma_{i}}^{(k)} := \lim_{\varsigma_{5} \to \varsigma_{i}} P_{5}^{(k)} = \frac{\partial}{\partial \varsigma_{i}} P_{4}^{(k)}, \tag{3.40}$$

Note that for $k+1-n \geq 0$, the general polynomial $P_{n}^{(k)}$ can be resummed into the rational form

$$P_{n}^{(k)} = \frac{\varsigma_{i}^{k}}{(\varsigma_{1} - \varsigma_{2})(\varsigma_{1} - \varsigma_{3})\ldots(\varsigma_{1} - \varsigma_{n})} + \text{cyc}(\varsigma_{1}, \varsigma_{2} \ldots \varsigma_{n}). \tag{3.41}$$

However, on four-point kinematics there are only four $\varsigma_{i}$ variables, hence why the higher order polynomials appear with doubled variables. It would be interesting to investigate what polynomials appear in the five-point and higher amplitudes, where we expect the spinor structure of the amplitudes to depend on more than the four variables $\varsigma_{1}, \varsigma_{2}, \varsigma_{3}, \varsigma_{4}$.

The first four terms of $M_{\text{Kerr}}$ are fixed by the cubic higher-spin theory. The last term is a contact term that we ansatz as a function of the polynomials $P_{n}^{(k)}$ and is fixed by a combination of higher-spin and classical constraints, which we discuss in Part III.

For $s \leq 2$ the first 5 terms contribute such that the amplitudes agree with $M_{\text{AHH}}$ defined in eq. (3.34). At $s = 5/2$ the contact terms in the last line do not contribute and the amplitude agrees with the $s = 5/2$ amplitude proposed in ref. [24]. For general spin the amplitude factorises correctly onto the Kerr three point amplitudes (2.10).

The parameter $\eta$ is introduced to match the results in ref. [9], which studies the classical scattering of gravitational waves off a Kerr black hole background. The matching is discussed in detail in Part III where we will also consider adding possible contact terms $C_{\alpha}^{(s)}$. 
4. Higher-spin amplitudes in spin-operator variables

As discussed in the introduction, the three-point amplitudes $M_{Kerr}$ can be related to the energy-momentum tensor for a Kerr black hole (2.9). However, these amplitudes are functions of spinors and the discrete spin quantum number $s$, while the energy momentum tensor of Kerr is a function of the continuous spin vector $a^\mu$. In order to relate the two we express the quantum amplitude in a basis of quantum spin-operators $\hat{a}^\mu$, which we will later identify with the classical parameter $a^\mu$, discussed in detail in Part III. Schematically the necessary steps are

$$M_{Kerr} \sim \frac{\langle 12 \rangle^{2s}}{m^2} \xrightarrow{\text{spin op. basis}} \langle e^{\hat{a}\cdot p_3} \rangle \xrightarrow{\text{cl. limit}} e^{a\cdot p_3} \sim T^{\mu\nu}.$$ (4.1)

In this chapter we will discuss the transformation to the spin operator basis. Note that, while motivated by the classical limit, we can encode the full quantum amplitude in this basis.

4.1 The spin operator and its properties

The quantum spin operator $\hat{a}^\mu$ is defined with respect to the massive momentum $p_1$ and acts on it’s little group representations,

$$(\hat{a}^\mu)_{\vec{a}\vec{b}} := \frac{1}{2m^2} \left( \langle 1 a_1 | \sigma^{\mu} | 1(b_1) \delta_{a_2}^{b_2} \ldots \delta_{a_{2s}}^{b_{2s}} \rangle + [1 a_1 | \sigma^{\mu} | 1(b_1) \delta_{a_2}^{b_2} \ldots \delta_{a_{2s}}^{b_{2s}} \rangle \right),$$ (4.2)

where $\vec{a} = a_1 \ldots a_{2s}$ and $\vec{b} = b_1 \ldots b_{2s}$ are multi-indices with $a_i, b_i \in \{1, 2\}$. This operator can be considered as a covariantisation of the quantum mechanical spin operator defined in three dimensions

$$[\hat{S}^i, \hat{S}^j] = -\epsilon^{ijk} \hat{S}^k.$$ (4.3)

The normalised spin-s four-dimensional operator, $\hat{a}^\mu = \hat{S}^\mu / m$, satisfies the following relations

$$[\hat{a}^\mu, \hat{a}^\nu] = \frac{1}{m^2} \varepsilon^{\mu\nu\rho\sigma} p_\rho \hat{a}_\sigma \quad m^2 \eta_{\mu\nu}(\hat{a}^\mu \cdot \hat{a}^\nu) = -s(s + 1) \mathbf{1}$$ (4.4)
where $\mathbf{1}$ is the $SU(2)$ unit-operator. In order to construct expectation values of the operator we can absorb the $SU(2)$ indices with the spinor wavefunctions such that

$$\langle \hat{a}^{(\mu_1 \ldots \mu_n)} \rangle := (\bar{z})^{2s} \cdot (\hat{a}^{(\mu_1 \ldots \mu_n)}) \cdot (z)^{2s}. \quad (4.5)$$

In particular, the expectation value of a single spin-$s$ operator is

$$\langle \hat{a}^{\mu} \rangle = (\bar{z})^{2s} \cdot (\hat{a}^{\mu}) \cdot (z)^{2s} = \frac{s}{2m} (\langle 1 | \sigma^{\mu} | 1 \rangle + \langle 1 | \sigma^{\mu} | 1 \rangle (\bar{z} a z a) \rangle^{2s-1}). \quad (4.6)$$

For finite spin-$s$ representations we can take at most $2s$ products of the spin operator as higher products collapse down to a polynomial of highest order $(\hat{a})^{2s}$. The general form of such a product where $n > 2s$ is

$$\langle \hat{a}^{(\mu_1 \ldots \mu_n)} \rangle = \sum_{i=0}^{s} d_i^{(n)} \langle \hat{a}^{(\mu_1 \ldots \hat{\mu}_i (\hat{a} \cdot \hat{a})^{s-i})} \rangle \prod_{k=i+1}^{s} P_{\mu_\nu} P_{\mu_k \nu_{k+1}} \quad (4.7)$$

where $P_{\mu\nu} = \eta_{\mu\nu} - \frac{v^\mu v^\nu}{m^2}$ is the spin-1 projector and $d_i^{(n)}$ is a combinatorial factor that can be computed by performing the little group contractions. For finite $s$, the self-contraction $\langle \hat{a} \cdot \hat{a} \rangle$ is proportional to the Casimir and contributes as an overall numerical factor $-s(s+1)(\bar{z} a z a)^{2s}$.

### 4.1.1 Relation to the Pauli-Lubanski operator

The Lorentz-covariant spin operator is often defined in terms of the Pauli-Lubanski operator $\hat{S}$ instead,

$$\hat{S}^\mu = \frac{i}{m} \epsilon^{\mu\nu\rho\sigma} p_\nu M_{\rho\sigma}. \quad (4.8)$$

The operator $\hat{S}$ is defined in terms of the spin-$s$ Lorentz generators $M_{\mu\nu}^{(s)}$, such that the operator acts on the Lorentz representations of the spin-$s$ particle as opposed to the little group. Thus we take the expectation of an integer spin-$s$ operator with respect to the spin-$s$ polarisations,

$$\langle \hat{S}^\mu \rangle := \bar{\varepsilon}^{(s)} \cdot S^\mu \cdot \varepsilon^{(s)}. \quad (4.9)$$

However, given our parameterisation of the massive polarisations in eq. (3.26), there exists a simple relation between the two definitions,

$$m(\hat{a}^\mu)_{\hat{a}} = \frac{1}{(2s)!^2} \left( \prod_{j=1}^{2s} \frac{\partial}{\partial z^a_j} \frac{\partial}{\partial z_{b_j}} \right) \langle \hat{S}^\mu \rangle. \quad (4.10)$$
We show some explicit low spin cases

\[ s = \frac{1}{2} : \quad m(\hat{a}^\mu)_{a}^{b} = \left( \frac{1_a}{m} \right) \hat{S} \left( \frac{1_b}{m} \right) \]

\[ s = 1 : \quad m(\hat{a}^\mu)_{a_1 a_2}^{b_1 b_2} = \left( \frac{1_{a_1}}{m} \otimes \frac{1_{a_2}}{m} \right) \hat{S} \left( \frac{1_{b_1}}{m} \otimes \frac{1_{b_2}}{m} \right), \quad (4.11) \]

where the massive spinors are contracted into the \( SL(2, \mathbb{C}) \) indices of the Lorentz generators. The \( \otimes \) notation implies a symmetric tensor product in the \( SL(2, \mathbb{C}) \) indices such that

\[ \langle 1_a | \otimes \langle 1_a | = \langle 1_a | (\alpha_1 \langle 1_a | \alpha_2). \]

Although the operator \( m\hat{a} \) acts on the little group \( SU(2) \) while \( \hat{S} \) acts on the \( SL(2, \mathbb{Z}) \) indices, the expectation values are equivalent \( \langle m\hat{a} \rangle = \langle \hat{S} \rangle \) so long as the expectation values are taken with respect to physical states.

### 4.2 Converting to spin variables

Converting scattering amplitudes written in covariant notation or massive spinor variables into the spin-operator basis involves two steps.

1. We first parametrise the outgoing degrees of freedom in terms of the incoming degrees of freedom. This is necessary as the spin operator is defined with respect to the little group of a single spin-\( s \) state, which we take to be particle 1.
2. Then we identify spin operators, either from combinations of the massive polarisations \( \varepsilon_1, \bar{\varepsilon}_1 \) or the massive spinors \( |1\rangle, |\bar{1}\rangle \).

#### 4.2.1 Boosted spinors

The amplitudes considered in this section are generalised Compton amplitudes featuring two massive spinning particles and \( n - 2 \) massless states

\[ A(1^s, 2^s, 3, \ldots, n). \quad (4.12) \]

This amplitude transforms under the little group of the massive particles 1 and 2. The spin operator, however, acts on the little group of particle 1, for which the massive bra and ket spinors \( |\bar{1}\rangle = (\langle 1|)^\dagger, |1\rangle \) form a basis. In order to express the amplitude as a function of \( \hat{a} \) we expand the degrees of freedom of particle 2 in terms of particle 1. This can be done via a Lorentz boost \( \Lambda \),

\[ |2\rangle = \Lambda |\bar{1}\rangle = \frac{1}{c_q} (|\bar{1}\rangle + \frac{1}{2m} q \cdot \sigma |\bar{1}\rangle). \quad (4.13) \]
The boost is defined such that
\[ \Lambda = \exp \left( \frac{i\zeta}{\sinh\zeta} q_{\mu}P_{\nu} \frac{M_{\mu\nu}}{m^2} \right), \quad p_2^\mu = \Lambda_{\mu\nu} p_1^\nu := -p_1^\mu - q^\mu, \] (4.14)

where \( q^\mu = \sum_{i=3}^n k_i^\mu \) and the parameter \( \zeta \) is defined by its relation to \( c_q \),
\[ c_q = \sqrt{1 - \frac{q^2}{4m^2}} = \cosh \frac{\zeta}{2}. \] (4.15)

Note that the Lorentz boost is well-defined for null \( q \) (\( \zeta = 0 \)) and corresponds to the boosts defined in [48] for spin-1/2 and spin-1 representations. For the Compton scattering considered in the thesis, we have \( q^2 \leq 0 \). Thus the boosts are well defined and we do not hit the singularity at \( q^2 = 4m^2 \).

4.2.2 Identifying expectations of spin operators

Covariant identities

Identities exist that map products of spin-s polarisations \( \varepsilon^{\mu_1...\mu_s} \bar{\varepsilon}^{\nu_1...\nu_s} \) to expectation values of the spin-s operators. However, the general form is cumbersome to derive. In paper I and in ref. [26], some low spin examples are given explicitly, for compactness we will take \( p = p_1, \varepsilon = \varepsilon_1 \) in the following examples. The relevant identity for \( s = 1 \) is,
\[ \bar{\varepsilon}^{\mu} \varepsilon^{\nu} = -m^2 \langle \hat{a}^{(\mu} \hat{a}^{\nu)} \rangle - \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} p_\rho \langle \hat{a}_\sigma \rangle - P^{\mu\nu} |z|^2. \] (4.16)

where \( P^{\mu\nu} \) is the spin-1 projector and we have used \( m^2 \langle \hat{a} \cdot \hat{a} \rangle = 2|z|^2 \) for \( s = 1 \). For \( s = 2 \) the identity already starts to become cumbersome,
\[ \bar{\varepsilon}^{\mu_1} \bar{\varepsilon}^{\mu_2} \varepsilon^{\nu_1} \varepsilon^{\nu_2} = \frac{m^4}{6} \langle \hat{a}^{(\mu_1} \hat{a}^{\mu_2} \hat{a}^{\nu_1} \hat{a}^{\nu_2)} \rangle - \frac{im^2}{6} \varepsilon^{\mu_1\nu_1\lambda\kappa} \lambda P_{\kappa} \langle \hat{a}^{(\lambda} \hat{a}^{\mu_2} \hat{a}^{\nu_2)} \rangle \\
+ \frac{m^2}{36} \left( P^{\mu_1\nu_2} \langle \hat{a}^{(\nu_1} \hat{a}^{\nu_2)} \rangle + P^{\nu_1\nu_2} \langle \hat{a}^{(\mu_1} \hat{a}^{\mu_2)} \rangle + 28 P^{\mu_1\nu_1} \langle \hat{a}^{(\mu_2} \hat{a}^{\nu_2)} \rangle \right) \\
- \frac{7i}{18} P^{\mu_1\nu_1} \varepsilon^{\mu_2\nu_2\lambda\kappa} \lambda P_{\kappa} \langle \hat{a}^{\lambda} \rangle + P^{\mu_1\nu_2\nu_1} |z|^4, \] (4.17)

where \( P^{\mu_1\mu_2\nu_1\nu_2} = \frac{1}{2} (P^{\mu\rho} P^{\nu\sigma} + P^{\mu\sigma} P^{\nu\rho} - \frac{2}{3} P^{\mu\nu} P^{\rho\sigma}) \) is the spin-2 projector [24]. Mapping from polarisations to spin operators is not very flexible given we do not have a general formula, furthermore they are not valid for half-integer spins. Instead, one can work at the level of the massive spinors, converting first to the spin-1/2 representation and then changing to the generic spin-s representation.

\(^1\)Strictly speaking this the Lorentz transformation also includes a reflection since in our conventions the momenta are all incoming.
Spin-1/2 identities

The one-particle expectation value of a spin-1/2 spin operator is

\[ \bar{\alpha}^\mu := \langle \hat{\alpha}^\mu \rangle = \frac{1}{2\tilde{m}^2} (\langle \bar{1}|\sigma^\mu|1 \rangle + \langle 1|\bar{\sigma}^\mu|1 \rangle). \quad (4.18) \]

We do not need to consider higher-order products of the spin-1/2 operator as they reduce down to

\[ \langle \hat{\alpha}^{\mu_1} \hat{\alpha}^{\mu_2} \ldots \hat{\alpha}^{\mu_n} \rangle \propto \left\{ \begin{array}{ll} |z|^2 P^{\mu_1 \mu_2} \ldots P^{\mu_{n-1} \mu_n} & \text{for even } n \\ \bar{\alpha}^{\mu_1} P^{\mu_2 \mu_3} \ldots P^{\mu_{n-1} \mu_n} & \text{for odd } n \end{array} \right. \quad (4.19) \]

where the proportionality constants are combinatorial factors depending on \( n \). The momentum and \( \bar{\alpha} \) variables are constructed from symmetric and anti-symmetric combinations of the massive spinors,

\[ |\bar{1}\rangle [1] + |1\rangle [\bar{1}] = m^2 \bar{\alpha} \cdot \sigma, \quad |\bar{1}\rangle [1] - |1\rangle [\bar{1}] = p_1 \cdot \sigma |z|^2. \quad (4.20) \]

It is sufficient to consider the expansion of the vectors

\[ \rho^\mu := \frac{1}{2} (\langle 1|\sigma^\mu|2 \rangle + \langle 2|\sigma^\mu|1 \rangle) = |\bar{\alpha}| \frac{m}{c_q} (p_1^\mu - p_2^\mu) - \frac{i}{c_q} \epsilon^\mu(p, q, \bar{\alpha}), \quad (4.21) \]

\[ \bar{\rho}^\mu := \frac{1}{2} (\langle 1|\sigma^\mu|2 \rangle - \langle 2|\sigma^\mu|1 \rangle) = 2m^2 c_q \bar{\alpha}^\mu - \frac{p_1^\mu q \cdot \bar{\alpha}}{c_q}, \quad (4.22) \]

since \( \rho \) and \( \bar{\rho} \) form a basis for the spinor variables, where we use \( \epsilon^\mu(p, q, \bar{\alpha}) = \epsilon^{\mu\nu\rho\sigma} p_\nu q_\rho \bar{\alpha}_\sigma \). When \( s = 1/2 \), we can choose freely when to substitute the identity \( |\bar{\alpha}| := \sqrt{-\bar{\alpha}^2} = |z|^2/(2m) \) in the expansion of \( \rho \). Our prescription will prioritise keeping an equal count of massless momenta and \( \bar{\alpha} \) at leading order, such that \( \rho \cdot p, \rho \cdot \bar{\rho} \sim |z|^2 \) and \( \rho \cdot k_3, \rho \cdot k_4 \sim |a| \). As an example we can expand the spinor structure of same-helicity amplitude (3.33),

\[ m\langle 2|1 \rangle = p_1 \cdot (\rho - \bar{\rho}) = \frac{m^2}{c_q} (|z|^2 + q \cdot \bar{\alpha} - |\bar{\alpha}| q^2). \quad (4.23) \]

We will often choose to normalise the \( z \) variables such that \( |z|^2 = 1 \). Alternatively one can redefine the expectation values \( \langle \hat{\alpha} \rangle \rightarrow \langle \hat{\alpha} \rangle / (z^a z_\alpha)^{2s} \) and factor out an overall \((z^a z_\alpha)^{2s}\) normalisation in the amplitudes. However, for now we will avoid either normalisation in order to keep the little group covariance of the following identities explicit.

### 4.2.3 Amplitudes in spin operator basis

The identities in eq. (4.21) map the massive spinors to the spin-operator basis, such that we can express a generic Compton amplitude \( \mathcal{A}(1^s, 2^s, 3, \ldots, n) \)
as polynomial in $\bar{a}^\mu$,

$$\mathcal{A}(1^s, 2^s, 3, \ldots, n) \sim \sum_{i=0}^{2s} c_i^{\mu_1 \ldots \mu_i} \bar{a}^{\mu_1} \ldots \bar{a}^{\mu_i} (z^a z_a)^{2s-i}. \quad (4.24)$$

The tensors $c_i$ are the theory dependent factors which depend on the momenta $p_1, k_3 \ldots k_n$ and the massless polarisations $\varepsilon_3 \ldots \varepsilon_n$. In order to maintain little group covariance, the amplitudes must be a polynomial of order $2s$ in the wavefunctions $z, \bar{z}$.

We can also consider expanding the amplitude into an alternative basis constructed from spin-operators in the spin-$s$ basis,

$$\mathcal{A}(1^s, 2^s, 3, \ldots, n) \sim \sum_{i=0}^{2s} \tilde{c}_i^{\mu_1 \ldots \mu_i} \langle \hat{a}_{(\mu_1} \ldots \hat{a}_{\mu_i)} \rangle. \quad (4.25)$$

this is a natural choice for an amplitude involving massive spin-$s$ states.

The map between the coefficients $c_i$ and $\tilde{c}_i$ relies on combinatorial factors generated by the change of the representation of operator $\hat{a}^\mu$.

