Static Hedging

Julie Loucks

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Handledare: Henrik Wanntorp, Swedbank
Examinator: Erik Ekström

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Abstract. The standard method for hedging an option is to use delta hedging. This method, however, is not only time consuming but also costly as it requires continuous rebalancing. We are therefore interested in alternative ways of hedging. Static hedging is a method in which the hedging portfolio needs only be rebalanced a finite number of times (or never). The aim of this thesis is to give a clear definition of static hedging and explain its advantages. We discuss a few types of contracts which we know can be hedged using a statically replicating portfolio. Detailed descriptions of some static hedging methods are given. We explain how the hedges are constructed and under which assumptions they replicate the contract. The hedging methods also give a price for the contracts and we compare these prices with the analytical prices. We end by discussing the accuracy of one of the hedging methods when applied to a specific contract.
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1. Introduction

The process of reducing investment risk is called hedging and is very important for investors. When an investment involves unwanted risk one would like to hedge away part (or all) of it. There is a trade-off between risk and reward, so hedging away risk usually also decreases reward. Hedging is usually done by adding additional instruments to a portfolio, which incurs additional costs. One can think of these costs as an investment with the return being reduced risk. Suppose we write a put option. There is a certain risk involved in this, namely that we must be able to buy the underlying for the price of the strike at maturity. We are exposed to the risk of the underlying falling below the strike at maturity. To eliminate the risk we would like to create a hedge for the put option which matches the payoff in any situation. This perfect hedge should pay nothing at expiration if the underlying lies above the strike, and should pay the difference between the strike and the underlying if the underlying lies below the strike. For some contracts (as we will see) we can create a hedge which not only hedges the risk of the contract, but also gives a price for the contract in terms of other (simpler) contracts.

The standard method for hedging an option is to use delta hedging. This method, however, is not only time consuming but costly as it requires continuous rebalancing. We are therefore interested in alternative ways of hedging such as static hedging. Static hedging is a method in which the hedging portfolio needs only be rebalanced a finite number of times (or never).

The aim of this thesis is to give a clear definition of static hedging and explain its advantages. We discuss a few types of contracts which we know can be hedged using a statically replicating portfolio. Detailed descriptions of some static hedging methods are given and we end by discussing the accuracy of one of the hedging methods when applied to a specific contract. In Section 2, we give a brief review of risk neutral valuation and dynamic hedging and then give an introduction to static hedging. In Section 3, a method is presented to replicate and price digital options. In Section 4, a static hedge and price are given for barrier options and the results are compared with other known pricing formulas. Section 5, treats the case of discrete barrier options and in the final section we focus on hedging a specific type of discrete barrier option and examine its accuracy.
2. Pricing and Hedging

In this thesis we refer often to the risk neutral value of a contract, so we say a few words about it here. It is intuitive that we value a contract as the discounted expected value of its payoff. For example, when valuing a forward contract written on $S$ with strike $K$ at time $t$ we would have

$$ F(S, t; K) = e^{-r(T-t)} E[S_T - K], \quad (1) $$

where $r$ is the risk free rate and the contract expires at $T$. If $S$ follows a geometric Brownian motion (i.e. $dS_t = \mu S_t dt + \sigma S_t dW_t$) then

$$ F(S, t; K) = e^{-r(T-t)} E[S_T - K] = e^{-r(T-t)} E[S_T] - e^{-r(T-t)} E[K] 
= e^{-r(T-t)} e^{\mu(T-t)} S_t - e^{-r(T-t)} K. $$

The forward contract can easily be replicated by buying one unit of the underlying and selling $K$ bonds each worth 1 at $T$ (which have the price $e^{-r(T-t)} K$ at time $t$). This position gives the payoff

$$ \Phi(S_T, T) = S_T - K $$

which perfectly matches the payoff of the forward contract. Thus for $t \leq T$ we must have

$$ F(S, t; K) = \Phi(S, t) $$

or

$$ e^{-(r-\mu)(T-t)} S_t - e^{-r(T-t)} K = S_t - e^{-r(T-t)} K $$

which holds only when $\mu = r$ (assuming $S_t \neq 0$). This implies that $S_t$ must have the dynamics

$$ dS_t = rS_t dt + \sigma S_t dW_t. $$

We denote the risk neutral value of a contract with payoff $X$ by

$$ e^{-r(T-t)} E^Q[X] $$

to indicate that the expectation is taken under $Q$.

In the argument above we used what is known as the law of one price. This states that any two contracts with identical future payoffs, no matter how the future turns out, should have identical current prices. If this law does not hold there exist arbitrage opportunities. That is, we can set up a portfolio at no cost, have a positive probability of making money and zero probability of losing money. We use the concept of the law of one price throughout the thesis.
2.1. Dynamic hedging. We will now give a review delta hedging. Working in the Black-Scholes framework it is assumed we have a stock which follows a geometric Brownian motion and that we can invest in a risk free asset (perhaps a bank account). That is, letting \( B \) denote the risk free asset, we have the model

\[
\begin{align*}
    dB &= rB dt \\
    dS &= \mu S dt + \sigma S dW,
\end{align*}
\]

where \( r \) is the risk free rate, \( \sigma \) and \( \mu \) are constants and \( W \) is a Wiener process. We also assume:

- The market is arbitrage free.
- There are no transaction costs.
- There are no bid-ask spreads.
- Trading takes place continuously.

Of course not all of these are realistic assumptions, we will discuss this later. The idea behind delta hedging is to eliminate the risk associated with an option. We focus on a European call option with payoff \( \max(S_T - K, 0) = (S_T - K)^+ \).

In order to be perfectly hedged we would ideally want to buy an identical call option. If we cannot do this we instead try to replicate one. Suppose we are short a call option on \( S \) with strike \( K \) which expires at time \( T \). If we receive \( C_0 = C(S_{t_0}, t_0) \) for the option at initial time \( t_0 \), we want to owe \( e^{r(T-t_0)}C_0 \) at maturity \( T \). This makes our portfolio risk free. Recall the Greek delta given by

\[
\Delta_C = \frac{\partial C}{\partial S}(S, t),
\]

which measures the sensitivity of the price to a change in the underlying. To achieve our goal of making the option risk free, we want to have a long position of \( \Delta_C \) units of the underlying at all times \( t \leq T \). That is, we want the position

\[
\Delta_C S - C(S, t).
\]

At initial time we borrow \( \Delta_{C_0}S_{t_0} \) from a bank account and use it to buy \( \Delta_{C_0} \) units of the underlying where

\[
\Delta_{C_0} = \frac{\partial C}{\partial S}(S_{t_0}, t_0).
\]

The value of our initial portfolio is thus given by

\[
\Delta_{C_0}S_{t_0} - C(S_{t_0}, t_0).
\]

We now hold the desired portfolio, but \( \Delta_C \) is not constant with respect to \( S \). As soon as the underlying changes we hold the portfolio

\[
\Delta_{C_0}S_{t_1} - C(S_{t_1}, t_1),
\]
but we need the portfolio
\[ \Delta_{C_1} S_{t_1} - C(S_{t_1}, t_1). \]

We must rebalance our hedging portfolio by buying \( \Delta_{C_1} - \Delta_{C_0} \) units of the underlying. We use the bank account as needed to borrow or lend money. Since the underlying is constantly changing, this hedging portfolio must be continuously rebalanced in this way. Suppose now that we continuously rebalance our portfolio so that at all times \( t \leq T \) we hold the portfolio
\[ \Delta_CS - C(S, t). \]

At expiration, \( T \), we have two possible scenarios. The first is when \( S_T \geq K \). Then
\[ \Delta_{C_T} = \frac{\partial C}{\partial S}(S_T, T) = \frac{\partial}{\partial S}(S_T - K) = 1, \]
so we are long one unit of the underlying. It can be shown (see e.g. Chriss [6]) that the cost of this hedging method at time \( T \) (i.e. what is in the bank account) is \( e^{r(T-t_0)}C_{t_0} + K \). We then have the payoff
\[ \Delta_{C_T} S_T - C(S_T, T) - \text{(costs)} = S_T - (S_T - K) - e^{r(T-t_0)}C_{t_0} - K \]
\[ = -e^{r(T-t_0)}C_{t_0}, \]
as desired. The other possibility is that \( S_T < K \). In this case, since
\[ C(S_T, T) = 0, \]
\( \Delta_C = 0 \) and it can be shown that the cost of hedging is \( e^{r(T-t_0)}C_{t_0} \).

Our hedging portfolio gives the final payoff
\[ \Delta_{C_T} S_T - C(S_T, T) - \text{(costs)} = 0 - 0 - e^{r(T-t_0)}C_{t_0} \]
\[ = -e^{r(T-t_0)}C_{t_0}, \]
as desired.

Unfortunately, this perfect hedge is not possible to achieve. In reality trading does not take place continuously so at best we rebalance the portfolio at discrete time points, perhaps on a daily basis. This introduces risk in between the rebalance times. Suppose that instead of continuously rebalancing, we rebalance the hedging portfolio at discrete times \( t_1 < t_2 < t_3 < \ldots < t_n < T \). We rebalance the portfolio in the same way as in the continuous case. At initial time, we have the portfolio
\[ \Delta_{C_0} S_{t_0} - C(S_{t_0}, t_0). \]

We receive \( C(S_{t_0}, t_0) \) from selling the call option and then borrow \( \Delta_{C_0} S_{t_0} \) to buy the underlying. We hold this portfolio until the next rebalance time \( t_1 \). Then the portfolio is worth
\[ \Delta_{C_0} S_{t_1} - C(S_{t_1}, t_1). \]
If $\Delta C_1 \neq \Delta C_0$, we rebalance the portfolio by buying $\Delta C_1 - \Delta C_0$ units of the underlying. We continue doing this at every rebalance time so that at time $t_i$ we hold the portfolio

$$\Delta C_i S_{t_i} - C(S_{t_i}, t_i).$$

In the limit, as we increase the frequency of rebalancing, this creates a perfect hedge, as described above.

