Anisotropy of Magnetohydrodynamic and Kinetic Scale Fluctuations through Correlation Tensor in Solar Wind at 0.8 au

Mirko Stumpo 1,†, Simone Benella 1,*,†, Pier Paolo Di Bartolomeo 1, Luca Sorriso-Valvo 2,3 and Tommaso Alberti 4

1 INAF—Istituto di Astrofisica e Planetologia Spaziali, Via del Fosso del Cavaliere 100, 00133 Rome, Italy; mirko.stumpo@inaf.it (M.S.)
2 Istituto per la Scienza e Tecnologia dei Plasmi (ISTP), Consiglio Nazionale delle Ricerche, Via Amendola 122/D, 70126 Bari, Italy
3 Swedish Institute of Space Physics (IRF), Ångström Laboratory, Lägerhyddsvägen 1, 75121 Uppsala, Sweden
4 Istituto Nazionale di Geofisica e Vulcanologia (INGV), Via di Vigna Murata, 605, 00143 Rome, Italy
* Correspondence: simone.benella@inaf.it
† These authors contributed equally to this work.

Abstract: Space plasma turbulence is inherently characterized by anisotropic fluctuations. The generalized k-th order correlation tensor of magnetic field increments allow us to separate the mixed isotropic and anisotropic structure functions from the purely anisotropic ones. In this work, we quantified the relative importance of anisotropic fluctuations in solar wind turbulence using two Alfvénic data samples gathered by the Solar Orbiter at 0.8 astronomical units. The results based on the joined statistics suggest that the anisotropic fluctuations are ubiquitous in solar wind turbulence and persist at kinetic scales. Using the RTN coordinate system, we show that their presence depends on the anisotropic sector under consideration, e.g., the RN and RT sectors exhibit enhanced anisotropy toward kinetic scales, in contrast with the TN. We then study magnetic field fluctuations parallel and perpendicular to the local mean magnetic field separately. We find that perpendicular fluctuations are representative of the global statistics, resembling the typical picture of magnetohydrodynamic turbulence, whereas parallel fluctuations exhibit a scaling law with slope $\sim 1$ for all the joined isotropic and anisotropic components. These results are in agreement with predictions based on the critical balance phenomenology. This topic is potentially of interest for future space missions measuring kinetic and MHD scales simultaneously in a multi-spacecraft configuration.

Keywords: anisotropic turbulence; solar wind; solar orbiter

1. Introduction

Magnetized and weakly collisional space plasmas undergo a great variety of physical processes devoted to scale-to-scale energy transfer, ion/electron excitation and energy dissipation [1–3]. While large-scale plasma dynamics consist of an energy injection from large solar driving and energy turbulent transfers toward smaller scales, small-scale dynamics are affected by the presence of intertwined kinetic processes due to wave–particle interactions, ion energization and magnetic reconnection. The predominant form of energy at large scales manifests as Alfvénic fluctuations in many solar wind streams [4–6]. Such fluctuations are characterized by variations in magnetic and velocity fields perpendicular to the average magnetic field direction [7] and are ubiquitous in both fast and slow solar wind conditions [8–10]. Theories of isotropic magnetohydrodynamic (MHD) turbulence introduced by Iroshnikov [11] and Kraichnan [12] predict an energy spectrum characterized by a typical slope of $-3/2$ for weak Alfvénic turbulence. There is a large consensus of the fact that velocity power spectra in the solar wind at 1 au exhibit scaling in agreement with isotropic MHD turbulence predictions (e.g., [13,14]), while the magnetic field exhibits a $-5/3$ scaling (e.g., [14,15]), which resembles the scaling predicted through isotropic
fluid turbulence theory introduced by Kolmogorov [16]. Recent observations of the radial evolution of magnetic field spectral slopes pointed out that closer to the Sun, it tends toward a Kraichnan-like spectrum [17–21]. However, the above-mentioned theories lie on the isotropy assumption, which is strongly violated in solar wind, since it is widely recognized that the magnetic field orientation is a source of anisotropy in plasma turbulence. Phenomenological theories that include the anisotropy of fluctuations to the direction of the local mean magnetic field, are generally based on the concept of critical balance [22]. This hypothesis consists of the equality between the timescale of the propagating Alfvénic fluctuations and the timescale of their nonlinear dissipation in order to generate an inertial range [22], which results in the wave number anisotropy \( k_\parallel \approx k_{\perp}^{2/3} \). As a consequence, the magnetic field power spectrum perpendicular to the local mean field is characterized by a Kolmogorov-like scaling, i.e., \( E(k_{\perp}) \sim k_{\perp}^{-5/3} \), while the parallel spectrum turns out to be steeper, i.e., \( E(k_{\parallel}) \sim k_{\parallel}^{-2} \), where \( k \) denotes the wave number. Several observations of solar wind turbulence support these predictions [15,23–26].

