High Accuracy Computations of the Hybrid Burnett Equations

Twana Ibrahim
Abstract

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A non-dimensional quantity that is used to describe fluid flows is the Knudsen numbers $Kn$. $Kn = \frac{l}{L}$ where $l$ is the mean free path of the particles and $L$ a typical length scale. For Knudsen number much smaller than 1 the particles can be described as a continuum. For these applications, Navier-Stokes equations are sufficient to describe the flow. For larger Knudsen values the second order Burnett equations can be used. Doing a linear stability analysis on the Burnett equations it can be shown that the equations are not stable for all wave numbers. A method of stabilizing the equations has been suggested by Lars Söderholm, see (Söderholm, 2006). This results in the Hybrid Burnett equations. For the purpose of investigating the ability for the Hybrid Burnett equations to describe physical flows, the phenomenon of shock waves will be numerically treated for different cases.

A high amplitude sinusoidal wave will be integrated forward in time using Runge-Kutta fourth order scheme as it travels along the x-axis. Fourier's method will be used for spatial discretization. This wave will with time form a shock wave. The high gradient in the wave profile will result in a reflected wave that is traveling in the opposite direction of the original wave. As the amplitude of the original wave and the self reflected wave decay we will have two small amplitude waves where non-linear effects can be neglected. From this old age wave the amplitude will be compared with the analytical solutions of the Burgers’ equation. The dependence of the amplitude of the self reflected wave of parameters such as the Reynolds number and Knudsen number will be investigated.
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1 Introduction

1.1 Background

The general equation describing a gas is the Boltzmann equation. It describes the evolution in time of the statistical distribution of the particles in the phase space. The practical usefulness of the equation is limited since influence from particle collisions are obtained through a 6-dimensional integral. Evaluating this collision integrals requires a large amount of computer power, limiting the applications to few idealized problems. But from the Boltzmann equation one can obtain the equations of conservation. Equations of conservation describe the density \( \rho \), velocity field \( \mathbf{v} \) and temperature \( T \) as they evolve with time \( t \).

\[
\frac{D \rho}{Dt} = -\rho \nabla \cdot \mathbf{v} \quad (1.1)
\]

\[
\rho \frac{D \mathbf{v}}{Dt} = -\nabla \cdot \mathbf{P} \quad (1.2)
\]

\[
\rho \frac{3k_B}{2m} \frac{D T}{Dt} = -\mathbf{P} : \nabla \mathbf{v} - \nabla \cdot \mathbf{q} \quad (1.3)
\]

Here \( k_B \) is the Boltzmann’s constant and \( m \) the molecular mass. The pressure tensor \( \mathbf{P} \) and the heat current \( \mathbf{q} \) can be developed into power series in the Knudsen number \( Kn \). \( Kn = l/L \) where \( l \) is the mean free path of the particles and \( L \) a typical length scale. In this case the wavelength \( \lambda \) will be used as \( L \). Developing \( \mathbf{P} \) and \( \mathbf{q} \) in the zeroth order of the Knudsen number one obtains the Euler equations, first order gives the Navier-Stokes equations and second order the Burnett equations. For Knudsen numbers much smaller than 1 the particles can be described as a continuum. For these applications, the Navier-Stokes equations are sufficient to describe the flow. When the Knudsen number is of the order 1 the mean free path is comparable to a typical size of the problem. For these types of problems statistical mechanics has to be used in order to describe the particles. In this thesis Knudsen values low enough for an expansion to be made but large enough to require a higher order equations will be considered.

From a linear stability analysis on the Burnett equations it can be shown that the equations are not stable for all wave numbers. A method of stabilizing the equations has been suggested by Lars Söderholm, (Söderholm, 2006). This method results in the Hybrid Burnett equations. The set of equations obtained agree with the Burnett equations to the second order in Knudsen number which also is the accuracy of the Burnett equations. The hybrid Burnett equations are stable for all wave numbers. Another set of equations suggested by Lars Söderholm to eliminate the instability is the 13 Moment Burnett equations. This set of equations is also second order in the Knudsen number.

In this thesis, high amplitude sinusoidal waves will be treated. A simple equation describing what will happen to a one dimensional sinusoidal wave profile \( v(x,t) \), where \( x \) is the physical space coordinate, is the inviscid Burgers’ equation which states.

\[
\frac{\partial v}{\partial t} + (c_0 + \varepsilon v) \frac{\partial v}{\partial x} = 0 \quad (1.4)
\]

The term \( (c_0 + \varepsilon v) \) in the equation can be regarded as a form of wave speed, \( c_0 \) is the speed of sound and \( \varepsilon \) a non linearity coefficient. From this we can see that the speed of the wave is larger for
large velocities compared to small velocities. The top of a harmonic wave will travel faster than the bottom of the wave, see Figure 1. The top of the wave will catch up with the bottom of the wave creating a shockwave.

![Figure 1, illustration of the buildup of a shock wave.](image)

The shock wave will generate a steep gradient in the density which will result in a reflected wave that is travelling in the opposite direction of the original wave. As the amplitude of the original wave and the self reflected wave decay we will have two small amplitude waves where non linear effects can be neglected, this stage of the wave will be referred to as the old age stage.

### 1.2 Purpose
The purpose of this thesis is to determine the ability of the Hybrid Burnett equations to describe a typical physical flow. To do this we will look at sinusoidal waves propagating in a moderately rarefied gas. We will also try to validate similar numeric results obtained for the 13 moment Burnett equations, see (Strömgren, 2002).

### 1.3 Method
As initial conditions the solutions to the linearized Hybrid Burnett equations will be used. This will have the form of a high amplitude sinusoidal wave that will be integrated forward in time using Runge-Kutta fourth order method as it travels along the x-axis. Fourier's method for spatial discretization will be used. The amplitude of the old age wave and the self reflected wave will be obtained through a least square fitting of an appropriate ansatz.

Results will be compared to the analytical solution of the Burgers’ equation and previous results obtained with the 13 moment Burnett equations. The dependence of the self reflected wave on the Knudsen and acoustic Reynolds number will also be investigated.
2 Physics

2.1 Burgers’ Equation

Burgers’ equation is accurate to $Ma^2$ and $MaKn$, $Ma$ is the Mach number. For small Mach and Knudsen values, Burgers’ equation gives a good description of the wave propagating through the medium. The advantage of the Burgers’ equation is that an analytical solution does exist. Although not accurate for larger Knudsen and Mach numbers, the basic structure of the solution is described. The Burgers’ equation in a coordinate system that follows the wave with the speed of sound $c_0$ for the background state is given by.

$$\frac{\partial v}{\partial t} + \frac{\gamma + 1}{2} v \frac{\partial v}{\partial x} = \frac{b}{2\rho_0} \frac{\partial^2 v}{\partial x^2}$$

(2.1)

The density is assumed to be constant with a value of $\rho_0$, $b$ is the diffusivity and $\gamma$ the ratio of specific heats. The diffusivity $b$ contains the viscosity and heat conductivity. By doing the Cole-Hopf transformation $v(x, t) = -2b \frac{\partial}{\partial x} \ln u$, solutions are obtained for $u(x, t)$ that satisfies the linear heat equation $u_t = \frac{b}{2\rho_0} u_{xx}$. With the initial conditions of the wave set to $v(x, 0) = v_0\sin(kx)$ the solution to Burgers’ equation is given below, here $k$ is the wave number.

$$v(x, t) = \frac{2}{(\gamma + 1) k} b \sum_{n=0}^{\infty} \frac{\beta_n}{\sum_{n=0}^{\infty} \beta_n} \exp \left(-\frac{bk^2}{2\rho_0} n^2 t\right) \left(\frac{\gamma + 1}{2} \frac{v_0\rho_0}{kb}\right)^n \exp \left(n(n-1)k\right)$$

(2.2)

$\beta_n = 1$ if $n = 0$ and 2 otherwise and $I_n$ is the $n$th modified Bessel function. One observation we can make from the solution of the Burgers’ equation is that the second term in the series decays a factor four faster than the first term. For large times the following holds.

$$\exp \left(-\frac{bk^2}{2\rho_0} n^2 t\right) \gg \exp \left(-\frac{bk^2}{2\rho_0} n^2 t\right) \quad \text{for} \quad n \geq 2$$

(2.3)

For large times, the $n = 0$ term will dominate in the denominator and the $n = 1$ term in the numerator. A good approximation of the solution is then.

$$v(x, t) = \frac{4}{(\gamma + 1) k} b \frac{I_1 \left(-\frac{\gamma + 1}{2} \frac{v_0\rho_0}{kb}\right)}{I_0 \left(-\frac{\gamma + 1}{2} \frac{v_0\rho_0}{kb}\right)} \exp \left(-\frac{bk^2}{2\rho_0} t\right) \sin(kx)$$

(2.4)

For large times, the wave will decay exponentially. Plotting $\ln v(x, t)$ against the time, we should have a straight line. We can in Figure 2 see how the Burgers’ solution strays back to the linear case. In the figure text of Figure 2 and in equation (2.5) the acoustic Reynolds number $Re_{ac} = \frac{v_0\rho_0}{kb}$ has been used, see section 3.5.2 for definition.
Figure 2, In \( \frac{v(x,t)}{v_0} \) against the number of periods \( \frac{\xi n}{T} \) for the case \( Re_{ac} = 10 \). The dashed line denotes the linear solution and the full line solution to Burgers’ equation.

For large acoustic Reynolds number equation (2.4) can be further simplified to the form.

\[
v(x, t) = \frac{2}{(y + 1)} \frac{b}{k} \left( 1 + O(Re_{ac}^{-1}) \right) \exp \left[ -\frac{bk^2}{2\rho_0} t \right] \sin(kx) \quad (2.5)
\]

The old age wave amplitude is independent of the amplitude of the initial wave. This phenomenon is called *acoustic saturation*.

A phenomenon that isn’t incorporated into Burgers’ description of waves is self reflection of the wave. When a shock wave is formed, together with the steep gradient in the velocity profile there will also be a sudden change in material properties such as density and temperature. As for waves travelling in any medium, changes in the medium will result in reflections of the wave. With Burgers’ equation the medium properties are not modeled sufficiently enough for a self reflected wave to be incorporated into the description. To describe the self reflected wave, higher order equations have to be used or through introducing a small disturbance to the Burgers’ equation that will capture the non uniform medium.