Recall the Greek gamma of an option is defined by

$$\Gamma_F = \frac{\partial^2 F}{\partial S^2}.$$  

The gamma gives a measure of how quickly the value of the option changes with respect to the underlying, or the rate of change of the delta with respect to a change in the underlying. Minimizing the gamma therefore decreases the frequency that the hedging portfolio needs to be rebalanced. To do this we use the method of delta-gamma hedging. Although delta-gamma hedging is an improvement on delta hedging, we still have the problem that $\Gamma_F$ is dependent on $S$. For large movements in $S$ the portfolio may need to be rebalanced, accruing large transaction costs.

Another thing to consider is that the price of an option may be affected by other factors than just the value of the underlying. For example, a change in volatility can affect the price of an option and delta hedging does not hedge against this. For this we would consider vega hedging. Recall the Greek vega gives the rate of change of the value of an option with respect to volatility:

$$\mathcal{V} = \frac{\partial C}{\partial \sigma}.$$  

We won’t discuss this method further.

A problem with dynamic hedging is that in the real world there are transaction costs. This makes this hedging method with many transactions very costly. We run into another problem with delta hedging when we attempt to delta hedge a barrier option. If we let $F(S, t)$ denote the price of a barrier option written on $S$, we want to hold

$$\Delta_F S - F(S, t)$$

at all times $t \leq T$. The pricing function is discontinuous at the barrier when the option is close to expiry. If the underlying lies near the barrier, the delta explodes making it unrealistic to hold $\Delta_F$ units of the underlying, see Figure 1.

We now take a look at the assumptions we made at the beginning of this section. We mentioned above that trading does not take place
continuously. We also mentioned that there are transactions costs. There are also bid-ask spreads which add to the total financial cost of delta hedging. We assume that the volatility of the underlying is constant when assuming it follows a geometric Brownian motion but in the market we observe a non-constant volatility. This is called the volatility skew. The assumption of an arbitrage free market is (sufficiently) realistic.

2.2. Static hedging. In the previous section we described a hedging method based only on the underlying. We will now assume that standard options (i.e, European calls and puts) are liquidly traded and can be used as hedging instruments. It then turns out that, for some derivatives, it is possible to construct a hedge that requires rebalancing only a few times (or never).

**Definition 2.1.** We call a hedge strong static if the hedge can be created at initial time and perfectly hedges the payoff of the option with no further trading.

**Definition 2.2.** A hedge is called weak static if the hedge requires a finite number of trades to perfectly hedge the option.

**Remark** If we have a hedging portfolio which does not create a perfect hedge but an arbitrarily close approximation we call this an almost strong (weak) static hedge.

The idea behind static hedging of an option is to create a portfolio of simpler options (generally vanilla puts and calls) which replicates the payoff of the option. Ideally one would be able to set up the portfolio at initial time and no more trading would be required. For most exotic options additional trading is needed. In the succeeding sections we describe hedging methods for certain exotic options which require at most one additional trade.
Example Suppose we want to replicate a European put option with strike $K$ expiring at $T$. The put option has the final payoff

$$X = (K - S_T)^+.$$ 
We can replicate this in the following way. We begin by shorting one unit of the underlying, this has payoff $-S_T$ at $T$. We then buy $K$ bonds each worth 1 at time $T$ so that our payoff is $K - S_T$. To make the portfolio worthless if $S_T > K$, we buy a call option with strike $K$. We now have the final payoff

$$K - S_T + (S_T - K)^+ = (K - S_T)^+$$
at time $T$. We set up this portfolio at initial time and it exactly matches the payoff of our put option without any further trading. We then hedge a long position in a put option by shorting this replicating portfolio. This gives us a strong static hedge which is model independent. Note that we not only get a hedge for the put option, but we also get the price as

$$P(S; K) = Ke^{-r(T-t)} - S + C(S; K),$$
where $C(S; K)$ and $P(S; K)$ represent the price of a call and put option on $S$ at time $t$ struck at $K$, respectively. This relation is the well known put-call parity

Example Consider a slightly more complicated payoff, a down-and-in call option with strike and barrier both equal to $K$ expiring at $T$. We assume $r = 0$. Using the indicator function the final payoff can be written as

$$X = (S_T - K)^+1_{S_t \leq K, \text{ for some } t \leq T}.$$ 
To hedge this option we begin by buying a put option with strike $K$, expiring at $T$. If or when $S_t = K$, we buy forward one unit of the underlying. The put and the forward contract together give the payoff

$$(K - S_T)^+ + (S_T - K) = (S_T - K)^+.$$ 
If the underlying never hits $K$, then the put option will expire worthless and we won’t enter the forward contract. This portfolio exactly replicates the payoff of the down-and-in call option with strike equal to the barrier. This hedging strategy requires at most one additional trade after the initial set up and hence is an example of a weak static hedge. This replication, of course, is only possible under the assumption that the underlying follows a continuous process so that it does not jump over the barrier $K$. In addition to the hedge, we get the price of this down-and-in call option. Since the forward contract is entered at zero cost, we get, rather surprisingly, that the price is given by

$$DIC(S; K) = P(S; K).$$
Example We now derive a hedge for an American digital put option. We work in the Black-Scholes framework under the assumption that the risk free rate is zero. An American digital option pays 1 at time
\( \tau \leq T \) where \( \tau \) is the first time a given barrier \( B \) is hit and pays nothing if \( S_t \neq B \) for all \( t \leq T \). We call the option a put when the underlying lies above the barrier at initial time. To hedge this option we want to find a portfolio with pricing function \( F(S, t) \) which satisfies

\[
\begin{cases}
F(S, T) = 0 & \text{if } S > B \text{ for all } t \leq T \\
F(B, t) = 1 & \text{for all } t \leq T
\end{cases}
\]  \hspace{1cm} (2)

Since an ordinary European digital put with strike \( B \) is worthless unless the barrier is hit we can use it in our hedge. Our first step in replicating the American digital put is to buy 2 units of an ordinary European digital put struck at \( B \). Using risk neutral valuation the price of a digital put is

\[
DP(S; B) = Q(S_T \leq B) = N(-d_2),
\]

where

\[
d_2 = \frac{\log \left( \frac{S}{B} \right) - \frac{\sigma^2}{2} (T - t)}{\sigma \sqrt{T - t}}
\]

and \( N(x) \) is the cumulative distribution function of the standard normal distribution evaluated at \( x \). When or if the underlying hits the barrier we want to receive 1 to replicate the American digital put. When \( S = B \) the digital puts are worth

\[
2DP(B; B) = 2N(-d_2) = 2N \left( \frac{-\log \left( \frac{B}{B} \right) - \frac{\sigma^2}{2} (T - t)}{\sigma \sqrt{T - t}} \right) = 2N \left( \frac{1}{2} \sigma \sqrt{T - t} \right). \hspace{1cm} (3)
\]

Since \( \frac{1}{2} \sigma \sqrt{T - t} \geq 0 \), we have that \( N \left( \frac{1}{2} \sigma \sqrt{T - t} \right) \geq \frac{1}{2} \) and hence the digital puts in (3) are worth more than 1. We must add another contract to get the correct payoff. That is, we need to find a contract with pricing function \( \Phi(S, t) \) so that

\[
2DP(B; B) + \Phi(B, t) = 1, \hspace{1cm} (4)
\]

or equivalently,

\[
2N(-d_2) - 1 = -\Phi(B, t).
\]
We calculate
\[
2N(-d_2) - 1 = N(-d_2) + N(-d_2) - 1 = N(-d_2) + 1 - N(d_2) - 1
\]
\[
= \frac{1}{B} (BN(-d_2) - BN(d_2))
\]
\[
= \frac{1}{B} \left( BN \left( \frac{1}{2} \sqrt{T-t} \right) - BN \left( -\frac{1}{2} \sqrt{T-t} \right) \right)
\]
\[
= \frac{1}{B} \left[ BN \left( \log \left( \frac{B}{S} \right) + \frac{1}{2} \sigma^2(T-t) \sigma \sqrt{T-t} \right) \right.
\]
\[
- BN \left( \log \left( \frac{B}{S} \right) - \frac{1}{2} \sigma^2(T-t) \sigma \sqrt{T-t} \right) \right].
\]
This is the price of either \( \frac{1}{B} \) puts or calls with strike \( B \) when the underlying is at the barrier. Thus the contracts
\[
- \Phi(S, t) = \frac{1}{B} P(S; B)
\]
and
\[
- \Phi(S, t) = \frac{1}{B} C(S; B).
\]
both solve equation (4). To replicate the American digital put, our replicating portfolio must also satisfy
\[
2DP(S_T; B) + \Phi(S_T, T) = 0 \text{ if } S_T > B \text{ for all } t \leq T
\]
so it should be clear that we must choose
\[
\Phi(S, t) = -\frac{1}{B} P(S; B).
\]
We now have a portfolio which is worth 1 when or if the underlying is at the barrier. At this point we can sell the portfolio and receive 1. If the underlying never hits the barrier it lies above the barrier at final time and thus the digital put and put options expire worthless, replicating the payoff of the American digital. Thus the portfolio
\[
F(S, t) = 2DP(S; B) - \frac{1}{B} P(S; B)
\]
satisfies the equations in (2). This hedge requires (at most) one additional trade after initial setup and thus gives a weak static hedge. Let \( AD(S; B) \) denote the value of an American digital put option on \( S \) with barrier \( B \). By the law of one price we also get the price of the option as
\[
AD(S; B) = 2DP(S; B) - \frac{1}{B} P(S; B).
\]
(5)
European digital puts and calls have become standardized on some underlyings, for example, on the OMXS30 index\(^1\). If they are not

\(^1\)The OMXS30 index is a weighted average of the 30 most traded stocks on the Stockholm Stock Exchange.
standardized, i.e. if the price is not directly observable, we can create an almost strong static hedge for a digital put as

$$DP(S; B) = \lim_{h \to 0} \frac{P(S; K) - P(S; K - h)}{h}. \quad (6)$$

We return to this approximation in Section 3. Using (6) we get an almost weak static hedge for the American digital put option by creating the portfolio

$$AD(S; B) \approx 2 \left( \frac{P(S; K) - P(S; K - h)}{h} \right) - \frac{1}{B} P(S; B),$$

for some small $h$.