In magnetized plasma, purely anisotropic turbulent fluctuations can be disentangled from the bulk of magnetic field fluctuations by resorting to the full correlation tensor. This idea was first introduced in the field of hydrodynamic turbulence and consists of decomposing the field correlations on a set of eigenfunctions of the SO(3) group of rotation. Such a computation relies on the full knowledge of three-dimensional fields, and it was applied on both experiments and simulations of neutral fluids (e.g., see [27]), as well as in magnetized plasma [28,29]. In situ data collected from single spacecraft inherently lack the necessary three-dimensional scope for a comprehensive SO(3) analysis, which necessitates a complete field representation across a three-dimensional volume. Nonetheless, based on mathematical and geometrical considerations, it is possible to separate a set of purely anisotropic field correlations from the undecomposed ones [30,31]. The aim of this work was to use high-cadence Solar Orbiter (SolO) magnetic field observations to quantify the importance of such anisotropic fluctuations in the Alfvénic solar wind. We analyzed their amplitude and their statistical properties, such as intermittency, and finally, we investigated the scaling properties of undecomposed and purely anisotropic fluctuations for the cases parallel and perpendicular to the local mean magnetic field.

2. Methods

Magnetic field fluctuations in solar wind are strongly influenced by the presence of a strong mean magnetic field pointing inward/outward from the Sun. In denoting the mean field with \( B_0 \), the total magnetic field vector observed at a given time can be written as

\[
B(t) = B_0 + \delta B(t)
\]

where \( \delta B(t) \) is a magnetic field fluctuation. Throughout this work, we resort to the correlation tensor approach, first introduced by Bigazzi et al. [30] in the context of space plasma. By combining magnetic field components, we can define a set of multiscale correlation tensors such as

\[
S^{(n)}_{\alpha_1 \ldots \alpha_n}(r) = \langle \delta_r B_{\alpha_1} \ldots \delta_r B_{\alpha_n} \rangle,
\]

where \( \delta_r B_{\alpha_i} = B_{\alpha_i}(x + r) - B_{\alpha_i}(x) \) is the increment of the magnetic field component \( \alpha_i \) on the scale separation vector \( r \). The functional dependence of the general correlation tensor reported in Equation (2) is on the separation vector \( r \) only, since the average is performed over all the locations \( x \) under the assumption of homogeneity. The condition of isotropy, indeed, has not been invoked yet, and the correlation described by Equation (2) comprises both isotropic and anisotropic contributions. Therefore, we can write the correlation tensor as the sum of the isotropic and anisotropic components as follows:

\[
S^{(n)}_{\alpha_1 \ldots \alpha_n}(r) = S^{(n),\text{iso}}_{\alpha_1 \ldots \alpha_n}(r) + S^{(n),\text{aniso}}_{\alpha_1 \ldots \alpha_n}(r)
\]
By setting \( n = 2, \alpha_1 = \alpha \) and \( \alpha_2 = \beta \) in Equation (2), for the isotropic part of the second-order structure function, we obtain

\[
S_{\alpha\beta}^{(2), \text{iso}} = f(r)\delta_{\alpha\beta} + g(r) r_{\alpha} r_{\beta},
\]

where \( f \) and \( g \) are two scalar functions of the separation amplitude \( r = |r| \), and \( \delta_{\alpha\beta} \) is the Kronecker delta. Similar, albeit more complex, expressions can be obtained for higher-order structure functions, as discussed in [30].

In this work, we were interested in assessing the roles of the isotropic and anisotropic fluctuations of the magnetic field and using high-cadence measurements to quantify their relative importance also at kinetic scales. Based on single-spacecraft measurements, we can take advantage of the natural reference frame offered by the radially expanding solar wind satisfying the Taylor hypothesis to take the separation vector aligned with the solar wind velocity vector, which, in essence, corresponds to the radial direction \( r = (r_R, 0, 0) \). In fact, under the conditions \( n = 2 \) and \( \alpha \neq \beta \), the isotropic part of correlation tensor (4) vanishes, thus leading to a set of three purely anisotropic structure functions. For even \( n \)-th order structure functions, it is necessary to consider odd powers of magnetic field increments in two directions to obtain a purely anisotropic structure function, viz.,

\[
S_{\alpha\beta}^{(n), \text{aniso}}(r_R) = \langle \delta_{r_R} B^p_{\alpha} \delta_{r_R} B^q_{\beta} \rangle,
\]

where \( p \) and \( q \) are odd, and \( p + q = n \). Purely anisotropic and undecomposed components of the correlation tensor are projections of the same fluctuation field in different sectors; hence, they are not independent of each other. We remark that, while it is possible to define a set of isotropic fluctuations, a completely anisotropic fluctuation field cannot exist.