### 2.2 Hybrid Burnett Equations

Developing the pressure tensor \( P \) and heat flow \( q \) in equations (1.2) and (1.3) to the second order of the Knudsen number one obtains the original Burnett equations.
\[
P = p \mathbf{1} - 2 \mu S + \alpha_1 \frac{\mu^2}{p} (\nabla \cdot \mathbf{v}) S + \alpha_2 \frac{\mu^2}{p} \left[ \frac{DS}{Dt} - 2 (S \cdot (\nabla \mathbf{v})) \right]
\]
\[
+ \alpha_3 \frac{\mu^2}{\rho T} (\nabla T) + \alpha_4 \frac{\mu^2}{\rho p T} (\nabla p \nabla T) + \alpha_5 \frac{\mu^2}{\rho T^2} (\nabla T \nabla T)
\]
\[
+ \alpha_6 \frac{\mu^2}{p} (\mathbf{S} \cdot \mathbf{S})
\]
\[
q = -\kappa \nabla T + \theta_1 \alpha_3 \frac{\mu^2}{\rho T} (\nabla \cdot \mathbf{v}) \nabla T + \theta_2 \frac{\mu^2}{\rho T} \left[ \frac{D (\nabla T)}{Dt} - (\nabla \mathbf{v}) \cdot \nabla T \right]
\] (2.6)
\[
(2.7)
\]

Where \((\nabla \mathbf{v})_{ij} = v_i \delta_{ij}\), \(S\) is the traceless rate of deformation tensor, \(\mu\) the dynamic viscosity, \(p\) the pressure, \(\kappa\) the heat conductivity coefficient, \(\alpha\) Burnett’s stress coefficients, \(\theta\) Burnett’s heat transfer coefficients and \(<...>\) is the symmetric traceless part of a tensor. Both the pressure tensor and the heat current contain \(\frac{\partial}{\partial T}\). These can be substituted by spatial derivatives from the zeroth order expansion, giving the Conventional Burnett equations.

\[
\frac{D ...}{Dt} \rightarrow \frac{D_0 ...}{Dt}
\]
(2.8)

Doing a linear stability analysis for the Conventional Burnett equations an unphysical instability was shown by Bobylev (Bobylev, 1982). This is done by adding a disturbance to the solution \(T = T_0\), \(\rho = \rho_0\) and \(\mathbf{v} = \mathbf{0}\). Choosing a disturbance of the form \(e^{\Delta t + ikx}\) for the density \(\rho\), temperature \(T\) and velocity \(\mathbf{v}\) one obtains a positive real part of \(\Lambda\) for large values of the wave number \(k\). The \(\Lambda\) value for growing \(k\) is illustrated in Figure 3 for a Maxwell gas. For comparison, the \(\Lambda\) value for Navier-Stokes equations is also plotted. The Navier-Stokes equations are stable for all values of the wave number \(k\). For the Burnett equations, large wave number \(k\) gives an unphysical instability with growing values. It can be shown that the instability does not arise from the substitution of the temporal derivatives to spatial derivatives. Also the original Burnett equations have the unphysical instability (Söderholm, 2006).

![Figure 3, linear stability analysis. \(\Lambda\) plotted on the complex plane for different wave numbers \(k\). Navier-Stokes equations in + and the Burnett equations in o. Values with positive real parts are unstable.](image-url)
If a hybrid between the actual temporal derivative and the spatial approximation is used instead of spatial approximations for the temporal derivatives, the equations can be stabilized.

\[
\frac{DS}{Dt} \rightarrow (1 - \alpha) \frac{DS}{Dt} + \alpha \frac{D_0 S}{Dt} \quad (2.9)
\]

\[
\frac{D(\nabla T)}{Dt} \rightarrow (1 - \beta) \frac{D(\nabla T)}{Dt} + \beta \frac{D_0 (\nabla T)}{Dt} \quad (2.10)
\]

With \( \alpha = \beta = 0 \) the original Burnett equations are obtained and \( \alpha = \beta = 1 \) the Conventional Burnett equations. A stability analysis shows that by choosing \( \beta \geq 1 \) and \( \alpha \geq \frac{c_3}{c_2} \) the equations are stabilized. For the Hybrid Burnett equations the two parameters have been chosen to \( \beta = 1 \) and \( \alpha = \frac{c_3}{c_2} \), resulting in the following expression for the pressure tensor and heat current.

\[
P = p \mathbf{1} - 2\mu S + \frac{c_1}{c} \frac{\mu^2}{p} (\nabla \cdot \mathbf{v}) S - \left( c_3 - c_2 \right) \frac{\mu^2}{p} \frac{DS}{Dt}
\]

\[
- 2c_2 \frac{\mu^2}{p} (S \cdot (\nabla \mathbf{v})) - \left( c_3 - c_2 \right) \frac{\mu^2}{p} (\nabla \nabla \rho)
\]

\[
+ c_3 \frac{\mu^2}{\rho T} \left( -\frac{1}{\rho} \nabla T \nabla \rho + \frac{T}{\rho^2} \nabla \rho \nabla \rho - \frac{p}{\rho T} (\nabla \mathbf{v})^2 \right)
\]

\[
+ c_4 \frac{\mu^2}{\rho p T} (\nabla p \nabla T) + c_5 \frac{\mu^2}{\rho T^2} (\nabla T \nabla T) + c_6 \frac{\mu^2}{p} (S \cdot S) \quad (2.11)
\]

\[
\mathbf{q} = -\kappa \nabla T + \frac{\mu^2}{\rho T} (\nabla \cdot \mathbf{v}) \nabla T + \frac{\mu^2}{\rho T} \left[ \frac{D(\nabla T)}{Dt} - (\nabla \mathbf{v}) \cdot \nabla T \right]
\]

\[
+ \frac{\mu^2}{\rho T} S \cdot \nabla p + \frac{\mu^2}{\rho} \nabla \cdot S + \frac{3\mu^2}{\rho T} S \cdot \nabla T \quad (2.12)
\]

This set of equations is stable for all wave numbers and agree with the Burnett equations to the order of \( Kn^2 \) which also is the accuracy of the Burnett equations. In the pressure tensor (2.11) \( c_3 - c_2 > 0 \), this plays an important role in the stability.
2.3 One Dimensional Weakly Non Linear Case

Using the pressure tensor (2.11) and heat current vector (2.12) in the set of equations formed by (1.1), (1.2) and (1.3) one obtains the full set of the Hybrid Burnett equations. In this thesis the one dimensional weakly non linear form of the Hybrid Burnett equations will be considered. This set of equations is obtained if the Burnett terms in the pressure tensor (2.11) and heat current vector (2.12) are linearized. The Burnett terms are the terms containing $\sigma$ and $\theta$. Linearizing these terms we obtain.

\[ \rho_t + \nu \rho_x = -\rho v_x \quad (2.13) \]
\[ \rho v_t + \nu \nu v_x = -p_x + \frac{4}{3}(\mu v_{xx} + \mu_x v_x) - (\sigma_2 - \sigma_3) \frac{2\mu_0^2}{3p_0} v_{xx} \]
\[ + \sigma_3 \frac{\mu_0^2}{\rho_0^2} \rho_{xxx} \quad (2.14) \]
\[ \rho \frac{3k_B}{2m} (T_t + \nu T_x) \]
\[ = -p v_x + \frac{2}{3} \mu v_{xx}^2 + \kappa T_{xx} + \kappa_x T_x \]
\[ + \frac{2}{3} (\theta_2 - \theta_4) \frac{\mu_0^2}{\rho_0^2} v_{xxx} \quad (2.15) \]

The set of equations formed by equations (2.13), (2.14) and (2.15) the terms $Kn^0$, $Kn^1$ to all orders in $Ma$ and $Kn^2Ma$ have been kept. This is the set that will be considered in this thesis.

2.4 Nondimensionalization

To minimize the numerical errors equation (2.13), (2.14) and (2.15) will be nondimensionalized according to the following.
\[ t = \frac{\lambda}{c_0} t^* \quad x = \lambda x^* \quad v = v_0 v^* \]

\[ \rho = \rho_0 + \rho_1 \rho^* \quad T = T_0 + T_1 T^* \quad p = p_0 + p_1 p^* \]

Here \( p_0 \) and \( T_0 \) are set in the beginning of the program. For \( \rho_0 \) the steady state value for an ideal gas is used. \( p_1, \rho_1 \) and \( T_1 \) are chosen so that the values used in numerical calculations \( \rho^*, T^* \) and \( p^* \) have a magnitude of one at the initial time. See section 3.5.1 for further details. From here on the nondimensional variables will be used without superscripts. Applying this on equation (2.13), (2.14) and (2.15) the following equations are obtained.

\[ \rho_t = -\frac{\rho_0 v_0}{\rho_1 c_0} v_x - \frac{v_0}{c_0} v \rho_x \quad (2.16) \]

\[ v_t - \frac{2}{3} (\alpha_3 - \alpha_2) \frac{\mu_0^2}{\lambda^2} \frac{1}{\rho_0 (\rho_0 + \rho_1 \rho)} v_{xxx} \]

\[ = - \frac{p_1}{(\rho_0 + \rho_1 \rho) v_0 c_0} p_x + \frac{4 \mu_0}{3 \lambda} \frac{1}{c_0 (\rho_0 + \rho_1 \rho)} (\mu v)_x \]

\[ + \alpha_3 \frac{\mu_0^2}{\lambda^2} \frac{\rho_1}{\rho_0^2 v_0 c_0 (\rho_0 + \rho_1 \rho)} (\rho_{xxx} - \frac{v_0}{c_0} v v_x) \quad (2.17) \]

\[ T_t = -\frac{2m}{3k_B} \frac{v_0}{T_1 c_0 \rho_0 + \rho_1 \rho} v_x \]

\[ + \frac{2}{3} (\theta_2 - \theta_4) \frac{2m \mu_0 v_0}{3k_B \lambda^2} \frac{1}{T_1 c_0 \rho_0 (\rho_0 + \rho_1 \rho)} v_{xxx} \]

\[ + \frac{2m \kappa_0}{3k_B \lambda c_0 (\rho_0 + \rho_1 \rho)} (\kappa T_x)_x \]

\[ + \frac{4 \mu v_0^2 \mu_0}{3 \lambda} \frac{\mu}{T_1 c_0 (\rho_0 + \rho_1 \rho)} v_x^2 - \frac{v_0}{c_0} v T_x \quad (2.18) \]

2.5 Linear Case

At the old age stage, the wave can be described by a small amplitude harmonic wave with wave number \( 2\pi \) in the nondimensional space. Here, the non linear effects can be neglected making it possible for the linearized set of equations to describe the wave with sufficient accuracy. The linearized set of equations is.

\[ \rho_t = -\frac{\rho_0 v_0}{\rho_1 c_0} v_x \quad (2.19) \]
Here, the pressure term has been linearized using the ideal gas law and the background values have been used for the viscosity $\mu$ and the heat coefficient $\kappa$. The linear set of equations can be put into the following form.