**Remark** To hedge an American digital call (i.e. when $S_0 < B$) we buy the portfolio

$$AD(S; B) = 2DC(S; B) - \frac{1}{B} C(S; B)$$

or

$$AD(S; B) \approx 2 \left( \frac{C(S; K) - C(S; K + h)}{h} \right) - \frac{1}{B} C(S; B)$$

for some small $h$. If the barrier is never hit, the call and digital call options expire worthless. If or when the barrier is hit we liquidate the portfolio and receive 1. The derivation is similar to that in the case of the American digital put, so we omit it here.

One of the first steps in the development of static hedging was taken by Carr, Ellis and Gupta [5] using the put-call symmetry to create a static hedge for barrier and lookback options. We will discuss this method later in detail. Another interesting result is given in the paper by Derman, Ergener and Kani [8] where an approximate method for hedging barrier options is presented. Although this method does not give a reasonable hedge (it requires a portfolio of infinitely many contracts), this method does provide an arbitrage free price for barrier options in the presence of a skewed volatility. In Brown, Hobson and Rogers [4], quite good upper and lower bounds are found on the price of barrier options. This method is useful because it is model independent. In Andersen, Andreasen and Eliezer [1] static hedges are derived for barrier options and discrete barrier options. In Joshi [11] we are given a static hedging method for Asian options and discrete barrier options. Later in this paper we will describe the method for discrete barrier option hedging in detail.
3. STATIC HEDGING OF DIGITAL OPTIONS

In this section we study how to statically hedge and price European digital options. We will also briefly discuss the volatility skew and see how it affects the price. A digital option is an option which pays either 1 or nothing, depending on the value of the underlying at maturity. In particular, a digital call will pay 1 at maturity if the underlying lies above some strike $K$, and nothing otherwise. The payoff can be written, using the indicator function, as

$$\Phi(S_T) = 1_{[S_T \geq K]}.$$  

Using risk neutral valuation we find that the price of the digital call is given by

$$DC(S; K) = e^{-r(T-t)} E^Q \left[ 1_{[S_T \geq K]} \right] = e^{-r(T-t)} Q(S_T \geq K)$$

where

$$d_2 = \frac{\log \left( \frac{S}{K} \right) + (r - q - \frac{a^2}{2})(T-t)}{a \sqrt{T-t}},$$

see Figure 2.

![Figure 2](image)

**Figure 2.** Solid line: Digital call option payoff. Dashed line: Price of the digital call before maturity.

The price can also be found using a replication argument, as follows. We construct a call spread by buying $\frac{1}{2h}$ call options on the underlying with strike $(K - h)$ for some $h$ and selling $\frac{1}{2h}$ call options with strike $(K + h)$. Letting $C(K - h, \sigma)$ and $C(K + h, \sigma)$ denote the price of the calls, the call spread is worth

$$\frac{C(K - h, \sigma) - C(K + h, \sigma)}{2h}$$

at time $t \leq T$. The final payoff for these two call options is shown in Figure 3. From the figure we see that our portfolio of call options (the
call spread) approaches a digital call as we let $h$ go to zero. Note that the factor $\frac{1}{2h}$ is needed to get the final payoff $1_{\{S_T \geq K\}}$. To further see that this is a replicating portfolio we look at what happens at expiry. If the underlying lies above $K + h$ the portfolio is worth

$$\frac{C(K - h, \sigma) - C(K + h, \sigma)}{2h} = \frac{(S - (K - h)) - (S - (K + h))}{2h} = \frac{2h}{2h} = 1.$$ 

If the underlying lies below $K - h$ at expiry, both the calls expire worthless. Letting $h \to 0$, the possibility that $K - h < S_T < K + h$ vanishes so that this portfolio replicates the payoff of the digital call.

In Figure 4 we see the value of the replicating portfolio and the option itself. The error is very small until close to expiry and the error at expiry is reduced by reducing $h$.

**Figure 3.** Call spread payoff

**Figure 4.** Left: The solid line is the final payoff of the digital call and the dashed line is the payoff of the replicating portfolio. Right: The value of the replicating portfolio and the option just before expiry ($T - t = 0.05$).
Now that we have shown that this is a replicating portfolio we focus on the price which is given by

\[
DC(S; K) = \lim_{h \to 0} \frac{C(K - h, \sigma) - C(K + h, \sigma)}{2h}.
\]  

(8)

Using a Taylor series expansion we find that

\[
C(K - h, \sigma) = C(K, \sigma) - h \frac{\partial C}{\partial K}(K, \sigma) + O(h^2)
\]

and

\[
C(K + h, \sigma) = C(K, \sigma) + h \frac{\partial C}{\partial K}(K, \sigma) + O(h^2).
\]

Plugging this into equation (8), we get

\[
DC(S; K) = \lim_{h \to 0} \frac{-2h \frac{\partial C}{\partial K}(K, \sigma) + O(h^2)}{2h} = \lim_{h \to 0} \left[ - \frac{\partial C}{\partial K}(K, \sigma) + O(h) \right] = - \frac{\partial C}{\partial K}(K, \sigma).
\]

(9)

Does this price correspond to the risk neutral price? We know that the Black-Scholes price of a call option is

\[
C(K, \sigma) = e^{-q(T-t)}SN(d_1) - e^{-r(T-t)}KN(d_2),
\]

where

\[
d_1 = \frac{\log \left( \frac{S}{K} \right) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T-t}.
\]

Using this price we get

\[
DC(S; K) = - \frac{\partial C}{\partial K}(K, \sigma) = - \frac{\partial}{\partial K} \left( e^{-q(T-t)}SN(d_1) - e^{-r(T-t)}KN(d_2) \right)
\]

\[
= e^{-r(T-t)}N(d_2).
\]

Comparing with equation (7) we see that we do indeed get the same price. In Table 1 we look at an example to examine how small \( h \) needs to be to get a good approximation. We see that even with a fairly large value of \( h \) we have quite a close approximation.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \frac{1}{2h}(C(K - h, \sigma) - C(K + h, \sigma)) )</th>
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<td>20</td>
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<tr>
<td>10</td>
<td>0.5462</td>
</tr>
<tr>
<td>5</td>
<td>0.5463</td>
</tr>
<tr>
<td>1</td>
<td>0.5464</td>
</tr>
</tbody>
</table>

Table 1. Speed of convergence of the approximation of the digital call price. \( S = 1060.9, K = 1040, r = 0.33\%, \sigma = 15.53\%, T = 35 \) days. Analytical price = 0.5464.
**Example** Suppose we want to price a digital call written on the OMXS30 index. Digital calls are commonly traded\(^2\) so some digital call prices are observable and we can compare these prices to the Black-Scholes price. In Table 2 we show the observed digital call price and the Black-Scholes price for a variety of strikes, using the implied volatility computed from the call with the same strike. The prices of traded calls are shown in Table 3. We see that the replicating price is generally lower than the observed price.

<table>
<thead>
<tr>
<th>strike</th>
<th>bid</th>
<th>mid</th>
<th>ask</th>
<th>replication price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1020</td>
<td>0.66</td>
<td>0.685</td>
<td>0.71</td>
<td>0.6859</td>
</tr>
<tr>
<td>1040</td>
<td>0.53</td>
<td>0.555</td>
<td>0.58</td>
<td>0.5464</td>
</tr>
<tr>
<td>1060</td>
<td>0.39</td>
<td>0.415</td>
<td>0.44</td>
<td>0.3850</td>
</tr>
<tr>
<td>1080</td>
<td>0.27</td>
<td>0.285</td>
<td>0.30</td>
<td>0.2337</td>
</tr>
<tr>
<td>1100</td>
<td>0.17</td>
<td>0.185</td>
<td>0.20</td>
<td>0.1205</td>
</tr>
</tbody>
</table>

Table 2. Observed prices and prices computed by replicating down-and-in calls on the OMXS30 index. The spot is 1060.9. Source: Trading system at Swedbank Markets.

3.1. **Non-constant volatility.** Since the stock market crash in 1987, it is generally observed in the market that volatility is not constant. This is most likely due to the realization of investors that the market could crash. In the equities market we now see what is called the volatility skew. When the implied volatility is plotted against the strike, the volatility is generally a downward sloping function of strike. This means that volatility is higher for lower strikes and lower for larger strikes. In Figure 5 we see the implied volatility skew of call options on the OMXS30 index.

Let’s examine what happens with the digital call price when we still quote call prices using the Black-Scholes model, but don’t assume the volatility is constant. We suppose the volatility depends on the strike price, that is, \( \sigma = \sigma(K) \). We can replicate the payoff in the same manner as before, by buying calls with strike \( K - h \), for some small \( h \), and selling calls with strike \( K + h \). We can then write the price of the digital call as

\[
DC_{\sigma}(S; K) = \lim_{h \to 0} \frac{C(K - h, \sigma(K - h)) - C(K + h, \sigma(K + h))}{2h}.
\]

\(^2\)In the Swedish market digital calls and puts are traded under the names "över" and "under."
<table>
<thead>
<tr>
<th>Strike</th>
<th>Bid</th>
<th>Mid</th>
<th>Ask</th>
<th>Implied Volatility (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>950</td>
<td>96.5</td>
<td>98.875</td>
<td>101.25</td>
<td>20.79</td>
</tr>
<tr>
<td>960</td>
<td>88.75</td>
<td>90</td>
<td>89.375</td>
<td>20.13</td>
</tr>
<tr>
<td>970</td>
<td>79.5</td>
<td>80.12</td>
<td>80.75</td>
<td>19.68</td>
</tr>
<tr>
<td>980</td>
<td>70.25</td>
<td>71.5</td>
<td>70.87</td>
<td>18.90</td>
</tr>
<tr>
<td>990</td>
<td>61.25</td>
<td>62.5</td>
<td>61.87</td>
<td>18.19</td>
</tr>
<tr>
<td>1000</td>
<td>52.75</td>
<td>54</td>
<td>53.37</td>
<td>17.74</td>
</tr>
<tr>
<td>1020</td>
<td>36.75</td>
<td>38</td>
<td>37.37</td>
<td>16.57</td>
</tr>
<tr>
<td>1040</td>
<td>23.25</td>
<td>24.25</td>
<td>23.75</td>
<td>15.53</td>
</tr>
<tr>
<td>1060</td>
<td>13</td>
<td>13.75</td>
<td>13.37</td>
<td>14.72</td>
</tr>
<tr>
<td>1080</td>
<td>6.5</td>
<td>6.75</td>
<td>6.62</td>
<td>14.19</td>
</tr>
<tr>
<td>1100</td>
<td>2.7</td>
<td>3</td>
<td>2.85</td>
<td>13.84</td>
</tr>
<tr>
<td>1120</td>
<td>1.05</td>
<td>1.25</td>
<td>1.15</td>
<td>13.83</td>
</tr>
<tr>
<td>1140</td>
<td>0.4</td>
<td>0.45</td>
<td>0.42</td>
<td>13.89</td>
</tr>
</tbody>
</table>

Table 3. Implied volatility of call options written on the OMXS30 index. Data observed on April 16th and options expired May 21st. Source: Trading system at Swedbank Markets.