**Example: three-point $\sqrt{\text{Kerr}}$ amplitudes**

We can expand the spin-$s$ three-point $\sqrt{\text{Kerr}}$ amplitudes into the spin-$1/2$ basis where the first step was done in eq. (4.23),

$$\mathcal{A}(1^s, 2^s, q^+) = \sqrt{2Q} p_1 \cdot \varepsilon_3^+ \left( \frac{\langle 21 \rangle}{m} \right)^{2s} = \sqrt{2Q} p_1 \cdot \varepsilon_3^+ \left( 1 + q \cdot \bar{a} \right)^{2s}$$

$$= \sqrt{2Q} p_1 \cdot \varepsilon_3^+ \sum_{n=0}^{2s} \frac{(2s)_n}{n!} (q \cdot \bar{a})^n. \quad (4.26)$$

To change to the spin-$s$ representation of the spin operator, we use the following three-point identity

$$(q \cdot \bar{a})^n = \frac{(2s - n)!}{(2s)!} \langle (q \cdot \hat{a})^n \rangle, \quad (4.27)$$

This identity can be derived explicitly from the definition of $\hat{a}^\mu$ in eq. (4.6) and the expectation values eq. (4.5). This identity provides the maps between the coefficients such that $\tilde{c}_i^{\mu_1 \ldots \mu_i} = \frac{(2s-n)!}{(2s)!} c_i^{\mu_1 \ldots \mu_i}$. The resulting amplitude is

$$\mathcal{A}(1^s, 2^s, q^+) = \sqrt{2Q} p_1 \cdot \varepsilon_3^+ \sum_{n=0}^{2s} \frac{1}{n!} \langle (q \cdot \hat{a})^n \rangle$$

$$= \sqrt{2Q} p_1 \cdot \varepsilon_3^+ \frac{c_q^{-2s}}{c_q} \langle e^{q \cdot \hat{a}} \rangle, \quad (4.28)$$

where we consider the exponential to truncate at order $2s$ in the operator.
The form of eq. (4.27) is governed by the properties of $SU(2)$ group. On general kinematics, the identity used to change representations has the form
\[
\bar{a}^{(\mu_1 \bar{a}^{\mu_2} \ldots \bar{a}^{\mu_n})} = \frac{(2s - n)!}{(2s)!} \langle \hat{a}^{(\mu_1 \hat{a}^{\mu_2} \ldots \hat{a}^{\mu_n})} \rangle + \mathcal{O}(\hat{a}^2),
\]
which has corrections proportional to powers of the Casimir $\hat{a}^2 \propto s(s+1)$, for example at quadratic order
\[
(q \cdot \bar{a})^2 = \frac{(2s - 2)!}{(2s)!} \left( \langle (q \cdot \hat{a})^2 \rangle - s s (s+1) \right) P^{\mu\nu} q_\mu q_\nu.
\]
Note that at three points the $\hat{a}^2$ correction vanishes as $P^{\mu\nu} q_\mu q_\nu = 0$, however for higher-point kinematics it does not. This structure persists for the corrections proportional to higher-order powers of $\hat{a}^2$. Each self-contraction of the spin-operator comes hand-in-hand with the spin-1 projector such that the tensor structure of the $\mathcal{O}(\hat{a}^{2i})$ correction is
\[
\langle \hat{a}^{(\mu_1} \ldots \hat{a}^{\mu_{n-2i}}(\hat{a}^2)^i) \rangle \prod_{k=n+1-2i}^{n-1} P^{\mu_k\mu_{k+1}}.
\]

The Casimir relation highlights a general ambiguity in the decomposition of the amplitude into products of spin-$s$ operators. Suppose we decompose $A(1^s, 2^s, 3, \ldots, n)$ into the spin-$s$ basis, as in eq. (4.25), and we generate an expression for the quadrupole, $\tilde{c}_{\mu_1\mu_2}$, and the hexadecapole, $\tilde{c}_{\mu_1\mu_2\mu_3\mu_4}$, tensors. The Casimir identity implies that the amplitude is invariant under the redefinition of the multipole tensors
\[
\tilde{c}_{\mu_1\mu_2} \rightarrow \tilde{c}_{\mu_1\mu_2} - s(s+1)\tilde{c}_{\mu_1\mu_2\nu_1\nu_2} \eta^{\nu_1\nu_2},
\]
\[
\tilde{c}_{\mu_1\mu_2\mu_3\mu_4} \rightarrow \tilde{c}_{\mu_1\mu_2\mu_3\mu_4} + s(s+1)\eta_{\mu_3\mu_4} \tilde{c}_{\mu_1\mu_2\nu_1\nu_2} \eta^{\nu_1\nu_2},
\]
such that clearly the quantum multipole tensors are not uniquely defined. As we will see in part III this introduces ambiguities in the finite spin classical limit where we need to introduce a prescription to relate the quantum multipole tensors to the classical coefficients. The identification turns out to be robust when the classical limits involve taking $s \rightarrow \infty$.

In this work we make use of both the spin-1/2 and the spin-$s$ bases; the choice is dependent on which classical limit procedure we implement. In general, whenever we discuss finite spin amplitudes we are implicitly using the spin-$s$ basis.
Part II: Constructing Consistent Massive Higher-Spin Theories

A massive spin-$s$ particle corresponds to an irreducible representation of the little group $SU(2)$ and has $2s + 1$ physical degrees of freedom in four-dimensions. In order to construct a field theory describing the interactions of this particle, we package up the degrees of freedom into a field $\Phi$. The field $\Phi$ is taken to transform in one of the spin-$s$ representations of the Lorentz group. In this work we consider two distinct constructions based on the $(s, s)$ and $(2s, 0)$ representations of the Lorentz group.

**Non-Chiral fields** are $(s, s)$-representations of the Lorentz group and correspond to totally symmetric tensor fields. We work with a tower of double traceless fields\(^2\) \{ $\Phi_{\mu(s)}$, $\Phi_{\mu(s-1)} \ldots \Phi_{\mu(0)}$ \} related by a massive gauge symmetry [61]. In principle one can construct theories for any half-integer representation $s$, however we will only discuss integer spin, i.e. massive spinning bosons. Fermionic constructions for $s = \frac{3}{2}$ in gauge theory and $s = \frac{5}{2}$ in gravity were studied in ref. [24].

**Chiral fields** are $(2s, 0)$-representations of the Lorentz group and correspond to a totally symmetric chiral field $\Phi_{\alpha_1 \ldots \alpha_{2s}}$, which contains $2s + 1$ degrees of freedom [50]. The chiral construction is well suited to describing both fermionic and bosonic interactions.

The descriptor ‘higher-spins’ refers to theories where the spin of the massive field $s$ is greater than the spin $|h|$ of the massless force carrier,\(^2\)

\(^2\)For compactness we introduce the shorthand notation for multi-indices $\Phi_{\mu(k)} := \Phi_{\mu_1 \ldots \mu_k}$ and trace $\tilde{\Phi}_{\mu(k-1)} := \Phi^{\alpha_1}_{\mu_1 \mu_2 \ldots \mu_k}$. Note $\Phi_{\mu(0)}$ corresponds to a scalar field and $\tilde{\Phi}_{\mu(k)} := 0$ for $k < 4$. 
in gauge theory $|h| = 1$ and in gravity $|h| = 2$. The distinction between ‘low’ and ‘high’ spins is inherited from the study of massless interacting spin-$s$ fields. The theories of massless low spins are the well-understood gauge and gravity theories, however theories of massless higher-spin theories are notoriously hard to construct consistently, see refs. [10, 52] for a review of the challenges involved.

We expect theories of massive high spins, in flat-space, to develop divergences in the high-energy limit given there is no corresponding consistent massless theory. In light of this, theories of interacting massive higher-spin fields should be considered as effective theories describing composite particles. Early work in refs. [39, 38, 26, 24] suggested that the theories with improved, albeit not finite, high-energy behaviour generate the known three-point Kerr amplitudes. Motivated by this connection, we did a systematic study of the gauge and gravity theories generating the three point amplitudes $A_{\sqrt{\text{Kerr}}}$ and $M_{\text{Kerr}}$ in papers II and III.

We have labelled the two constructions for the higher-spin theories by the chirality properties of the primary ingredients, the spin-$s$ fields. In section 6 we will discuss how to introduce non-minimal interactions to restore parity in the chiral approach. In this section, we will first discuss the non-chiral approach making use of some illustrative low spin examples.
5. Non-chiral construction

5.1 Free theory and minimal coupling

We can package the physical spin-$s$ degrees of freedom into a totally symmetric tensor field $\Phi^{\mu(s)}$. We will also impose the field to be double-traceless, $\Phi^{\nu_1 \nu_2 \nu_3 \ldots \nu_{s-4}} = 0$. In four-dimensions, the field $\Phi^{\mu_1 \ldots \mu_s}$ for $s > 0$ contains

$$\frac{1}{3!} (s + 1)_3 - \frac{1}{3!} (s - 3)_3$$

(5.1)

degrees of freedom, where $(b)_n := \Gamma(b + n)/\Gamma(b)$ is the Pochhammer symbol. The second term corresponds to the degrees of freedom killed by the double tracelessness property of the fields and it vanishes for $s = 1, 2, 3$. In order to reduce to the physical $2s + 1$ degrees of freedom, we add auxiliary fields coupled to the physical field by a generalised gauge symmetry.

Following Zinoviev’s approach [61], we add a tower of symmetric, double-trace-free Stückelberg fields $\{\Phi^{\mu(s-1)}, \Phi^{\mu(s-2)}, \ldots, \Phi^{\mu(1)}, \Phi^{\mu(0)}\}$. The fields are subject to a gauge symmetry,

$$\delta_0 \Phi^{\mu(k)} = \partial(\mu_1 \xi_{\mu_2 \ldots \mu_k}) + m \alpha_k \xi_{\mu_1 \ldots \mu_k} + m \beta_k \eta(\mu_1 \mu_2 \xi_{\mu_3 \ldots \mu_k}) ,$$

(5.2)

which depend on the totally symmetric traceless gauge parameters $\xi_{\mu(k)}$ and the numerical coefficients

$$\alpha_k = \sqrt{\frac{(s - k)(s + k + 1)}{2(k + 1)(k + 1)}}, \quad \beta_k = \frac{k \alpha_{k-1}}{2k - 2}.$$  

(5.3)

The degrees of freedom in the full tower of fields is

$$1 + \sum_{k=1}^{s} \frac{1}{3!} [(k + 1)_3 - (k - 3)_3] = \frac{1}{4!} (s + 1)_4 - \frac{1}{4!} (s - 3)_4 ,$$

(5.4)

where we factored out the $k = 0$ term corresponding to the degrees of freedom of the scalar field. However the gauge symmetry can be used to remove the unphysical degrees of freedom. The tower of traceless gauge parameters contain a total of

$$1 + \sum_{k=1}^{s-1} \frac{1}{3!} [(k + 1)_3 - (k - 1)_3] = \frac{1}{4!} (2s)_3 ,$$

(5.5)
degrees of freedom, where once again the first term corresponds to the degree of freedom of $\xi_{k=0}^0$. Fixing the gauge symmetry removes double the degrees of freedom of the gauge parameters. This is a general feature, which one can motivate by considering the massless gauge variations of the photon fields, $\delta_0 A_\mu = \partial_\mu \alpha$ where $\alpha$ is the gauge parameter. By fixing Lorenz gauge $\partial \cdot A = 0$, one can remove a single scalar degree of freedom. The Lorenz condition leaves a remaining scalar degree of freedom unfixed, given that $\Box \alpha = 0$ trivialises $\delta (\partial \cdot A) = \Box \alpha$. This scalar mode decouples in physical processes.

Thus, the total count of degrees of freedom for the massive spin-$s$ theory corresponds to

$$\text{fields } - 2 \times \text{gauge params. } = \frac{1}{4!} ((s + 1)_4 - (s - 3)_4 - 2(2s)_3) = 2s + 1,$$

and confirms that the field content of the theory contains the correct degrees of freedom. There exist other packagings of the $2s + 1$ degrees of freedom where the field content is altered, for example using the partially gauge fixed approach of Singh-Hagen [55], or using traceful fields [35]. However, we find the Zinoviev approach is well suited to a generic spin approach and gives simple computational rules.

The original presentation of the free Lagrangian can be found in ref. [61]. In paper III we worked out a simpler form using complexified fields. The Lagrangian is decomposed into a Feynman-gauge part and the corresponding gauge-fixing terms,

$$L_2 = L_F - L_{gf},$$

where $L_F$ is diagonal in the fields

$$L_F = - \sum_{k=0}^{s} (-1)^{k+1} \left[ \Phi_\mu(k)(\Box + m^2)\Phi_\mu(k) - \frac{k(k-1)}{4} \Phi_\mu(k-2)(\Box + m^2)\Phi_\mu(k-2) \right],$$

The transverse and off-diagonal quadratic interactions belong to

$$L_{gf} = - \sum_{k=0}^{s-1} (-1)^k (k+1) G_\mu(k) $$

which is quadratic in the gauge-fixing the functions $G_\mu(k)$

$$G_\mu(k) := \partial_\lambda \Phi_\lambda(k) - \frac{k}{2} \partial_\mu \Phi_\mu(k-1) + m_\alpha k \Phi_\mu(k)$$

$$- m_\alpha k + \frac{k+1}{2} \Phi_\mu(k) - m_\alpha \frac{k-1}{4} \eta_\mu \Phi_\mu(k-2).$$
Both terms are necessary for the free theory to be invariant under the massive gauge transformations 5.2 such that \( \delta_0 \mathcal{L}_2 = 0 \). However the decomposition anticipates our gauge choice, \( \mathcal{L}_{\text{gf}} = 0 \) is Feynman gauge, which is closely related to Lorenz gauge \( G^{(k)} = 0 \). As discussed previously, this gauge fixing removes half of the unphysical degrees of freedom. Since the variation of the constraint, \( \delta_0 G^{(k)} = (\Box + m^2)\xi^{(k)} \), is trivialized by \( (\Box + m^2)\xi^{(k)} = 0 \), the residual gauge symmetry removes the second half of the unphysical degrees of freedom.

In Feynman gauge, the massive propagators \( \Delta^{(s)} \) for a spin-\( s \) state can be obtained by inverting \( \mathcal{L}_F \). For unconstrained fields in Feynman gauge this is well defined. However, recall that the field \( \Phi^{(s)} \) is double traceless, hence the propagator has to be a projector that enforces this constraint. Combining the two conditions, being an inverse of the kinetic term and a double-traceless projector, one can show that the propagator is unique. These propagators can be packaged into the following generating function

\[
\Delta(\epsilon, \bar{\epsilon}) = \sum_{s=0}^{\infty} (\epsilon)^s \Delta^{(s)}(\epsilon)^s = \frac{i}{p^2 - m^2 + i0} \frac{1 - \frac{1}{4} \epsilon^2 \bar{\epsilon}^2}{1 + \epsilon \cdot \bar{\epsilon} + \frac{1}{4} \epsilon^2 \bar{\epsilon}^2}.
\]

(5.11)

We can extract the spin-1 massive propagator by acting on \( \Delta(\epsilon, \bar{\epsilon}) \) with derivatives,

\[
\Delta^{(1)}_{\mu\nu} = \frac{\partial^2 \Delta}{\partial \epsilon_\mu \partial \bar{\epsilon}_\nu} \bigg|_{\epsilon^2 \to 0} = -\frac{i\eta^{\mu\nu}}{p^2 - m^2 + i0}.
\]

(5.12)

This is the mass deformed version of the usual spin-1 Feynman-gauge propagator. For spin-2, the massive propagator is

\[
\Delta^{(2)}_{\mu\nu;\rho\sigma} = \frac{1}{(2!)^2} \frac{\partial^4 \Delta}{\partial \epsilon_\mu \partial \epsilon_\nu \partial \epsilon_\rho \partial \epsilon_\sigma} \bigg|_{\epsilon^2 \to 0} = \frac{1}{2} \frac{i \eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\nu} \eta^{\rho\sigma}}{p^2 - m^2 + i0},
\]

(5.13)

which corresponds to a mass deformation of the de Donder propagator (2.2). The propagator \( \Delta^{(k)} \) is a valid for both the physical and auxiliary spin-\( k \) fields. This follows from the \( \mathcal{L}_F \), which makes no distinction between physical and auxiliary fields.

**Minimal coupling**

With the free theory at hand, we can investigate what happens when we couple either to electromagnetism\(^1\) or gravity. In order to couple the theory consistently we have to, at minimum, upgrade all partial

\(^1\)See paper II for coupling to a non-abelian gauge field.
derivatives in $\mathcal{L}_2$ and $\delta_0$ to the relevant covariant derivatives

\[
\text{electromagnetism: } \partial_\nu \Phi^{\mu(k)} \rightarrow D_\nu \Phi^{\mu(k)} = \partial_\nu \Phi^{\mu(k)} + iQA_\nu \Phi^{\mu(k)} \\
\delta_0 \Phi^{\mu(k)} \rightarrow \delta_0 \Phi^{\mu(k)} = D(\mu_1 \xi_{\mu_2 \ldots \mu_k} + \ldots) \\
\text{gravity: } \partial_\nu \Phi^{\mu(k)} \rightarrow \nabla_\nu \Phi^{\mu(k)} = \partial_\nu \Phi^{\mu(k)} + k\Gamma^{(\mu_1}_{\rho \nu \Phi^{\rho \mu_2 \ldots \mu_k)} \\
\delta_0 \Phi^{\mu(k)} \rightarrow \delta_0 \Phi^{\mu(k)} = \nabla(\mu_1 \xi_{\mu_2 \ldots \mu_k}) + \ldots
\] (5.14)

There are different prescriptions for minimal couplings since before co-

variantisaton the derivatives commute but not after, i.e. $[\partial_\mu, \partial_\nu] = 0$ but $[D_\mu, D_\nu] = F_{\mu \nu}$. Consistent minimal coupling to gravity also requires replacing the flat metric with the dynamical one $\eta_{\mu \nu} \rightarrow g_{\mu \nu}$ and using the proper volume form $d^4x\sqrt{-g}$.

In this thesis we work with the free Lagrangian presented in eq. (5.9). The three-point amplitudes generated by minimal Lagrangian are

\[
\mathcal{A}_{\text{min}}(1^s, 2^s, 3^+) = x\langle 12 \rangle^s[12]^s, \\
\mathcal{M}_{\text{min}}(1^s, 2^s, 3^+) = x^2\langle 12 \rangle^s[12]^{s-1}(s\langle 12 \rangle - (s - 1)[12]).
\] (5.15)

While these amplitudes are well-behaved, given they transform covari-

antly under the little group transformations of the massive and massless particles, the underlying theories are not consistent. Indeed, the minimal coupling induces an explicit breaking of the massive gauge symmetry such that

\[
\delta_0 \mathcal{L}_2 \neq 0.
\] (5.16)

This is a problem as the gauge invariance was used to enforce the correct degrees of freedom. Thus the breaking induced by minimal coupling implies unphysical degrees of freedom now propagate.

**Massive spin-1 electromagnetism**

The breaking of the massive gauge symmetry is easiest to see at low spins, for example consider a massive spin-1 field, here denoted by $W^\mu$, minimally coupled to electromagnetism,

\[
\mathcal{L}_2^{(s=1)} = -\frac{1}{2}|W_{\mu \nu}|^2 + |mW_\mu - D_\mu \varphi|^2,
\] (5.17)

where $W_{\mu \nu} = 2D_{[\mu}W_{\nu]}$ and, as before, fields that couple to electro-

magnetism are complex. The minimal coupling extension of the massive gauge variations are

\[
\delta_0 W_\mu = D_\mu \xi, \quad \delta_0 \varphi = m\xi, \quad \delta_0 A_\mu = 0.
\] (5.18)

Note that this minimal coupling prescription differs from the one used for the general-spin free Lagrangian in eq. (5.9).
Using this gauge symmetry we could fix $\varphi = 0$ to obtain the usual Proca action, however we will keep gauge unfixed in order to use gauge invariance as a constraint. At cubic order in the fields, massive gauge invariance requires

$$\delta_0 L_2 = \mathcal{O}(Q^2), \quad (5.19)$$

While order $Q^0$ is guaranteed to vanish by the massive gauge invariance of the free theory. It suffices to consider the variations of the unbarred field $\Phi$, giving

$$\delta_{0,Q^0} L_{2,Q^0}^{(s=1)} = iQ \xi \left[ (\partial_\mu \partial \cdot \bar{W}) A^\mu - (\partial^2 \bar{W}_\mu) A^\mu - m^2 \bar{W} \cdot A + m^2 (\partial \varphi) \cdot A \right]$$

$$\delta_{0,Q^1} L_{2,Q^1}^{(s=1)} = - \delta_{0,Q^1} L_{2,Q^0}^{(s=1)} + 2iQ \xi \partial_{[\mu} \bar{W}_{\nu]} \partial^\mu A^\nu. \quad (5.20)$$

Where we distinguish the free theory variations and Lagrangian, $\delta_{0,Q^0}$ and $L_{2,Q^0}$, from the non-linear interactions introduced by the covariant derivatives, $\delta_{0,Q^1}$ and $L_{2,Q^1}$. Thus adding the two contributions gives

$$\delta_0 L_{2}^{s=1} = 2iQ \xi \partial_{[\mu} \bar{W}_{\nu]} \partial^\mu A^\nu, \quad (5.21)$$

implying that the massive gauge symmetry is broken in the minimally coupled theory. Since the gauge invariance was necessary to encode the physical degrees of freedom, the breaking induced by minimal coupling implies that unphysical degrees of freedom can now propagate. Note, the propagator is still $\eta_{\mu\nu}/(p^2 - m^2)$.