$$v_t - \frac{2}{3}(\sigma_3 - \sigma_2) \frac{\mu_0^2}{\lambda^2} \frac{1}{\rho_0 \rho_0} v_{xxt} = -\frac{p_1}{\rho_0 v_0 c_0} \left( \frac{k_B}{m} \rho_0 T_x + \frac{k_B T_0 \rho_x}{m} \right) + \frac{4 \mu_0}{3 \lambda} \frac{1}{\rho_0 c_0} v_{xx}$$

(2.20)

$$+ \sigma_3 \frac{\mu_0^2}{\lambda^2} \frac{\rho_1}{\rho_0^3} \rho_{xxx}$$

$$T_t = -\frac{2m}{3k_B} \frac{p_0 v_0}{T_1 c_0 \rho_0} v_x + \frac{2}{3}(\theta_2 - \theta_4) \frac{2m}{3k_B} \frac{\mu_0}{\lambda c_0 T_1 \rho_0} v_{xxx}$$

(2.21)

$$+ \frac{2m}{3k_B} \kappa \frac{1}{\lambda c_0 \rho_0} T_{xx}$$

Assigning the vector $\mathbf{y} = (\rho, v, T)$ and seeking solutions of the form $\mathbf{y} = c e^{i(\omega t + 2\pi x)}$ differentiation in space and time turn into the multiplications $i2\pi$ and $i\omega$ respectively. This makes it possible to write equation (2.22) in matrix form.

$$\begin{bmatrix}
\rho_t \\
\frac{2}{3}(\sigma_3 - \sigma_2) \frac{\mu_0^2}{\lambda^2} \frac{1}{\rho_0 \rho_0} v_{xxt} \\
T_t
\end{bmatrix} = \mathbf{L}_1 \begin{bmatrix}
\rho_x \\
v_x \\
T_x
\end{bmatrix} + \mathbf{L}_2 \begin{bmatrix}
\rho_{xx} \\
v_{xx} \\
T_{xx}
\end{bmatrix} + \mathbf{L}_3 \begin{bmatrix}
\rho_{xxx} \\
v_{xxx} \\
T_{xxx}
\end{bmatrix}$$

(2.22)

Equation (2.24) is an eigenvalue problem with three solutions for $\omega$. The three modes will correspond to two acoustic waves travelling in opposite directions along the $x$ axis and one entropy mode that will remain stationary and decay. The three eigenvalues are given by.

$$\omega_{1,2} = \mp \xi_a + iv_a$$

(2.25)

$$\omega_3 = iv_s$$

(2.26)

Where $\omega_{1,2}$ denote the right and left moving acoustic wave and $\omega_3$ the entropy mode. Since only the wave traveling along the $x$-axis will be considered here only $\omega_1$ needs to be taken into account. The eigenvalue of the self reflected wave is $\omega_2$. Due to symmetry, $\xi_a$ and $v_a$ will have the same value for $\omega_1$ and $\omega_2$. 
3  Numerical Methods

3.1  Spatial Discretization

3.1.1  Fourier Method

In the problem considered in this report, the solution is periodic in space with periodical derivatives. The solution can then be approximated by a discrete Fourier transform. For a function $u$ in $[0,1]$ known on the set of points $x_j, j = 1, 2, ..., N$, the discrete Fourier transform takes the form.

$$u(x_j, t) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{u}_n(t)e^{i2\pi k_n x_j}$$  \hspace{1cm} (3.1)

Here $\tilde{u}_n$ is the $n$-th discrete Fourier coefficient. The discrete Fourier coefficients can easily be calculated from the set of values for $u$ through.

$$\tilde{u}_n(t) = \frac{1}{N} \sum_{j=1}^{N} u(x_j, t) e^{-i2\pi k_n x_j}$$  \hspace{1cm} (3.2)

The spatial derivation operator is turned into a multiplication. Knowing the discrete Fourier coefficients of the function, the coefficients of the spatial derivative are given by.

$$\frac{du}{dx}\bigg|_{x=x_j} = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} i2\pi k_n \tilde{u}_n(t)e^{i2\pi k_n x_j}$$  \hspace{1cm} (3.3)

Transformations between the Fourier space and spatial space are done with Fast Fourier Transformation algorithms. Here, the built in functions FFT and IFFT of Matlab have been used. The number of operations of such a transformation is $\frac{N}{2} \log_2 N$ compared to the $2N^2$ operations required for a matrix-vector multiplication.

The accuracy of the description according to equation (3.1) depends on the properties of the function. The accuracy can be evaluated through Fourier analysis which states that a periodic function always can be described with a Fourier series, equation (3.4). If the function is infinitely smooth with all its derivatives periodic as well the $n$-th Fourier coefficient $\hat{u}_n$ of the function decays faster than any negative power of $n$. This rapid decay of the Fourier coefficients $\hat{u}_n$ is often referred to as exponential decay.

$$u(x, t) = \sum_{n=-\infty}^{\infty} \hat{u}_n(t)e^{i2\pi k_n x}$$  \hspace{1cm} (3.4)

The Fourier series in equation (3.4) is an exact description of the continuous function $u(x, t)$ for all $x$. When evaluating a discrete Fourier transformation, $N$ discrete Fourier coefficients are evaluated from the function value on $N$ nodes. The coefficients obtained are not the same as in the truncated Fourier series consisting of the coefficients $-\frac{1}{2}N$ to $\frac{1}{2}N - 1$ in equation (3.4). On the set of nodes from which the discrete Fourier series is evaluated the $n$-th Fourier mode and the $(n + Nm)$ -th
Fourier mode are indistinguishable. The relation between the Fourier coefficient \( \hat{u}_n \) and the discrete Fourier coefficient \( \tilde{u}_n \) is given by:

\[
\tilde{u}_n = \hat{u}_n + \sum_{m \neq 0}^{+\infty} \hat{u}_{n+Nm}
\]  

(3.5)

The error that arises from this is called aliasing error. But if the function that is evaluated fulfills the criterions for the \( n \)-th Fourier coefficient \( \hat{u}_n \) to decay faster than any negative power of \( n \). Also the discrete Fourier transform will show exponential accuracy often referred to as spectral accuracy.

In practice spectral accuracy is only obtained for \( N \) values larger than a certain value, before this value the basic characteristics of the function is not described. This gives a poor approximation of the function. Spectral accuracy needs to be validated in order to confirm that the solution is of such type as described in this section. If so, Fourier’s method gives a fast discretization that for relatively few nodes gives sufficient accuracy.

### 3.1.2 Spatial Resolution

The wave profile can be divided into three stages in time, see Figure 5. An initial stage before the shock is created, a shock stage and an old age stage where the shock has died out. At the initial and old age stage the wave profile is close to a harmonic function. The wave profile at the shock stage has a high gradient in the wave profile.

![Figure 5](image)

*Figure 5, the wave profile at the three stages with the nondimensional speed \( v \) and non-dimensional position \( x \). Results have been obtained numerically with 148 Fourier modes.*

A function close to a cosine or sine function requires less Fourier modes to describe the profile sufficiently than for a function with a sharp gradient, seen in Figure 6.

![Figure 6](image)

*Figure 6, the Fourier spectra for the wave profiles in Figure 5.*
It's possible to peel off the majority of the spectra at the initial and old age wave and still have a sufficient description of the wave profile. Since our choice of the number of Fourier modes $N$ will remain constant throughout the program the spectra at the shock stage will determine the number of Fourier coefficients needed to describe the wave.

The number of Fourier coefficients needed to describe the shockwave is determined by the thickness of the shock. The thickness of the shock is inversely proportional to the acoustic Reynolds number $Re_{ac}$. A general way of choosing $N$ is to determine the number of points $n_0$ one wants to describe the shockwave with and through that calculate $N$ needed. To do this, the proportionality constant $\alpha$ needs to be determined.

\[
\Delta x_{\text{shock}} = \frac{\alpha}{Re_{ac}} \quad (3.6)
\]

\[
N = \frac{1}{\Delta x_{\text{shock}}} n_0 = \frac{Re_{ac} n_0}{\alpha} \quad (3.7)
\]

$\Delta x_{\text{shock}}$ has been measured for different $Re_{ac}$ in order to determine $\alpha$. Least square fitting of the results in Table 1 gives a value of $\alpha = 0.7$.

<table>
<thead>
<tr>
<th>$Re_{ac}$</th>
<th>3.66</th>
<th>4.88</th>
<th>9.14</th>
<th>20.0</th>
<th>30.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta x_{\text{shock}}$</td>
<td>0.16</td>
<td>0.13</td>
<td>0.070</td>
<td>0.039</td>
<td>0.026</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.57</td>
<td>0.61</td>
<td>0.64</td>
<td>0.78</td>
<td>0.78</td>
</tr>
</tbody>
</table>

Table 1, $\Delta x_{\text{shock}}$ measured for different $Re_{ac}$ in order to calculate $k_q$.

Figure 7, measured $\Delta x_{\text{shock}}$ values plotted against $Re_{ac}$. Also the fitted line for $\alpha = 0.7$.  

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3.2 Temporal Discretization

3.2.1 Runge-Kutta Method

When choosing a numerical solution method efficiency and stability are the two key aspects that have to be taken into consideration. For the temporal discretization the alternatives are a simple first order method, a higher order explicit method or an implicit method. An implicit method has the advantage of being stable for all choices of step length allowing a large step length. On the other hand it requires a system of equations to be solved for each step. That computer power can be used to take several steps with an explicit method and obtain a higher accuracy. This solution method should only be used when explicit methods require very small step lengths for slow changing system, which is not the case. First order methods lacks in efficiency, in order to obtain accurate results the step length has to be chosen small. Using the fourth order Runge-Kutta method step lengths can be chosen near the stability limit and still obtain accurate results.

Given the set of points \( y_n \) at the time \( t_n \) and the time derivative \( f(t, y) \) of \( y \), the set \( y_{n+1} \) at the time \( t_{n+1} = t_n + h \) can be calculated with Runge-Kutta method according to the equations below.

\[
y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\]

(3.8)

\[
k_1 = f(t_n, y_n)
\]

(3.9)

\[
k_2 = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right)
\]

(3.10)

\[
k_3 = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right)
\]

(3.11)

\[
k_4 = f(t_n + h, y_n + hk_3)
\]

(3.12)

The higher order of accuracy is obtained by evaluating the slope not only at \( t = t_n \) but by using a weighted average at \( t = t_n, t = t_n + \frac{1}{2}h \) and \( t = t_n + h \). This method gives a good balance between accuracy and simplicity in the program.