![Figure 5](image.png)

Figure 5. Black-Scholes implied volatility of OMXS30 call options expiring in 35 days, with spot 1060.9, risk free rate 0.33%, dividend yield 14.01% using an average of the bid and ask prices.

A second order Taylor series expansion of the call option prices yields

\[
C(K - h, \sigma(K - h)) = C(K, \sigma(K)) - h \frac{\partial C}{\partial K}(K, \sigma(K)) \\
- h \frac{\partial C}{\partial \sigma}(K, \sigma(K)) \frac{\partial \sigma}{\partial K}(K) + O(h^2)
\]
and
\[ C(K + h, \sigma(K + h)) = C(K, \sigma(K)) + h \frac{\partial C}{\partial K}(K, \sigma(K)) + \frac{h}{2} \frac{\partial^2 C}{\partial \sigma \partial K}(K, \sigma(K)) \sigma(K) \partial_K \sigma(K) + O(h^2). \]

We can then write the digital call price as
\[
DC_\sigma(S; K) = \lim_{h \to 0} \frac{C(K - h, \sigma(K - h)) - C(K + h, \sigma(K + h))}{2h} = \lim_{h \to 0} \frac{-2h \frac{\partial C}{\partial K}(K, \sigma(K)) - 2h \frac{\partial C}{\partial \sigma}(K, \sigma(K)) \frac{\partial \sigma}{\partial K}(K) + O(h^2)}{2h} = \lim_{h \to 0} \left[ -\frac{\partial C}{\partial K}(K, \sigma(K)) - \frac{\partial C}{\partial \sigma}(K, \sigma(K)) \frac{\partial \sigma}{\partial K}(K) \right].
\]

The first term we recognize from equation (9) as the Black-Scholes price of a digital call. The term \( \frac{\partial C}{\partial \sigma} \) is the Greek vega and the third term is the slope of the volatility skew. In Figure 6 we see the price of the digital call with non-constant volatility for a few different slopes of the volatility skew. A more negatively skewed volatility skew gives a higher price.

![Figure 6. Price of a digital call when volatility is non-constant for different slopes of the volatility skew.](image)

We have now derived the digital call price
\[
DC_\sigma(S; K) = -\frac{\partial C}{\partial K}(K, \sigma(K)) - \frac{\partial C}{\partial \sigma}(K, \sigma(K)) \frac{\partial \sigma}{\partial K}(K). \quad (10)
\]

Note that this incorporates the case when volatility is constant because this implies
\[
\frac{\partial \sigma}{\partial K} = 0.
\]
Let’s look more closely at equation (10). The vega is given by

$$\frac{\partial C}{\partial \sigma}(K, \sigma(K)) = e^{-r(T-t)} S \sqrt{T-t} \varphi(d_1),$$

where $\varphi(\cdot)$ is the standard normal probability density function. Therefore, $\varphi(d_1) \geq 0$ and as mentioned earlier we generally have $\frac{\partial \sigma}{\partial K} < 0$ (recall Figure 5). We thus get

$$-\frac{\partial C}{\partial \sigma}(K, \sigma(K)) \frac{\partial \sigma}{\partial K} \sigma(K) = -e^{-r(T-t)} S \sqrt{T-t} \varphi(d_1) \frac{\partial \sigma}{\partial K} \geq 0$$

or

$$DC_\sigma(S; K) = -\frac{\partial C}{\partial K}(K, \sigma) - \frac{\partial C}{\partial \sigma}(K, \sigma(K)) \frac{\partial \sigma}{\partial K} \geq -\frac{\partial C}{\partial K}(K, \sigma) = DC(S; K),$$

which says that the price of the digital call assuming non-constant volatility is higher than the price when volatility is assumed constant.

**Remark** Using the same reasoning as above, we get a replicating portfolio for a digital put as

$$DP(S; K) \approx P(S; K + h) - P(S, K - h)$$

for some small value of $h$. We also get the price for a digital put by noting that if we are long both a digital call and digital put, we are guaranteed to receive 1 at maturity. Thus we have the relation

$$e^{-r(T-t)} = DC(S; K) + DP(S; K)$$

or

$$DP(S; K) = e^{-r(T-t)} - DC(S; K),$$

for all $S, K$ and $t \leq T$. Since we have derived the price of the digital call, we also know the price of the digital put.

3.2. **Comparison with market data.** Consider a digital call written on the OMXS30 index with strike $K = 1020$, risk free rate 0.33% and dividend yield 14.01% which expires in 35 days. The current value of the underlying is 1060.9. We can write equation (10) as

$$DC_\sigma(S; K) = -\frac{\partial C}{\partial K} - \frac{\partial C}{\partial \sigma} \frac{\partial \sigma}{\partial K}$$

$$= e^{-r(T-t)} N(d_2) - e^{-q(T-t)} S \sqrt{T-t} \varphi(d_1) \cdot \frac{\partial \sigma}{\partial K}. \quad (11)$$

Both terms $e^{-r(T-t)} N(d_2)$ and $e^{-q(T-t)} S \sqrt{T-t} \varphi(d_1)$ can be calculated using the information provided by the market but we do not know the
slope of the volatility skew. We use the information in Table 3 for the two nearest strikes to estimate the slope:

\[
\frac{\partial \sigma}{\partial K} \approx \frac{\sigma(K + h) - \sigma(K - h)}{2h} = \frac{\sigma(1040) - \sigma(1000)}{2 \cdot 20} = -0.00055067,
\]

so equation (11) becomes, for constant volatility,

\[
DC(1060.9; 1020) = e^{-0.0033(35/365)}(0.6862) = 0.6859
\]

and for non-constant volatility

\[
DC_\sigma(1060.9; 1020) = e^{-0.0033(35/365)}(0.6862) - (111.992)(-0.00055067) = 0.7476.
\]

We compare the replicating prices to the observed market prices for a variety of strikes in Table 4. One would expect that the replication price with non-constant volatility more closely matches what is seen in the market.

<table>
<thead>
<tr>
<th>strike</th>
<th>bid</th>
<th>mid</th>
<th>ask</th>
<th>replication price constant volatility</th>
<th>replication price non-constant volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>990</td>
<td>0.83</td>
<td>0.85</td>
<td>0.87</td>
<td>0.8330</td>
<td>0.8776</td>
</tr>
<tr>
<td>1020</td>
<td>0.66</td>
<td>0.685</td>
<td>0.71</td>
<td>0.6859</td>
<td>0.7476</td>
</tr>
<tr>
<td>1040</td>
<td>0.53</td>
<td>0.555</td>
<td>0.58</td>
<td>0.5464</td>
<td>0.6054</td>
</tr>
<tr>
<td>1060</td>
<td>0.39</td>
<td>0.415</td>
<td>0.44</td>
<td>0.3850</td>
<td>0.4270</td>
</tr>
<tr>
<td>1080</td>
<td>0.27</td>
<td>0.285</td>
<td>0.30</td>
<td>0.2337</td>
<td>0.2562</td>
</tr>
<tr>
<td>1100</td>
<td>0.17</td>
<td>0.185</td>
<td>0.20</td>
<td>0.1205</td>
<td>0.1267</td>
</tr>
</tbody>
</table>

Table 4. Digital call prices on OMXS30 index, expiring in 35 days, risk free rate is 0.33% and dividend yield 14.01%, spot 1060.9, using implied volatility. Source: Trading system at Swedbank Markets.
4. Static hedging of barrier options

Barrier options are the most commonly traded exotic option. There are two general types of barrier options, knock in and knock out options. Knock out options give a specified payoff unless the underlying crosses a given barrier before expiry. If the barrier is crossed the option pays nothing or perhaps a rebate. Knock in options pay nothing (or possibly a rebate) unless the barrier is crossed before expiry, in which case the option pays a specified payoff.

We can price barrier options using risk-neutral valuation. Consider a down-and-in option which has payoff

\[ X = \Phi(S_T) \cdot 1_{[S_t \leq B \text{ for some } t \leq T]}, \]

where \( B \) is the barrier level. The price of this option is

\[ F(S, t) = e^{-r(T-t)}E^Q[X]. \]

For an example we give the price for a down-and-in call option. Letting \( DIC(S; K, B) \) denote the price of a down-and-in call on \( S \) with strike \( K \) and barrier \( B \), after a bit of work, we get the price

\[ DIC(S; K, B) = \left( \frac{B}{S} \right)^{\frac{2\bar{r}}{\sigma^2}} C \left( \frac{B^2}{S}; K \right), \] (12)

when \( B \leq K \), where \( \bar{r} = r - \frac{1}{2}\sigma^2 \). For \( B > K \) we get the price

\[ DIC(S; K, B) = \left( \frac{B}{S} \right)^{\frac{2\bar{r}}{\sigma^2}} \left[ C \left( \frac{B^2}{S}; B \right) + (B - K)DC \left( \frac{B^2}{S}; B \right) \right] \]

\[ - (B - K)DC(S; B) + C(S; K) - C(S; B). \] (13)

These pricing formulas can be found in e.g. Björk [2]. We have a price for the option, but we don’t have any apparent way of hedging this option besides using delta hedging.