We can restore the gauge invariance by introducing non-minimal interactions in the Lagrangian and massive gauge variations. The lowest derivative non-minimal extension of the massive spin-1 theory in eq. (5.17) that leaves the minimal coupling unchanged is

$$L_3 = -ic_1 Q \bar{W}^\mu F_{\mu\nu} W^\nu, \quad \delta_1 W_\mu = 0 = \delta_1 \varphi, \quad \delta_1 A_\mu = ic_2 Q [\bar{W}_\mu \xi - \xi W_\mu]. \quad (5.22)$$

Imposing massive gauge invariance at cubic order fixes the coefficients uniquely to $c_1 = c_2 = 1$. This non-minimal contribution corresponds to the cubic interactions of $W$ bosons in the standard model. The three point amplitude,

$$\mathcal{A}(1^{s=1}, 2^{s=1}, 3) = -2Q(\varepsilon_1 \cdot \varepsilon_2 \varepsilon_3 \cdot p_1 + \varepsilon_2 \cdot \varepsilon_3 \varepsilon_1 \cdot p_2 + \varepsilon_3 \cdot \varepsilon_1 \varepsilon_2 \cdot p_3), \quad (5.23)$$

corresponds to the $s = 1 \sqrt{Kerr}$ amplitude in (3.20). Notably, for spin-1, this lowest derivative non-minimal interaction is required for the theory to have a smooth massless limit, in which case we obtain the non-abelian Yang-Mills theory with gauge group $SU(2)$. The Higgs field and self-interactions of the spin-1 field $W$ are not considered in this set up, see paper III for details.

The breaking of the gauge symmetry is a property of minimally coupled theories of generic spins $s \geq 1$ coupled to gauge theory and $s \geq 2$.  

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in gravity. In the next section we will discuss how to restore it systematically.

5.2 Adding non-minimal interactions

Returning to the spin-$s$ case, we conclude that minimal coupling of the free Lagrangians introduces propagating unphysical degrees of freedom. In order to restore the massive gauge invariance, we will introduce non-minimal interactions order by order in the fields,

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \ldots,$$

$$\delta = \delta_0 + \delta_1 + \delta_2 + \ldots.$$  \hfill (5.24)

Where $\mathcal{L}_2$ and $\delta_0$ are minimally coupled extensions of the free theory Lagrangian (5.9) and gauge variations (5.2). The non-minimal interactions $\mathcal{L}_{n>2}$ and $\delta_{n>0}$ are constructed via an anstaz and constrained by massive gauge invariance of the full theory $\delta \mathcal{L} = 0$. We will also impose further constraints on the interactions, they are

(SI) interactions involve at most two massive fields, as the self-interactions of massive fields are suppressed in the classical limit of the effective theory,

(PS) parity symmetry is imposed on the interactions. This implies we will not include interactions involving the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ or (anti-)self-dual field strengths,

(MC) minimal-coupling interactions and linearized gauge transformations $\delta_0$ will not be modified,

(PC) power counting in the $\mathcal{L}_3$ interactions have at most $s+s' - |h|$ derivatives, where $s$ and $s'$ are the ranks of the massive fields, e.g. rank 1 for $W_\mu$ and rank 0 for $\varphi$ and where $|h| = 1, 2$ is the spin of the photon or graviton respectively.

(PC2) power counting in the $\delta_1$ gauge variations have at most $s+s' - |h|$ derivatives, where $s$ and $s'$ are the ranks $\delta_1 \Phi$ and $\xi$, respectively.

The last two constraints (PC) and (PC2) correspond to the lowest-derivative solutions compatible with the other constraints. In order to describe a general higher-spin theory these power-counting constraints should be loosened.

These constraints, along with massless gauge invariance, imply the following structure for the non-minimal interactions and gauge variations

**electromag.:**

$$\mathcal{L}_n \sim \Phi \phi(F_{\mu\nu})^{n-2}, \quad \delta_n \Phi \sim \xi (F_{\mu\nu})^n,$$

$$\delta_n A^\mu \sim \xi (A_\nu)^{n-1},$$ \hfill (5.25)

**gravity:**

$$\mathcal{L}_n \sim \Phi \phi(R_{\mu\nu\rho\sigma})^{n-2}, \quad \delta_n \Phi \sim \xi (R_{\mu\nu\rho\sigma})^n,$$

$$\delta_n h^{\mu\nu} \sim \xi (h_{\rho\sigma})^{n-1}. $$
where derivatives are not included. Note that $A_\mu$, $h_{\mu\nu}$ also picks up variations that are non-linear in the fields. These depend on the massive gauge parameters $\xi$ and are not related to the gauge parameters of the massless gauge symmetry.

### 5.2.1 Cubic order: Lagrangian approach

By including non-minimal cubic interactions $L_3$, we now have a hope of restoring massive gauge invariance at cubic order in the coupling $g$

$$(\delta_0 + \delta_1)(L_2 + L_3) = \mathcal{O}(g^2). \quad (5.26)$$

where $g = Q$ in electromagnetism and $g = \kappa = \sqrt{32\pi G_N}$. The right-hand side is linear in $g$ given $L_2$ and $\delta_0$ are fixed by the free theory and they only $L_3$ and $\delta_1$ need to be constructed via ansatz.

In papers II and III, we studied the full off-shell gauge invariance of massive $s = 2$, $s = 3$ fields coupled to electromagnetism and $s = 3$ coupled to gravity. In each case, it was found that massive gauge invariance of the full off-shell Lagrangian coupled with the constraints (SI), (PS), (MC), (PC) and (PC2) is enough to uniquely fix the free parameters that appear in the three-point amplitudes $\mathcal{A}(1^s, 2^s, 3)$ and $\mathcal{M}(1^s, 2^s, 3)$.

In particular these amplitudes correspond to the $\sqrt{\text{Kerr}}$ amplitudes (3.20). While the theories describing these amplitudes are not minimal due to massive gauge invariance constraints, the results from papers II and III suggest that they correspond to lowest derivative non-minimal theories.

As the spin $s$ increases the number of free parameters in the interactions and gauge variations increases steeply. This renders it somewhat inconvenient to study higher-spin theories using the full Lagrangian variations, in particular when moving on to higher multiplicity. Therefore we move on to Ward identities that makes studying higher spins more tractable.

### 5.2.2 Cubic order: Ward identity approach

The non-linear variations of the free Lagrangian are proportional to the equations of motion

$$\delta_1 L_2 = \frac{\delta L_2}{\delta \Phi^{\mu(k)}} \delta_1 \Phi^{(k)} \longrightarrow 0,$$

such that it vanishes on shell. The massive gauge invariance constraint in eq. (5.26) simplifies to

$$\delta_0 (L_2 + L_3) \big|_{\text{on-shell}} = \mathcal{O}(g^2).$$

(5.28)
This constraint no longer depends on the non-linear variation \( \delta_1 \), which we previously needed to construct via ansatz.

Let us discuss the practical implementation of the above constraints. We work in momentum space, where vertices \( V(\ldots) \) are generated by the Lagrangian or constructed as an ansatz. At cubic order the vertices can effectively be read off from the Lagrangian using identifications

\[
\partial_{\mu} \to -i \not{p}_\mu, \quad \Phi^{\mu(k)} \to \epsilon^{\mu_1} \ldots \epsilon^{\mu_k},
\]

(5.29)

\[
A^\mu \to \epsilon^{\mu}, \quad \xi^{\mu(k)} \to \epsilon^{\mu_1} \ldots \epsilon^{\mu_k}.
\]

(5.30)

The vectors \( \epsilon_{\mu} \) can be considered off-shell placeholders for the physical polarisations, such that, in general, \( \epsilon_{\mu} \) does not satisfy the usual transverse and null constraints required for on-shell states. In momentum space we put a particle label on the momenta and polarisations, therefore there is no ambiguity in what \( \epsilon^\mu \) means for the different fields. The on-shell massive state satisfies

\[
\not{p}_i^2 = m^2, \quad p_i \cdot \epsilon_i = 0, \quad \epsilon_i^2 = 0,
\]

(5.31)

while the massless state \( i \) satisfies

\[
k_i^2 = 0, \quad k_i \cdot \epsilon_i = 0, \quad \epsilon_i^2 = 0.
\]

(5.32)

Therefore putting the particle \( i \) on-shell corresponds to swapping \( \epsilon_i \to \epsilon_i \) or \( \epsilon_i \) depending on whether the particle is massive or not. The notation \( V(\ldots)|_{(i)} \) indicates that we put particle \( i \) on shell.

The three point vertex receives a contribution from the minimal coupling, in this case to electromagnetism, of the free Lagrangian and the unfixed non-minimal cubic interactions

\[
V(\Phi_1^k \Phi_2^s A_3) = V_{\text{min}}(\Phi_1^k \Phi_2^s A_3) + V_{\text{non-min.}}(\Phi_1^k \Phi_2^s A_3).
\]

(5.33)

We will mainly consider the case where leg 2 is the highest spin-\( s \) field \( \Phi^s := \Phi^{\mu(s)} \) and leg 1 is one of the fields in the tower with \( k \leq s \). The vertices are Lorentz-invariant polynomials in the polarisations \( (\epsilon_1)^k, (\epsilon_2)^s, \epsilon_3 \) and momenta \( p \in \{ p_1, p_2 \} \), with schematic form

\[
V_{\text{min}}(\Phi_1^k \Phi_2^s A_3) \sim m(\epsilon_1)^k (\epsilon_2)^s \epsilon_3 \left( \frac{p}{m} + 1 \right),
\]

\[
V_{\text{non-min.}}(\Phi_1^k \Phi_2^s A_3) \sim m(\epsilon_1)^k (\epsilon_2)^s \epsilon_3 \sum_{n=0}^{s+k-1} \left( \frac{p}{m} \right)^n.
\]

(5.34)

The vertex \( V_{\text{min}} \) is given by the free theory while \( V_{\text{non-min.}} \) is constructed as an ansatz and constrained by the previously introduced constraints (MC), (PS), (SI) and (PC).
The variation of the Lagrangian (5.28) corresponds to the constraint on the cubic interactions $V_3 = (L_2 + L_3)|_{3pt},$

$$\delta_0 V_3 = \sum_{k=0}^{s} \frac{\delta V_3}{\delta \Phi^\mu(k)} \delta_0 \Phi^\mu(k). \quad (5.35)$$

Using the $\epsilon, p$ variables, the variations translate to

$$\frac{\delta V_3}{\delta \Phi^\mu(k)} \to \frac{1}{k!} \prod_{l=1}^{k} \frac{\partial}{\partial \epsilon_1^{\mu_l}} V(\Phi_1^k \Phi_2^s A_3),$$

$$\delta_0 \Phi^\mu(k) \to m \alpha_k \epsilon_1^{\mu(k)} - \frac{1}{k} i p_1^{\mu_k} \epsilon_1^{\mu(k-1)} + \frac{m \alpha_{k-1}}{2(k-1)^2} \eta^{\mu_{k-1} \mu_k} \epsilon_1^{\mu(k-2)}. \quad (5.36)$$

The function $\delta_0 V_3$ is an inhomogeneous polynomial in $(\epsilon_1)^k, 0 \leq k \leq s.$ It is useful to extract the homogeneous pieces, which we denote by

$$V(\xi_1^k \bar{\Phi}_2^s A_3) := m \alpha_k V(\Phi_1^k \Phi_2^s A_3) - \frac{1}{k+1} i p_1 \cdot \frac{\partial}{\partial \epsilon_1} V(\Phi_1^{k+1} \Phi_2^s A_3)$$

$$+ \frac{m \alpha_{k+1}}{2(k+1)^2} \frac{\partial}{\partial \epsilon_1} V(\Phi_1^{k+2} \Phi_2^s A_3). \quad (5.37)$$

The gauge invariance constraint in eq. (5.28) corresponds to imposing $\delta_0 V_3$ such that we supplement the list of constraints (SI), (PS), (MC), (PC) with the following:

(WI) Massive Ward identities $V(\xi_1^k \bar{\Phi}_2^s A_3)|_{(2,3), \epsilon_1^2=0} = 0,$ where we set $\epsilon_1^2 = 0$ in order to impose the tracelessness constraint on the gauge parameter $\tilde{\xi}^\mu(k-2) = 0.$

So far we have discussed the Ward identities in the context of electromagnetism. There is an analogous construction in gravity where we adjust the power-counting in $V_{\text{non-min.}}$ to reflect the constraint (PC). All the equations above hold up to swapping $A_\mu \to h_{\mu \nu}, \epsilon_3^\mu \to \epsilon_3^{\mu \nu}$ as the structure of the Ward identity is inherited by the free theory gauge variations.

**Results**

Since the Ward identity construction is more economical, we are able to explore the theories that satisfy the constraints (SI), (PS), (MC), (PC) and (WI) up to $s = 10.$ Up to $s = 1$ in electromagnetism, and up to $s = 2$ in gravity, this gives unique theories. Free parameters appear in the three-point amplitudes starting at $s = 2$ in electromagnetism

$$A(\Phi_1^s \bar{\Phi}_2^s A_3^+) = A_0 \left( \frac{12}{m^2 s} \right)^{2s} \left\{ 1 + \sum_{k=1}^{s-1} b_k \left( \frac{[12]^k}{(12)^k} - 1 \right) \right\}, \quad (5.38)$$
and \( s = 3 \) in gravity

\[
\mathcal{M}(\Phi_1^s \Phi_2^s h_3^+) = \mathcal{M}_0 \left( \frac{12}{m^{2s}} \right)^{2s} \left\{ 1 + \left( 1 - \frac{12}{\langle 12 \rangle} \right)^{2s-4} \sum_{k=0}^{s-4} \tilde{b}_k \frac{\langle 12 \rangle^k}{\langle 12 \rangle ^k} \right\}.
\] (5.39)

The parameters \( b_k \) are related to the free parameters in the general three-point amplitudes in eq. (3.31) by

\[
c_0 = 1 - \sum_{k=1}^{s-1} c_{k>0} = b_{k>0},
\]

likewise for \( \tilde{c}_k \) and \( \tilde{b}_k \). Therefore clearly the higher-spin constraints (SI), (PS), (MC), (PC) and (WI) fix \( s + 1 \) of the \( 2s - 1 \) free parameters.

The loss of uniqueness in comparison to the off-shell approach can be traced back to the translation between off-shell massive gauge invariance and constraint on the currents in (WI). Notably, we did not construct or impose the Ward identities generated by vertices \( V(\Phi^k \tilde{\Phi}^{k'} A_3) \) with \( k, k' < s \), such that neither massive fields are the top spin field. These interactions do generate constraints when varying the full off-shell Lagrangian, especially since the interactions are constrained by the power counting (PC). The obstruction to including such terms is that these vertices include contain no on shell states, hence we always consider \( k' = s \).

To constrain the remaining \( s \) parameters and obtain unique amplitudes, introduce two the additional constraints:

(CC) Current constraint is motivated by high-energy unitarity, as discussed in ref. [24]:

\[
p_1 \cdot \frac{\partial}{\partial \epsilon_1} V(\Phi_1^s \tilde{\Phi}_2^s A_3) \bigg|_{(2,3), \epsilon_1^2=0} = \mathcal{O}(m).
\] (5.40)

(ND) Near-diagonal interactions of two massive fields. The cubic vertices differ at most by one unit:

\[
V(\Phi_1^{s-k} \tilde{\Phi}_2^s A_3^{|h|}) \big|_{k>|h|} = 0,
\] (5.41)

where \(|h| = 1, 2 \) in electromagnetism and gravity respectively.

The first constraint (CC) has been previously discussed in higher spin literature, in ref. [34] it is sufficient to fix the tree-level gyromagnetic ratio to \( g = 2 \) for theories of general-spin particles. In ref. [27, 51] it is introduced to avoid violations of tree-level unitarity for theories of massive higher-spins. As motivation consider the spin-1 four point amplitude \( \mathcal{A}(1^{s=1}, 2^{s=1}, 3, 4) \). Unitary gauge fixes \( \phi = 0 \) in eq. (5.17) giving the Proca Lagrangian. Using the unitary gauge spin-1 propagator, we can construct one of massive channels

\[
\left. \frac{\partial}{\partial \epsilon_p} V(W_1 \bar{W}_p A_3) \frac{\eta^{\mu\nu} - p^\mu p^\nu}{p^2 - m^2} \frac{\partial}{\partial \epsilon_p} V(W_p \bar{W}_2 A_4) \right|_{(1,2,3,4)}.
\] (5.42)

Adding the constraint (CC) is sufficient to fix all the free parameters in eq. (5.39) uniquely to \( \tilde{b}_k = 0 \), giving a unique Kerr amplitude. Combining (CC) with the second constraint (ND) is necessary to fix \( b_k = 0 \)
in eq. (5.38) giving a unique $\sqrt{Kerr}$ amplitude. The latter constraint is a property of the minimal Lagrangian and amounts to imposing stronger power counting constraints on certain vertices.

The current constraint is a remarkably powerful constraint at low spins. In ref. [24], it was used to uniquely fix the three point amplitudes up to $s = 3/2$ in gauge theory and $s = 5/2$ in gravity.

5.2.3 Adding quartic interactions

In principle, extending the analysis to quartic order is straightforward and corresponds to solving

$$(\delta_0 + \delta_1 + \delta_2)(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4) = \mathcal{O}(g^3),$$

where we include non-minimal interactions that are quartic in the fields $\mathcal{L}_4$ and quadratic in the fields for $\delta_2$. We can fix all the physical parameters in $\delta_1$ and $\mathcal{L}_3$ by cubic order gauge invariance. Any remaining unfixed parameters at $\mathcal{O}(g)$ correspond to field re-definitions and can be set to zero without loss of generality. Therefore the quartic order massive gauge invariance is a linear constraint.

In practice, the full off-shell analysis is intractable for high spins; in paper III we were able to implement up to $s = 3$ in electromagnetism. The constraint in eq. (5.43) is a system of $s - 1$ equations that are highly coupled and depend tensor structures that increase in complexity as the spin grows.

The Ward identity formulation provides an efficient approach to the quartic order analysis. In this approach we can neglect the non-linear gauge variations and construct currents corresponding to interactions where all-but-one of the fields in the interactions satisfy on-shell constraints.

At three points, the Ward identities act on currents generated from cubic vertices where two fields are taken on-shell. At quartic order, the relevant current resembles a four point amplitude with one massive leg off-shell,

$$\langle \Phi_1^k \bar{\Phi}_2^s A_3 A_4 \rangle_{(2,3,4)} := \left( \Gamma_t(\Phi_1^k \bar{\Phi}_2^s A_3 A_4) + \Gamma_u(\Phi_1^k \bar{\Phi}_2^s A_3 A_4) + V_4(\Phi_1^k \bar{\Phi}_2^s A_3 A_4) \right)_{(2,3,4)}. \quad (5.44)$$

In electromagnetism, the current has contributions from the $t$- and $u$-channel graphs $\Gamma_t, \Gamma_u$ and from the quartic vertices $V_4(\Phi_1^k \bar{\Phi}_2^s A_3 A_4)$ generated by the minimal coupling of $\mathcal{L}_2$ and $\mathcal{L}_3$ as well as the unfixed quartic interactions $\mathcal{L}_4$. In the non-abelian gauge theory and gravity...
theories there is also a massless channel since the massless fields self-interact. The four point amplitude corresponds to setting leg-1 on-shell too, $\mathcal{A}(1^s, 2^s, 3, 4) := \langle \Phi^s \Phi^s A_3 A_4 \rangle_{(1,2,3,4)}$.