The theoretical framework introduced is rather general and can, in principle, provide important quantitative information about fundamental characteristic of magnetic field fluctuations in the solar wind such as the amplitude of purely anisotropic fluctuations and intermittency. On the other hand, a conventional framework for studying anisotropy in plasma turbulence and kinetic dynamics relies on the definition of the local mean magnetic field [24–26,32,33]. In invoking the Taylor hypothesis [34], time-scale separation yields direct information about spatial scale separation \( r \) through the relation \( r = (r_R, 0, 0) \), where \( V_0 \) indicates the bulk solar wind velocity vector. This hypothesis is generally assumed to be valid with solar wind since the bulk medium speed moving past the spacecraft is usually several times larger than the velocity at which perturbations propagate in a magnetized plasma, i.e., the Alfvén speed. Hence, temporal fluctuations observed by a single spacecraft reflect, with good approximation, spatial variations [35,36]. From a single-spacecraft perspective, the radially expanding solar wind is sampled along the radial direction \( R \), thus setting the spatial separation direction as \( r = (r_R, 0, 0) \) in the increment calculation. Information about the local mean magnetic field angle can be useful for investigating the amplitude of purely anisotropic structure functions with respect to the diagonal terms of the correlation tensor, which comprise both isotropic and anisotropic contributions.

3. Data

In this work, we aimed to investigate both MHD and kinetic fluctuations of the interplanetary magnetic field. To this purpose, we used high-resolution observations of bulk plasma parameters and magnetic field gathered by the SolO mission at a heliocentric distance of 0.8 au. We focused on two independent Alfvénic data samples gathered at the same heliocentric distance: the first one is a 13 h dataset starting on 13 May 2022 at 15:00 UT, and the second one is 23 h long and starts on 1 December 2022 at 19:00 UT. An overview of the data is shown in Figure 1. Magnetic field data are provided with a time resolution of 64 samples/s using the Magnetometer instrument in burst mode with an optimal signal-to-noise ratio [37]. The average bulk parameters were measured with a Proton and Alpha Particle Sensor, which is one of the sensors of the Solar Wind Analyser plasma suite [38].
and the parameters are reported in Table 1 for the selected period. As highlighted in Table 1, Taylor’s hypothesis can be reasonably assumed in our analysis since the bulk solar wind speed is approximately 10 times the Alfvén speed in both intervals. Hence, in the following, we associate the spatial scale $r$ to the time scale $\tau$ using the relation $r = V_0 \tau$, where $V_0$ indicates the average bulk flow speed between the samples: $V_0 = 536 \text{ km s}^{-1}$. Both data samples exhibit clear Alfvénic fluctuations as pointed out by the superimposed velocity and magnetic field components in Figure 1. We computed the correlation between the normal components $V_N$ and $B_N$ of the velocity and magnetic fields, and on average, we obtained $C_{VB} = 0.93$ for the 2022 May data sample and $C_{VB} = -0.91$ for the 2022 December interval.

Table 1. Average magnetic field and bulk plasma parameters.

<table>
<thead>
<tr>
<th></th>
<th>13 May 2022, 15:00–14, 04:00</th>
<th>1 December 2022, 19:00–2, 18:00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_0$ (nT)</td>
<td>10.1</td>
<td>5.2</td>
</tr>
<tr>
<td>$B_{\text{max}}$ (nT)</td>
<td>14.3</td>
<td>5.8</td>
</tr>
<tr>
<td>$V_0$ (km s$^{-1}$)</td>
<td>567</td>
<td>504</td>
</tr>
<tr>
<td>$V_{\text{max}}$ (km s$^{-1}$)</td>
<td>707</td>
<td>611</td>
</tr>
<tr>
<td>$\rho_{p}$ (cm$^{-3}$)</td>
<td>11.8</td>
<td>4.7</td>
</tr>
<tr>
<td>$T_p$ (10$^5$ K)</td>
<td>2.3</td>
<td>1.5</td>
</tr>
<tr>
<td>$d_p$ (km)</td>
<td>66</td>
<td>104</td>
</tr>
<tr>
<td>$\rho_p$ (km)</td>
<td>45</td>
<td>69</td>
</tr>
<tr>
<td>$V_A$ (km$^{-1}$)</td>
<td>65</td>
<td>53</td>
</tr>
<tr>
<td>$C_{VB}$</td>
<td>0.93</td>
<td>-0.91</td>
</tr>
</tbody>
</table>

Figure 1. SoO data for the two Alfvénic solar wind intervals analyzed in this work: the 13–14 May 2022 data sample (left) and the 1–2 December 2022 data sample (right). Panels depict the magnetic field (blue) and solar wind bulk velocity (red) $R$, $T$ and $N$ components from top to bottom.