In this section we describe in detail the weak static hedging method introduced in Carr, Ellis and Gupta [5] and Bowie and Carr [3]. A method is described in these papers for valuing and hedging barrier and lookback call and put options. This method can be extended to more complex final payoffs. We assume that markets are frictionless, arbitrage free and that a pricing function exists for European call options with any strike. We assume that the underlying follows a continuous process and we also assume no carrying costs under the martingale
measure. The assumption of no carrying costs generally means

\[ r = 0 \] if there are no dividends,
\[ r - q = 0 \] where \( q \) denotes the dividend yield,
\[ r_d - r_f = 0 \] for foreign exchange rate options.

We handle the case of non-zero carrying costs at the end of this section. Lastly, we assume that the volatility of the underlying satisfies a certain symmetry condition. This conditions assumes that the volatility is a function of the underlying and time. Denote this by \( \sigma(S_t, t) \) and the following symmetry condition must hold:

\[ \sigma(S_t, t) = \sigma(S^2/S_t, t), \text{ for all } t \leq T, \]

where \( S \) is the current value of the underlying. This assumption is needed for the put-call symmetry which we will use. This condition is trivially satisfied in the Black-Scholes model since volatility is assumed constant. This condition is also satisfied in models where the volatility is only a function of time. Stochastic volatility models do not satisfy this symmetry condition. Throughout the remainder of this section we work in the Black-Scholes framework.

**Lemma 4.1.** The put-call symmetry gives the relation

\[ C(S; K) = \left( \frac{K}{S} \right) P \left( S; \frac{S^2}{K} \right). \]

**Proof.** See Carr et al. [5].

We will focus on finding a weak static hedge for the down-and-in call option. In order to hedge the option we want to find a portfolio with price \( F(S, t) \) such that

\[
\begin{cases}
F(S, T) = 0 & \text{if } S_t > B \text{ for all } t \leq T, \\
F(B, t) = C(B; K).
\end{cases}
\]

If the barrier is never hit, the portfolio expires worthless. Otherwise, if the barrier is hit we can immediately sell the portfolio and use the proceeds to purchase a call with strike \( K \). First consider the case when \( B \leq K \) (recall that we showed the case \( B = K \) in the introduction of this paper, using a slightly simplified derivation). When the underlying hits the barrier \( (S = B) \), the put-call symmetry gives the relation

\[ C(B; K) = \frac{K}{B} P \left( B; \frac{B^2}{K} \right). \]

We start the hedge at initial time by buying \( \frac{K}{B} \) puts struck at \( \frac{B^2}{K} \) expiring at \( T \). Since \( B < K \) we have \( \frac{B^2}{K} < B \) so if the underlying never hits the barrier the put options will expires worthless. On the other hand, if the underlying hits the barrier we sell our put option and have exactly what it takes to buy a call option with strike \( K \), replicating
the payoff of the down-and-in call. We now have the desired portfolio, \( F(S, t) \), in (14):

\[
F(S, T) = \frac{K}{B} P\left( S; \frac{B^2}{K} \right) = 0 \quad \text{if} \quad S_t > B \quad \text{for all} \quad t \leq T
\]

and

\[
F(B, t) = \frac{K}{B} P\left( B; \frac{B^2}{K} \right) = C(B; K).
\]

Not only do we get a hedge, but we get the price of the option:

\[
DIC(S; K, B) = \frac{K}{B} P\left( S; \frac{B^2}{K} \right), \quad \text{when} \quad B \leq K.
\]  (15)

**Remark** Market practice when pricing barrier options is to use two different volatilities (corresponding to \( K \) and \( B \)). Using the above hedging method to price the option we see that only the volatility corresponding to \( \frac{B^2}{K} \) is used.

The case when \( B > K \) is a bit more complicated. When \( S = B \), the put-call parity gives

\[
C(B; K) = B - K + P(B; K).
\]

We begin our hedge at initial time by buying \( B - K \) American digital puts struck at \( B \) and a regular put struck at \( K \). If the underlying never hits the barrier both the American digitals and the put option will expire worthless since \( B > K \). When or if \( S = B \) the American digitals gives us \( B - K \). By the put-call parity, we can sell our portfolio and buy a call option struck at \( K \) giving us a perfect hedge for the down-and-in call. We now have the desired portfolio, \( F(S, t) \), in (14):

\[
F(S, T) = (B - K) AD(S; B) + P(S; K) = 0 \quad \text{if} \quad S_t > B \quad \text{for all} \quad t \leq T
\]

and

\[
F(B, t) = (B - K) AD(B; B) + P(B; K) = C(B; K).
\]

We also get the price for the down-and-in call:

\[
DIC(S; K, B) = (B - K) AD(S; B) + P(S; K).
\]

Using the price we found for the American digital in Section 2, (equation (5)) we get

\[
DIC(S; K, B) = (B - K) \left( 2DP(S; B) - \frac{1}{B} P(S; B) \right) + P(S; K). \quad (16)
\]

We have now derived a weak static hedge for a down-and-in call. This is a weak hedge since we need to liquidate the portfolio if the barrier is hit. The method is model dependent due to the symmetry assumption on the volatility.
Remark This method also gives us the price of (and hedging method for) a down-and-out call by the parity relation

\[ DIC(S; K, B) + DOC(S; K, B) = C(S; K). \]

We get the prices

\[ DOC(S; K, B) = C(S; K) - \frac{K}{B} P(S; \frac{B^2}{K}) \] (17)

if \( B \leq K \) and

\[ DOC(S; K, B) = C(S; K) - (B - K)AD(S; B) - P(S; K) \] (18)

if \( B > K \). To create a hedge for the case \( B \leq K \), we begin by buying the portfolio in (17). By the put-call symmetry, if or when the underlying hits the barrier \((S = B)\), we have the equality

\[ \frac{K}{B} P\left( B; \frac{B^2}{K} \right) = C(B; K). \]

We then liquidate the portfolio and are left with nothing. If the barrier is never hit, the put struck at \( \frac{B^2}{K} \) will expire worthless and our payoff is a call option struck at \( K \), just as desired.

For the case when \( B > K \), we begin our hedge at initial time by buying the portfolio in (18). By the put-call parity, if or when the underlying hits the barrier we have the equality

\[ C(B; K) = B - K + P(B; K) \]

so we can liquidate the portfolio at no cost. If the underlying never hits the barrier, the American digital put struck at \( B \) and the put struck at \( K \) both expire worthless and we are left with a call struck at \( K \), just as desired.

Remark We handle up options in a similar way. The price of an up-and-out call option is

\[ UOC(S; K, B) = C(S; K) - \left( \frac{K}{B} \right) C\left( S; \frac{B^2}{K} \right) - (B - K)AD(S; B) \] (19)

for \( B > K \). To hedge a short position in the option we buy the portfolio in (19). If the barrier is never hit \((S < B\) for all \( t \leq T\)), the American digital call and the regular call struck at \( \frac{B^2}{K} \) expire worthless and we’re left with the call option struck at \( K \). If or when the barrier is hit \((S = B)\) the put-call symmetry gives

\[ \left( \frac{K}{B} \right) C\left( B; \frac{B^2}{K} \right) = P(B; K) \]
so that
\[
UOC(B; K, B) = C(B; K) - P(B; K) - (B - K)AD(B; B)
\]
\[
= C(B; K) - P(B; K) - (B - K) = 0
\]
by the put-call parity. At this time we liquidate the portfolio at no cost and have replicated the payoff of the up-and-out call. We also get the price for the up-and-in call option:
\[
UIC(S; K, B) = C(S; K) - UOC(S; K, B)
\]
\[
= \left( \frac{K}{B} \right) C\left( S; \frac{B^2}{K} \right) + (B - K)AD(S; B). \tag{20}
\]
We hedge a long position in an up-and-in call by shorting the portfolio in (20). When or if the underlying is at the barrier, we sell it to buy a call option struck at \( K \).

**Remark** The replication method described in this section can be generalized to all standard barrier options, lookback options and their extensions. Replicating portfolios for multi barrier options, roll down calls, ratchet calls and lookback calls are given in Carr et al. [5]. A roll down call is an option with two barriers \( (B^2 < B_1) \), both below the initial spot and strike. If neither barrier is hit, the payoff is that of a standard call option at maturity. If the first barrier, \( B_1 \), is hit the option becomes a standard down-and-out call option, with barrier \( B_2 \) and strike \( B_1 \). We can write the final payoff as
\[
X = (S_T - K)^+ 1_{S_t > B_1 \text{ for all } t \leq T} + (S_T - B_1)^+ 1_{S_t \leq B_1 \text{ for some } t \leq T; S_t > B_2 \text{ for all } t \leq T}.
\]
We can extend this option, so that when \( B_1 \) is hit, the new strike becomes \( K_1 \) where \( K_1 \in [B_1, K] \), not necessarily equal to the first barrier. If the second barrier is hit, then the strike rolls down to a new level \( K_2 \) where \( K_2 \in [B_2, K_1] \) and a new out barrier \( B_3 \) becomes active, where \( B_3 < B_2 \). This process repeats an arbitrary number of times.