Note that the only free parameters in eq. (5.44) appear in contact terms $V_4(\Phi_k \Phi_{\bar{s}} A_3 A_4)$ which include contributions from the ansatzed the non-minimal quartic interactions using (MC) and (PS). We will discuss the relevant power-counting shortly. The contributions of the massive channels are constructed out of the cubic currents using the massive Feynman-like propagators defined in eq. (5.11),

$$
\Gamma_t(\Phi_1^k \Phi_{\bar{s}} A_3 A_4) := \sum_{l=0}^{s} \Delta \left( \frac{\partial}{\partial \epsilon_p}, \frac{\partial}{\partial \bar{\epsilon}_p} \right) V(\Phi_1^k \Phi_p A_4) V(\Phi_{-p} \Phi_{\bar{s}} A_3) \bigg|_{\epsilon_p \to 0, \bar{\epsilon}_p \to 0}.
$$

(5.45)

In analogy to the cubic Ward identities, we can now consider the variation of field $\Phi^k$ and isolate the constraints generated by each independent gauge parameter $\xi^k$. The resulting quartic Ward identities are

$$
\langle \xi_1^k \Phi_{\bar{s}} A_3 A_4 \rangle_{(2,3,4), \epsilon_1^2 = 0} := \left[ m \alpha_k \langle \Phi_1^k \Phi_{\bar{s}} A_3 A_4 \rangle - \frac{i p_1}{k+1} \frac{\partial}{\partial \epsilon_1} \langle \Phi_1^{k+1} \Phi_{\bar{s}} A_3 A_4 \rangle \right]_{(2,3,4), \epsilon_1^2 = 0}
$$

$$
+ \frac{m}{2} \beta_{k+2} \left( \frac{\partial}{\partial \epsilon_1} \right)^2 \langle \Phi_1^{k+2} \Phi_{\bar{s}} A_3 A_4 \rangle_{(2,3,4), \epsilon_1^2 = 0}.
$$

(5.46)

Imposing the Ward identities corresponds to solving a system of $s$ linear equations, which constraints the quartic contact terms in $\mathcal{A}(1^s, 2^s, 3, 4)$. In paper III, we present discuss the calculation for spins $s = 2, 3$ in the non-abelian gauge theory.

**Spin-2 electromagnetism**

The first non-trivial result at quartic order is the spin-2 electromagnetism analysis. In paper III, there is a detailed discussion of the cubic order analysis using both off-shell and Ward identity approaches. At quartic order we ansatz the non-minimal interactions that enter in the massive Ward identities, $V(\Phi_1^k \Phi_{\bar{s}} A_3 A_4)$ for $k \leq 2$.

The lowest-possible derivative counting of the four point amplitude $\mathcal{A}(1^{s=2}, 2^{s=2}, 3, 4)$ is fixed by the exchange of the highest spin-state. For spin-2, the minimum derivative count in the four-point amplitude is $4s - 4$ since (PC) implies that each cubic vertex can contribute $2s + 1$ derivatives, while the propagator in eq. (5.11) removes two.
The power-counting used in our ansatz for the non-minimal interactions is

\[ V(\Phi_1^2 \Phi_2^2 A_3 A_4) \sim p^4, \]
\[ V(\Phi_1^1 \Phi_2^2 A_3 A_4) \sim p^5, \]
\[ V(\Phi_0^0 \Phi_2^2 A_3 A_4) = 0, \]

where the last vertex vanishes, satisfying the (ND) constraint. Each vertex is constructed explicitly out of field strengths for the photons, \( F_{\mu\nu} \rightarrow k_{3[\mu} \epsilon_{3\nu]} \). Note that the diagonal spin-2 interaction has the lowest power-counting \( 4s - 4 = 4 \).

Imposing the quartic order Ward identities introduces constraints on the parameters in the ansatz. However it is not enough to uniquely fix the four-point amplitudes, such that the resulting amplitudes are

\[
A(1^{s=2}, 2^{s=2}, 3^+, 4^+) = \frac{\langle 12 \rangle^4 [34]^2}{m^2 t_{13} t_{14}} + C_{++}^{(2)},
\]

\[
A(1^{s=2}, 2^{s=2}, 3^-, 4^+) = \frac{\langle 31 \rangle^2 [14]^2}{t_{13} t_{14}} P_1^{(2)} - \frac{\langle 13 \rangle [31] [42]}{m^2 t_{13}} P_2^{(2)}
\]

\[
+ \frac{\langle 13 \rangle [32] [14] [42]}{m^4} P_2^{(1)} + C_{-+}^{(2)},
\]

where \( C_{++}^{(2)} \) and \( C_{-+}^{(2)} \) contain unfixed parameters. We note these parameters remain unfixed even when we consider the full off-shell variation of the Lagrangian as done in paper III. The opposite-helicity amplitude has three unfixed free parameters,

\[
C_{-+}^{(2)} = c_1 \langle 31 \rangle [13] [14] [42] (s_3 + s_4)^2 + \frac{c_2}{m^4} \langle 13 \rangle [32] [14] [42] (s_3 - s_4)
\]

\[
+ \frac{c_3}{m^8} \left( m \langle 31 \rangle [21] (s_3 + s_4) (\langle 13 \rangle [23] [14] + \langle 23 \rangle [14] [42])
\]

\[
+ \langle 31 \rangle^2 [12] [12]^3 + s_{12} [12] [12] [13] [32] [14] [42],
\]

while the same helicity sector has five unfixed parameters,

\[
\]

\[
\]

In Part III we constrain the contact terms further by imposing classical consistency constraints. The spin-2 \( \sqrt{Kerr} \) amplitudes introduced in eq. (3.33) and proposed in eq. (3.36) correspond to the fixing the coefficients \( c_1 = -c_2 = 1 \) and \( c_{n>2} = 0 \).
6. Chiral construction

6.1 Minimal theory

In this approach we will use fields in the \((2s,0)\) chiral representation of the Lorentz group. In this representation, the fields are totally symmetric 2s-spinor fields \(\Phi_{\alpha(2s)}\), such that the multi-index \(\alpha(2s) = \alpha_1 \ldots \alpha_{2s}\) is a string of spacetime \(SL(2, \mathbb{C})\) indices each taking values \(\alpha_i = 1,2\). In contrast to the fields in the \((s,s)\)-representation, the fields in this chiral representation contain exactly \(2s + 1\) degrees of freedom. Therefore one can construct the free theory for a spin-\(s\) particles from a single field [50]

\[
\mathcal{L}^{(s)} = \langle \partial_\mu \Phi | \partial^\mu \Phi \rangle - m^2 \langle \Phi | \Phi \rangle.
\] (6.1)

We have introduced a short-hand notation to suppress the spinor indices,

\[
|\Phi\rangle := \Phi_{\alpha(2s)}, \quad |\partial^\mu \Phi\rangle := \partial^\mu \Phi_{\alpha(2s)}, \quad \langle \Phi | \Phi \rangle := \Phi^{\alpha(2s)}_{\alpha(2s)}.
\] (6.2)

While in the \((s,s)\) representation the natural on-shell wavefunctions for a spin-\(s\) tensor are the spin-\(s\) polarisations \(\Phi_{\mu(s)} \rightarrow \epsilon_{\mu_1} \ldots \epsilon_{\mu_{2s}}\), the natural on-shell wavefunctions for the chiral field are the left-handed massive spinors \(\Phi_{\alpha(2s)} \rightarrow |p\rangle_{\alpha_1} \ldots |p\rangle_{\alpha_{2s}}/m^s [50]\).

In order to add interactions, we can minimally couple the theory to electromagnetism and gravity using the same prescription as in section 5.1.

**Minimal coupling to electromagnetism**

We couple to electromagnetism by promoting the partial derivatives in the free Lagrangian to \(\partial_\mu \rightarrow D_\mu = \partial_\mu + iQA_\mu\),

\[
\mathcal{L} = \frac{1}{2} \langle D_\mu \Phi | D^\mu \Phi \rangle - \frac{m^2}{2} \langle \Phi | \Phi \rangle.
\] (6.3)

In this thesis I will only introduce the electromagnetic theory and we assume \(\Phi\) carries electric charges, which is not one-to-one with the complex properties of the fields given the spinor representation is itself complex. The general non-abelian treatment was introduced in [50] and discussed in paper III.

We compute the three-point vertex \(V_2(\Phi_1^s \Phi_2^s A_3)\) as a function of the massive spinors

\[
V_{\text{min}}(\Phi_1^s \Phi_2^s A_3) = Q(p_1 - p_2) \cdot \epsilon_3 \frac{(21)^{2s}}{m^{2s}}.
\] (6.4)
where the spinors are assumed to not yet satisfy the coupled Weyl equations in eq. (3.7). Hence, the three point amplitude for the minimally coupled theory is practically the same equation,

$$\mathcal{A}(1^s, 2^s, 3^\pm) = V_2(\Phi_1^s \Phi_2^s A_3)\bigg|_{(1,2,3^\pm)} = \sqrt{2}Q p_1 \cdot \varepsilon_3^+ \frac{(21)^s}{m^{2s}}. \quad (6.5)$$

Unsurprisingly, the three point amplitudes are break chiral symmetry since

$$\mathcal{A}(1^s, 2^s, 3^+) \neq \mathcal{A}(1^s, 2^s, 3^-)\bigg|_{\varepsilon_3^+ \rightarrow \varepsilon_3^-} \cdot (6.6)$$

This asymmetry in the dependence on right- and left-handed spinors is inherited from the Lagrangian, which is constructed from the chiral left-handed field $\Phi_{\alpha(2s)}$. We will restore parity at the level of the amplitudes by including a tower of non-minimal interactions to $\mathcal{L}$.

Note that $V_{\text{min}}$ factorises into the scalar vertex and spin-dependent term, $V_{\text{min}}(\Phi_1^s \Phi_2^s A_3) = V_{\text{min}}(\Phi_1^0 \Phi_2^0 A_3)\langle 21 \rangle^{2s}/m^{2s}$. This a feature of the minimally coupled Lagrangian, which behaves as a scalar Lagrangian given the contractions of the spinor fields are trivial.

**Minimal coupling to gravity**

The Lagrangian for the chiral spin-$s$ theory minimally coupled to gravity is

$$\mathcal{L} = \sqrt{-g} \left\{ \frac{1}{2} \langle \nabla_\mu \Phi | \nabla^\mu \Phi \rangle - \frac{m^2}{2} \langle \Phi | \Phi \rangle \right\}. \quad (6.7)$$

Consistent minimal coupling causes a proliferation of interactions compared to the gauge theory. In this section we will use the parametrisation of the metric $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. There are two cubic interactions, one from the expansion of $\sqrt{-g} = 1 + \frac{1}{2} h + \ldots$

$$V_{\text{min,}\sqrt{g}}(\Phi_1^s, \Phi_2^s, h_3) = -\frac{\kappa}{2} \varepsilon_3^2 (p_1 \cdot p_2 - m^2) \frac{(21)^s}{m^{2s}}. \quad (6.8)$$

Note that this vertex exhibits the same factorisation into a scalar vertex and a spin-dependent factor as the vertex in electromagnetism. However this is not the case for the other vertex, generated by the expansion of the spin connection $\omega_{\mu,\alpha \beta} := \frac{1}{4} \omega_{\mu \nu} \tilde{\rho}^\nu \sigma_{\mu,\alpha \gamma} \delta^\beta_\tilde{\rho}$ that appears in the covariant derivative

$$\nabla_\mu \Phi_{\alpha_1 \ldots \alpha_2} = \partial_\mu \Phi_{\alpha_1 \ldots \alpha_2} + 2s \omega_{\mu,(\alpha_1 \beta} \Phi_{\alpha_2 \ldots \alpha_2 s)} \beta \cdot (6.9)$$
The resulting off-shell three-point vertex

\[
V_{2,\nabla}(\Phi_1^s, \Phi_2^s, h_3) = \frac{\kappa}{2} \left[ - (\epsilon_3 \cdot p_1)^2 \langle 21 \rangle^{2s} - s (\epsilon_3 \cdot p_1) \langle 21 \rangle^{2s-1} \langle 2 | p_3 \epsilon_3 | 1 \rangle \\
+ \frac{1}{2} \langle 21 \rangle^{2s-1} (\epsilon_3 \cdot p_3)(2(s-1)(\epsilon_3 \cdot p_1) \langle 21 \rangle - s \langle 2 | p_3 \epsilon_3 | 1 \rangle) \\
+ \frac{1}{2} s (\epsilon_3 \cdot p_3)^2 \langle 21 \rangle^{2s} \right].
\]

(6.10)

Putting the graviton on-shell greatly simplifies the vertex such that only the first line contributes. The full on-shell three point amplitude is

\[
\mathcal{M}(1^s, 2^s, 3^\pm) = -\kappa (p_1 \cdot \epsilon_3^\pm) \frac{\langle 21 \rangle^{2s}}{m^{2s}} - s\kappa (p_1 \cdot \epsilon_3^\pm) \frac{\langle 21 \rangle^{2s-1}}{m^{2s}} \langle 2 | p_3 \epsilon_3^\pm | 1 \rangle.
\]

(6.11)

This corresponds to the spin-s Kerr amplitude for positive helicities given \( \langle 2 | p_3 \epsilon_3^\pm | 1 \rangle = 0 \). However, as was the case in the minimally coupled gauge theory, the negative helicity amplitudes are incorrect and are not related to \( \mathcal{M}(1^s, 2^s, 3^+) \) by a parity transformation. Parity can be restored by introducing non-minimal interactions to the Lagrangians.

6.2 Parity invariant non-minimal theories

Although this construction discriminates between left- and right-handed fields, one can still use it to describe parity invariant theories. A concrete example is given in ref. [23] where the covariant spin-1 Proca action describing electroweak bosons is mapped to the chiral representation via a non-linear field redefinition. This example is discussed in detail in paper III and the resulting chiral Lagrangian

\[
\mathcal{L}^{(1)} = \langle \Phi \left| \mathcal{D} \cdot \mathcal{D} \otimes \frac{1}{1 + \frac{iQ}{m^2} |F_-|} \right| \Phi \rangle - m^2 \langle \Phi | \Phi \rangle + \mathcal{O}(\Phi^4),
\]

(6.12)

reproduces the non-chiral physics of the interacting W bosons. The non-local term should be interpreted as an infinite series of interactions in the anti-self-dual strength \( |F_-| := F_{-\alpha}^\beta := \frac{1}{2}(\sigma^\mu \bar{\sigma}^\nu)_{\alpha}^\beta F_{\mu\nu} \). The first non-minimal interaction is

\[
\langle \Phi \left| \mathcal{D} \cdot \mathcal{D} \otimes |F_-| \right| \Phi \rangle := (D_\mu \Phi^{\alpha_1 \alpha_2})(\sigma^\mu \bar{\sigma}^\nu)_{\alpha_1}^{\beta_1} (F_-)_{\alpha_2}^{\beta_2} (D_\nu \Phi_{\beta_1 \beta_2}),
\]

(6.13)

where the covariant derivatives are contracted into the Pauli matrices and the arrows indicate which massive field they act on. The notation \( \otimes \) should be interpreted as a tensor product such that, for general spin, \( \mathcal{D} \cdot \mathcal{D} \otimes |F_-| \sim \mathcal{D} \cdot \mathcal{D} |\alpha_1 \beta_1 (F_-)_{\alpha_2}^{\beta_2} \delta_{\alpha_3}^{\beta_3} \ldots \delta_{\alpha_2}^{\beta_2} | \) is 2s-by-2s spinor matrix.
This spin-1 gauge theory example illustrates how additional non-minimal interactions are necessary to restore parity at each order in the fields. For general spin, the field redefinitions relating representations \((s, s)\) and \((2s, 0)\) are not known. Instead we will choose to restore parity order by order in the interactions.

6.2.1 Cubic order

Gauge theory

Working with general spin, we can restore parity at cubic order by adding a generalisation of the non-minimal term added at spin-1 (6.13),

\[
\mathcal{L}_{\text{Kerr}} = \langle D_\mu \Phi | D^\mu \Phi \rangle - m^2 \langle \Phi | \Phi \rangle + \sum_{k=0}^{2s-1} \frac{i Q}{m^{2k}} \langle \Phi | \left\{ | D | D |^k \otimes | F_- \rangle \right\} | \Phi \rangle.
\]  

We use the notation \(\otimes\) to indicate a symmetrised tensor product such that \(\langle \Phi \rangle_{\alpha_1} \ldots \langle \Phi \rangle_{\alpha_k} \). This non-minimal interaction generates a vertex of the type

\[
\mathcal{V}_{\text{non-min}}(\Phi^s \Phi^s A_3) = - \sum_{k=0}^{2s-1} \frac{Q}{m^{2s+k-1}} \langle 2|p_2 p_1|1 \rangle^k \langle 2|p_3 \varepsilon_3|1 \rangle \langle 21 \rangle^{2s-k-1}.
\]

Note that this interaction does not contribute for positive helicity photons since the spinor \(\varepsilon_3|_{\alpha} \beta := p_3 \mu \varepsilon_3 \nu (\sigma^\mu \sigma^\nu)|_{\alpha} \beta\) satisfies the helicity dependent identities

\[
\varepsilon_3|_{\alpha} \beta = 0 \quad \text{and} \quad (p_3 \varepsilon_3)|_{\alpha} \beta = \sqrt{2} \langle 3 |_{\alpha} \langle 3 |_{\beta}.
\]

Therefore \(\mathcal{V}_{\text{non-min}}\) only corrects the negative helicity three-point amplitude by

\[
\mathcal{V}_{\text{non-min}}(\Phi_1^s \Phi_2^s A_3) \big|_{\langle 1, 2, 3 \rangle^{-}} = -Q \langle 23 \rangle \langle 31 \rangle \sum_{k=0}^{2s-1} \frac{1}{m^{2s}} \frac{[21]^k \langle 21 \rangle^{2s-k-1}}{[21] - \langle 21 \rangle}.
\]

Note that this spinor structure is intimately related to the polynomials first introduced in section 3.2.1,

\[
P_1^{(2s)}(\varsigma_3, \varsigma_4) = \frac{[21]^{2s} - \langle 21 \rangle^{2s}}{[21] - \langle 21 \rangle}.
\]

Where the polynomial \(P_2^{(2s)}(\varsigma_3, \varsigma_4)\) corresponds to the polynomial defined in (3.38) but swapping \(\varsigma_1, \varsigma_2 \rightarrow \varsigma_3, \varsigma_4\). This is the first suggestion that
the polynomials discussed in section 3.2.1 have a higher-spin origin given their relation to interactions of the chiral Lagrangian. In order to reconstruct the full negative helicity three-point amplitude we employ the following three-point relation

$$\langle 2|p_3\varepsilon^-|1\rangle = \sqrt{2}\langle 23\rangle\langle 31\rangle = \sqrt{2}p_1 \cdot \varepsilon^- ([21] - \langle 21\rangle),$$  \hspace{1cm} (6.19)

such that the Lagrangian in eq. (6.14) generates the correct negative-helicity $\sqrt{\text{Kerr}}$ amplitudes

$$A_{\sqrt{\text{Kerr}}} (1^s, 2^s, 3^-) = \sqrt{2}Q p_1 \cdot \varepsilon^- \frac{[21]2s}{m^{2s}}.$$  \hspace{1cm} (6.20)

The chiral Lagrangian in eq. (6.14) is not the most general parity invariant cubic Lagrangian. In principle one could add cubic operators of the form

$$\Delta L_3 = \sum_{k=1}^{2s} d_k \left( m^{-2k-4s} \left\{ |D|D|^{\odot k-1} |D|F_+|D\rangle \right\} |\Phi\rangle \right)$$

$$+ m^{2k-4s} \left\{ |D|D|^{\odot 2s-k} |F_-\rangle \right\} |\Phi\rangle \right)$$  \hspace{1cm} (6.21)

which depend on $|F_-|$ and the self-dual field strength $|F_+| := F^\alpha_+ \beta := \frac{1}{2} (\tilde{\sigma}^{\mu} \sigma^{\nu})^{\alpha}_{\beta} F_{\mu\nu}$. The relative coefficient between the operators containing anti- and self-dual field strengths is fixed by parity. The mass factors are fixed by dimensional analysis and are a clear indication that the power-counting for the interactions involving $F_-$ and $F_+$ are inversely related in the chiral Lagrangian.