4. Results

4.1. Global Correlation Tensor Analysis

The strength of the anisotropic magnetic field fluctuations was assessed by calculating the second-order structure functions, shown in Figure 2 (left panel). To increase the statistics, we joined the independent sets of magnetic field increments $\delta r \delta B_\alpha$ obtained from the two data samples. Thanks to the high-resolution measurements provided by the MAG instrument, the statistical error associated with the estimation of the structure functions lies within the symbols for the global time interval of 36 h. The diagonal terms $S_{\alpha\alpha}^{(2)}$ exhibit a similar trend resembling the typical scaling $r^{1/2}$ predicted through the Iroshnikov–Kraichnan phenomenology of MHD turbulence [11,12]. Magnetic field fluctuations, especially during
Alfvénic intervals, develop mostly in the direction perpendicular to the mean field; then, we observed that the highest power is associated with increments in the $N$-direction, whereas the lowest was observed on the radial increments. As stated in Section 2, these structure functions enclose both isotropic and anisotropic fluctuations. To estimate the strength of purely anisotropic fluctuations, we calculated the mixed second-order structure functions $S_{\alpha\beta}^{(2)}$, $\alpha \neq \beta$. For $\alpha = R$ and $\beta = T$, we obtained a scaling that resembles the one observed in the diagonal terms, with a slightly steeper slope and, in general, values that are lower by approximately one order of magnitude. The same was observed in the cases $\alpha = R$ and $\beta = N$, though with a break in the power law below $r \sim 10^4$ km, whereas the term perpendicular to the radial direction, viz., $\alpha = T$ and $\beta = N$, displays an interruption in the power law trend due to the sign inversion of the odd power of magnetic field increments for scales $< 10^4$ km. Our results suggest that purely anisotropic correlations have a smaller amplitude with respect to the diagonal terms of the correlation tensor, indicating that the isotropic contribution plays an important role. We also note that the purely anisotropic correlations decay slightly faster toward small scales, i.e., isotropic fluctuations may be dominant at kinetic scales below the proton inertial length. These results are in agreement with previous findings based on Ulysses [30], Helios-2 [31] and Cluster [39,40]. Nonetheless, in order to evaluate the strength of the anisotropic fluctuations, which may be rare and strong, thus affecting high-order statistics, we computed the fourth-order diagonal correlations $S_{\alpha\alpha\alpha\alpha}^{(4)}$ and purely anisotropic fourth-order terms $S_{\alpha\beta\beta\beta}^{(4)}$, shown in Figure 2 (right panel). In this case, we note that the strength of the anisotropic fluctuations is comparable to that of the diagonal terms of the correlation tensor; see, e.g., $S_{RNNN}^{(4)}$ compared to the radial term $S_{RRRR}^{(4)}$. Hence, high-order statistics confirm the fundamental role played by anisotropic fluctuations in solar wind both at MHD and kinetic scales [30].

![Figure 2](image.png)

**Figure 2.** (Left): Second-order structure functions. The diagonal terms of the correlation tensor are represented by the components $S_{\alpha\alpha}^{(2)}$ with $\alpha = R$ in dark red, $\alpha = T$ in red and $\alpha = N$ in yellow. The mixed, purely anisotropic terms $S_{\alpha\beta}^{(2)}$ are indicated by blue ($\alpha = R, \beta = T$), cyan ($\alpha = R, \beta = N$) and purple ($\alpha = T, \beta = N$) circles. The straight line indicates the K41 slope $r^{1/2}$. (Right): Fourth-order structure functions. The diagonal terms of the correlation tensor are represented by the component $S_{\alpha\alpha\alpha\alpha}^{(4)}$ with $\alpha = R$ in dark red, $\alpha = T$ in red and $\alpha = N$ in yellow. The mixed, purely anisotropic terms $S_{\alpha\beta\beta\beta}^{(4)}$ are indicated by blue ($\alpha = R, \beta = T$), cyan ($\alpha = R, \beta = N$) and purple ($\alpha = T, \beta = N$) circles. In both panels, vertical solid lines indicate the proton inertial lengths of the two data samples: the 2022 May interval in black and the 2022 December in gray.

It is widely recognized that both magnetic and velocity fields in solar wind exhibit strong intermittency in their fluctuations, which represents an additional degree of complexity. Structure functions generally exhibit a power law behavior as a function of the scale, viz.,

$$S_{\alpha}^{(n)}(r) \sim r^p.$$  

(6)
Theories of homogeneous and isotropic turbulence predict a linear law for the scaling exponent as a function of the order of the structure functions, $\zeta_n \propto n$. The deviation from this linear trend is referred to as anomalous scaling, and it is the signature of small-scale intermittency [41]. A standard procedure for analyzing intermittency/anomalous scaling is the inspection of the kurtosis as a functions of the scale. Here, we define the kurtosis starting from the diagonal terms of the correlation tensor as

$$K_\alpha^{(4)}(r) = \frac{S_{\alpha\alpha\alpha\alpha}(r)}{[S_{\alpha\alpha}(r)]^2},$$