3.2.2 Numerical Treatment of the Mixed Derivatives

Numerically, the term with the mixed derivatives pose a problem. In order for our set of equations to be time stepped forward in time using Runge-Kutta method the equations have to be given in the form below.

\[
\frac{dy}{dt} = f(y, t)
\]

(3.13)

This is not the case for the equation of motion given by equation (2.17), rewritten below. For simplicity only the terms with time derivatives have been included, all other terms are denoted with (...)..

\[
v_t + \frac{2}{3}(\omega_2 - \omega_3)\frac{\mu_2}{h^2} \frac{1}{p_0(p_0 + \rho_1\rho)} v_{xxt} = (...) 
\]

(3.14)
To overcome this problem, the new variable \( w \) is chosen in such a way that the terms on the left hand side of equation (3.14) can be written in the form \( w \). This gives a set of equations that can be stepped forward in time with the Runge-Kutta method, seen below.

\[
\rho_t = (...) \quad (3.15)
\]
\[
w_t = (...) \quad (3.16)
\]
\[
T_t = (...) \quad (3.17)
\]

The right hand side of the equation of motion is still expressed in \( v \) and not \( w \) since there is no possible choice of \( w \) that eliminates \( v \) from both the left and right hand side of the equation. Therefore, \( w \) will be converted into \( v \) at each calculation of \( w \). To simplify the conversion the background value has been chosen for the density, this simplification is consistent with the weakly non linear approximation. With a linear relation between \( v \) and \( w \), one can from the fouriercoefficients of \( w \) calculate the Fourier coefficients of \( v \). If the Fourier coefficients \( V_n \) of \( v \) is known, the Fourier coefficients \( W_n \) of \( w \) is obtained through equation (3.19).

\[
w = v + \frac{2}{3} (\alpha_2 - \alpha_3) \frac{\mu^2}{\lambda^2} \frac{1}{p_0 \rho_0} v_{xx}
\]

\[
W_n = \left(1 + \frac{2}{3} (\alpha_2 - \alpha_3) \frac{\mu^2}{\lambda^2} \frac{1}{p_0 \rho_0} (i2\pi k_n)^2 \right) V_n \quad (3.19)
\]

Approximating the density with a uniform density \( \rho_0 \) gives an error in the conservation of momentum. To estimate the size of the error we will look at the dimensional momentum equation in the Hybrid Burnett equations, seen below.

\[
\rho \frac{Dv}{Dt} + \nabla \cdot \left( (\alpha_2 - \alpha_3) \frac{\mu^2}{p} \frac{DS}{D} \right) = (...) \quad (3.20)
\]

For simplicity terms not containing temporal derivatives have been denoted to (...). Considering the one dimensional case and doing some simplification the following is obtained.

\[
\left(1 + (\alpha_2 - \alpha_3) \frac{\mu^2}{\rho p} \frac{\partial^2}{\partial x^2} \right) \rho \frac{Dv}{Dt} = (...) \quad (3.21)
\]

The term with the mixed derivatives is in the order of.

\[
\frac{\mu^2}{\rho p \lambda^2} \nabla p \quad (3.22)
\]

Using the background value for the density results in an error in \( \frac{\partial v}{\partial t} \) of the order.

\[
\left(1 - \frac{1}{\rho_0} \right) \frac{\mu^2}{\rho p \lambda^2} \nabla p \sim Ma^2 Kn^2 \frac{c^2}{\lambda} \quad (3.23)
\]

This error will after one period result in an error in \( \rho v \) of the order.
This error is the price for simplifying the momentum equation in such a way that the system of differential equations can be solved with a standard numerical scheme.

3.2.3 Time Step
The fourth order Runge-Kutta method has well balanced properties when it comes to simplicity, stability and accuracy. This method is stable for $\lambda h$ in the region seen in Figure 8 where $\lambda$ is the dominating eigenvalue of the differential operator and $h$ the time step.

\[ M a^2 Kn^2 \rho c \lambda \]  
\[ (3.24) \]

In order for the eigenvalue to be calculated for the equations (2.16), (2.17) and (2.18) the set of equations has to be expressed in matrix form.

\[
\begin{bmatrix}
\rho_t \\
v_t \\
T_t
\end{bmatrix} = A(\rho, v, T) \begin{bmatrix}
\rho \\
v \\
T
\end{bmatrix}
\]  
\[ (3.25) \]

Here the vectors $\rho$, $v$ and $T$ are the vectors containing density, velocity and temperature at a given number of nodes. For simplicity, the term with the mixed derivatives is disregarded and background values have been used for the viscosity and the heat coefficient, see appendix A for details.

The dominant eigenvalue is calculated using Matlabs built in function eigs. The sensitivity of $\lambda$ due to changes in the wave is small. Therefore a new time step is only calculated every $10N$ steps during the calculations.

The time step will be calculated according to:

\[ \Delta t = \Gamma \frac{\sqrt{8}}{2\pi \lambda_{dom}} \]  
\[ (3.26) \]

Here $\Gamma$ is the safety factor. In this program a safety factor of $\Gamma = 0.7$ have been used.
3.2.4 End Time
The purpose of the numerical calculations is to obtain high accuracy numerical values of the amplitude of the old wave. For this purpose the end time has to be chosen large enough so that the shock wave has died out and only a small amplitude sinusoidal wave remains. For the Burgers’ equation, where the analytical solution is known, a measure of this can be taken by comparing the second and first Fourier mode in the solution. \( t_{\text{end}} \) can be chosen in such a way that.

\[
\exp \left[ -\frac{bk^2}{2\rho_0} t_{\text{end}} \right] = \varepsilon \exp \left[ -\frac{bk^2}{2\rho_0} 4t_{\text{end}} \right]
\]  

(3.27)

Here \( \varepsilon \) is chosen small. \( t_{\text{end}}(\varepsilon) \) is given by

\[
t_{\text{end}}(\varepsilon) = -\frac{4 \rho_0 \ln \varepsilon}{3bk^2}
\]  

(3.28)

After the time \( t_{\text{end}} \) the non linearity can be neglected and the solution to the wave can be described by a simple cosine or sine function. The calculation of the end time is based on the Burgers’ equation which differs from the Hybrid Burnett equations. The actual end time have to be reassured by checking the spectra of the solutions obtained numerically at the end time. In this thesis, \( \varepsilon \) has been set to \( 10^{-5} \).

3.3 Initial and Boundary Conditions
To physically define the problem the steady state pressure \( p_0 \), background temperature \( T_0 \), the Mach number \( Ma \) and frequency of the wave \( f_0 \) will be set at the beginning of the program. From these parameters the steady state density \( \rho_0 \) can be obtained through the ideal gas law.

\[
\rho_0 = \frac{m}{k_B} \frac{p_0}{T_0}
\]  

(3.29)

For the initial conditions a perturbation from the rest state \( T_0, \rho_0 \) with zero speed of the form below will be considered.

\[
T_{\text{physical}} = T_0 + T_1 T
\]  

(3.30)

\[
\rho_{\text{physical}} = \rho_0 + \rho_1 \rho
\]  

(3.31)

\[
v_{\text{physical}} = v_0 v
\]  

(3.32)

In (3.32), \( v_0 \) is obtained from the Mach number. \( \rho_1, \rho, T_1, T \) and \( v \) are obtained from the linear set of equations. Searching for solutions of the form \( [\rho_1 \rho, v, T_1, T]^T = ce^{i(2\pi x - \omega t)} \), \( c \) and \( \omega \) are obtained as eigenvectors and eigenvalues with the method described in the section 2.5. Doing this, three sets of eigenvalues with corresponding eigenvectors will be obtained. As initial conditions the acoustic wave travelling along the x axis will be considered.

The eigenvector \( c \) will be normalized in such a way that the component corresponding to the speed has the value one, this gives a velocity amplitude of \( v_0 \), \( \rho_1 \) and \( T_1 \) will be chosen as the first and third component of the normalized eigenvector.
Periodical boundary conditions will be chosen. The numerical implementation is especially easy since all the Fourier the modes are periodical in $\lambda$. The nondimensional boundary conditions take the form.

\[ T(0, t) = T(1, t) \]  \hfill (3.33)
\[ \rho(0, t) = \rho(1, t) \]  \hfill (3.34)
\[ v(0, t) = v(1, t) \]  \hfill (3.35)

### 3.4 Old Age Wave

#### 3.4.1 Hybrid Burnett Equations

When the numeric solution has reached the old age stage the non linear effects can be neglected. The wave will contain a sinusoidal wave travelling along the $x$-axis and a self reflected wave travelling in the opposite direction. The first Fourier coefficient will contain the following information.

\[ V_1(t) = \frac{1}{2} B e^{-i\omega t} e^{i(\text{Re}(\omega) t + 1)} + \frac{1}{2} b_R e^{-i\omega t} e^{i(\text{Re}(\omega) t + 2)} \]  \hfill (3.36)

Here $B$ is the amplitude of the wave and $b_R$ the amplitude of the self reflected wave. $\frac{1}{2} B e^{ia}$ and $\frac{1}{2} b_R e^{i\beta}$ are obtained with the least square method. With the least square method the residual $R$ is minimized by choosing $\tilde{B} = B e^{ia}$ and $\tilde{b}_R = b_R e^{i\beta}$.

\[ R = \frac{1}{n} \sum_{n} |V_1(t_n) - V_n|^2 \]  \hfill (3.37)

In order to monitor the error $e_a$ is introduced.

\[ e_a = \max_n \frac{|V_1(t_n) - V_n|}{|V_n|} \]  \hfill (3.38)

#### 3.4.2 Comparison to Burgers’ Equation

The old age amplitude $B_B$ for Burgers’ equation is obtained from equation (2.4).

\[ B_B = \frac{4}{(\gamma + 1) k} \int_{0}^{1} \left( \frac{v_0 \rho_0}{k b} \right) \]  \hfill (3.39)

The old age amplitudes are normalized with $C_B$.

\[ C_B = \frac{4}{(\gamma + 1) k} \]  \hfill (3.40)
3.5 Numerical Values

3.5.1 Initial State

In order to define constants and properties of the gas the gas argon has been chosen. The pressure $p_0$, temperature $T_0$, the Mach number $Ma$ and frequency $f_0$ is set in the beginning of the program. The amplitude of the wave is obtained through the Mach number.

$$v_0 = Ma \, c_0$$  \hspace{1cm} (3.41)

Where the speed of sound $c_0$ is obtained through the assumption of constant entropy.

$$c_0 = \sqrt{\frac{k_B}{m} T_0}$$  \hspace{1cm} (3.42)

The background density is obtained from the ideal gas law.

$$\rho_0 = \frac{m \, p_0}{k_B \, T_0}$$  \hspace{1cm} (3.43)

The background value of the dynamic viscosity is obtained from Sutherland’s formula.

$$\mu_0 = K_4 \frac{T_0^{3/4}}{T_0 + S}$$  \hspace{1cm} (3.44)

Where the two empirical constants $K_4$ and $S$ for the gas argon are given by.

$$K_4 = 1.9749 \cdot 10^{-6} \, kg \, m^{-1} \, s^{-1} \, K^{-1/2}$$  \hspace{1cm} (3.45)

$$S = 1.9749 \cdot 10^{-6} \, K$$  \hspace{1cm} (3.46)