The overall coefficients $d_k$ are free and are related to the $2s-1$ free parameters in the general three-point amplitude (3.31) via the relation $c_{k>0} = d_{k+1} - d_k$, where $d_{k>2s} = 0$. The first coefficient, $c_0 = 1$, is fixed by the minimal coupling.

In the non-chiral construction, the $\sqrt{\text{Kerr}}$ amplitudes were generated by cubic interactions with power counting of $2s-1$. In this case the $\sqrt{\text{Kerr}}$ chiral Lagrangian discriminates between interactions with $|F_-|$ which have power counting $2s-1$, and those involving $|F_+|$ which are constrained to vanish.

**Gravity theory**

In the gravitational case, the cubic order parity-invariant chiral theory is

$$L_{\text{Kerr}} = \sqrt{-g} \left\{ \frac{1}{2} \langle \nabla_\mu |\Phi| \nabla^\mu |\Phi\rangle - m^2 \langle |\Phi|\Phi\rangle \right. $$

$$\left. - \frac{1}{4} \sum_{k=0}^{2s-2} \frac{2s-k-1}{m^{2k}} \langle |\nabla|\nabla|^{\odot k} |R_-\rangle \right\} |\Phi\rangle \right\},$$  \hspace{1cm} (6.22)

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where we introduce the chiral Riemann curvature spinor
\[ R_{\alpha \beta}^{\gamma \delta} := \frac{1}{4} R^{\hat{\lambda} \hat{\mu} \hat{\nu} \hat{\sigma}} \sigma^{\alpha \hat{\epsilon}} \bar{\sigma}^{\hat{\mu} \hat{\epsilon} \beta} \sigma^{\hat{\nu} \gamma} \bar{\sigma}^{\hat{\sigma} \hat{\delta}}. \]  

(6.23)

At cubic order, this corresponds to including a non-minimal vertex
\[ V_{R_-}^{\hat{\alpha}} (\Phi_1^s \Phi_2^s A_3) = -\frac{\kappa}{8} (\langle 2|p_3 \epsilon_3|1 \rangle - p_3 \cdot \epsilon_3 \langle 21 \rangle)^2 \times \]
\[ \sum_{k=0}^{2s-2} \frac{2s-k-1}{m^{2k}} \langle 21 \rangle^{2s-k-2} \langle 2|p_2 p_1|1 \rangle^k. \]  

(6.24)

The resummed on-shell vertex has the form
\[ V_{R_-}^{\hat{\alpha}} (\Phi_1^s \Phi_2^s h_3) \big|_{(1,2,3)} = \frac{\kappa}{8} \langle 2|p_3 \epsilon_3|1 \rangle^2 \left( \frac{\langle 21 \rangle^{2s} - [21]^{2s}}{\langle 21 \rangle^2 - [21]^2} - \frac{2s \langle 21 \rangle^{2s-1}}{\langle 21 \rangle - [21]} \right) \]
\[ = \frac{\kappa}{8} \langle 2|p_3 \epsilon_3|1 \rangle^2 P_3^{(2s)}(s_3, s_3, s_4), \]  

(6.25)

where \( P_3^{(2s)}(s_3, s_3, s_4) \) corresponds to a three-variable polynomial, which can be constructed as a limit. From the overall factor \( \langle 2|p_3 \epsilon_3|1 \rangle \), it is clear this vertex vanishes for positive-helicity gravitons, such that \( M(1^s, 2^s, 1^+) \) is unchanged.

In the negative helicity case the amplitude \( M(1^s, 2^s, 1^-) \) receives contributions from the three vertices \( V_{\sqrt{-g}}^{\hat{\alpha}}, V_{\nabla} \) and \( V_{R_-} \). Making use of the identity in eq. (6.19), we find that the second contribution from \( V_{R_-} \) cancels exactly against the on-shell contribution of spin-connection \( V_{\nabla} \). Meanwhile the first term of \( V_{R_-} \) corrects the chiral \( V_{\sqrt{-g}} \) term. The result is the expected Kerr amplitude
\[ M_{\text{Kerr}}(1^s, 2^s, 1^-) = \kappa (p_1 \cdot \epsilon_3)^2 \frac{[21]^{2s}}{m^{2s}}. \]  

(6.26)

In principle one can add further cubic interactions proportional to \( \mathcal{L}_{\text{Kerr}} \) the anti-chiral Riemann tensor, \( R_{\alpha \beta}^{\gamma \delta} \). This would be necessary to describe a theory of a generic spin-s particle. However the theories corresponding to Kerr amplitudes do not require such operators at cubic order.

### 6.2.2 Quartic order

We can extend the analysis of the chiral Lagrangian to quartic order. In order to investigate the parity properties of the quartic Lagrangian we will compute the four-point Compton amplitude \( A(1^s, 2^s, 3^\pm, 4^\pm) \) in electromagnetism. The gravity calculation will not be presented.
In order to compute $\mathcal{A}(1^s, 2^s, 3^\pm, 4^\pm)$ we need the massive propagator,

$$\frac{1}{p^2 - m^2} = \frac{i\delta^{(\beta_1 \ldots \beta_{2s})}}{p^2 - m^2} .$$

(6.27)

Given the free chiral Lagrangian (6.1) is effectively a Lagrangian for a free massive scalar, the massive propagator resembles a scalar propagator with the $SL(2, \mathbb{C})$ unit operator in the propagator numerator.

**Quartic order gauge theory**

As discussed previously, the Lagrangian (6.14) generates two cubic vertices $V_{\text{min}}(\Phi_1^s, \Phi_2^s, A_3)$, $V_{\text{non-min}}(\Phi_1^s, \Phi_2^s, A_3)$ from the minimal coupling and the non-minimal parity completion. At quartic order, the cubic Lagrangian generates two quartic vertices,

$$V_{\text{min}}(\Phi_1^s, \Phi_2^s, A_3, A_4) = Q^2 \epsilon_3 \cdot \epsilon_4 \frac{\langle 21 \rangle^{2s}}{m^{2s}} ,$$

$$V_{\text{non-min}}(\Phi_1^s, \Phi_2^s, A_3, A_4) =$$

$$- Q^2 \sum_{k=0}^{2s-1} \frac{\langle 21 \rangle^{2s-k-1}}{m^{2s+k}} \langle 2|p_3 \epsilon_3|1 \rangle \left\{ \frac{\langle 2|\epsilon_4|1 \rangle}{\langle 2|4|1 \rangle} \left( \langle 2|1+4|1 \rangle^k - m^k \langle 21 \rangle^k \right) \right\}$$

$$+ (3 \leftrightarrow 4) + \text{off-shell in } \Phi_1, \Phi_2 ,$$

(6.28)

where we only write explicitly the terms in $V_{\text{non-min}}$ valid for on-shell fields $\Phi_1, \Phi_2$.

The four-point amplitudes have two massive channels $\Gamma_t$ and $\Gamma_u$. In electromagnetism there are no self-interactions of the massless field, so there is no channel where a photon propagates. We construct the massive channel $\Gamma_t$

$$\Gamma_t(\Phi_1^s, \Phi_2^s, A_3^+, A_4^+) = \frac{1}{(2s)!} \left( \frac{\partial^2}{\partial z_p^a \partial z_{p,a}} \right)^{2s} \frac{V(\Phi_1^s, \Phi_p^s, A_4) V(\Phi_p^s, \Phi_2^s, A_3)}{t_{14}}$$

(6.29)

where the differential operator acts on the massive spinors and such that $SL(2, \mathbb{C})$ indices are trivially contracted, as imposed by the massive propagator in eq. (6.27). The other massive channel is related by $\Gamma_u = \Gamma_t|_{3\leftrightarrow4}$.

Since the photons will always be on-shell, the non-minimal vertices $V_{\text{non-min}}$ will only contribute to amplitudes with at least one external negative helicity photon. Therefore the all-plus amplitude only depends
on the interactions of the minimally coupled Lagrangian,
\[
\mathcal{A}(1^s, 2^s, 3^+, 4^+) = \left\{ \Gamma_t(\Phi_1^s, \Phi_2^s, A_3^+, A_4^+) + \Gamma_u(\Phi_1^s, \Phi_2^s, A_3^+, A_4^+) \right\}^{(1,2,3^+,4^+)}
+ V_{\text{min}}(\Phi_1^s, \Phi_2^s, A_3^+, A_4^+)
= Q^2(t_{14}p_1 \cdot \epsilon_3^+ p_2 \cdot \epsilon_4^+ + t_{13}t_{14} \epsilon_3^+ \cdot \epsilon_4^+) \langle \mathbf{21} \rangle^{2s} + (3 \leftrightarrow 4)
= Q^2 \frac{m^2[34]^2 \langle \mathbf{21} \rangle^{2s}}{t_{13}t_{14} m^{2s}}.
\]
(6.30)

As expected the minimally coupled Lagrangian generates the all-plus sector of the AHH amplitudes as shown in ref. [50]. Although we did the explicit calculation, this is clear from the form of the minimally coupled chiral Lagrangian. The minimally coupled terms in eq. (6.14) have trivial spinor contractions such that the spin-dependence of the interactions can be factored out. Therefore, in the all-plus helicity sector, the theory is effectively the theory of minimally coupled scalars and the \( n \)-point amplitudes
\[
\mathcal{A}(1^s, 2^s, 3^+, \ldots n^+) = \mathcal{A}(1^0, 2^0, 3^+, \ldots n^+) \frac{\langle \mathbf{21} \rangle^{2s}}{m^{2s}}
\]
(6.31)

factor into the scalar amplitudes \( \mathcal{A}(1^0, 2^0, 3^+, \ldots n^+) \) and a spin-dependent term. The factorisation property does not hold in any other helicity sector, given the non-minimal interactions in eq. (6.14) have non-trivial spinor contractions.

The all-negative helicity amplitudes generated by \( \mathcal{L}_{\sqrt{Kerr}} \) do not correspond to the \( \sqrt{Kerr} \) amplitudes in eq. (3.33), indicating that the chiral Lagrangian is not parity invariant at quartic order. In order to restore parity, we need to introduce interactions of form \( \propto |F^-|^2 \). However, we leave the identification of these operators to future work since we are mostly concerned with the opposite-helicity amplitudes in this thesis.

Remarkably the opposite-helicity amplitudes \( \mathcal{A}(1^s, 2^s, 3^-, 4^+) \) generated by the Lagrangian respect parity directly, without any need to add additional quartic operators \( \propto |F^-||F^+| \). The construction of the amplitude follows the approaches of the all-plus amplitudes but in this case the non-minimal vertices \( V_{\text{non-min}} \) will also contribute. We suppress the details of the calculation and present the result
\[
A(1^s, 2^s, 3^-, 4^+) = \frac{\langle 3|1|4 \rangle^2_{t_{13}t_{14}} P_1^{(2s)}}{t_{13}} - \frac{\langle 13 \rangle \langle 3|1|4 \rangle [42] P_2^{(2s)}}{t_{13}}
+ \langle 13 \rangle \langle 32 |14 \rangle [42] \left( P_2^{(2s-1)} - s_3 s_4 P_4^{(2s-1)} \right).
\]
(6.32)
This amplitude matches the $s = 0$ and $s = 1/2$ AHH amplitudes in eq. (3.34) and $\sqrt{\text{Kerr}}$ amplitudes in eq. (3.36). The mismatch with the $\sqrt{\text{Kerr}}$ corresponds to the contact term

$$C^{(s)} := A_{\sqrt{\text{Kerr}}} (1^s, 2^s, 3^-, 4^+) - A (1^s, 2^s, 3^-, 4^+)$$

$$= - \frac{\langle 13 \rangle \langle 32 \rangle [14][42]}{2m^4} (s_3 + s_4) \left( P_4^{(2s)} - P_2^{(2s-2)} \right) \quad (6.33)$$

The missing contact term corresponds to a specific choice of additional four-point interactions which have the form

$$\Delta L_4 = - \sum_{k \leq l=0}^{2s-4} \sum_{j=0}^{2s-3-l} \frac{Q^2}{m^6} \langle \Phi \{ |D^{j,k,l}| \circ |F_-| \circ |D|F_+|D| \} | \Phi \rangle,$$

$$D^{j,k,l} = \frac{1}{m^2(j+l)} \left( |D|D| + m^2 \right) \circ |D|D|^{\circ j} \circ |D|D|^{\circ k} \circ |D|D|^{\circ (l-k)} \right). \quad (6.34)$$

While $C^{(s)}$, and in turn $\Delta L_4$, is not fixed using solely higher-spin constraints, we motivate it by a series of classical constraints introduced in Part III.
7. String theory

In the field theories considered previously, the fundamental object was a point-particle, while in string theory it is an extended object, the string. String theory provides a fully consistent theory of interacting higher-spin states corresponding to vibration modes of the string.

The classical string is described by the Polyakov action \[ S = -\frac{1}{4\pi\alpha'} \int d^2 \sigma \partial_a X^\mu \partial^a X_\mu, \] (7.1)

where the classical space-time coordinate \( X^\mu \) satisfies two additional constraints. When quantising the bosonic open string, \( X^\mu \) is expanded into oscillators \( \alpha_n^\mu \) corresponding to the creation \((n < 0)\) and annihilation \((n > 0)\) operators. We can thus construct an arbitrary state in the Fock space by acting with \( \alpha_{-n}^\mu \) on the vacuum \( |0; p\rangle \)

\[ \alpha_{-n_1}^\mu \ldots \alpha_{-n_s}^\mu |0; p\rangle. \] (7.2)

By the state-operator correspondence, we can map the string state to an operator in the conformal field theory on the worldsheet

\[ : \partial^{n_1} X^{\mu_1} \ldots \partial^{n_s} X^{\mu_s} e^{ip \cdot X} : \] (7.3)

constructed from the conformal primary \( \partial X := \partial_1 X + i \partial_2 X \) and it’s descendants. Physical string states must satisfy the Virasoro constraints, inherited from the constraints on \( X \) in the classical theory.

The leading Regge trajectory corresponds to taking states of maximum spin \( s = n - 1 \) at a given level \( n = \sum n_i \). In the bosonic open string, this corresponds the states

\[ \epsilon_{\mu_1 \ldots \mu_s} \alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_s} |0; p\rangle \leftrightarrow \epsilon_{\mu_1 \ldots \mu_s} : \partial X^{\mu_1} \ldots \partial X^{\mu_s} e^{ip \cdot X} :. \] (7.4)

The Virasoro constraints translate to the expected constraints on the momentum \( p \) and polarisation tensors \( \epsilon_{\mu_1 \ldots \mu_s} = \epsilon_{\mu_1} \ldots \epsilon_{\mu_s} \)

\[ p^2 = (s - 1)/\alpha', \quad p^\nu \epsilon_{\nu \mu_2 \ldots \mu_s} = 0, \quad \eta^{\nu_1 \nu_2} \epsilon_{\nu_1 \nu_2 \mu_3 \ldots \mu_s} = 0. \] (7.5)

Therefore the level-\( n \) state in the leading trajectory has the same degrees of freedom of a spin-\( s \) particle, where \( s = n - 1 \). Therefore we can interpret the scattering of string states in the leading Regge trajectory.
as the scattering of higher-spin particles with non-minimal interactions that are preselected by the string theory.

A natural question to ask is whether the $\sqrt{\text{Kerr}}$ and Kerr three-point amplitudes have a string theory origin. The quantum three-point amplitudes for the scattering of leading Regge states in the bosonic string and the superstring are known [57, 53, 54] and do not correspond to the quantum $\sqrt{\text{Kerr}}$ and Kerr amplitudes, as shown in paper I. Furthermore, we confirmed that the classical limit of the superstring amplitudes do not generate the classical $\sqrt{\text{Kerr}}$ and Kerr amplitudes.

However there are also candidate spin-$s$ states lying in the subleading trajectories. In principle these states can be constructed either in light-cone gauge [37, 36] or using the Del Giudice - Di Vecchia - Fubini construction [29, 21], however the analysis of the scattering amplitudes is complicated by the explicit dependence on a reference momentum. A covariant approach was recently introduced in ref. [47], which can construct whole trajectories at a time. We leave the study of such amplitudes and their classical limits to future work.

In the bosonic theory, the lowest spin state state in the leading trajectory has $m^2 < 0$. In order to avoid this tachyonic state we studied the scattering of states in the leading Regge trajectory of the superstring.

**Leading Regge states in the open superstring**

Introducing supersymmetry removes the tachyon state and also lowers the critical string dimension to $d = 10$. In principle, there are several variations of the superstring corresponding to how the fermions are included. In this work we only consider the scattering of bosonic states which remain unchanged.

In the supersymmetric theory, a spin-$s$ state on the leading trajectory is

$$\alpha^{(\mu_1} \ldots \alpha^{\mu_{s-1}} \psi^{\mu_s)}_{-1/2}(0; p),$$

where $\psi^{\mu}_{-1/2}$ is a worldsheet fermion. The corresponding vertex operator\(^1\) is

$$V^{(-1)}_{s-1}(\epsilon, k) = \frac{1}{(\sqrt{2\alpha'}^{s-1})} : (\epsilon \cdot i \partial X)^{s-1} (\epsilon \cdot \psi) e^{-\phi} e^{ik \cdot X} :$$

where the operator carries a superghost charge $-1$ [54, 17]. Since tree-level superstring amplitudes must carry overall ghost charge $-2$, we also

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\(^1\)We have suppressed the dependence on the Chan-Paton factor $T$.  

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need to introduce the zero ghost charge vertex operator,

\[ V_{s-1}(\epsilon, k) = \frac{1}{(\sqrt{2\alpha'})^{s-2}} : (\epsilon \cdot i\partial X)^{s-2} \left[ (s-1)\epsilon \cdot \partial \psi \epsilon \cdot \psi + k \cdot \psi \epsilon \cdot \psi \epsilon \cdot i\partial X \right. \]

\[ \left. + \frac{1}{2\alpha'} (\epsilon \cdot i\partial X)^2 \right] e^{ik \cdot X} : \] (7.8)

The polarisation tensors corresponding to the level-\((s-1)\) state is totally symmetric such that \(\epsilon_{\mu_1 \ldots \mu_s} = \epsilon_{\mu_1} \ldots \epsilon_{\mu_s}\). As in the bosonic string, \(\epsilon_{\mu_1 \ldots \mu_s}\) is constrained by BRST invariance to be transverse and traceless [17], such that it corresponds to the physical polarisation tensor for a spin-\(s\) particle. In the open string, the states on the leading Regge trajectory satisfy open string \(\alpha' = (s-1)/m^2\), such that the lowest spin state is massless vector and corresponds to the photon.

The three-point amplitude between two spin-\(s\) states and the photon can be computed from the correlation function

\[ \mathcal{A}(1^s, 2^s, 3) = \langle cV_{s-1}^{(-1)}(\epsilon_1, p_1) cV_{s-1}^{(-1)}(\epsilon_2, p_2) cV_{0}^{(0)}(\epsilon_3, k_3) \rangle, \] (7.9)

and was done in ref. [54]. While the vertex operators are defined in \(d = 10\), we can trivially compactify down to \(d = 4\) via \(\epsilon_i = (\epsilon_i, 0)\) where \(\epsilon_i\) are the 4d massive polarisations for the massive legs \(i = 1, 2\). Likewise we compactify the massless polarisation \(\epsilon_3 = (\epsilon_3, 0)\) and the momenta \(p_i^{d=10} = (p_i, 0)\), \(k_3^{d=10} = (k_3, 0)\). In four-dimensions we can present the resulting amplitude in massive spinor variables,

\[ \mathcal{A}_{L.R.}(1^s, 2^s, 3^+) = \sqrt{2} Q p_1 \cdot \epsilon_3^+ \sum_{n=0}^{s} \sum_{l=0}^{s} c_{s,n,l} \frac{\alpha'^s-n-1}{m^{2n+2}} (\langle 12 \rangle - [12])^{2s-l-n}, \] (7.10)

where the coefficients are

\[ c_{s,n,l} = -\frac{(s-1)(s-1)! (s-n-1)}{l!(n-l)![(s-n+1)!]^2} (s l - n(n-1)). \] (7.11)

The covariant formula is presented in paper I. Note that taking the field theory limit, \(\alpha' \to 0\), does not generate \(\mathcal{A}_{\sqrt{\text{Kerr}}}\) amplitudes. We do find an instance when \(\mathcal{A}_{L.R.} = \mathcal{A}_{\sqrt{\text{Kerr}}}\), corresponding to the scattering of spin-2 states, where the identity only holds after substituting the relation \(m^2\alpha' = s-1\).