(7)

which corresponds to the kurtosis of the longitudinal (transverse) structure functions for $\alpha = R$ ($\alpha = T, N$). The quantity in Equation (7) provides information about the intermittency associated with fluctuations comprising both isotropic and anisotropic contributions. It is usually observed in the turbulent regime that this quantity scales as a power law, viz., $K_\alpha^{(4)}(r) \sim r^{\kappa_\alpha}$, where $\kappa_\alpha = \zeta_4 - 2\zeta_2$. Following Bigazzi et al. [30], we introduce the purely anisotropic kurtosis based on the definition of a mixed high-order structure functions as follows:

$$K_{\alpha\beta}^{(4),\text{aniso}}(r) = \frac{S_{\alpha\beta\beta\beta}(r)}{[S_{\alpha\beta}(r)]^2}.$$  

(8)

Also in this case, we aimed for a scaling dependence of purely anisotropic kurtosis by introducing the mixed-term scaling exponent $\kappa_{\alpha\beta}$ such that $K_{\alpha\beta}^{(4)}(r) \sim r^{\kappa_{\alpha\beta}}$. The values of the kurtosis obtained from SolO data samples considered in this work are reported in Figure 3. In the left panel, we plotted the values of the kurtosis obtained from the diagonal terms of the correlation tensor. We highlight that they exhibit a similar trend on the three components, with a well-defined slope at MHD scales. Subsequently, a maximum is reached at scales compatible with the typical transition region observed between the MHD and kinetic scales [42,43]. The fitting power law exponents, listed in Table 2, are consistent with previous observations in the solar wind (e.g., see [44]). On the contrary, by computing the purely anisotropic kurtosis, we found stronger intermittency to be associated with anisotropic fluctuations (see the values of the scaling exponents $\kappa_{\alpha\beta}$ reported in Table 2). These scaling exponents were fitted at scales larger than $10^4$ km, but sometimes, they became even steeper at smaller scales. Note that the values of the anisotropic kurtosis at all scales are much larger than the ordinary kurtosis, since in this case, the fourth-order moment was normalized to the square of a signed second-order moment, which can be very small.

![Figure 3](image-url)

**Figure 3.** (Left): Kurtosis $K_\alpha^{(4)}$ as a function of the time scale $r$ for $\alpha = R$ (dark red), $\alpha = T$ (red) and $\alpha = N$ (yellow) components. The horizontal solid line indicates the Gaussian reference value of 3. (Right): Purely anisotropic kurtosis $K_{\alpha\beta}^{(4)}$ indicated by blue ($\alpha = R$, $\beta = T$), cyan ($\alpha = R$, $\beta = N$) and purple ($\alpha = T$, $\beta = N$) circles. For both panels, the slopes represented by solid lines are listed in Table 2. Vertical solid lines indicates the proton inertial lengths of the two data samples: the 2022 May interval in black and the 2022 December in gray.
Table 2. Scaling exponent of undecomposed \((\kappa_{aa})\) and purely anisotropic \((\kappa_{a\beta})\) kurtosis.

<table>
<thead>
<tr>
<th>(\kappa_{RR})</th>
<th>(\kappa_{TT})</th>
<th>(\kappa_{NN})</th>
<th>(\kappa_{RT})</th>
<th>(\kappa_{RN})</th>
<th>(\kappa_{TN})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-0.31 \pm 0.01)</td>
<td>(-0.28 \pm 0.02)</td>
<td>(-0.31 \pm 0.02)</td>
<td>(-0.58 \pm 0.02)</td>
<td>(-0.74 \pm 0.12)</td>
<td>(-1.19 \pm 0.11)</td>
</tr>
</tbody>
</table>

The above results suggest that the intermittency of both undecomposed and purely anisotropic fluctuations tend to increase toward small scales and indicate that anisotropic ones are the most intermittent. On the other hand, undecomposed structure functions are shown to be larger than purely anisotropic ones at both MHD and kinetic scales. By introducing the ratio between the \(n\)-the order structure function and the \(n/2\) power of the second-order one \([40,45,46]\), viz.,

\[
F^{(4)}_{a\beta}(r) = \frac{S^{(4)}_{a\beta\beta\beta}(r)}{[S^{(2)}_{aa}(r)]^2}, \quad F^{(6)}_{a\beta}(r) = \frac{S^{(6)}_{aaa\beta\beta}(r)}{[S^{(2)}_{aa}(r)]^3}, \tag{9}
\]

we aimed to assess the relative importance of undecomposed and purely anisotropic fluctuations. The top panels of Figure 4 report the fourth- and sixth-order functions \(F^{(n)}_{a\beta}(r)\) calculated on SolO data in the three anisotropic sectors. The anisotropic sector \(RN\) is characterized by the stronger anisotropy in magnetic field fluctuations, as highlighted by the maximum reached in the vicinity of the proton inertial length. Conversely, fourth- and sixth-order ratios estimated in the \(TN\) anisotropic sector present a tendency toward isotropization, i.e., they exhibit a decreasing trend at scales \(<10^3\) km, though the correct sign convergence of the purely anisotropic structure functions is not achieved at scales comparable to \(d_p\). The anisotropic sector \(RT\) exhibits intermediate behavior.