A ratchet call is similar to the extended roll down call. When a barrier is hit it acts as the new strikes, that is, the option is never knocked out. If the option has barriers \( B_1, B_2, \ldots \), the option gives the payoff
\[
X = (S_T - B)^+
\]
where
\[
B = \min_i \{ B_i : \min_{t \leq T} (S_t < B_i) \}
\]

4.1. **Comparison with the traditional Black-Scholes price.** In this section we verify that the prices derived using the risk neutral valuation (equations (12) and (13)) match the prices given above (when volatility is constant). We begin with the case \( B \leq K \).
The price for a down-and-in call given in equation (12) is

\[ DIC(S; K, B) = \left( \frac{B}{S} \right)^{2r} C \left( \frac{B^2}{S}; K \right). \]  

(21)

We’re assuming \( r = 0 \) so

\[ \frac{2r}{\sigma^2} = \frac{2(r - \frac{1}{2}\sigma^2)}{\sigma^2} = \frac{2(-\frac{1}{2}\sigma^2)}{\sigma^2} = -1. \]

Thus equation (21) becomes

\[ DIC(S; K, B) = \frac{S}{B} C \left( \frac{B^2}{S}; K \right). \]  

(22)

Next we use the put-call symmetry. In equation (22) we have the price of a call option struck at \( K \) which we want to rewrite as a put option. Note that an alternative way to write the put-call symmetry is

\[ C \left( \frac{B^2}{S}; K \right) = \left( \frac{K}{A} \right)^{1/2} P \left( \frac{B^2}{S}; A \right) \]

when \( (KA)^{1/2} = \frac{B^2}{S} \),

so we find the strike \( A \) which satisfies

\[ (KA)^{1/2} = \frac{B^2}{S} \quad \text{or} \quad A = \frac{B^4}{K^2 S^2}. \]

We then rewrite the call as

\[ C \left( \frac{B^2}{S}; K \right) = \left( \frac{K}{B^2 S^2} \right)^{1/2} P \left( \frac{B^2}{S}; \frac{B^4}{K S^2} \right) = \frac{KS}{B^2} P \left( \frac{B^2}{S}; \frac{B^4}{K S^2} \right) \]

and equation (22) becomes

\[ DIC(S; K, B) = \frac{S}{B} \left[ KS P \left( \frac{B^2}{S}; \frac{B^4}{K S^2} \right) \right] \]

\[ = \frac{K}{B} \cdot \frac{S^2}{B^2} P \left( \frac{B^2}{S}; \frac{B^4}{K S^2} \right) \]

\[ = \frac{K}{B} \cdot \frac{S^2 B^2}{B^2 S^2} P \left( \frac{S}{B}; \frac{B^2}{K} \right) \]

\[ = \frac{K}{B} P \left( \frac{S}{B}; \frac{B^2}{K} \right) \]  

(23)

where the third equality holds due to the homogeneity of the pricing function, i.e.

\[ P(aS; aK) = aP(S; K) \quad \text{for any constant } a. \]

Our result in (23) is exactly the value we have in equation (15) using the method in Carr et al. [5].
**Remark** The formula in (22) gives us the same price as the price we derived through our hedge, but it does not give us a hedge for the option.

The case $B > K$ is rather lengthy so the reader is referred to Appendix A.

**Remark** We found a misprint in Björk [2] in the price of both the down-and-in call and the down-and-out call, when the barrier lies above the strike. The error is easiest to see when looking at the value of a down-and-out call option. It should read

$$DOC(S; K, B) = C(S; B) + (B - K)DC(S; B)$$

Using the parity relation, we get the price given in (13) for the down-and-in call:

$$DIC(S; K, B) = C(S; K) - DOC(S; K, B)$$

$$= \left( \frac{B}{S} \right)^{\frac{2\bar{a}}{\sigma^2}} \left[ C \left( \frac{B^2}{S}; B \right) + (B - K)DC \left( \frac{B^2}{S}; B \right) \right]$$

$$- (B - K)DC(S; B) - C(S; B) + C(S; K).$$

### 4.2. Comparison of three different pricing formulas.

We mentioned above that the pricing formulas we derived do not coincide with the pricing formulas commonly used in the market. These formulas can be found in e.g. Heynen and Kat [10]. We assume that we have an option written on an underlying which follows a geometric Brownian motion and the volatility at the barrier may differ from the volatility at the strike. Denote these volatilities by $\sigma_K$ and $\sigma_B$, respectively. When we assume there are no carrying costs, we get the price for a down-and-in call as

$$DIC(S; K, B) = S \left( \frac{B}{S} \right)^{\frac{\sigma_K/\sigma_B}{\sigma^2}} \left[ N\left( e_1 \right) - K \left( \frac{B}{S} \right)^{\frac{\sigma_K/\sigma_B}{\sigma^2}} N\left( e_2 \right) \right]$$

where

$$e_1 = \log \left( \frac{S}{K} \right) + \frac{1}{2} \sigma_K \sigma_B (T - t) + 2 \frac{\sigma_K}{\sigma_B} \log \left( \frac{B}{S} \right),$$

$$e_2 = e_1 - \sigma_K \sqrt{T - t}.$$

We now have three pricing formulas for a down-and-in call. If we don’t assume constant volatility, these price don’t all coincide. In Tables 5 and 6 we compare the three prices which are the replication price we derived, the traditional Black-Scholes price and the price used in the market, given above.
traditional price  traditional price
\begin{tabular}{cccc}
\hline
B \leq K & (using \(\sigma_K\)) & (using \(\sigma_B\)) & market practice  & replication price \\
0.8907 & 1.1766 & 1.1198 & 1.4919 \\
\hline
\end{tabular}

Table 5. Down and in call prices when \(K = 100, r = q, S = 100, B = 80, \sigma_K = 30\%, \sigma_B = 32\%\) and \(\sigma_{B^2/K} = 34\%\).

\begin{tabular}{cccc}
\hline
B > K & (using \(\sigma_K\)) & (using \(\sigma_B\)) & market practice  & replication price \\
17.8630 & 16.3894 & 15.9346 & 17.2320 \\
\hline
\end{tabular}

Table 6. Down and in call prices when \(K = 100, r = q, S = 140, B = 120, \sigma_K = 35\%, \sigma_B = 32\%\) and \(\sigma_{B^2/K} = 30\%\).

4.3. **Verification of the put-call symmetry.** The hedging method in this section requires the use of the put-call symmetry which make a certain assumption on the volatility of the underlying. It is not clear if this assumption holds in reality. To see if it might hold, we check the put-call symmetry relation using data observed in the market. The data was taken on the OMXS30 index when the spot was 1060.9. The risk free rate was 0.33\%, dividend yield 14.01\% and time to maturity 35 days. The traded options had strikes in increments of 10 or 20 so if the price for the put with a certain strike was not available linear interpolation was used to estimate the price. Considering the possibility of carrying costs, if the put-call symmetry holds we should have

\[ C(S; K) - \left( \frac{K}{e^{(r-q)(T-t)}} \right) P \left( S; \frac{(e^{(r-q)(T-t)} S)^2}{K} \right) = 0. \]

In Table 7 we use the market data to evaluate the put-call symmetry for a few different strikes. The error looks quite small.

To have a reference, we also look at the put-call parity which should hold by an arbitrage argument. The put-call parity says that we should have

\[ C(S; K) - P(S; K) - e^{-q(T-t)} S + e^{-r(T-t)} K = 0. \]

Using the same data as above on the OMXS30 index, we compute the put-call parity in Table 8 for a variety of strikes.

4.4. **Relaxing the assumption of no carrying costs.** When we relax the assumption of no carrying costs we can no longer get a perfect hedge or price for the down-and-in call, but we can get tight bounds
strike (K) | $C(S; K) - \left(\frac{K}{e^{r-q(T-t)}}\right) P\left(S\left(\frac{e^{(r-q)(T-t)}}{K}\right)^2\right)$ | error
--- | --- | ---
960 | 89.4 - (960/1047.07)95.1 | = 2.1
970 | 80.13 - (970 / 1047.07)83.6 | = 2.6
980 | 70.9 - (980 / 1047.07)72.5 | = 3
990 | 61.9 - (990 / 1047.07)62.3 | = 3
1000 | 53.4 - (1000 / 1047.07)52.7 | = 3.1
1020 | 37.4 - (1020 / 1047.07)36 | = 2.3
1040 | 23.8 - (1040 / 1047.07)23.5 | = 0.4
1060 | 13.4 - (1060 / 1047.07)14.9 | = -1.7
1080 | 6.6 - (1080 / 1047.07)9.3 | = -2.9
1100 | 2.85 - (1100 / 1047.07)5.8 | = -3.2

Table 7. Put-call symmetry for options which expire in 35 days, $S = 1060.9$, $r = 0.33\%$, $q = 14.01\%$.

strike (K) | $C(S; K) - P(S; K) - e^{-q(T-t)}S + e^{-r(T-t)}K$ | error
--- | --- | ---
960 | 89.4 - 2.25 - 1046.7 + (0.9997)960 | = 0.0784
970 | 80.125 - 2.925 - 1046.7 + (0.9997)970 | = 0.1502
980 | 70.9 - 3.75 - 1046.7 + (0.9997)980 | = 0.0720
990 | 61.9 - 4.875 - 1046.7 + (0.9997)990 | = -0.0561
1000 | 53.4 - 6.25 - 1046.7 + (0.9997)1000 | = 0.0657
1020 | 37.4 - 10.25 - 1046.7 + (0.9997)1020 | = 0.0594
1040 | 23.8 - 16.75 - 1046.7 + (0.9997)1040 | = -0.0720
1060 | 13.4 - 26.25 - 1046.7 + (0.9997)1060 | = 0.0467
1080 | 6.6 - 39.375 - 1046.7 + (0.9997)1080 | = 0.1654
1100 | 2.85 - 55.625 - 1046.7 + (0.9997)1100 | = 0.1341

Table 8. Put-call parity for options which expire in 35 days, $S = 1060.9$, $r = 0.33\%$, $q = 14.01\%$.

on the price. These bounds are given in Bowie and Carr [3] and Carr et al. [5], but a proof is not given so we present one here. Let’s first look at the case when the barrier lies below the strike.

**Lemma 4.2.** The price of a down-and-in call option is monotone increasing in the barrier. That is, if $B_1 \leq B_2$ then

$$DIC(S; K, B_1) \leq DIC(S; K, B_2).$$

**Proof.** Table 9 shows the payoffs of the two options $DIC(S; K, B_1)$ and $DIC(S; K, B_2)$. Since the payoff of the option with the lower barrier, $B_1$, is less than or equal to that of the option with the higher barrier, $B_2$, no matter what, we must have that $DIC(S; K, B_1) \leq DIC(S; K, B_2)$.