However beyond spin-2, this equality does not hold. Therefore the corresponding effective chiral Lagrangian in eq. (6.14) thus requires additional cubic interactions \(\Delta \mathcal{L}_3\) with coefficients that are explicitly dependent on \(s\). From Ward identities, such amplitudes satisfy (WI) and (PC) however, they are ruled out by the current constraint (CC). Given
the amplitudes are unique from the Lagrangian level approach at power counting $s + s' + 1$ it suggests that the EFT for the leading Regge superstring must have slightly loosened power counting constraints, for example the power counting on $\delta_1$ could be increased or else the grading of the power counting when looking at the unphysical interactions could be tailored.

Closed string three-point amplitude
Having studied the open string amplitude, we can use the Kawai-Lewellen-Tye relations [45] to generate the closed string amplitude,

$$M(1^s, 2^s, 3) = \frac{\kappa}{2} \left( A(1^s, 2^s, 3) \bigg| \begin{array}{c} \alpha' \to \alpha'/4 \\ Q \to 1 \end{array} \right)^2. \quad (7.12)$$

We will not present an explicit formula, however we note that for $s = 4$ the amplitude coincides with the Kerr amplitude. As in the open string case, this matching no longer holds for spins $s > 4$. 
Part III:
Constructing Classical Amplitudes
8. Scattering amplitudes from classical physics

Scattering amplitudes are typically not direct observables, instead they are stepping stones towards predictions, also for classical physics. In section 2, it was shown that the the two-to-two scattering amplitude is an intermediate step in the computation of the effective potential that governs the conservative dynamics the binary black hole system.

The Compton amplitudes considered in this thesis do have a corresponding well-defined classical process. In general, one can consider amplitudes of the form $\mathcal{M}(1,2,3,\ldots n)$, which involve two massive states that, in the classical limit, represent the classical source and perturbations are represented by the gravitons. The classical amplitudes are direct limits of the ones computed using field theory methods. The classical solutions we are interested in are characterised by a mass parameter, $m$, and a spin vector $a^\mu$, in particular, the perturbations to the Kerr solution.

The three-point gravitational amplitude can be computed from the effective energy-momentum tensor $T^{\mu\nu}$, which sources the classical solution. The four-point amplitudes correspond to the scattering of a gravitational wave in the classical background solution, as was done explicitly in ref. [8, 9] for the Kerr solution.

The classical amplitudes in electromagnetism have an analogous definition. The three-point amplitudes can be computed from the effective current $j^\mu$, while the four-point amplitude corresponds to scattering of a single electromagnetic wave off of the classical background.

8.1 Classical amplitudes at three points

We consider a classical solution in electromagnetism or general relativity sourced by an effective current $j^\mu$ or energy-momentum tensor $T^{\mu\nu}$ at linear order in the coupling. The resulting linearised equations of motion are $\Box A^\mu \sim j^\mu$ in electromagnetism and $\Box h_{\mu\nu} \sim T_{\mu\nu} - \eta_{\mu\nu} T/2$ in gravity, using harmonic gauge. The interactions between the source and the field are

$$A_\mu j^\mu, \quad h_{\mu\nu} T^{\mu\nu}. \quad (8.1)$$
The classical three-point amplitudes are obtained by evaluating these interactions on-shell,

\[ A_{cl} = \varepsilon_\mu \tilde{j}^\mu, \quad M_{cl} = \varepsilon_\mu \varepsilon_\nu \tilde{T}^{\mu\nu}. \quad (8.2) \]

Where the sources are Fourier transformed and on-shell polarisations introduced.

The amplitudes can be expressed using momenta \( p^\mu, q^\mu \), spin \( a^\mu \) and the massless polarisation \( \varepsilon_\mu^q \). On the three-point kinematics there are two independent Lorentz contractions \( p \cdot \varepsilon \) and \( q \cdot a \) such that the general amplitudes have the form,

\[ A_{cl} = Q p_1 \cdot \varepsilon_3 \sum_{n=0}^{\infty} c_n (q \cdot a)^n, \quad M_{cl} = \frac{\kappa}{2} (p_1 \cdot \varepsilon_3)^2 \sum_{n=0}^{\infty} c_n (q \cdot a)^n. \quad (8.3) \]

The spin-multipole coefficients \( c_n \) describe the nature of the source.

**Kerr black hole**

Ref. [59] computed the linearised energy-momentum tensor from the Kerr metric,

\[ \tilde{T}^{\mu\nu}_{\text{Kerr}}(q) = \int d^4x e^{i q \cdot x} T^{\mu\nu}_{\text{Kerr}}(x) = \delta(p \cdot q) p^{(\mu} (e^{a* q})^{\nu)} p_1^\rho, \quad (8.4) \]

where \((a \ast q)^{\mu\nu} = i \varepsilon^{\mu\nu\rho\sigma} a^\rho q^\sigma\). This exponential structure is reflected in the three-point amplitudes

\[ M_{\text{Kerr}} = \kappa (p \cdot \varepsilon^\pm)^2 e^{\pm q \cdot a}, \quad (8.5) \]

which were originally computed in refs. [39, 26].

**Electromagnetic \( \sqrt{\text{Kerr}} \) solution**

The interpretation of the \( \sqrt{\text{Kerr}} \) electromagnetic solution is not properly understood, although at leading order in \( Q \) it is known to correspond to the electromagnetic part of the Kerr-Newman solution, a charged and rotating black hole [25]. The authors in ref. [25] derive the effective current \( j^\mu_{\sqrt{\text{Kerr}}}(x) \), such that

\[ \tilde{j}^\mu_{\sqrt{\text{Kerr}}}(q) = \int d^4x e^{i q \cdot x} j^\mu_{\sqrt{\text{Kerr}}}(x) \]

\[ = Q \delta(p \cdot q) \left( p^\mu \cosh(q \cdot a) + (a \ast q)^\mu \nu p_1^\nu \frac{\sinh(q \cdot a)}{(q \cdot a)} \right). \quad (8.6) \]

The resulting classical amplitude is

\[ A_{\sqrt{\text{Kerr}}} = Q p \cdot \varepsilon^\pm e^{\pm q \cdot a}. \quad (8.7) \]
This amplitude can also be constructed from a Newman-Janis shift of the Coloumb solution, such that the effective source is a rotating disk with a certain charge distribution [5].

Note that the spin-multipole coefficients of the Kerr and $\sqrt{\text{Kerr}}$ amplitudes are both $c_n = 1/(n!)$. The amplitudes also satisfy a classical double copy relation

$$M_{\text{Kerr}} = A_{\sqrt{\text{Kerr}}} \times A_{\sqrt{\text{Kerr}}} \Big|_{a \to a/2, \quad Q^2 \to \frac{2}{\kappa^2}}. \quad (8.8)$$

**Classical string solutions**

We can construct classical amplitudes for various other types of sources. For example consider the solution to the Maxwell equations corresponding to a rotating rigid rod of radius $r$ with a charged $Q$ at one end point, as shown in figure 8.1. The explicit solution is given in detail in paper I such that the corresponding current is

$$j^\mu_{\text{rod}} = \frac{Q}{r} \delta(\rho - r)\delta(\phi - \tau/r)\delta(z) \, n^\mu \quad (8.9)$$

given in polar coordinates $(\rho, \phi)$ such that the rod is rotating in the $xy$-plane, $n^\mu := (1, \hat{\phi}) = (1, -\sin(\tau/a), \cos(\tau/a), 0)$.

This configuration is of interest as it is a solution of the classical string action (7.1). In particular, we expect these classical solutions to be related to the leading Regge trajectory given that the angular momentum is quadratic in total energy

$$E = \frac{r}{2\alpha'}, \quad J = \frac{r^2}{4\alpha'} \quad \Rightarrow \quad J = \alpha' E^2. \quad (8.10)$$

This corresponds to the large spin limit of the defining relation $n = s - 1$ for the open string states on the leading Regge trajectory where the energy is the rest mass $E = m = \sqrt{n/\alpha'}$, see section 7. The orbital angular momentum of the charged rotating rod $J$ is identified with the spin of the effective point particle, such that $a^\mu = (0, 0, 0, J/E) = (0, 0, 0, r/2)$ in the rest frame where the spin is aligned with the $z$-axis. The corresponding solution for the leading Regge closed string states is a rotating rigid massive folded loop.

In paper I, we compute the effective current $j^\mu_{\text{str}}$ and energy-momentum tensor $T^{\mu\nu}_{\text{str}}$ explicitly for the open and closed string solutions. This allows us to compute the full classical amplitudes,

$$A_{\text{cl string}} = (p \cdot \varepsilon^+) \left( I_0(2q \cdot a) + I_1(2q \cdot a) \right)$$

$$M_{\text{cl string}} = (p \cdot \varepsilon^+)^2 \left( I_0(q \cdot a) + I_1(q \cdot a) \right)^2. \quad (8.11)$$
As opposed to the simple exponential structure of the Kerr and $\sqrt{\text{Kerr}}$ amplitudes, the spin-multipoles for the classical string solutions resum to two modified Bessel functions of the first kind, defined by the series expansions

$$I_0(2x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(k!)^2}, \quad I_1(2x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!(k + 1)!}.$$  \hspace{1cm} (8.12)

The classical amplitudes $A_{\text{cl, string}}$ and $M_{\text{cl, string}}$ are related by the same classical double copy relation (8.8) as the $\sqrt{\text{Kerr}}$ and Kerr amplitudes.

We expect the classical amplitudes $A_{\text{cl, strings}}$ and $A_{\sqrt{\text{Kerr}}}$ to differ significantly given the effective currents that source the static solutions differ, the former corresponding to a single point charge rotating orbit while the latter corresponds to a rotating disk of charge [5].

### 8.2 Classical Compton amplitudes

At four point, the classical Compton amplitudes correspond to the linearised scattering of plane waves off an exact background sourced by the classical solution. This is a tractable computation for Kerr backgrounds, given the linearised scattering is described by the separable Teukolsky
The corresponding classical Compton amplitudes have been computed up to $O(a^8)$ in refs. [9, 7]. For a general background, the equations of motion for linearised perturbations may not be separable. As of yet, the classical Compton amplitudes for $\sqrt{\text{Kerr}}$ have not been computed from linearised perturbations of the Kerr-Newman solutions. Furthermore, it is not clear if $\sqrt{\text{Kerr}}$ beyond three-point is still related to Kerr-Newman. Therefore there is less classical data to compare our proposed $\sqrt{\text{Kerr}}$ amplitudes to.

The classical amplitude $M(1,2,3,4)$ corresponds to the scattering of an incoming wave with momenta $k_3^\mu$ and the incoming massive black hole $p_1^\mu$ to the outgoing outgoing wave with momenta $-k_4$ and the deflected massive black hole $-p_2^\mu = p_1^\mu + k_3^\mu + k_4^\mu$.

As discussed in section 2, the classical regime corresponds to the low frequency regime of the waves, such that $k_3, k_4$ are soft. This corresponds to a small deflection of the black hole. In practice we will scale the massless momenta with $\hbar$

$$p^\mu := p_1^\mu \sim 1 \ , \ k^\mu \sim \hbar ,$$

(8.13)

such that the classical limit corresponds to taking $\hbar \to 0$ in the final amplitude. At three points the quantum kinematics are essentially classical given all the momentum invariants vanish. Although the amplitude does not necessarily physical given we use complexified momenta at three-points.

At four points, there are non-trivial momentum invariants such that we will need to take an explicit $\hbar \to 0$ limit on the amplitudes. The relevant soft momenta for the opposite-helicity amplitudes are $q^\mu := (k_3 + k_4)^\mu$, $q_\perp^\mu := (k_4 - k_3)^\mu$ and $\chi^\mu := \langle 3|\sigma^\mu|4 \rangle$. Thus, the four momentum invariants that parameterise the scalar opposite-helicity amplitude scale as

$$p^2 = m^2 \sim 1 \ , \ q_\perp^2 = 2p\cdot q = -q^2 \sim \hbar^2 \ , \ p\cdot q_\perp \sim \hbar \ , \ p\cdot \chi \sim \hbar .$$

(8.14)

Given the Compton amplitudes are defined with all-incoming states, the opposite-helicity amplitude corresponds to the helicity-conserving classical process, where both the incoming wave $k_3$ and outgoing wave $-k_4$ have negative helicity.

One can consider non-spinning backgrounds, corresponding to the Coulomb potential in electromagnetism and the Schwarzschild black hole in general relativity. As mentioned in section 2, the dynamics of these solutions are modelled by a minimally coupled massive scalar. The scalar, opposite-helicity, classical Compton amplitudes thus correspond to

$$A_0 = Q^2 \frac{(p\cdot \chi)^2}{(p\cdot q_\perp)^2} \ , \ M_0 = \left( \frac{\kappa}{2} \right)^2 \frac{(p\cdot \chi)^4}{q^2(p\cdot q_\perp)^2} .$$

(8.15)
When we consider the scattering of backgrounds with spin, the general classical amplitude is expressed as a multipole expansion in the classical spin variable $a^\mu$. For the four point helicity-conserving scattering, this introduces four new spin-dependent variables

$$x = a \cdot q_\perp, \quad y = a \cdot q, \quad z = |a| \frac{p \cdot q_\perp}{m}, \quad w = \frac{a \cdot \chi}{p \cdot \chi}.$$  \hfill (8.16)

These variables are not independent and are related by a Gram determinant

$$\xi^{-1} := \frac{m^2 q_\perp^2}{(p \cdot q_\perp)^2} = \frac{(w - x)^2 - y^2}{z^2 - w^2},$$  \hfill (8.17)

where $\xi$ is the optical parameter [9]. In order for these variables to scale classically, the spin vector must scale $a^\mu \sim \hbar^{-1}$.

Extracting these spin-dependent variables from the quantum amplitudes can be subtle and is discussed in detail in section 9. In this section, the classical amplitudes are constructed as covariant ansätze that are matched to classical computations. The form of the general classical four-point amplitude is discussed in detail in paper III. Here we note that the classical non-spinning amplitudes can be factored out such that

$$A(1, 2, 3^-, 4^+) = A_0 f(x, y, z, w),$$

$$M(1, 2, 3^-, 4^+) = M_0 f(x, y, z, w).$$  \hfill (8.18)

The classical amplitudes are invariant under exchange and complex-conjugation of the massless bosons, implying $f$ is an even function in variable $y$. Likewise, if the classical amplitude is symmetric under exchange of the massive leg, $f$ is even in $z$. This is symmetry is broken for dissipative effects as the incoming and outgoing legs are no longer related by time-reversal symmetry. Such dissipative effects appear in the classical amplitudes $M_{Teuk}$ calculated in ref. [9], which we discuss in detail.

**Scattering off a Kerr black hole**

The classical amplitude $M_{Teuk}$ was computed up to $O(a^6)$ by matching solutions of the Teukolsky equation to an covariant ansatz [9]. The amplitude will be used both as input and as a check for the classical amplitude $M_{Kerr}$ proposed in paper IV. We present $M_{Teuk}$ in the form

$$M_{Teuk} = M_0 \left[ e^{x - w} + P(x, y, z, w) + P_\eta(x, y, z, w) + \alpha^2 \frac{z^2}{\xi} P_\alpha(x, y, z, w) + \eta \frac{wz}{\xi} P_{\alpha\eta}(x, y, z, w) + O(a^7) \right],$$  \hfill (8.19)
where the functions $P, P_\eta, P_\alpha, P_{\alpha\eta}$ are polynomials that are even in variables $y$ and $z$. The parameters $\alpha$ and $\eta$ are bookkeeping parameters that appear in ref. [9].

The $\eta$ terms are associated to near-zone effects given they are sensitive to the boundary conditions at the horizon. In $\mathcal{M}_{\text{Teuk}}$ such dissipative terms are odd in variable $z \propto |a|$, such that the exchange symmetry on the massive legs is broken.

The $\alpha$ tag was introduced to keep track of terms that arose from the expansion of polygamma functions when taking the super-extremal limit $|a| \gg Gm$. This super-extremal limit was taken in order to identify the tree-level $\mathcal{O}(G)$ Compton dynamics from the Teukolsky solutions, which are generically Laurent series in $G$. However there are alternative definitions of the $\mathcal{O}(G)$ physics, for example ref. [7] defines $\mathcal{M}_{\text{Teuk}}$ after implementing a near- and far-zone splitting of the Teukolsky solutions, such that the super-extremal limit is not necessary. Since it is not clear if this splitting procedure is a unique, there are ambiguities in how to define the classical amplitude $\mathcal{M}_{\text{Teuk}}$.

Nonetheless, in paper IV we present an all-orders-in-spin classical amplitude $\mathcal{M}_{\text{Kerr}}$ that predicts the $\alpha$-independent terms of $\mathcal{M}_{\text{Teuk}}$ defined in ref. [9]. The classical amplitude $\mathcal{M}_{\text{Kerr}}$ is generated by the classical limit of the quantum amplitude $\mathcal{M}_{\text{Kerr}}$ presented in eq. (3.39). Therefore, before we can present our classical amplitude we have to introduce how to extract classical physics from the quantum amplitudes.
9. Classical Kerr and $\sqrt{\text{Kerr}}$ Amplitudes

The goal of this section is to present the classical $\sqrt{\text{Kerr}}$ and Kerr Compton amplitudes proposed in papers III and IV. These classical amplitudes are generated as classical limits of the quantum amplitudes proposed in section 3.2.1. Therefore, we start by discussing the consistent classical limits for extracting classical spin variables from the quantum scattering amplitude.

9.1 Classical limits for spin

Extracting the dependence on classical spin $a^\mu$ from the quantum scattering amplitudes introduces subtleties. We will discuss three prescriptions for obtaining classical amplitudes from quantum scattering amplitudes of spinning particles,

(i) finite spin, $\hbar \to 0$ limit of a fixed spin amplitude
(ii) large quantum spin, take $s \to \infty$ of the generic spin $s$ amplitude
(iii) coherent spin states as introduced in ref. [3].

In each case we will make use of the spin-operator basis introduced in section 4. If we return to the schema,

$$
\mathcal{M}_{\text{Kerr}} \sim \frac{\langle 12 \rangle^{2s}}{m^{2s}} \xrightarrow{\text{spin op. basis}} \langle e^{\hat{a} \cdot p_3} \rangle \xrightarrow{\text{cl. limit}} e^{a \cdot p_3} \sim T^{\mu \nu}, \quad (9.1)
$$

the operation $\xrightarrow{\text{cl. limit}}$ corresponds to implementing one of the three limits (i), (ii), (iii).

As we will discover, (i) is not, strictly speaking, a valid classical limit and suffers from ambiguities in defining the classical amplitude. In comparison, (ii) and (iii) prove to be more robust approaches for extracting classical amplitudes with classical spin.

9.1.1 (i) finite spin classical limit

The guiding principle of the finite-spin approach is that one can generate the first $2s$ classical multipole orders of a classical solution from the scattering of a quantum spin-$s$ particle [58]. This suggests there exists a map from some suitably defined quantum spin multipoles, e.g. $c^{(s)}_{\mu_1...\mu_i}$ in
eq. (4.25) to the classical spin multipoles $c_{\mu_1...\mu_i}$. However, as discussed in section 4, the expansion of the quantum amplitude into the spin operator basis is ambiguous such that the quantum multipole tensors are not unique (4.32).

Furthermore, in order for the classical amplitude to be well defined, we expect the result to be the same whether we scatter spin-$s$ or spin-$s'$ states in the quantum theory. This implies the quantum amplitudes $\mathcal{A}(1^s, 2^s, 3^{h_3}, \ldots, n^{h_n})$ have to exhibit a particular property,

$$\lim_{\hbar\to 0} \tilde{c}^{(s=1)}_{\mu_1...\mu_i} = \lim_{\hbar\to 0} \tilde{c}^{(s=2)}_{\mu_1...\mu_i} = \cdots = c_{\mu_1...\mu_k},$$

which we call spin-universality. This is a restrictive property on the family of spin-$s$ quantum amplitudes and is not true for generic theories.