Other useful quantities that allow us to measure the relative strength of anisotropic fluctuations with respect to the undecomposed ones are the generalized flatness and the hyperflatness (indicating the normalized sixth-order moment):

\[
G^{(4)}_{a\beta}(r) = \frac{S^{(4)}_{a\beta\beta\beta}(r)}{S^{(4)}_{aa\beta\beta}(r)}, \quad G^{(6)}_{a\beta}(r) = \frac{S^{(6)}_{aaa\beta\beta}(r)}{S^{(6)}_{aa\alpha\alpha\alpha}(r)}. \tag{10}
\]

In a perfectly isotropic fluctuation field, the quantities defined in (9) and (10) vanish. The generalized flatness \(G^{(n)}_{a\beta}\) differs from the previous one for the normalization factor, which is here represented by a structure function of the same order of the one in the numerator. Thus, the difference observed between them reflects the rare and burst-like character of the anisotropic fluctuations, responsible for the high numerical values of \(F^{(n)}_{a\beta}\) reached in Figure 4. Previous studies have shown how these quantities in solar wind tend to decrease when approaching the proton inertial length, indicating that the isotropic contribution is the leading one at small scales \([30]\). In using near-Earth observations gathered using the Cluster spacecraft \([40]\), the quantity of Equation (10) tends to increase for different paired components toward the kinetic range. In this work, we confirmed this tendency on SolO data, as shown in the bottom panels of Figure 4. In fact, by considering \(\alpha = R\) and \(\beta = N\), we obtained a trend \(<1\) at MHD scales slightly increasing toward \(d_p\). Similar to the previous case, the anisotropic sector \(TN\) shows a tendency toward isotropization at small scales, whereas the sector \(RT\) displays an intermediate trend. The essence of these results is that the most isotropic content is observed in the plane perpendicular to the sampling direction \(R\), suggesting that the principal source of anisotropic fluctuations is represented by the anisotropic sectors coupling radial and transverse directions, i.e., \(RT\) and \(RN\).
4.2. Correlation Tensor Analysis with Respect to the Local Mean Magnetic Field

As discussed in Section 4.1, we used the full correlation tensor to obtain information about sectors where anisotropic fluctuations tend to principally appear in solar wind turbulence. The procedure has been applied to RTN magnetic field measurements, without considering the role of the local mean magnetic field in the generation of anisotropy. In the following, we present an analysis of the undecomposed and purely anisotropic contributions of the correlation tensor, taking into account the local direction of the mean magnetic field. The aim of this study was to observe how the magnitude and slope of undecomposed and anisotropic fluctuations change with respect to the local mean field.

Since the solar wind is radially expanding, the sampling direction of the longitudinal increment is the radial direction. The local mean magnetic field vector can be defined as follows [47]:

\[
B_{\text{loc}}(x,r) = \frac{B(x) + B(x+r)}{2},
\]

this being a function of both position \(x\) and scale separation vector \(r\). Note that the use of the local mean field for estimating anisotropic structure functions is largely popular, although some caveats should be considered. For example, for the local mean field, being itself a scale-dependent quantity, the order of the structure functions may be altered (see, e.g., [48,49]). By computing the angle \(\theta_B\) between the local mean field vector \(B_{\text{loc}}\) and the sampling (radial) direction at a fixed scale, we obtain the probability distributions reported in Figure 5 for the two solar wind data samples considered in this work. We set \(r = 17\) km for visualization purposes, although the distribution exhibits a weak dependence on the scale, especially toward large values of \(r\). The PDF associated with the first sample (blue) shows a peak between 30° and 60°, which is compatible with the angle 35° expected from the nominal Parker spiral at 0.8 au. The same argument is valid for the second data sample, likewise at 0.8 au, showing a peak in the PDF of \(\theta_B\) between 125° and 150°, which is in agreement with the angle 141° of an inward Parker spiral. Angles derived from the nominal Parker spiral are indicated by vertical lines in Figure 5 and were calculated as \(\tan^{-1}(r\Omega_\odot/V_0)\), where \(r\) is the heliocentric distance, \(\Omega_\odot = 2.86 \times 10^{-6}\) rad s\(^{-1}\) is the...
angular velocity of the solar synodic rotation, and \(V_0\) is the bulk solar wind velocity reported in Table 1.

\[
r(x, r) = 17 \text{ km}
\]

Figure 5. Probability distribution of the angle \(\theta_B\) for the two data samples analyzed in this study: 2022 May (blue) and 2022 December (red) at a fixed scale of \(r = 17\) km. Vertical dashed lines indicate the nominal Parker spiral angle to the radial direction for both time intervals.