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\[ \begin{array}{c|cc} 
\min_{t \leq T} \{S_t\} > B_2 & 0 & 0 \\
B_1 < \min_{t \leq T} \{S_t\} < B_2 & 0 & (S_T - K)^+ \\
\min_{t \leq T} \{S_t\} < B_1 & (S_T - K)^+ & (S_T - K)^+ 
\end{array} \]

**Table 9.** Payoff of the down-and-in call options \( DIC(S; K, B_1) \) and \( DIC(S; K, B_2) \) where \( B_1 \leq B_2 \).

**Proposition 4.3.** Given the assumptions of this section the following relation holds (even when carrying costs are not zero) for a down-and-in call option when the barrier lies below the strike:

\[
\frac{K}{B} P \left( S, \frac{B^2}{K} \right) \leq DIC(S; K, B) \leq \frac{K}{B} P \left( S, \frac{\hat{B}^2}{K} \right),
\]

where \( \hat{B} = e^{rT} B \).

**Proof.** We first derive the lower bound. It is assumed that \( S \) follows a geometric Brownian motion, so we can write

\[
S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t} = S_0 e^{rT} H_t.
\]

Using risk neutral valuation, at initial time we have

\[
DIC(S; K, B) = e^{-rT} E^Q \left[ (S_T - K)^+ 1_{[S_t < B \text{ for some } t \leq T]} \right]
\]

\[
= e^{-rT} E^Q \left[ (S_0 e^{rT} H_T - K)^+ 1_{[S_t < B \text{ for some } t \leq T]} \right]
\]

\[
= E^Q \left[ (S_0 H_T - e^{-rT} K)^+ 1_{[S_t < e^{-rT} B \text{ for some } t \leq T]} \right]
\]

\[
\geq E^Q \left[ (S_0 H_T - e^{-rT} K)^+ 1_{[S_t < e^{-rT} B \text{ for some } t \leq T]} \right]
\]

\[
= \frac{e^{-rT} K}{e^{-rT} B} E^Q \left[ \left( \frac{e^{-rT} B^2}{e^{-rT} K} - S_0 H_T \right)^+ \right]
\]

\[
= e^{-rT} K \frac{B^2}{B} E^Q \left[ \left( \frac{B^2}{K} - S_0 e^{rT} H_T \right)^+ \right]
\]

\[
= \frac{K}{B} P \left( S, \frac{B^2}{K} \right).
\]
where the fourth equality follows from the put-call symmetry, which holds since $S_0 HT$ has no drift. That is, 
\[
E^Q \left[ (S_0 HT - e^{-rT} K)^+ 1_{[S_0 H_t < e^{-rT} B \text{ for some } t \leq T]} \right] \\
= DIC(S_0 HT; e^{-rT} K, e^{-rT} B) \\
= \frac{K}{B} P \left( S_0 HT, \frac{e^{-rT} B^2}{K} \right) \\
= \frac{K}{B} E^Q \left[ \left( \frac{e^{-rT} B^2}{K} - S_0 HT \right)^+ \right].
\]

To get the upper bound we calculate:
\[
DIC(S; K, B) = e^{-rT} E^Q \left[ (S_T - K)^+ 1_{[S_t < B \text{ for some } t \leq T]} \right] \\
= e^{-rT} E^Q \left[ (S_0 e^{rT} H_T - K)^+ 1_{[S_0 e^{rT} H_t < B \text{ for some } t \leq T]} \right] \\
= E^Q \left[ (S_0 H_T - e^{-rT} K)^+ 1_{[S_0 H_t < e^{-rT} B \text{ for some } t \leq T]} \right] \\
\leq E^Q \left[ (S_0 H_T - e^{-rT} K)^+ 1_{[S_0 H_t < B \text{ for some } t \leq T]} \right] \\
= e^{-rT} \frac{K}{B} E^Q \left[ \left( \frac{B^2}{e^{-rT} K} - S_0 H_T \right)^+ \right] \\
= e^{-rT} \frac{K}{B e^{rT}} E^Q \left[ \left( \frac{(Be^{rT})^2}{K} - S_0 e^{rT} H_T \right)^+ \right] \\
= \frac{K}{B} P \left( S, \frac{B^2}{K} \right) .
\]

**Proposition 4.4.** When the barrier lies above the strike, $B > K$, we get the following bounds on a down-and-in call option:
\[
(B-K) DIB(S, \hat{B}) + P(S, K) \leq DIC(S; K, B) \leq (B-K) DIB(S, B) + P(S, K)
\]
where $DIB(S, B)$ denotes a down-and-in bond on $S$ with strike $B$.

The down-and-in bond pays 1 at expiry if the barrier has been hit at some time prior to expiry. See Carr et al. [5] and Bowie and Carr [3] for motivation for the proof. For a table which shows the tightness of the bounds for various parameter values see Bowie and Carr [3].

If the price of the down-and-in call lies outside of these bounds an arbitrage opportunity exists. To see an arbitrage opportunity, suppose that $B \leq K$ and the price of the down-and-in call lies below the lower bound, that is,
\[
DIC \left( S; K, B \right) \leq \frac{K}{B} P \left( S, \frac{B^2}{K} \right) .
\]
If this is the case we buy the down-and-in call and sell $\frac{K}{B}$ puts on $S$ with strike $\frac{B^2}{K}$. By our assumption, we have money left over at initial time. If the underlying stays above the barrier $B$, both contracts expire worthless and we’ve made an arbitrage. On the other hand, if or when the underlying hits the barrier, we buy back our puts and sell a call with strike $K$, expiring at $T$. Computing the analytical price we see that we also make a profit off of this. At maturity, we are both long and short a call struck at $K$ whose values cancel, so we have just made an arbitrage. This arbitrage is summarized in Tables 10 and 11.

<table>
<thead>
<tr>
<th>Initial Setup</th>
<th>Maturity, $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-DIC(S; K, B)$</td>
<td>0</td>
</tr>
<tr>
<td>$+\frac{K}{B} P(S, \frac{B^2}{K})$</td>
<td>0</td>
</tr>
<tr>
<td>$&gt; 0$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 10. Arbitrage possibility. Scenario when the barrier is not crossed.

<table>
<thead>
<tr>
<th>Initial Setup</th>
<th>$S_t = B$</th>
<th>Maturity, $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-DIC(S; K, B)$</td>
<td>-</td>
<td>$(S_T - K)^+$</td>
</tr>
<tr>
<td>$+\frac{K}{B} P(S, \frac{B^2}{K})$</td>
<td>$-\frac{K}{B} P(B, \frac{B^2}{K}) + C(B, K)$</td>
<td>$(S_T - K)^+$</td>
</tr>
<tr>
<td>$&gt; 0$</td>
<td>$\geq 0$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 11. Arbitrage possibility. Scenario when the barrier is crossed.
5. Static hedging of discrete barrier options

In this section we describe in detail a hedging method for discrete barrier options introduced in Andersen, Andreasen and Eliezer [1] and is more explicitly described in Joshi [11]. The method can be applied to all standard barrier options but for clarity we focus on a discrete down-and-out barrier option. This option pays a certain amount at time $T$ unless the value of the underlying lies below a given barrier at any one of the predetermined times $t_1 < t_2 < \cdots < t_n$, where $t_n < T$. The payoff function is

$$X = \Phi(S_T)1_{[S_{t_1}>B;\ldots;S_{t_n}>B]},$$

where $\Phi$ is the final time payoff as a function of the underlying at time $T$. This replication method creates an almost weak static hedge for a discrete barrier option using plain vanilla options which requires buying and selling only at the initial time and at the time of knock-out or expiration. We present the method through a specific example.

Consider a discrete down-and-out call option expiring in 1 year, with barrier $B = 95$ and strike $K = 90$. Assume that the current spot is $S_0 = 100$, the risk free rate is $r = 3\%$ and volatility is at $\sigma = 30\%$. For simplicity, suppose there is only one reset time, $t_1 = 0.5$. The final payoff of this option is

$$X = (S_T - K)^+1_{[S_{t_1}>B]} = (S_T - 90)^+1_{[S_{0.5}>95]}.$$ 

To replicate this payoff it suffices to find a portfolio with pricing function $\Pi$ which satisfies

$$\Pi(S_T, T) = (S_T - 90)^+$$

and

$$\Pi(S, t_1) = 0$$

for all $S \leq 95$.

If we have such a portfolio we can liquidate it at zero cost at the reset time, $t_1$, if the underlying lies below the barrier (i.e. if $S \leq 95$). If $S > 95$ at $t_1$, we hold the portfolio and receive the payoff of the call option at maturity. This contract perfectly replicates the down-and-out call. To obtain this hedging portfolio, we begin at time $t_0$ by creating the desired payoff at time $T$, assuming the underlying was not below the barrier at $t_1$. That is, we buy a European call option struck at $K = 90$, with expiry in one year. Using the Black-Scholes price, the price of our replicating portfolio, only consisting of the call option thus far, is

$$\Pi_0(S, t) = SN(d_1) - 90e^{-0.03(1-t)}N(d_2),$$

where

$$d_1 = \log \left( \frac{S}{K} \right) + \frac{(0.03 + \frac{(0.3)^2}{2})(1-t)}{0.3\sqrt{1-t}}$$

and

$$d_2 = d_1 - 0.3\sqrt{1-t}. $$
We now have a portfolio which replicates our down-and-out option when \( t > t_1 \). We move next to time \( t = t_1 \). The value of the portfolio at time \( t_1 = 0.5 \) (i.e. \( \Pi_0(S_{0.5}, 0.5) \)) is shown in Figure 7. Our replicating portfolio matches the value of the option when \( S_{t_1} > B \), but is non-zero when the underlying lies below the barrier \( (S_{t_1} \leq B) \). Our next step is to make the value of this portfolio zero on the interval \([0, B] = [0, 95]\) at time \( t_1 = 0.5 \). To get the value of the portfolio to be zero at \( B = 95 \), we short a digital put with strike 95, expiring at time \( t_1 = 0.5 \) and with notional

\[
\Pi_0(95, 0.5) = 11.3327.
\]