Neglecting these ambiguities for now, we can implement this limit by, first expanding the quantum amplitude in the spin-$s$ spin operator basis, as discussed in section 4 and then scaling the kinematics according to eq. (8.14).

However, we must scale the spin operators as $\hat{a} \sim \mathcal{O}(\hbar^{-1})$, in order for the spin-dependence to survive as $\hbar \to 0$. This scaling is suggested by the correspondence principle where the macroscopic spin emerges from a large quantum spin limit. We will identify the expectation values $\langle \hat{a}^{(\mu_1} \ldots \hat{a}^{\mu_k)} \rangle$ with the product of classical ring radius $a^{\mu_1} \ldots a^{\mu_k}$.

### Three points

Let us study spin-universality in more detail at the three point level. We can study the classical limit of the $\sqrt{Kerr}$ amplitudes in eq. (3.20) and the open string amplitudes in eq. (7.10). At three points, the tensor structure is unique such that $c^{(s)}_{\mu_1...\mu_k} = c^{(s)}_{k} q_{(\mu_1} \cdots q_{\mu_k)}$.

Using the identities in section 4, we can write the quantum general-spin three-point $\sqrt{Kerr}$ amplitudes as

$$\mathcal{A}(1^s, 2^s, q^+) = A_0 \sum_{k=0}^{2s} \frac{1}{k!} \langle (q \cdot \hat{a})^k \rangle.$$

These amplitudes exhibit spin universality given the spin-multipole coefficients $c^{(s)}_{k} = 1/k!$ are independent of the quantum number $s$. Note that the Kerr three-point amplitudes are also spin-universal since the spin-structure is identical.

However the open-string amplitudes, $\mathcal{A}_s := \mathcal{A}_{open}(1^s, 2^s, q^+)$ defined in eq. (7.10), do not exhibit this property. The classical amplitudes defined by the scattering of the first few massive states in on the leading
Regge trajectory are

\[
\begin{align*}
\lim_{\hbar \to 0} \mathcal{A}_2/\mathcal{A}_0 = & 1 + (q \cdot a) + \frac{1}{2} (q \cdot a)^2 + \frac{1}{6} (q \cdot a)^3 + \frac{1}{24} (q \cdot a)^4 \\
\lim_{\hbar \to 0} \mathcal{A}_3/\mathcal{A}_0 = & 1 + (q \cdot a) + \frac{9}{30} (q \cdot a)^2 + \frac{3}{10} (q \cdot a)^3 + \frac{13}{120} (q \cdot a)^4 + \mathcal{O}(a^5) \\
\lim_{\hbar \to 0} \mathcal{A}_4/\mathcal{A}_0 = & 1 + (q \cdot a) + \frac{5}{7} (q \cdot a)^2 + \frac{5}{14} (q \cdot a)^3 + \frac{39}{280} (q \cdot a)^4 + \mathcal{O}(a^5).
\end{align*}
\]

(9.4)

Clearly scattering spin-2, spin-3 or spin-4 states define three different classical amplitudes, none of which reproduce the classical amplitude \( \mathcal{A}_{\text{cl string}} \), computed from classical field theory methods in eq. (8.11).

While the monopole and dipole coefficients are universal \( c_0^{(s)} = c_1^{(s)} = 1 \), the higher multipole coefficients depend explicitly on the spin of the scattered state, for example

\[
c_2^{(s)} = \frac{4s^2 - 7s + 4}{2s(2s - 1)}, \quad c_3^{(s)} = \frac{2s - 3}{2(2s - 1)}.
\]

(9.5)

Given the amplitudes do not exhibit spin universality it is ambiguous how to define the classical amplitude. Indeed, for the leading Regge amplitudes, the \( s \to \infty \) limit is required to match classical multipoles after taking \( s \to \infty \), such that \( \lim_{s \to \infty} c_2^{(s)} = \frac{1}{4} \) is the quadrupolar coefficient in the expansion of the classical amplitude \( \mathcal{A}_{\text{cl. string}} \) in eq. (8.11).

**Same-helicity Compton amplitudes for Kerr**

Given the three-point Kerr amplitudes exhibit spin universality, we should investigate whether this property holds at four-points. Let us study the same-helicity amplitudes in eq. (3.33) given we expect them to be the correct Kerr amplitudes.

These four-point amplitudes have the same spinorial structure as the three-points amplitude, which we expand into \( \tilde{a} \) variables in eq. (4.23). After changing representation using eq. (4.29), the resulting amplitude is

\[
\mathcal{M}(1^s, 2^s, 3^+, 4^+) = \mathcal{M}_0 \langle e^{\tilde{a} \cdot q} \rangle + \mathcal{O} \left( \frac{1}{s} q^2 \tilde{a}^2 \right) + \mathcal{O}(\hbar).
\]

(9.6)

As expected, the same-helicity amplitude has the same exponential structure as the three-point amplitude plus a correction term proportional to \( q^2 \tilde{a}^2 \). At finite spin there is an ambiguity on how such terms scale due to the Casimir identity,

\[
q^2 \tilde{a}^2 = -s(s + 1)q^2,
\]

(9.7)

which suggests that the term can either scale classically as \( q^2 \tilde{a}^2 \sim 1 \), or be suppressed, given \( s \) remains finite \( q^2 s^2 \sim h^{-2} \). Note that this
ambiguity disappear if we allow the spin quantum number to scale as $s \sim \hbar^{-1}$.

We can choose a prescription that restores spin-universality in the same-helicity amplitude. This corresponds to always implementing $\hat{a}^2 = -s(s+1)\mathbf{1}$. Using this prescription we can generate a consistent classical amplitude

$$\mathcal{M}_{\text{cl}}(1, 2, 3^+, 4^+) = \mathcal{M}_0 e^{q\cdot a}. \quad (9.8)$$

**Four-point opposite-helicity**

The prescription used in the same-helicity amplitude also restores spin-universality of the opposite-helicity AHH Compton amplitudes in eq. (3.34). We expand the spin-dependent term into spin-operator variables,

$$\langle \langle 13 \vert [42] + \langle 23 \vert [41] \rangle \vert 3 \vert 1 \vert 4 \rangle \vert = (1 + \bar{w} - \bar{x})^2s + O(\hbar) = \langle e^{w-x}\rangle + O(q^2 \bar{a}^2) + O(\hbar),$$

where the variables $\bar{w}$ and $\bar{x}$ defined in terms of $\bar{a}$. In the second line we change representation to the spin-$s$ operators, picking up a correction term. However, if we substitute the Casimir identity, this correction is suppressed when $\hbar \to 0$.

Implementing this substitution universally does not generate the correct classical physics for generic amplitudes. To illustrate this, we can study the first two terms of $A_{\sqrt{Kerr}}$ in eq. (3.36), $P_1^{(2s)} + t_{14} \frac{\rho \cdot \chi}{p \cdot \chi} P_2^{(2s)}$. We can expand the combination in terms of the spin-$1/2$ spin operators,

$$P_1^{(2s)} + t_{14} \frac{\rho \cdot \chi}{p \cdot \chi} P_2^{(2s)} \approx \sum_{n=0}^{s} \left( \frac{2s}{2n} \right) (1 + q_\perp \cdot \bar{a})^{2s-2n}((p \cdot q_\perp)^2 \bar{a}^2)^n$$

where $\approx$ implies we have dropped quantum corrections $O(\hbar)$. In this case the Casimir appears due to the expansion of the spinor variables (4.21). However it generates the same ambiguities as we need to decide whether to keep or drop such terms. We can either use the Casimir to remove such terms completely such that the term contributes classically as

$$P_1^{(2s)} + t_{14} \frac{\rho \cdot \chi}{p \cdot \chi} P_2^{(2s)} \xrightarrow{\hbar \to 0} e^x,$$  \quad (9.11)

or choose to keep such $\bar{a}^2$ terms,

$$P_1^{(2s)} + t_{14} \frac{\rho \cdot \chi}{p \cdot \chi} P_2^{(2s)} \xrightarrow{\hbar \to 0} e^x \cosh z. \quad (9.12)$$

Therefore clearly the classical multipoles are dependent on the prescription for the $\bar{a}^2$ terms. A priori the correct prescription is not clear, however we check it against classical computations and consistency with
other classical limits, such that we find that the correct prescription for this term is given in eq. (9.12).

In principle we find a consistent prescription for the finite spin classical limit is to use the Casimir identity to neglect \( \mathcal{O}(\hat{a}^2) \) corrections from the representation change (4.27), this is required by keeping the exponential form of the classical same helicity amplitude. However we find that we should not reduce any \( \bar{a}^2 \) terms generated by the spinor expansions, in order to be consistent with limits (ii) and (iii) where we have better control of such contributions.

We emphasise that this ambiguity arises since we keep \( s \) finite. The classical limits (ii) and (iii) provide a robust treatment of \( \hat{a} \).

### 9.1.2 (ii) Infinite spin classical limit

As indicated by the name, this prescription involves taking explicit \( s \to \infty \) limits of the quantum amplitude. Therefore, in this case, it is meaningless to speak of the classical limit of a spin-1 amplitude. Instead we have to work with general spin-\( s \) amplitudes, for example the Kerr three point amplitudes (2.10). Indeed, the original classical analysis of the Kerr amplitudes did include taking a large spin limit [39, 26].

The large spin limit can be implemented by scaling \( s \sim \hbar^{-1} \), which diverges when we take \( \hbar \to 0 \). From the definition of \( \langle \hat{a}^\mu \rangle \) (4.6) clearly the classically finite spin-dependent variables are

\[
\langle q \cdot \hat{a} \rangle \sim 1, \quad \langle q_\perp \cdot \hat{a} \rangle \sim 1, \quad \langle \chi \cdot \hat{a} \rangle \sim 1.
\]

In the classical limit, we can make a consistent identification between \( \langle \hat{a} \rangle \) and the classical ring radius \( a \) given the variance of the spin operators vanishes

\[
\langle \prod_{i=1}^n \hat{a}^{\mu_i} \rangle - \prod_{i=1}^n \langle \hat{a}^{\mu_i} \rangle \to 0.
\]

In practice we implement this classical limit in two stages; a \( \hbar \to 0 \) limit when the amplitude is expressed in spin-1/2 operators \( \bar{a}^\mu \), followed by a \( s \to \infty \) limit after we convert to the spin-\( s \) operators \( \hat{a}^\mu \).

When taking the dual limit we must impose \( \hbar s \sim \mathcal{O}(1) \) by hand. Therefore we have to carefully to examine subleading orders in the \( \hbar \) expansion as they can ultimately contribute classically if there are compensating \( s \) factors.

The classical spin-multipoles are therefore defined after taking the \( s \to \infty \) limit. This was confirmed in paper I by studying the amplitudes generated by the scattering of open and closed superstring states. As discussed in the finite spin analysis, the amplitudes \( A_{\text{open}} \) and \( M_{\text{open}} \) are not spin-universal. Only in the \( s \to \infty \) limit do we generate the
correct classical spin multipoles,

\[
\text{open string: } c_n = \begin{cases} 
\frac{1}{(k!)^2} & \text{if } n = 2k \text{ with } k \in \mathbb{N}, \\
\frac{1}{k!(k+1)!} & \text{if } n = 2k + 1 \text{ with } k \in \mathbb{N}.
\end{cases}
\] (9.15)

\[
\text{closed string: } c_n = \begin{cases} 
\frac{(2k+1)!}{4^k(k+1)!(k!)^2} & \text{if } n = 2k \text{ with } k \in \mathbb{N}, \\
\frac{4^k((k+1)!)^2}{(2k+1)!} & \text{if } n = 2k + 1 \text{ with } k \in \mathbb{N}.
\end{cases}
\] (9.16)

By inserting these multipole coefficients into eq. (8.3) and resumming, we recover the modified Bessel functions in the classical amplitudes in eq. (8.11) computed from classical field theory methods.

**Quartic same-helicity amplitude revisited**

As we saw in the finite spin case, the quartic same-helicity amplitude is sensitive to corrections \(O(q^2 \hat{a}^2)\) generated by the representation change (4.27). We can no longer suppress such terms using the Casimir identity as \(s\) is no longer finite. However, for all the amplitudes considered in this thesis, the combinatorics of such terms collude such that they are suppressed when \(s \to \infty\). For example, consider the quadrupolar term generated by \(\langle 21 \rangle^2 s = (1 + q \cdot \bar{a})^2 s + \mathcal{O}(\hbar)\). Changing the representation from spin-1/2 to spin-\(s\) operators generates the \(\hat{a}^2\) correction

\[
\left( \frac{2s}{2} \right) (q \cdot \bar{a})^2 = \frac{1}{2!} \left( \langle (q \cdot \hat{a})^2 \rangle - \frac{s}{4} \frac{\langle \hat{a}^2 \rangle}{s(s+1)} P^{\mu \nu} q_\mu q_\nu \right),
\] (9.17)

\[
\longrightarrow \frac{1}{2!} \langle (q \cdot \hat{a})^2 \rangle + \mathcal{O}(s^{-1}).
\]

However in the \(s \to \infty\) limit this correction is suppressed. In paper III, we investigated up to dotriacontapole \(\mathcal{O}(a^5)\) and found that all contributions from the representation change are suppressed. This confirms that the subleading terms generated by the representation change of the spin-operator are not classically relevant.

**9.1.3 (iii) Coherent state classical limit**

As opposed to scattering two massive particles of massive spin, the authors in ref. [3] consider the scattering of coherent spin-states, which we schematically denote as

\[
|\text{coherent}\rangle = e^{-|z|^2/2} \sum_{s=0}^{\infty} \frac{1}{\sqrt{(2s)!}} |s, \{z\}\rangle.
\] (9.18)

Ref. [3] follows the Schwinger construction, where the spin eigenstates \(|s, \{z\}\rangle \sim (z^a a_\dagger^a)^{\otimes 2s} |0\rangle\) are constructed from the repeated action of the
creation operator \((a^\dagger_a)\) on the scalar state \(|0\rangle\). Once again we absorb the little group SU(2) indices with the wavefunctions \(z\).

The spin-operator in the Schwinger construction is defined \((\hat{a}^\mu)^a_b := \frac{1}{2m}a^\dagger_a \sigma^\mu a^b\), where \(a^b\) is the annihilation operator. While we are overloading the notation \(\hat{a}\), however we are permitted to do so given the expectation of this operator with respect to the spin-eigenstates matches exactly the definition of \(\langle \hat{a} \rangle\) in eq. (4.6),

\[
\langle s, \{z\}|(\hat{a}^\mu)^a_b|s, \{z\}\rangle \equiv \langle \hat{a}^\mu \rangle = \bar{s}a^\mu (z^a z_a)^{2s-1},
\]

where we note that \(\hat{a}\) can be constructed from the massive spinors (4.20).

These details are necessary to derive the expectation of the spin-operator with respect to the coherent states,

\[
\langle \text{coherent}|\hat{a}^\mu|\text{coherent}\rangle = e^{-|z|^2} \sum_{s=0}^{\infty} \frac{1}{(2s)!} (\hat{a}^\mu) = \bar{a}^\mu.
\]

We scale the wave functions \(z_a, \bar{z}_a \sim O(\hbar^{-1/2})\), such that the variance of the operators satisfy a similar relation as in the large spin limit (9.14). This once again allows us to consistently identify the expectation values \(\langle \text{coherent}|\hat{a}^\mu|\text{coherent}\rangle\) with the classical spin variable \(a^\mu\).

We will define the classical amplitude as the scattering of two coherent massive states. This corresponds to taking a weighted infinite sum of the general spin-\(s\) amplitudes,

\[
\mathcal{A}(1, 2, 3^\pm, 4^\pm) = \lim_{\hbar \to 0} e^{-|z|^2} \sum_{s=0}^{\infty} \frac{1}{(2s)!} \mathcal{A}(1^s, 2^s, 3^\pm, 4^\pm).
\]

In practice, we first resum the general spin amplitudes and then expand the result in spin-1/2 variables. Given we will take an \(\hbar \to 0\) limit we can identify \(\bar{a}^{\mu_1} \ldots \bar{a}^{\mu_n} \approx \langle \text{coherent}|\hat{a}^{\mu_1} \ldots \hat{a}^{\mu_n}|\text{coherent}\rangle\) up to irrelevant \(\hbar\) corrections.

For example the classical limit of the same-helicity quartic amplitude is almost trivial

\[
\mathcal{A}(1, 2, 3^+, 4^+) = \lim_{\hbar \to 0} e^{-|z|^2} \sum_{s=0}^{\infty} \frac{1}{(2s)!} m^2[34]^2 t_{13} t_{14} \left(\frac{21}{m}\right)^{2s} e^{q \cdot a}.
\]

In the first line, the amplitudes are resummed and, in the second, expanded in spin-1/2 variables in the second keeping only leading order in \(\hbar\) contributions. There is no ambiguity how to treat \(\hat{a}^2\) terms nor do we need to implement any representation changes.
Note that the coherent state approach requires knowledge of the scattering of general-spin particles. For generic quantum amplitudes, these infinite sums could prove intractable. However, for the amplitudes studied in this paper, we the resummation is trivial. Indeed the amplitudes proposed in paper III and paper IV have relatively simple dependence on the spin number $s$ encoded by the polynomials $P_n^{(k)}$, which resum to rational functions of exponentials. For example,

$$
\sum_{k=0}^{\infty} \frac{1}{k!} P_1^{(k)} = e^{s_1}, \quad \sum_{k=0}^{\infty} \frac{1}{k!} P_2^{(k)} = \frac{e^{s_1}}{s_1 - s_2} + (s_1 \leftrightarrow s_2). \quad (9.23)
$$

### 9.2 Classical constraints for Kerr and $\sqrt{\text{Kerr}}$ amplitudes

In this section we will present our candidate classical Compton amplitudes for $\sqrt{\text{Kerr}}$ and Kerr, which are generated by taking classical limits of the quantum amplitudes presented in section 3.2.1. We use the coherent state classical limit (iii) for ease however, the resulting classical amplitudes are equivalent to those generated by the infinite-spin limit (ii) as shown in paper III.

We will also discuss the combination of higher-spin and classical constraints used to constrain the contact terms. However, we will first introduce the classical functions that will appear in the classical amplitudes.

**Classical functions**

Since the polynomials $P_n^{(k)}$ form a basis of the $\sqrt{\text{Kerr}}$ and Kerr quantum amplitudes, the functional form of the corresponding classical amplitudes is governed by the classical limits of these polynomials. The pole terms of the quantum amplitudes $A_{\sqrt{\text{Kerr}}}$, $M_{\text{Kerr}}$ in eq. (3.36) depend on the three polynomials,

$$
\begin{align*}
\lim_{\hbar \to 0} \sum_{2s=0}^{\infty} \frac{e^{-|z|^2}}{(2s)!} P_1^{(2s)} &= e^{x+z} \\
\lim_{\hbar \to 0} \sum_{2s=0}^{\infty} \frac{e^{-|z|^2}}{(2s)!} P_2^{(2s)} &= e^x \sinh z =: e^x \sinhc z \\
\lim_{\hbar \to 0} \sum_{2s=0}^{\infty} \frac{e^{-|z|^2}}{(2s)!} P_4^{(2s)} &= \frac{2x \cosh y + (x^2+y^2-z^2) \sinhc y}{((x-y)^2-z^2)((x+y)^2-z^2)} + (y \leftrightarrow z) e^x \\
&=: \tilde{E}(x, y, z)
\end{align*}
$$

(9.24)

The resulting classical functions are all entire functions that can be expanded as an infinite tower of spin multipoles. For example, the first
spin-multipole orders of the $\tilde{E}$ are

$$\tilde{E}(x, y, z) = \frac{1}{6} + \frac{1}{12} x + \frac{1}{120} (3x^2 + y^2 + z^2) + \frac{1}{360} x(2x^2 + y^2 + 2z^2) + \mathcal{O}(a^4).$$  

(9.25)

The Compton amplitudes, $A_{\sqrt{\text{Kerr}}}$ and $M_{\text{Kerr}}$ also depend on polynomials with degree shifted by an integer $i$, $P^{(2s)}_n \to P^{(2s+i)}_n$. The classical limit of these shifted polynomials are identical up to a penalty of $(\hbar m)^i(2z)^{-i}$, such that the leading order is suppressed and appears at $\mathcal{O}(\hbar^i)$. Such shifted polynomials can, and do, still contribute classically if they appear with prefactors that have non-trivial $\hbar$ scaling.