In order to investigate the fluctuation anisotropy in a three-dimensional space using single-spacecraft observations, we introduce a spherical reference frame that takes into account the sampling direction, the local mean magnetic field and the magnetic field fluctuations \([50, 51]\). In addition to \(\theta_B(x, r)\), we introduce the direction of the local perpendicular fluctuations as \(B_{\text{loc}} \times (\delta_r B \times B_{\text{loc}})\), and then we define the angle \(\phi_{\delta B \perp}(x, r)\) between the local perpendicular fluctuation and the component of the sampling direction vector perpendicular to the local mean field. According to Chen et al. \([50]\), we can classify magnetic field fluctuations as parallel if the local mean field is aligned with the sampling direction, \(\theta_B = 0^\circ\), and the local perpendicular fluctuation forms the angle \(\phi_{\delta B \perp} = 90^\circ\) with the perpendicular component of the sampling direction, and perpendicular if the angle between the local mean field and the sampling direction is \(\theta_B = 90^\circ\) with \(\phi_{\delta B \perp} = 90^\circ\).

We chose a \(10^\circ\) angle bin width, and we consider fluctuations in the intervals \(0^\circ < \theta_B < 10^\circ\) and \(0^\circ < \phi_{\delta B \perp} < 90^\circ\) to be parallel and fluctuations in the intervals \(80^\circ < \theta_B < 90^\circ\) and \(80^\circ < \phi_{\delta B \perp} < 90^\circ\) to be perpendicular. Note that all angles \(> 90^\circ\) were remapped in the interval \([0^\circ, 90^\circ]\). We then computed, for each angle bin, all the components of the correlation tensor as a function of the scale, as shown in Figure 6. The action of conditioning magnetic field fluctuations in angle bins decreases the statistics, so it is not always possible to observe a clean power law for purely anisotropic correlations. This statistical argument explains, for instance, why some purely anisotropic components of the correlation tensor of Figure 6 appear with a negative sign. However, interesting considerations can be drawn by examining the relative amplitude of purely anisotropic to undecomposed correlations, and the slope of the power laws observed for parallel and perpendicular fluctuations. For parallel fluctuations, the power of undecomposed structure functions is approximately the same in the kinetic range. When moving toward MHD scales, the transverse structure functions acquire higher levels than the longitudinal ones, which turn out to have magnitudes comparable to those of purely anisotropic sectors. This result confirms the well-established property of anisotropic MHD turbulence, where large fluctuations develop mostly in the direction perpendicular to the local mean field. This feature is also confirmed in the perpendicular case (see Figure 6, right panel). An interesting result is the clear tendency of both undecomposed and purely anisotropic components to follow the same power laws, which are different for parallel and perpendicular fluctuations. Perpendicular fluctuations exhibit the typical trend predicted by the Kolmogorov theory \([16]\), viz., \(S^{(2)}_{\perp, \parallel} \sim r^{2/3}\), at MHD scales, with a steepening below the proton inertial length. Conversely, parallel fluctuations
show a steeper slope in the MHD range, viz., $S_{\alpha\beta}^{(2)}_{\text{par}}(r) \sim r$, which remains consistent also toward kinetic scales. Moreover, the slope associated with the longitudinal undecomposed structure function in this case seems to have a slope $< 1$. A scenario characterized by the scaling exponents of the second-order structure function of 1 for parallel fluctuations and $2/3$ for the perpendicular case is associated with power spectral density slopes of $-2$ and $-5/3$, respectively, thus resembling the typical predictions of strong MHD turbulence under the critical balance condition [22,52–54]. Using the correlation tensor, we observe that such scaling laws are not a prerogative of longitudinal fluctuations but extend also to transverse undecomposed structure functions and purely anisotropic fluctuations.

**Figure 6. (Left):** Second-order structure functions $S_{\alpha\alpha}^{(2)}$ and purely anisotropic structure functions $S_{\alpha\beta}^{(2)}$ for magnetic field increments sampled on a direction nearly parallel to the local mean magnetic field, $0^\circ < \theta_B < 10^\circ$ and $0^\circ < \phi_{\delta B_\perp} < 90^\circ$. The slope of 1 is also illustrated. **(Right):** Second-order structure functions $S_{aa}^{(2)}$ and purely anisotropic structure functions $S_{\alpha\beta}^{(2)}$ for magnetic field increments sampled on a direction nearly perpendicular to the local mean magnetic field, $80^\circ < \theta_B < 90^\circ$ and $80^\circ < \phi_{\delta B_\perp} < 90^\circ$, along with the $2/3$ slope. In both panels, vertical solid lines indicate the proton inertial lengths of the two data samples: the 2022 May interval in black and the 2022 December in gray.