The heat coefficient is given by.

$$\kappa_0 = \frac{\mu_0 c_p}{T_0}$$  \hspace{1cm} (3.47)

In the program the diffusivity, pressure, dynamic viscosity and heat coefficient are used. These are calculated through.

$$b = \frac{4}{3} \mu + \kappa \left( \frac{1}{c_v} - \frac{1}{c_p} \right)$$  \hspace{1cm} (3.48)

$$p_0 + p_1 p = \frac{k_B}{m} (T_0 + T_1 T)(\rho_0 + \rho_1 \rho)$$  \hspace{1cm} (3.49)

$$\mu = K_4 \frac{(T_0 + T_1 T)^{3/4}}{T_0 + T_1 T + S}$$  \hspace{1cm} (3.50)
\[ \kappa = \frac{\mu c_p}{T_0 + T_1 T} \]  

(3.51)

### 3.5.2 Nondimensional Numbers

The acoustic Reynolds Number gives a ratio between the nonlinear and dissipative forces in a flow. Flows dominated by non linear effects have an acoustic Reynolds number larger than one and flows dominated by dissipative effects an acoustic Reynolds number smaller than one.

\[ Re_{ac} = \frac{\rho_0 v_0}{bk} \]  

(3.52)

The Knudsen number \( Kn \) gives a ratio between the mean free path and a typical length scale.

\[ Kn = \frac{l}{L} \]  

(3.53)

Where \( l \) is the mean free path for a hard sphere and \( L \) the wave length.

\[ l = \frac{\mu_0}{c_0 \rho_0} \]  

(3.54)

### 3.5.3 Constants

Numeric values for constants used.

- \( k_B = 1.380654 \cdot 10^{-6} \text{ J K}^{-1} \)
- \( Pr = 6.651 \cdot 10^{-1} \)
- \( m = 6.634 \cdot 10^{-26} \)
- \( \gamma = \frac{5}{3} \)
- \( c_v = \frac{3k_B}{2m} \)
- \( c_p = \gamma c_v \)

\( Pr \) is the Prandtl number, \( c_v \) the specific heat at constant volume and \( c_p \) the specific heat at constant pressure. For a hard sphere model, Burnett second order heat transfer and stress coefficients are given by.

- \( \omega_2 = 2.029 \)
- \( \omega_3 = 2.415 \)
- \( \theta_2 = 5.826 \)
- \( \theta_4 = 2.415 \)
4 Algorithm

4.1 Program Construction

The choice of programming language for this work has been Matlab. Although previous experiences suggest that C++ can be as much as five times faster than Matlab, the later has been chosen for its user friendliness when programming. Most of the time will be spent writing the program code, computational efficiency is therefore not as important as user friendliness.

In order to have a program that can be modified easily the code has been divided into three main programs. These three parts are directed by a main program. The problem parameters are defined in the main program.

From the main program Constants is run, here the constants required are calculated from the problem parameters and initial conditions are set in order to start the solution algorithm.

The solution part starts with an initial vector $y_1 = [\rho \ w \ T]^T$ and steps forward in time using the Runge-Kutta's fourth order scheme. The vectors $\rho$, $w$ and $T$ contain the physical values at the $N$ nodes. To calculate the time derivative of the three vectors Function is used.

![Figure 9, basic program construction implemented in Matlab.](image)

4.1.1 Function

Here the temporal derivative of the system of equations is calculated from the vector $y$.

Given $y$:

1. The pressure, viscosity and the heat coefficient is calculated according to equation (3.48), (3.49), (3.50) and (3.51).
2. $v$ is calculated from $w$ according to (3.18).
3. $\rho_t$, $w_t$ and $T_t$ are calculated from $\rho$, $v$ and $T$ according to (2.16), (2.17) and (2.18).

This is accomplished with 10 Fourier transforms, 10 inverse Fourier transforms and $30N$ multiplications.

4.1.2 Solution

This part of the program is an implementation of the Runge-Kutta scheme.
Given \( y_n \):

1. \( \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \) and \( \mathbf{k}_4 \) is calculated using Function.
2. \( y_{n+1} \) is calculated according to equation (3.8).
3. \( t_{n+1} = t_n + \Delta t \)
4. \( t_{n+1} > t_{end} \) quit, else go to step 1.

Including the operations in function, this is done using 40 Fourier transforms, 40 inverse Fourier transforms and 80N multiplications for each step.

### 4.2 Validation

#### 4.2.1 Fourth Order Convergence

The local error \( e_l \) in Runge-Kutta’s fourth order method is \( O(h^5) \). For each time step a local error of \( O(h^5) \) is accumulated into the global error \( e_g \). The number of time steps required to reach the final time \( t_f \) is proportional to \( h \) which gives a global error \( e_g \) of \( O(h^4) \).

The relation between the value \( y_{t,h,\text{num}} \) at time \( t \) obtained numerically with step size \( h \) and the analytical value \( y_{t,\text{ana}} \) is given by.

\[
y_{t,h,\text{num}} = y_{t,\text{ana}} + O(h^4)
\]

Using half the time step, the accuracy of the numerical solution increases by a factor \( 2^4 \).

\[
\frac{y_{t,h,\text{num}} - y_{t,\text{ana}}}{y_{t,\frac{h}{2},\text{num}} - y_{t,\text{ana}}} = \frac{O(h^4)}{O(h^4)} = 2^4
\]

Since the analytical solution is unknown a forcing function has been used. The idea is that an arbitrary function \( g(x, t) \) that satisfies the boundary conditions is chosen, in our case \( g(x, t) = \sin(2\pi x - t) \). The program is then modified to solve \( \frac{\partial y}{\partial t} = f(y, t) - f(g(x, t), t) + \frac{\partial g(x, t)}{\partial t} \) instead of \( \frac{\partial y}{\partial t} = f(y, t) \). The solution of \( y \) will then converge to \( g(x, t) \) as \( h \) and \( \frac{1}{N} \) goes to zero. The number of Fourier coefficients \( N \) is chosen so that the numerical errors due to spatial discretization are negligible compared to the numerical error due to temporal discretization.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( |y_{t,h,\text{num}} - y_{t,\text{ana}}|_\infty )</th>
<th>( \log_2 \left( \frac{|y_{t,h,\text{num}} - y_{t,\text{ana}}|<em>\infty}{|y</em>{t,\frac{h}{2},\text{num}} - y_{t,\text{ana}}|_\infty} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.50E-02</td>
<td>7.19E-08</td>
<td>4.02E+00</td>
</tr>
<tr>
<td>1.25E-02</td>
<td>4.43E-09</td>
<td>4.01E+00</td>
</tr>
<tr>
<td>6.25E-03</td>
<td>2.75E-10</td>
<td>4.01E+00</td>
</tr>
<tr>
<td>3.13E-03</td>
<td>1.71E-11</td>
<td></td>
</tr>
</tbody>
</table>

Table 2, convergence of numerical error when step size is halved using a forcing function.

Results presented in Table 2 confirm that the program is fourth order accurate.
4.2.2 Spectral Accuracy

For a method of p-th order the numerical error is reduced by a factor of $2^p$ every time the resolution is doubled. For a spectrally accurate method, the order of accuracy increases with the resolution for a spectrally accurate method.

To measure the numerical error due to spatial discretization, the assumption that this error is negligible for $N$ equals 512 compared $N$ values lower than 128. With this assumption the results from $N$ equals 512 can be used as analytical values. To ensure that the time step $h$ is small enough for the numerical error to be predominated by the error due to spatial discretization the error due to temporal discretization can be measured by comparing the results from different step sizes.

Results presented in Table 3 do indicate that the program is spectrally accurate. The order of accuracy increases with $N$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|v_N - v_{512}|_\infty$</th>
<th>$\log_2 \frac{|v_N - v_{512}|<em>\infty}{|v</em>{2N} - v_{512}|_\infty}$</th>
<th>Error from time step size.</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>3.70E-01</td>
<td>1.34E+00</td>
<td>5.04E-02</td>
</tr>
<tr>
<td>16</td>
<td>1.46E-01</td>
<td>2.67E+00</td>
<td>1.78E-04</td>
</tr>
<tr>
<td>32</td>
<td>2.29E-02</td>
<td>5.64E+00</td>
<td>7.62E-07</td>
</tr>
<tr>
<td>64</td>
<td>4.60E-04</td>
<td>1.15E+01</td>
<td>2.98E-09</td>
</tr>
<tr>
<td>128</td>
<td>1.64E-07</td>
<td></td>
<td>1.44E-11</td>
</tr>
</tbody>
</table>

Table 3, comparison of results for different $N$ to verify that the program is spectral accurate.
5 Results

Three series have been calculated numerically. Each case is defined by a Mach number, frequency, temperature and pressure. Each case have been numbered \( S*K* \), where the number after \( S \) denotes which series it belongs to and the number after \( K \) the number in the series. Example, \( S1K3 \) is the third case in the first series of cases. The initial steady state pressure and temperature are constant for all calculations. For each series, a different frequency has been used.

\[
p_0 = 1.01 \cdot 10^5 \text{ Pa} \quad \quad T_0 = 300 \text{ K}
\]

5.1 Series 1

\( f_0 = 5 \cdot 10^6 \quad Kn = 6.8 \cdot 10^{-4} \)

<table>
<thead>
<tr>
<th>( Ma )</th>
<th>S1K1</th>
<th>S1K2</th>
<th>S1K3</th>
<th>S1K4</th>
<th>S1K5</th>
<th>S1K6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Re_{ac} )</td>
<td>5.02e+00</td>
<td>1.00e+01</td>
<td>1.51e+01</td>
<td>2.01e+01</td>
<td>2.52e+01</td>
<td>3.01e+01</td>
</tr>
</tbody>
</table>

Table 4, Mach and acoustic Reynolds number.

<table>
<thead>
<tr>
<th>( B )</th>
<th>S1K1</th>
<th>S1K2</th>
<th>S1K3</th>
<th>S1K4</th>
<th>S1K5</th>
<th>S1K6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B/B_C )</td>
<td>4.445E+00</td>
<td>4.657E+00</td>
<td>4.753E+00</td>
<td>4.835E+00</td>
<td>4.917E+00</td>
<td>5.007E+00</td>
</tr>
<tr>
<td>( B_{B}/B_{C} )</td>
<td>9.230E-01</td>
<td>9.671E-01</td>
<td>9.870E-01</td>
<td>1.004E+00</td>
<td>1.021E+00</td>
<td>1.040E+00</td>
</tr>
<tr>
<td>( b_R )</td>
<td>5.840E-03</td>
<td>1.310E-02</td>
<td>1.816E-02</td>
<td>1.40E-01</td>
<td>3.040E-01</td>
<td>6.241E-01</td>
</tr>
</tbody>
</table>

Table 5, results obtained for the first series.