The Kerr amplitude in eq. (3.39) also includes limits of higher-order polynomials. Taking a limit of the polynomial $P^{(k)}_{n+1}$ is equivalent to taking derivatives of the polynomial $P^{(k)}_n$, such that

$$\lim_{\varsigma \to \varsigma_1} P^{(2s)}_5 = \frac{\partial}{\partial \varsigma_1} P^{(2s)}_4.$$  

(9.26)

Therefore the resulting classical functions can be related to $\tilde{E}$ by the partial derivatives

$$\lim_{\hbar \to 0} \sum_{2s=0}^{\infty} e^{-|z|^2} \frac{(2s)!}{(2s)!} \left( \frac{\partial}{\partial \varsigma_1} + \frac{\partial}{\partial \varsigma_2} \right) P^{(2s)}_4 = \frac{\partial}{\partial x} \tilde{E}(x, y, z)$$

and

$$\lim_{\hbar \to 0} \sum_{2s=0}^{\infty} e^{-|z|^2} \frac{(2s)!}{(2s)!} \left( \frac{\partial}{\partial \varsigma_1} - \frac{\partial}{\partial \varsigma_2} \right) P^{(2s)}_4 = \frac{\partial}{\partial z} \tilde{E}(x, y, z).$$  

(9.27)

In the majority of terms in the $\sqrt{\text{Kerr}}$ and Kerr, the leading order of the polynomials is the classically significant term. However the electromagnetic and gravity amplitudes contain a pair of terms that are individually divergent but cancel such that the subleading terms are classically relevant. In the gravity amplitude (3.36), these terms are pole terms

$$\frac{(13) [32] [14] [42]}{m^{4s-4} s_{12}} \rho \cdot p \cdot \chi \left( \frac{\rho \cdot \chi}{p \cdot \chi} P^{(2s-1)}_4 - \varsigma_3 \varsigma_4 P^{(2s-2)}_4 \right).$$  

(9.28)

Working to leading order in $\rho \cdot \chi \approx 2z/(p \cdot q_\perp)$, the classical limit is

$$\lim_{\hbar \to 0} \sum_{2s=0}^{\infty} e^{-|z|^2} \frac{1}{(2s)!} \frac{\rho \cdot \chi}{4 p \cdot \chi} \left( \frac{\rho \cdot \chi}{p \cdot \chi} P^{(2s-1)}_4 - \varsigma_3 \varsigma_4 P^{(2s-2)}_4 \right)$$

$$= e^y - e^x \cosh z + \frac{(x-y)e^x \sinh z}{(x-y)^2 - z^2} + (y \to -y)$$

$$=: E(x, y, z),$$  

(9.29)

\footnote{In paper III $\frac{\partial}{\partial x} \tilde{E}(x, y, z)$ is called $\mathcal{E}$ and $\frac{\partial}{\partial z} \tilde{E}(x, y, z)$ is called $\tilde{E}$ but we will refrain from using such notation here.}
where individual $O(h^{-1})$ divergences cancel. The rational function $E$ is once again an entire function and can be related to $\tilde{E}$ by the following differential equation

$$E(x, y, z) = \frac{\partial}{\partial \lambda} \lambda^3 \tilde{E}(\lambda x, \lambda y, \lambda z). \quad (9.30)$$

The full treatment of the term in eq. (9.28) includes taking into account the subleading term in $\rho \cdot \chi \approx 2z/(p \cdot q_\perp) + h(x - w)$, which combines with the super-classical $h^{-1}$ piece of $P_4^{(2s-1)}$ such that the contribution is classical.

### 9.2.1 Candidate classical Compton amplitude for $\sqrt{\text{Kerr}}$

We define our classical amplitude as the classical limit of the quantum $\sqrt{\text{Kerr}}$ amplitude given in eq. (3.36), such that

$$A_{\sqrt{\text{Kerr}}}(1, 2, 3^-, 4^+) = A_0 \left(e^x \cosh z - w e^x \text{sinhc} z + \frac{w^2 - z^2}{2} E(x, y, z)\right), \quad (9.31)$$

where we have implemented either classical limit (ii) or (iii).

Therefore fixing the quantum $\sqrt{\text{Kerr}}$ amplitude is equivalent to fixing the classical amplitude. The quantum amplitude given in eq. (3.36) follows from the chiral higher-spin $\sqrt{\text{Kerr}}$ Lagrangian, given in eq. (6.14) to cubic order. However the resulting general-spin Compton amplitude (6.32) has unfixed contact terms $C^{(s)}$. We now list the combination of higher-spin and classical constraints that are used to fix $C^{(s)}$.

#### Higher-spin constraints

Our study of the non-chiral constructions for spin-2 and spin-3 fields produced a set of constraints on $C^{(s)}$. Notably by a combination of minimal coupling (MC), power counting (PC) and Ward identity (WI) constraints we can constrain $C^{(s<2)} = 0$, while at $C^{(s=2)}$ has is constrained to be a linear combination of three possible contact terms (5.51). The same analysis at spin-3, constrains $C^{(s=3)}$ to be a linear combination of 21 possible contact terms.

#### Classical consistency constraints

While the above higher-spin constraints restrict the space of possible contact terms, we reduce the parameter space further by imposing certain classical consistency constraints.

The first constraint is that $A_{\sqrt{\text{Kerr}}}$ must converge in the classical limit. This is non-trivial given the cubic Lagrangian generates the contact term

$$C^{(s)}_{L_3} = -\frac{\langle 13 \rangle \langle 32 \rangle [14][42]}{m^4} \epsilon_{334} P_4^{(2s-1)} \quad (9.32)$$
which diverges in the classical limit. This divergence appears in both the large spin and coherent spin approaches. Studying $C_{\mathcal{L}_3}^{(s)}$ in the coherent state approach we find the spin-dependent term scales as

$$
\sum_{2s=0}^{\infty} \frac{e^{-|z|^2}}{(2s)!} P_4^{(2s-1)}(x,y,z) = \hbar \frac{1}{2z} \tilde{E}(x,y,z) + \mathcal{O}(\hbar^2),
$$

(9.33)

while the coefficient $\langle 13 \rangle \langle 32 \rangle \langle 42 \rangle \varsigma_3 \varsigma_4 \sim \hbar^{-2}$, such that $C_{\mathcal{L}_3}^{(s)}$ contributes super-classically $\mathcal{O}(\hbar^{-1})$. This suggest we must add further contact terms $C^{(s)}$ in order to generate a classical amplitude with consistent $\mathcal{O}(\hbar^0)$ scaling.

We construct a general spin contact term with the form

$$
C^{(s)} \propto \langle 13 \rangle \langle 32 \rangle \langle 42 \rangle \left[ 14 \right] \left[ 42 \right] \times \left\{ P_4^{(2s-n)}, P_2^{(2s-n)} \right\},
$$

(9.34)

where we have constrained the helicity structure of the massless spinors to match $C_{\mathcal{L}_3}^{(s)}$. Note this is consistent with the massive gauge symmetry constraints at spin-2, given two of the terms in $C_{\mathcal{L}_3}^{(2)}$ exhibit this structure (5.51). We also fix the dependence on $s$ to be encoded in the polynomials $P_{2,4}^{(2s-i)}$ where we allow the degree to be shifted by integer $i$. We will also impose that $C^{(s)}$ respects charge, parity and time-reversal symmetry, such that $A^{\sqrt{\text{Kerr}}}$ contains no dissipation effects.

Even after imposing the constraints above, it is clear there are many possible quantum contact terms of this form. However, the majority of this freedom is washed out in the classical limit such that there is a single free parameter $\delta$ in the classical amplitude

$$
\lim_{\hbar \to 0} \sum_{2s=0}^{\infty} \frac{e^{-|z|^2}}{(2s)!} \left( C_{\mathcal{L}_3}^{(s)} + C^{(s)} \right) = (1 - \delta) \frac{w^2 - z^2}{2} E(x,y,z).
$$

(9.35)

This free parameter $\delta$ enters at quadrupole order $\mathcal{O}(a^2)$ in the multipole expansion of the amplitude. Therefore we can fix it by requiring the quadrupole to be fixed by the finite spin limit of the spin-1 amplitude,

$$
A_{1s=1,2s=1,3^-,4^+}/A_0 \xrightarrow{\text{limit } (i)} 1 + (w - x) + \frac{1}{2} (w - x)^2,
$$

(9.36)

see section 9.1.1. This corresponds to setting $\delta = 0$, such that the resulting classical amplitude, given in eq. (9.31), is unique.

The simplest quantum contact term that satisfies the combined higher-spin and classical constraints is

$$
C^{(s)} = -\frac{\langle 13 \rangle \langle 32 \rangle \langle 42 \rangle}{2m^4} \left( \varsigma_3 + \varsigma_4 \right) \left( P_4^{(2s)} - P_2^{(2s-2)} \right).
$$

(9.37)

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Although one can easily deform the quantum contact term $C(s)$ without impacting the classical amplitude.

The resulting classical amplitude coincides with the classical limit of $\mathcal{A}_{\text{AHH}}$ in eq. (3.34) up to quadrupolar order, $a^2$. As mentioned in section 8.2, we expect $A^\text{Kerr}_{\nu}$ to correspond to the scattering of electromagnetic waves of a Kerr-Newman background induced by the electromagnetic charge of the black hole, although the classical field theory calculation is yet to be done.

### 9.2.2 Candidate classical Compton amplitudes for Kerr

The classical Kerr amplitude is, likewise, defined as the classical limit of the quantum Kerr amplitude, proposed in eq. (3.39),

\[
M^\text{Kerr}(1, 2, 3^-, 4^+) = M_0 \left[ e^x \cosh z - w e^x \frac{w^2 - z^2}{2} E + (w^2 - z^2)(x-w) \tilde{E} - \frac{(w^2 - z^2)^2}{2\xi} \left( \frac{\partial \tilde{E}}{\partial x} + \eta \frac{\partial \tilde{E}}{\partial z} \right) \right] + \alpha C^{(\infty)}_\alpha, \tag{9.38}
\]

where $E = E(x, y, z)$ and $\tilde{E} = \tilde{E}(x, y, z)$ are the entire functions defined in eq. (9.24) and eq. (9.29). The first line of the classical amplitude is generated by the pole structure of the quantum amplitude in eq. (3.39).

The second line of the classical amplitude is generated by the contact term in eq. (9.39),

\[
C^{(s)} = \frac{(13)^2(32)^2[14]^2[42]^2}{2m^6} \varsigma_3 \varsigma_4 \left[ (1+\eta) P_{5|s_1}^{(2s-2)} + (1-\eta) P_{5|s_2}^{(2s-2)} \right]. \tag{9.39}
\]

This contact term is fixed by the following combination of higher-spin and classical constraints.

#### Higher-spin constraints

From the analysis of the non-chiral higher-spin theories, we find that compatibility with massive gauge symmetry requires $C^{(s<3)} = 0$.

We also require the amplitude to have improved high-energy behaviour such that, in the massless limit, $M^\text{Kerr}$ is finite for $s \leq 2$ and $M^\text{Kerr} \sim m^{-4s+4}$ otherwise. Given $P_{5|s_1}^{(2s-2)}$ is degree $2s - 6$ polynomial in the variables $\varsigma_i$, the combination $\varsigma_3 \varsigma_4 P_{5|s_1}^{(2s-2)}$ scales as $m^{-4s+8}$ in the massless limit. Therefore in the massless limit the contact term defined in eq. (9.39) scales as $C^{(s)} \sim m^{-4s+4}$.  

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Classical constraints
In contrast to the $\sqrt{Kerr}$ amplitudes, the Kerr amplitude is automatically well-behaved in the classical limit, therefore it is not a meaningful constraint.

However, in this case we have access to classical data generated by studying linear perturbations of a Kerr black holes, see section 8.2. The classical amplitude $M_{Teuk.}$ computed in ref. [9] contains results up to $a^6$, which can be split into sectors according to their dependency on the book-keeping parameters $\alpha$ and $\eta$.

In analogy to the $\sqrt{Kerr}$ constraint, we impose that the classical hexadecapole $a^4$ is fixed by the classical limit of the $s = 2$ amplitude defined in eq. (3.34). This is a constraint compatible with the results of ref. [9] since the $s = 2$ amplitude matches the first orders of $M_{Teuk.}$ up to order $a^4$.

This constraint is enough to fix the $\eta$-independent contributions of contact term $C^{(s)}$ in eq. (9.39). The contact term $C^{(s)}$ generates an infinite tower of classical multipoles given the classical limit generates the dependence on the entire function $\tilde{E}(x, y, z)$, and it’s derivatives. Remarkably, we find that these classical multipoles correctly predict all the $\alpha = \eta = 0$ terms in $M_{Teuk.}$, which is known up to $a^6$ in ref. [9].

We can also construct quantum contact terms to capture the $\eta$-dependent contact terms in $M_{Teuk.}$. These contact terms are built from the same spinor-helicity variables and polynomials, however, since $\eta$ tags terms are dissipative, the quantum contact terms must break time-reversal symmetry. This corresponds to breaking the symmetry when swapping legs 1 and 2.

We can fix the contact term by fixing the classical $a^5$ multipole given in $M_{Teuk.}$, such that the combined $\alpha$-independent contact term is given in eq. (9.39). Note how the $\eta$-dependent terms are not symmetric under swapping legs 1 $\leftrightarrow$ 2.

Our resulting classical amplitude accurately predicts all given orders of $M_{Teuk}$ up to the $\alpha$-dependent terms [9],

$$M_{Kerr} - M_{Teuk} \bigg|_{\alpha=0} = \mathcal{O}(a^7). \quad (9.40)$$

We can also reproduce the $\alpha$-dependence given an adequate choice of $C^{(s)}_\alpha$ shown in paper IV. However we find that such contributions must be added at each spin multipole order, such that the classical amplitude is not predictive.
10. Summary of Results and Outlook

In Part I we fixed the cubic order EFTs for $\sqrt{\text{Kerr}}$ and Kerr with a series of higher-spin constraints. In the non-chiral construction, the consistency requirements on massive higher-spin theories correspond to imposing massive gauge invariance, which was either at the level of the Lagrangian or through Ward identities. This, combined with restrictive power counting on the non-minimal interactions was sufficient to fix the cubic order and other constraints was enough to fix the quantum three-point amplitudes $A_{\sqrt{\text{Kerr}}}$ and $M_{\text{Kerr}}$, defined in eq. (3.20) and eq. (2.10) uniquely. In the chiral construction, the amplitudes follow from imposing parity invariance on the cubic Lagrangian and not introducing dependence on the self-dual field strength or Riemann spinor.

At quartic order, we found that the higher-spin constraints were not enough to fix the four-point Compton amplitudes uniquely. Therefore, we supplemented them with a set of classical constraints. Uplifting classical constraints to a constraint on the quantum contact terms is only possible if we have well-defined classical limits. Therefore, in section 9, we discussed in detail two consistent formulations of the classical limit, the infinite spin classical limit and the coherent state classical limit. While the two approaches agree, we predominantly use the coherent state limit in this thesis as it is particularly well-suited to amplitudes involving the polynomials $P_n^{(k)}$.

The nature of the classical constraints differs according to whether we consider $\sqrt{\text{Kerr}}$ or Kerr since there is a large discrepancy in the state of knowledge of the classical amplitudes for the $\sqrt{\text{Kerr}}$ and Kerr cases. For $\sqrt{\text{Kerr}}$ there are no explicit four-point results from classical field theory, therefore we only rely on classical consistency requirements to fix the free contact terms in the Compton amplitude. The resulting classical amplitude is unique given the constraints imposed. It would be interesting to compare the result to future classical computations, for example the scattering of electromagnetic waves off a Kerr-Newman black hole.

For Kerr, the classical data up to $a^6$ has been available in the literature for some time [9]. In ref. [9], the classical amplitude is split into conservative terms, dissipative terms and terms generated from the expansion of non-analytic functions, which are bookmarked with $\alpha$. We fix the contact terms corresponding the conservative sector at order $a^4$ and find that they correctly predict the known higher-orders. By introducing
time-reversal breaking contact terms, which we fix at $a^5$, the resulting amplitude predicts the results in ref. [9], up to the terms proportional to $\alpha$.

While we are able to construct an explicit contact term $C^{(s)}_\alpha$ that reproduces them up to $a^6$, it is not clear that the contact term will capture higher order terms correctly. There are still open questions on whether these $\alpha$ terms are physically relevant terms or not given that they are sensitive to the basis of poly-gamma functions used in classical calculation [9, 7].

Nonetheless, our classical Kerr amplitude is a closed form expression that contains all-orders in the spin multipole, such that it can be used to generate new predictions for classical observables, such as the leading impulse and spin-kick and the leading-order waveform.

Alternatively, we could return to the study of the Lagrangians, with the goal of treating the dissipative terms seriously. The Lagrangians studied in this thesis are not able to generate the time reversal breaking contact terms that captured the dissipative physics of the scattering. A possible resolution would be to study a theory describing the interactions of particles with different spin.

We can also extend the study to higher-point amplitudes, which are necessary for corrections to the observables at higher orders in the post-Minkowski expansion. We expect the higher-point classical Compton amplitudes to describe non-linear perturbations of Kerr black holes.
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12. Svensk Sammanfattning


Den statiska lösningen för Kerrs svarta hål är välkänd sedan drygt 60 år tillbaka, men dynamiken hos dessa svarta hål är mycket komplicerad och beskrivs av störningar av metriken som uppfyller Einsteins ickelinjära fältekvationer. Av särskilt intresse är dynamiken hos bundna binära system, dvs. system där två svarta hål kretsar kring varandra i krympande omloppsbantor för att slutligen småta samman till ett större svart hål. Gravitationsvågorna som sänds ut under denna process kan observeras av de markbaserade detektorerna LIGO, Virgo samt KAGRA.

Gravitationsvågorna kodar information om sina astofysikaliska källor och utrönandet av underliggande parametrar är beroende av matchning med teoretiska modeller som håller hög precision. På grund av den allmänna relativitetsteorins ickelinjära natur är det inte möjligt att finna exakta lösningar som är relevanta för observationer. Istället konstrueras teoretiska vågformerna vanligtvis via en kombination av numeriska och analytiska metoder som uppfyller kravet att de är goda approximativa beskrivningar.

Under den tidiga delen av den krympande omloppsbanan är de svarta hålen fortfarande väl separerade och fysiken styrs av svaga växelverkningar. Därför kan vi lösa fältekvationerna med hjälp av störningsteori serieutvecklad i gravitationskonstanten G, som även kallad post-Minkowski (PM) utvecklingen. För att ytterligare förenkla analysen kan man studera en så kallad spridningsamplitud för två förbipasserande svarta hål där gravitationens växelverkan är ett övergående fenomen, i motsats till det bundna systemet där svarta hål ständigt växelverkar. I många fall kan spredningsprocessen matematiskt relateras till den bundna oloppssbanan, till exempel genom att beräkna en effektiv gravitatspotential.
I spridningsscenariot kan vi dra nytta av de många analytiska verktyg som ursprungligen utformades för partikelfysik, genom att modellera svarta hål som elementarpartiklar med stor massa och spinn som beskrivs av en kvantfältteori. Att detta tillvägagångssätt fungerar är anmärkningsvärt med tanke på att vi för närvarande inte har någon komplett beskrivning av gravitationskraften som en kvantmekanisk teori. Kvантfältmetodens framgång bygger på att dess verkan modellerar en effektiv teori som är vid låga energier förenlig med allmänna relativitetsteorin. Vi kan undvika de oändligheter som förekommer i kvantgravitations ofullständiga formuleringar genom att begränsa oss till energiskalor under Planckenergin, vilket är fullt tillräckligt för de svarta hål som observeras genom gravitationsvägdetektorer.


en Comptonamplitud, studeras i detalj och en fullständig formel för alla kvantspinn presenteras.

References


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