### 5. Discussion and Conclusions

Anisotropic fluctuations play a crucial role in solar wind turbulence. By using the representations of the SO(3) group of rotations, it is possible to inspect the full correlation tensor of magnetic field fluctuations to disentangle their anisotropic content [30,31,55,56]. A complete separation in isotropic and anisotropic fluctuations can be achieved only in numerical simulations where the knowledge of the three-dimensional structure of the fluctuation field is complete. For the free solar wind, we exploit the Taylor hypothesis ($r = V_0 \tau$) and, therefore, the direction of the expanding flow to fix the sampling direction and then separate the diagonal undecomposed terms, containing isotropic and anisotropic fluctuations, from the off-diagonal ones, comprising purely anisotropic fluctuations. The main results of the present study, based on SolO observations of Alfvénic solar wind at 0.8 au, can be summarized as follows:

- By comparing undecomposed and purely anisotropic MHD-scale fluctuations, we observed that by increasing the order of the structure functions, the purely anisotropic components tend to be comparable to the undecomposed ones. This indicates that anisotropic fluctuations mostly appear as rare bursty events, which develop in a strong intermittent way [30]. This is also evident from the trend observed in the purely anisotropic kurtosis.

- Anisotropic fluctuations persist at kinetic scales, as highlighted through a higher-order structure function analysis. This result is in agreement with previous observations by Bigazzi et al. [30] and Sorriso-Valvo et al. [31].
• Using the generalized flatness, we observed that the most important anisotropic contribution at small scales is associated with sectors combining the sampling direction with the transverse plane, viz., $RT$ and $RN$. Anisotropic fluctuations in sector $TN$ tend toward a more isotropic state toward kinetic scales.

• By introducing a spherical coordinate system formed by the sampling direction, the local mean magnetic field angle $\theta_B$ and the angle of the local perpendicular fluctuation $\phi_{\delta B \perp}$, we can separate parallel and perpendicular fluctuations to the local mean magnetic field. In the case of parallel fluctuations, through the second-order correlation tensor, we observed that turbulence tends to develop larger perpendicular fluctuations with respect to the parallel ones and that both undecomposed ad purely anisotropic component share the same scaling law, resembling a spectral slope of $-2$. Such scaling is almost unchanged at kinetic scales. In the case of perpendicular fluctuations, again, we computed the second-order correlation tensor, and we observed that perpendicular fluctuations are more developed than parallel ones. All diagonal and mixed structure functions follow the same scaling law, showing a Kolmogorov-like spectrum $-5/3$ at MHD scales and a steeper slope in the kinetic range.

In a previous work, Sorriso-Valvo et al. [31] studied the radial evolution of the scaling laws associated with the SO(3) correlation tensor in the inner Heliosphere by means of Helios 2 observations. Values reported for the scaling exponent of the kurtosis in the case of fast solar wind at a distance of 0.9 au, thus comparable with the SolO data samples considered in the present work, are well in agreement for the diagonal terms, which are $\kappa_{xx} = [-0.34 \pm 0.04]$ for Helios 2 observations. The scaling exponent associated with the purely anisotropic components is slightly higher, viz., $\kappa_{xZ} = [-0.45 \pm 0.11]$, confirming the trend found in the present study. However, the values obtained from SolO are larger than those calculated from Helios 2, which is representative of an enhanced intermittency of the anisotropic fluctuations present in our data samples.

The investigation of anisotropy in plasma turbulence depends on the reliable estimation of the spatial structure of magnetic field fluctuations with respect to the local mean magnetic field orientation. In this work, we took advantage of Taylor’s hypothesis and of the radial sampling direction to obtain information about the spatial scale dynamics of undecomposed and purely anisotropic correlations. However, the optimal setup used to precisely achieve these observations is provided by a multi-point spacecraft configuration [57,58]. The importance of the local mean magnetic field as a source of anisotropy has been deeply emphasized during recent decades. Multi-spacecraft studies on magnetic field anisotropy based on the cluster constellation have been carried out to investigate both local mean field anisotropy, [15,47], and have been complemented by anisotropic turbulence studies on both solar wind and magnetosheath using magnetospheric multiscale observations [42,59]. Comparisons between single-spacecraft and multi-spacecraft estimations of the SO(3) correlation tensor have been carried out using cluster data by Yordanova et al. [40], pointing out some differences between them, especially toward kinetic scales. Exploring the SO(3) correlation tensor from a multi-point perspective, including the effect of the local mean field on the scaling of purely anisotropic fluctuations, is left as challenging future research. This topic would potentially be of interest for future space missions, such as the proposed Plasma Observatory of the European Space Agency, which will allow for simultaneous measurements of kinetic and fluid scales in a multi-spacecraft configuration in diverse near-Earth plasma environments [60,61].

The main result of this work consists of showing the influence of the local mean field on a full correlation tensor. Undecomposed and purely anisotropic structure functions seem to share the same scaling laws at both MHD and kinetic scales. Such scaling laws are in agreement with predictions based on the critical balance phenomenology and are valid also for purely anisotropic fluctuations, a fact that could potentially stimulate new theoretical research on this topic.
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Abbreviations
The following abbreviations are used in this manuscript:

- MHD: Magnetohydrodynamics
- PDF: Probability Distribution Function
- RTN: Radial–Tangential–Normal
- SolO: Solar Orbiter
- UT: Universal Time

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