<table>
<thead>
<tr>
<th>( \Delta m )</th>
<th>S1K1</th>
<th>S1K2</th>
<th>S1K3</th>
<th>S1K4</th>
<th>S1K5</th>
<th>S1K6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta p )</td>
<td>8.95E-03</td>
<td>1.24E-02</td>
<td>1.19E-02</td>
<td>1.06E-02</td>
<td>2.30E-02</td>
<td>2.25E-02</td>
</tr>
<tr>
<td>( \Delta E )</td>
<td>7.21E-06</td>
<td>3.47E-06</td>
<td>4.46E-06</td>
<td>1.64E-06</td>
<td>6.36E-06</td>
<td>1.32E-06</td>
</tr>
</tbody>
</table>

Table 6, error estimates.

<table>
<thead>
<tr>
<th>( \Delta p ) after one period</th>
<th>S1K1</th>
<th>S1K2</th>
<th>S1K3</th>
<th>S1K4</th>
<th>S1K5</th>
<th>S1K6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Ma^2 Kn^2 \rho )</td>
<td>4.54E-12</td>
<td>2.74E-11</td>
<td>1.54E-10</td>
<td>8.34E-10</td>
<td>2.29E-09</td>
<td>3.15E-09</td>
</tr>
</tbody>
</table>

Table 7, the increase in momentum after one period.
5.2 Series 2

\[ f_o = 1 \cdot 10^7 \quad Kn = 1.3 \cdot 10^{-3} \]

<table>
<thead>
<tr>
<th>Series 2K</th>
<th>S2K1</th>
<th>S2K2</th>
<th>S2K3</th>
<th>S2K4</th>
<th>S2K5</th>
<th>S2K6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Ma )</td>
<td>5.00E-02</td>
<td>1.00E-01</td>
<td>1.50E-01</td>
<td>2.00E-01</td>
<td>2.50E-01</td>
<td>3.00E-01</td>
</tr>
<tr>
<td>( Re_{ac} )</td>
<td>2.51E+00</td>
<td>5.02E+00</td>
<td>7.54E+00</td>
<td>1.00E+01</td>
<td>1.26E+01</td>
<td>1.51E+01</td>
</tr>
</tbody>
</table>

Table 8, Mach and acoustic Reynolds number.

<table>
<thead>
<tr>
<th>Series 2K</th>
<th>S2K1</th>
<th>S2K2</th>
<th>S2K3</th>
<th>S2K4</th>
<th>S2K5</th>
<th>S2K6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B )</td>
<td>8.025E+00</td>
<td>8.918E+00</td>
<td>9.245E+00</td>
<td>9.464E+00</td>
<td>9.664E+00</td>
<td>9.865E+00</td>
</tr>
<tr>
<td>( B/C_B )</td>
<td>8.331E-01</td>
<td>9.259E-01</td>
<td>9.598E-01</td>
<td>9.825E-01</td>
<td>1.003E+00</td>
<td>1.024E+00</td>
</tr>
<tr>
<td>( B_B )</td>
<td>8.022E+00</td>
<td>8.881E+00</td>
<td>9.139E+00</td>
<td>9.265E+00</td>
<td>9.340E+00</td>
<td>9.389E+00</td>
</tr>
<tr>
<td>( b_R )</td>
<td>1.119E-02</td>
<td>2.268E-02</td>
<td>4.270E-02</td>
<td>8.597E-02</td>
<td>1.804E-01</td>
<td>2.226E-01</td>
</tr>
</tbody>
</table>

Table 9, results obtained for the second series.

<table>
<thead>
<tr>
<th>Series 2K</th>
<th>S2K1</th>
<th>S2K2</th>
<th>S2K3</th>
<th>S2K4</th>
<th>S2K5</th>
<th>S2K6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_a )</td>
<td>1.47E-02</td>
<td>1.77E-02</td>
<td>1.53E-02</td>
<td>2.04E-02</td>
<td>3.17E-02</td>
<td>1.34E-02</td>
</tr>
<tr>
<td>( e_E )</td>
<td>1.34E-06</td>
<td>3.24E-07</td>
<td>7.67E-06</td>
<td>1.35E-06</td>
<td>3.02E-07</td>
<td>3.27E-07</td>
</tr>
<tr>
<td>( \Delta m )</td>
<td>1.04E-20</td>
<td>1.50E-20</td>
<td>1.93E-20</td>
<td>4.93E-20</td>
<td>6.77E-20</td>
<td>8.11E-20</td>
</tr>
<tr>
<td>( \Delta p )</td>
<td>1.07E-09</td>
<td>3.82E-09</td>
<td>8.66E-09</td>
<td>1.53E-08</td>
<td>2.32E-08</td>
<td>3.19E-08</td>
</tr>
<tr>
<td>( \Delta E )</td>
<td>2.17E-07</td>
<td>8.81E-07</td>
<td>2.03E-06</td>
<td>3.74E-06</td>
<td>6.02E-06</td>
<td>8.97E-06</td>
</tr>
</tbody>
</table>

Table 10, error estimates.

<table>
<thead>
<tr>
<th>Series 2K</th>
<th>S2K1</th>
<th>S2K2</th>
<th>S2K3</th>
<th>S2K4</th>
<th>S2K5</th>
<th>S2K6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta p ) after one period</td>
<td>5.04E-10</td>
<td>1.14E-09</td>
<td>1.38E-09</td>
<td>2.34E-09</td>
<td>3.15E-09</td>
<td>3.99E-09</td>
</tr>
<tr>
<td>( Ma^2 Kn^2 \rho c \lambda )</td>
<td>7.77e-11</td>
<td>3.11e-10</td>
<td>6.99e-10</td>
<td>1.24e-09</td>
<td>1.94e-09</td>
<td>2.80e-09</td>
</tr>
</tbody>
</table>

Table 11, the increase in momentum after one period.
### 5.3 Series 3

\[ f_0 = 5 \cdot 10^7 \quad \quad Kn = 6.8 \cdot 10^{-3} \]

<table>
<thead>
<tr>
<th>( S3K1 )</th>
<th>( S3K2 )</th>
<th>( S3K3 )</th>
<th>( S3K4 )</th>
<th>( S3K5 )</th>
<th>( S3K6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Ma )</td>
<td>5.00E-02</td>
<td>1.00E-01</td>
<td>1.50E-01</td>
<td>2.00E-01</td>
<td>2.50E-01</td>
</tr>
<tr>
<td>( Re_{ac} )</td>
<td>5.02E-01</td>
<td>1.00E+00</td>
<td>1.51E+00</td>
<td>2.01E+00</td>
<td>2.51E+00</td>
</tr>
</tbody>
</table>

Table 12, Mach and acoustic Reynolds number.

<table>
<thead>
<tr>
<th>( S3K1 )</th>
<th>( S3K2 )</th>
<th>( S3K3 )</th>
<th>( S3K4 )</th>
<th>( S3K5 )</th>
<th>( S3K6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B )</td>
<td>1.479E+01</td>
<td>2.582E+01</td>
<td>3.273E+01</td>
<td>3.691E+01</td>
<td>3.965E+01</td>
</tr>
<tr>
<td>( B/C_B )</td>
<td>3.072E-01</td>
<td>5.360E-01</td>
<td>6.796E-01</td>
<td>7.696E-01</td>
<td>8.233E-01</td>
</tr>
<tr>
<td>( B_B )</td>
<td>1.529E+01</td>
<td>2.668E+01</td>
<td>3.368E+01</td>
<td>3.772E+01</td>
<td>4.011E+01</td>
</tr>
<tr>
<td>( B_B/C_B )</td>
<td>3.174E-01</td>
<td>5.539E-01</td>
<td>6.993E-01</td>
<td>7.832E-01</td>
<td>8.328E-01</td>
</tr>
<tr>
<td>( b_R )</td>
<td>5.936E-02</td>
<td>1.269E-01</td>
<td>2.087E-01</td>
<td>3.175E-01</td>
<td>4.529E-01</td>
</tr>
</tbody>
</table>

Table 13, results obtained for the third series.

<table>
<thead>
<tr>
<th>( S3K1 )</th>
<th>( S3K2 )</th>
<th>( S3K3 )</th>
<th>( S3K4 )</th>
<th>( S3K5 )</th>
<th>( S3K6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_a )</td>
<td>2.08E-02</td>
<td>1.81E-02</td>
<td>1.83E-02</td>
<td>1.29E-02</td>
<td>1.81E-02</td>
</tr>
<tr>
<td>( e_E )</td>
<td>6.96E-07</td>
<td>5.65E-06</td>
<td>5.65E-06</td>
<td>4.22E-07</td>
<td>4.66E-07</td>
</tr>
<tr>
<td>( \Delta m )</td>
<td>1.03E-19</td>
<td>1.12E-19</td>
<td>1.13E-19</td>
<td>1.27E-19</td>
<td>1.78E-19</td>
</tr>
<tr>
<td>( \Delta p )</td>
<td>7.40E-09</td>
<td>9.44E-09</td>
<td>1.29E-08</td>
<td>1.76E-08</td>
<td>2.35E-08</td>
</tr>
<tr>
<td>( \Delta E )</td>
<td>3.00E-07</td>
<td>1.21E-06</td>
<td>2.74E-06</td>
<td>4.94E-06</td>
<td>7.81E-06</td>
</tr>
</tbody>
</table>

Table 14, error estimates.

<table>
<thead>
<tr>
<th>( S3K1 )</th>
<th>( S3K2 )</th>
<th>( S3K3 )</th>
<th>( S3K4 )</th>
<th>( S3K5 )</th>
<th>( S3K6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta p ) after one period</td>
<td>1.81E-09</td>
<td>2.17E-09</td>
<td>4.48E-09</td>
<td>7.12E-09</td>
<td>1.23E-08</td>
</tr>
<tr>
<td>( Ma^2Kn^2\rho c \lambda )</td>
<td>3.88e-10</td>
<td>1.55e-09</td>
<td>3.50e-09</td>
<td>6.21e-09</td>
<td>9.71e-09</td>
</tr>
</tbody>
</table>

Table 15, the increase in momentum after one period.
6 Discussion

6.1 Previous Results

The same type of waves was numerically treated using the 13 Moment Burnett equations in (Strömgren, 2002). Results from this thesis can be seen in appendix B. Both the 13 Moment Burnett equations and the Hybrid Burnett equations are second order in the Knudsen number. The differences between calculations made in (Strömgren, 2002) and the ones conducted in this thesis are the initial conditions and the set of equations used. As initial conditions, analytical solution of the inviscid Euler equations was used in (Strömgren, 2002). Here, the solution to the linearized Hybrid Burnett equations has been used.

In Figure 10, Figure 11 and Figure 12 the results the 13 moment Burnett equations and Hybrid Burnett equations are plotted in the same graph.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{\(\frac{B}{C_B}\) plotted as dots, \(\frac{B_B}{C_B}\) as rings and \(\frac{B}{C_B}\) from (Strömgren, 2002) as triangles for the first series plotted against the acoustic Reynolds number. The Knudsen number is \(Kn = 6.8 \cdot 10^{-4}\).}
\end{figure}
Figure 11, $B/C_B$ plotted as dots, $B/C_B$ as rings and $B/C_B$ from (Strömgren, 2002) as triangles for the second series plotted against the acoustic Reynolds number. The Knudsen number $Kn = 1.3 \cdot 10^{-3}$.

Figure 12, $B/C_B$ plotted as dots, $B/C_B$ as rings and $B/C_B$ from (Strömgren, 2002) as triangles for the third series plotted against the acoustic Reynolds number. The Knudsen number $Kn = 6.8 \cdot 10^{-3}$.

From these figures, it can be seen that the results obtained coincide with the results of (Strömgren, 2002). This is a strong indication that both these results and precious results have been correctly calculated. The relative small difference between the results is probably due to the difference in initial conditions.
6.2 Self Reflected Wave

In Figure 13, Figure 14 and Figure 15 $b_R/C_R$ have been plotted as a function of the acoustic Reynolds number.

Figure 13, $b_R/B$ as a function of acoustic Reynolds number. The Knudsen number $K_n = 6.8 \cdot 10^{-4}$.

Figure 14, $b_R/B$ as a function of acoustic Reynolds number. The Knudsen number $K_n = 1.3 \cdot 10^{-3}$.

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An increase in the amplitude of the self reflected wave can be observed. Also here, results coincide with results in (Strömgren, 2002).

6.3 Acoustic Saturation

In Figure 16, $\ln(v_0 \nu)$ has been plotted against the dimensionless time to illustrate the phenomenon of acoustic saturation. The difference between the two cases plotted is the initial velocity amplitude, for S1K6 $Ma = 0.3$ and for S1K1 $Ma = 0.05$. Even though the initial amplitudes are significantly different we can see that old age amplitudes are the same. The roughness of the two curves is due to the poor sampling.
6.4 Increase in Momentum

In the term with the mixed derivatives, the density value has been approximated with the steady state density. This will result in an error in the momentum conservation, as discussed in section 3.2.2. The magnitude of the error in the momentum after one period in time has been shown to be of the order $Ma^2Kn^2 \rho c \lambda$ as predicted.

A typical increase in momentum for one wavelength as a function of the dimensionless time is illustrated in Figure 17.

![Figure 17](image_url)

*Figure 17, deviation from initial momentum for one wavelength plotted against dimensionless time for S2K3. This type of deviation was observed for all simulations.*

A drastic increase in momentum has been observed at the beginning of the simulation where nonlinear effects are strong. When the nonlinear effects fade away, the term with the mixed derivative will lose its importance. This will stabilize the momentum. This has been observed for all simulations made.
7 Conclusion
The Hybrid Burnett equations suggested by Lars Söderholm at the department of Mechanics at KTH, see (Söderholm, 2006), have been used to describe high amplitude acoustic waves in moderately rarified gas. The weakly non linear version of the equations has been solved with a high order accurate Fourier method with the fourth-order Runge-Kutta method.

As initial condition, a sinusoidal wave traveling in one direction has been considered. As the wave is subjected to non linear effects, the wave deforms into a shock wave. The jump in the density, due to the shock wave, results in a so called self reflected wave. As the amplitudes decay, non linear effects can be neglected and the initial wave and the self reflected wave will assume the shape of two sinusoidal waves. The amplitudes of the two waves for large times are obtained through a least square fitting of the anzats below to the numerical results.

\[ V_1(t) = \frac{1}{2} B e^{-\text{Re}(\omega_2)t} e^{i(-\text{Re}(\omega_2)t+\alpha)} + \frac{1}{2} b_R e^{-\text{Re}(\omega_2)t} e^{i(\text{Re}(\omega_2)t+\beta)} \] (7.1)

Here \( B \) is the amplitude of the original wave and \( b_R \) the amplitude of the self reflected wave.

It have been concluded that.

- The program used is fourth-order accurate in time.
- The program is spectrally accurate in space.
- The thickness of the shock wave is inversely proportional to the acoustic Reynolds number.
- The second Fourier mode of the calculated old age wave is small compared to the first Fourier mode at the end time.
- The relative error of the least square fitted function to the numerical results is small.
- Mass is conserved.
- The momentum increases systematically. But the increase in the momentum is in the order of the error due to simplifications made in the equations.

From the results we have drawn the conclusions

- The phenomenon acoustic saturation is observed in results.
- Results coincide with analytical solution to Burgers’ equation for parameter choices where Burgers’ equation is expected to be valid, with a systematic deviation otherwise.
- Results from (Strömgren, 2002) have been validated.
8 Acknowledgements

I would like to thank my two supervisors, their help and support have been of great value. In alphabetical order they are Gunilla Kreiss at the department of Information Technology at Uppsala University and Lars Söderholm at the department of Mechanics at Royal Institute of Technology.
9 References


Appendix A

Eigenvalue

In order to calculate the stability region for a certain differential operator when integrated with a numerical scheme, the eigenvalue of the operator have to be calculated. To do this, the system of equations formed by equation (2.16), (2.17) and (2.18) has to be put into matrix form.

\[ f_i = A f \]

In order to do this, the term with mixed derivatives have been neglected and background values have been used for the viscosity and heat coefficient.

Having a vector \( \mathbf{v} \) with the velocities at \( x_n \), where \( n = 1, 2, ..., N \), the Fourier coefficients of the function and the spatial derivative of the function can be calculated through.

\[
V_n(t) = \frac{1}{N} \sum_{j=1}^{N} v(x_j, t) e^{-i2\pi k_n x_j}
\]

\[
\frac{dv}{dx}_{x=x_j} = \sum_{n=-\frac{2}{2N}}^{\frac{2}{2N}-1} i2\pi k_n V_n(t) e^{i2\pi k_n x_j}
\]

The Fast Fourier Algorithm can be regarded as a linear transformation from the physical to the Fourier space and be written with a transformation matrix \( \mathbf{T} \).

\[
\mathbf{V} = \mathbf{T} \mathbf{v}
\]

Since the transformation is linear we have.

\[
\mathbf{v} = \mathbf{T}^{-1} \mathbf{V}
\]

The derivative can be calculated through.

\[
\frac{dv}{dx} = \mathbf{T}^{-1} \mathbf{D} \mathbf{T} \mathbf{v}
\]

Using these reformulations an expression for the matrix \( \mathbf{A} \) can be obtained.

\[
\begin{bmatrix}
\rho_t \\
\mathbf{v}_t \\
\mathbf{T}_t
\end{bmatrix} =
\begin{bmatrix}
A_1 & A_2 & A_3 \\
A_4 & A_5 & A_6 \\
A_7 & A_8 & A_9
\end{bmatrix}
\begin{bmatrix}
\rho \\
\mathbf{v} \\
\mathbf{T}
\end{bmatrix}
\]

Where we have.

\[
A_1 = -\frac{\rho_0}{c_0} v \mathbf{T}^{-1} \mathbf{D} \mathbf{T}
\]

\[
A_2 = -\frac{\rho_0}{\rho_1 c_0} v \mathbf{T}^{-1} \mathbf{D} \mathbf{T}
\]

\[
A_3 = 0
\]
\[ A_4 = \omega_3 \frac{\mu^2}{\lambda^2 \rho_0^3 \nu_0 c_0} T^{-1} D^3 T - \frac{T_0 \rho_1}{\rho_0 v_0 c_0} k_B \mathbf{1} \]

\[ A_5 = \frac{4 \mu_0}{3 \lambda \rho_0} T^{-1} D^2 T - \frac{v_0}{c_0} \mathbf{v} T^{-1} D \mathbf{T} \]

\[ A_6 = -\frac{p_0}{T_0 v_0 c_0} T^{-1} D \mathbf{T} \]

\[ A_7 = 0 \]

\[ A_8 = -\frac{2m}{3k_B T_1 c_0 \rho_0} T^{-1} D \mathbf{T} + \frac{2}{3} (\theta_2 - \theta_4) \frac{2m}{3k_B \lambda^2 T_1 c_0 \rho_0^2} \frac{v_0}{c_0} T^{-1} D^3 T \]

\[ + \frac{4}{3} \frac{2m}{3k_B T_1 c_0 \rho_0} \frac{\mu_0}{\lambda} (T^{-1} D \mathbf{v})(T^{-1} D \mathbf{T}) \]

\[ A_9 = \frac{2m}{3k_B \lambda c_0 \rho_0} T^{-1} D^2 T - \frac{v_0}{c_0} \mathbf{v} T^{-1} D \mathbf{T} \]
Appendix B

Previous results

The following three pages are pages 48 to 50 in (Strömgren, 2002).

\[ f = 5 \cdot 10^6 \text{ Hz} \]

\[ \tilde{K}n \approx 1.1 \cdot 10^{-3}. \]

![Graph showing coefficients scaled by \( C_B \) as a function of \( Ma \).]

**Figure 4.10.** The coefficients \( B \) and \( B_B \) scaled by \( C_B \), as a function of \( Ma \).

<table>
<thead>
<tr>
<th>( Ma )</th>
<th>( Re_{ac} )</th>
<th>( B )</th>
<th>( B/C_B )</th>
<th>( B_B )</th>
<th>( B_B/C_B )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>5.02</td>
<td>4.4451</td>
<td>0.9230</td>
<td>4.403</td>
<td>0.9220</td>
<td>0.0059</td>
</tr>
<tr>
<td>0.10</td>
<td>10.05</td>
<td>4.6578</td>
<td>0.9672</td>
<td>4.6325</td>
<td>0.9619</td>
<td>0.0129</td>
</tr>
<tr>
<td>0.15</td>
<td>15.07</td>
<td>4.7549</td>
<td>0.9873</td>
<td>4.6945</td>
<td>0.9748</td>
<td>0.0182</td>
</tr>
<tr>
<td>0.20</td>
<td>20.09</td>
<td>4.8353</td>
<td>1.0040</td>
<td>4.7252</td>
<td>0.9812</td>
<td>0.1137</td>
</tr>
<tr>
<td>0.25</td>
<td>25.12</td>
<td>4.9177</td>
<td>1.0211</td>
<td>4.7435</td>
<td>0.9850</td>
<td>0.3060</td>
</tr>
<tr>
<td>0.30</td>
<td>30.14</td>
<td>5.0077</td>
<td>1.0398</td>
<td>4.7556</td>
<td>0.9875</td>
<td>0.6240</td>
</tr>
</tbody>
</table>

**Table 4.6.** Computed results.

<table>
<thead>
<tr>
<th>( Ma )</th>
<th>( Re_{ac} )</th>
<th>Time(s)</th>
<th>( \Delta m )</th>
<th>( \Delta P )</th>
<th>( \Delta E )</th>
<th>( e_E )</th>
<th>( e_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>5.02</td>
<td>1411</td>
<td>2.3e-19</td>
<td>1.7e-12</td>
<td>4.0e-09</td>
<td>8.4e-05</td>
<td>2.4e-04</td>
</tr>
<tr>
<td>0.10</td>
<td>10.05</td>
<td>2143</td>
<td>1.4e-19</td>
<td>9.0e-11</td>
<td>2.1e-07</td>
<td>8.7e-05</td>
<td>2.3e-04</td>
</tr>
<tr>
<td>0.15</td>
<td>15.07</td>
<td>4921</td>
<td>2.6e-19</td>
<td>6.9e-10</td>
<td>6.5e-07</td>
<td>8.3e-05</td>
<td>2.7e-04</td>
</tr>
<tr>
<td>0.20</td>
<td>20.09</td>
<td>6014</td>
<td>2.6e-19</td>
<td>1.4e-09</td>
<td>1.1e-06</td>
<td>8.1e-05</td>
<td>3.8e-04</td>
</tr>
<tr>
<td>0.25</td>
<td>25.12</td>
<td>8434</td>
<td>2.2e-19</td>
<td>2.0e-09</td>
<td>1.6e-06</td>
<td>7.1e-05</td>
<td>7.6e-04</td>
</tr>
<tr>
<td>0.30</td>
<td>30.14</td>
<td>11699</td>
<td>3.0e-19</td>
<td>1.3e-09</td>
<td>2.4e-06</td>
<td>5.4e-05</td>
<td>1.4e-03</td>
</tr>
</tbody>
</table>

**Table 4.7.** Error estimates.
\begin{align*}
f & = 10^7 \text{ Hz} \\
\tilde{K}n & \approx 2 \cdot 10^{-3}.
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure.png}
\caption{The coefficients $B$ and $B_R$ scaled by $C_B$, as a function of $Ma$.}
\end{figure}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$Ma$ & $Re_{ac}$ & $B$ & $B/C_B$ & $B_R$ & $B_R/C_B$ & $b$ \\
\hline
0.05 & 2.51 & 8.0207 & 0.8327 & 8.0216 & 0.8328 & 0.0112 \\
0.10 & 5.02 & 8.9167 & 0.9258 & 8.8806 & 0.9220 & 0.0248 \\
0.15 & 7.54 & 9.2420 & 0.9595 & 9.1392 & 0.9489 & 0.0426 \\
0.20 & 10.05 & 9.4637 & 0.9826 & 9.2651 & 0.9619 & 0.0860 \\
0.25 & 12.56 & 9.6635 & 1.0033 & 9.3397 & 0.9697 & 0.1808 \\
0.30 & 15.07 & 9.8672 & 1.0244 & 9.3891 & 0.9748 & 0.2251 \\
\hline
\end{tabular}
\caption{Computed results.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
$Ma$ & $Re_{ac}$ & Time & $\Delta m$ & $\Delta P$ & $\Delta E$ & $e_E$ & $e_a$ \\
\hline
0.05 & 2.51 & 718 & 1.2e-19 & 3.9e-13 & 2.7e-10 & 6.9e-05 & 9.2e-04 \\
0.10 & 5.02 & 718 & 8.1e-20 & 2.4e-11 & 1.6e-08 & 8.2e-05 & 9.8e-04 \\
0.15 & 7.54 & 731 & 8.8e-20 & 2.6e-10 & 1.8e-07 & 7.8e-05 & 9.4e-04 \\
0.20 & 10.05 & 1104 & 6.8e-20 & 6.9e-10 & 5.1e-07 & 7.4e-05 & 9.3e-04 \\
0.25 & 12.56 & 1302 & 7.5e-20 & 1.0e-09 & 7.3e-07 & 6.3e-05 & 1.2e-03 \\
0.30 & 15.07 & 2548 & 6.1e-20 & 9.6e-10 & 1.0e-06 & 5.7e-05 & 1.8e-03 \\
\hline
\end{tabular}
\caption{Error estimates.}
\end{table}
\( f = 5 \cdot 10^7 \text{ Hz} \)

\( Kn \approx 1 \cdot 10^{-2} \).

![Graph](image)

**Figure 4.12.** The coefficients \( B \) and \( B_R \) scaled by \( C_B \), as a function of \( Ma \).

<table>
<thead>
<tr>
<th>( Ma )</th>
<th>( Re_{oc} )</th>
<th>( B )</th>
<th>( B/C_B )</th>
<th>( B_R )</th>
<th>( B_R/C_B )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.50</td>
<td>14.7898</td>
<td>0.3071</td>
<td>15.2874</td>
<td>0.3174</td>
<td>0.0590</td>
</tr>
<tr>
<td>0.10</td>
<td>1.00</td>
<td>25.8373</td>
<td>0.5365</td>
<td>26.6765</td>
<td>0.5539</td>
<td>0.1265</td>
</tr>
<tr>
<td>0.15</td>
<td>1.51</td>
<td>32.7442</td>
<td>0.6799</td>
<td>33.6790</td>
<td>0.6993</td>
<td>0.2095</td>
</tr>
<tr>
<td>0.20</td>
<td>2.01</td>
<td>36.9285</td>
<td>0.7668</td>
<td>37.7169</td>
<td>0.7832</td>
<td>0.3174</td>
</tr>
<tr>
<td>0.25</td>
<td>2.51</td>
<td>39.6650</td>
<td>0.8236</td>
<td>40.1079</td>
<td>0.8328</td>
<td>0.4527</td>
</tr>
<tr>
<td>0.30</td>
<td>3.01</td>
<td>41.6851</td>
<td>0.8656</td>
<td>41.6215</td>
<td>0.8643</td>
<td>0.6174</td>
</tr>
</tbody>
</table>

**Table 4.10.** Computed results.

<table>
<thead>
<tr>
<th>( Ma )</th>
<th>( Re_{oc} )</th>
<th>( \text{Time(s)} )</th>
<th>( \Delta m )</th>
<th>( \Delta P )</th>
<th>( \Delta E )</th>
<th>( \epsilon_E )</th>
<th>( \epsilon_\alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.50</td>
<td>167</td>
<td>1.2e-20</td>
<td>4.0e-15</td>
<td>1.7e-12</td>
<td>9.8e-06</td>
<td>2.3e-02</td>
</tr>
<tr>
<td>0.10</td>
<td>1.00</td>
<td>169</td>
<td>8.5e-21</td>
<td>3.6e-14</td>
<td>1.8e-11</td>
<td>2.7e-05</td>
<td>2.3e-02</td>
</tr>
<tr>
<td>0.15</td>
<td>1.51</td>
<td>167</td>
<td>1.4e-20</td>
<td>1.9e-13</td>
<td>9.7e-11</td>
<td>4.3e-05</td>
<td>2.3e-02</td>
</tr>
<tr>
<td>0.20</td>
<td>2.01</td>
<td>167</td>
<td>8.5e-21</td>
<td>6.8e-13</td>
<td>3.6e-10</td>
<td>4.5e-05</td>
<td>2.3e-02</td>
</tr>
<tr>
<td>0.25</td>
<td>2.51</td>
<td>169</td>
<td>1.2e-20</td>
<td>1.9e-12</td>
<td>1.0e-09</td>
<td>4.8e-05</td>
<td>2.4e-02</td>
</tr>
<tr>
<td>0.30</td>
<td>3.01</td>
<td>169</td>
<td>1.2e-20</td>
<td>4.4e-12</td>
<td>2.4e-09</td>
<td>4.0e-05</td>
<td>2.4e-02</td>
</tr>
</tbody>
</table>

**Table 4.11.** Error estimates.
## Appendix C

### Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>Diffusivity</td>
<td>$kg \ m^{-1} \ s^{-1}$</td>
</tr>
<tr>
<td>$c$</td>
<td>Speed of sound</td>
<td>$m/s$</td>
</tr>
<tr>
<td>$c_p$</td>
<td>Specific heat at constant pressure</td>
<td>$J \ kg^{-1} \ K^{-1}$</td>
</tr>
<tr>
<td>$c_v$</td>
<td>Specific heat at constant volume</td>
<td>$J \ kg^{-1} \ K^{-1}$</td>
</tr>
<tr>
<td>$f_0$</td>
<td>Frequency</td>
<td>$s^{-1}$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Ratio of specific heats</td>
<td>$1$</td>
</tr>
<tr>
<td>$k_B$</td>
<td>Boltzmann’s constant</td>
<td>$J \ K^{-1}$</td>
</tr>
<tr>
<td>$K_s$</td>
<td>Sutherland’s formula coefficient</td>
<td>$kg \ m^{-1} \ s^{-1} \ K^{-1/2}$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Heat conductivity coefficient</td>
<td>$J \ K^{-1} \ m^{-1} \ s^{-1}$</td>
</tr>
<tr>
<td>$Kn$</td>
<td>Knudsen number</td>
<td>$1$</td>
</tr>
<tr>
<td>$l$</td>
<td>Mean free path</td>
<td>$m$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Wave length</td>
<td>$m$</td>
</tr>
<tr>
<td>$m$</td>
<td>Molecular mass</td>
<td>$kg$</td>
</tr>
<tr>
<td>$Ma$</td>
<td>Mach number</td>
<td>$1$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Dynamic viscosity</td>
<td>$kg \ m^{-1} \ s^{-1}$</td>
</tr>
<tr>
<td>$p$</td>
<td>Pressure</td>
<td>$kg \ m \ s^{-2}$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Burnett second-order stress coefficients</td>
<td>$1$</td>
</tr>
<tr>
<td>$P$</td>
<td>Pressure tensor</td>
<td>$kg \ m \ s^{-2}$</td>
</tr>
<tr>
<td>$Pr$</td>
<td>Prandtl number</td>
<td>$1$</td>
</tr>
<tr>
<td>$q$</td>
<td>Heat flux</td>
<td>$kg \ s^{-3}$</td>
</tr>
<tr>
<td>$Re_{ac}$</td>
<td>Acoustic Reynolds number</td>
<td>$1$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density</td>
<td>$kg \ m^{-3}$</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
<td>$s$</td>
</tr>
<tr>
<td>$T$</td>
<td>Temperature</td>
<td>$K$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Burnett second-order heat transfer coefficients</td>
<td>$1$</td>
</tr>
<tr>
<td>$v$</td>
<td>Velocity</td>
<td>$m/s$</td>
</tr>
<tr>
<td>$x$</td>
<td>Physical space</td>
<td>$m$</td>
</tr>
</tbody>
</table>