Recall that the payoff of this digital put is

\[
X = -11.3327 \cdot 1_{\{S_{0.5} \leq 95\}}.
\]

We now have a portfolio consisting of the call option and the digital put whose price we denote by \( \Pi_{01}(S, t) \). More specifically,

\[
\Pi_{01}(S, t) = \Pi_0(S, t) - 11.3327DP(S, t),
\]

see Figure 8. Now our portfolio is worth zero at \( B = 95 \), but is not worth zero on the interval \([0, B]\). We approximate \( \Pi_{01} \) on this interval using straight lines. To do this we divide the interval into segments. It should become clear that increasing the number of segments will decrease the error in the approximation, but for simplicity we partition the interval as follows:

\[
[0, B) = \left[0, \frac{1}{2}B\right) \cup \left[\frac{1}{2}B, \frac{3}{4}B\right) \cup \left[\frac{3}{4}B, B\right).
\]

We work next on the last interval \( \left[\frac{3}{4}B, B\right) \) and approximate the value on the interval using a put option. The idea is to make the value of the portfolio zero at both \( B \) and \( \frac{3}{4}B \). If we buy a put with strike \( B \)
Figure 8. $\Pi_0^0(S_{0.5}, 0.5)$, the replicating portfolio consisting of the call option and digital put option at $t_1$.

expiring at $t_1 = 0.5$ and add it to our portfolio $\Pi_0^0$, we will still have zero value at $B$ since the put is worth zero at $B$. But, we need to choose a notional to ensure the value is zero at $\frac{3}{4}B = 71.25$. That is, we want to find the notional $n$ such that

$$n(B - \frac{3}{4}B) + \Pi_0^0(\frac{3}{4}B, t_1) = 0.$$  

Using equation (24) we find that

$$n = -\frac{\Pi_1^0(\frac{3}{4}B, t_1)}{(B - \frac{3}{4}B)} = -\frac{\Pi_1^0(71.25, 0.5)}{(95 - 71.25)} = 0.4216.$$

Adding this option to our portfolio, we have a new portfolio consisting of the call option, the digital put and a put option whose price we denote by $\Pi_1^1(S, t)$. This portfolio has value zero at $B = 95$ and at $\frac{3}{4}B = 71.25$, and is close to zero on $(\frac{3}{4}B, B)$. This portfolio can be written

$$\Pi_1^1(S, t) = \Pi_0^0(S, t) + 0.4216P(S, t)$$

$$= \Pi_0^0(S, t) - 11.3327DP(S, t) + 0.4216P(S, t),$$

see Figure 9. Now we move backward to the next interval $[\frac{1}{2}B, \frac{3}{4}B)$ and repeat this process once again, this time shorting a put with strike $\frac{3}{4}B = 71.25$ expiring at $t_1 = 0.5$ and with notional equal to

$$n = \frac{\Pi_1^1(47.5, 0.5)}{(71.25 - 47.5)} = -0.3663.$$

Adding this option to our portfolio we have a new portfolio which we denote $\Pi_1^2(S, t)$. This portfolio is worthless at $\frac{1}{2}B$, $\frac{3}{4}B$, and $B$, see Figure 10. We repeat this process for the last interval, $[0, \frac{1}{2}B)$, buying
Figure 9. $\Pi_1(S, t_1)$, the replicating portfolio consisting of the call option, digital put and put option.

Figure 10. $\Pi_2^1(S, t_1)$, the replicating portfolio consisting of the call option, digital put and two put options (the figure on the right is a zoomed version).

A put with strike $\frac{1}{2}B = 47.5$ expiring at $t_1 = 0.5$ and with notional

$$n = -\frac{\Pi_2^1(0, 0.5)}{(47.5 - 0)} = -0.0552.$$  

Adding this last put we have a portfolio which we denote $\Pi_1(S, t)$. We now have a portfolio which is almost zero on $[0, B]$ and is worth zero at the points $0, \frac{1}{2}B, \frac{3}{4}B$ and $B$. Since $t_1$ is the only reset time, $\Pi_1$ is our complete replicating portfolio, see Figure 11. To summarize, our replicating portfolio is listed in Table 12.

Alternatively, we can price this particular option using risk neutral valuation which gives the price

$$F(S, t) = e^{-r(T-t)}E_Q^{S,t} \left[ (S_T - K)^+ \cdot 1_{[S_T > B]} \right].$$  

The reader is referred to Appendix B for the derivation of this price. In Figure 12 we see the risk neutral price compared to the price of
Figure 11. On the left, $\Pi_1(S, t_1)$, the complete replicating portfolio and on the right, the actual value of the option at time $t_1$.

<table>
<thead>
<tr>
<th>option</th>
<th>strike</th>
<th>maturity</th>
<th>position</th>
</tr>
</thead>
<tbody>
<tr>
<td>call</td>
<td>90</td>
<td>1 year</td>
<td>1</td>
</tr>
<tr>
<td>digital put</td>
<td>95</td>
<td>6 months</td>
<td>-11.3327</td>
</tr>
<tr>
<td>put</td>
<td>95</td>
<td>6 months</td>
<td>0.4216</td>
</tr>
<tr>
<td>put</td>
<td>71.25</td>
<td>6 months</td>
<td>-0.3663</td>
</tr>
<tr>
<td>put</td>
<td>47.5</td>
<td>6 months</td>
<td>-0.0552</td>
</tr>
</tbody>
</table>

Table 12. Replicating portfolio.

our replication portfolio at initial time. We see that the two prices are quite similar.

Figure 12. Comparison of the risk neutral and replication pricing at initial time. The figure to the right is zoomed in.

5.1. Scenarios. To further see how this replicating method works, let us continue with our explicit example and consider two scenarios.
Scenario 1
Consider the case when the down-and-call is not knocked out. Suppose $S_{0.5} = 110$, that is, the underlying lies above the barrier, $B = 95$, at the reset time, $t_1$. Considering we are long the replicating portfolio, all of the puts in our replicating portfolio will expire out of the money. What is left in our portfolio is the call option which expires at $T$, see Table 13. This payoff exactly matches the payoff of the discrete barrier down-and-out call.

<table>
<thead>
<tr>
<th>option, $S_{0.5} = 105$</th>
<th>strike</th>
<th>maturity</th>
<th>value at $t_1$</th>
<th>value at $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>call</td>
<td>90</td>
<td>$T$</td>
<td>$\Pi_0(S_{0.5}, 0.5)$</td>
<td>$(S_T - 90)^+$</td>
</tr>
<tr>
<td>digital put</td>
<td>95</td>
<td>$t_1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>put</td>
<td>95</td>
<td>$t_1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>put</td>
<td>71.25</td>
<td>$t_1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>put</td>
<td>47.5</td>
<td>$t_1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td>$\Pi_0(S_{0.5}, 0.5)$</td>
<td>$(S_T - 90)^+$</td>
</tr>
</tbody>
</table>

**Table 13.** Hedging portfolio when the barrier lies above the barrier at the reset time.

down-and-out call.

Scenario 2
Now consider a scenario where the option is knocked out. Suppose $S_{0.5} = 70$ so the underlying lies below the barrier at the reset time. Considering we are long the replicating portfolio, we would complete our static hedge by selling the portfolio at time $t_{0.5}$. The value of our portfolio $\Pi_1$ is small, so we liquidate at little cost. Recall that increasing the number of partitions of $[0, B]$ (and hence increasing the number of put options in the hedge) would make the value of $\Pi_1$ closer to zero, making this payoff arbitrarily close to the payoff of the discrete barrier down-and-out call.

<table>
<thead>
<tr>
<th>option</th>
<th>strike</th>
<th>maturity</th>
<th>value at $t_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>call</td>
<td>90</td>
<td>$T$</td>
<td>1.1112</td>
</tr>
<tr>
<td>digital put</td>
<td>95</td>
<td>$t_1$</td>
<td>-11.3327</td>
</tr>
<tr>
<td>put</td>
<td>95</td>
<td>$t_1$</td>
<td>10.5396</td>
</tr>
<tr>
<td>put</td>
<td>71.25</td>
<td>$t_1$</td>
<td>-0.4578</td>
</tr>
<tr>
<td>put</td>
<td>47.5</td>
<td>$t_1$</td>
<td>0</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td>-0.1397</td>
</tr>
</tbody>
</table>

**Table 14.** Hedging portfolio when the barrier lies below the barrier at the reset time, $S_{0.5} = 70$.

Remark It is important to note that the difference between the value of the replicating portfolio and the down-and-out call option at time $t_1$ also depends on the value of the underlying. Consider the case when
$S_{0.5} = 90$. This value is below the barrier so we will liquidate our portfolio, see Table 15. The value of the replicating portfolio at $t_1$ is

<table>
<thead>
<tr>
<th>option</th>
<th>strike</th>
<th>maturity</th>
<th>value at $t_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>call</td>
<td>90</td>
<td>T</td>
<td>8.2345</td>
</tr>
<tr>
<td>digital put</td>
<td>95</td>
<td>$t_1$</td>
<td>-11.3327</td>
</tr>
<tr>
<td>put</td>
<td>95</td>
<td>$t_1$</td>
<td>2.1079</td>
</tr>
<tr>
<td>put</td>
<td>71.25</td>
<td>$t_1$</td>
<td>0</td>
</tr>
<tr>
<td>put</td>
<td>47.5</td>
<td>$t_1$</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>-0.9903</td>
</tr>
</tbody>
</table>

**Table 15.** Hedging portfolio when the barrier lies above the barrier at the reset time, $S_{0.5} = 90$.

Further from zero than when $S_{0.5} = 70$. This can be seen in Figure 12 where the value of the hedging portfolio differs from the analytical value of the option.

This method assumes that given the value of the underlying, the strike, risk free rate and volatility the price of European calls and puts are known at all times $t \leq T$. Therefore, this method cannot be used when we consider a stochastic volatility model, i.e. when the future volatility cannot be determined today. Here we give a brief description of some of the models which can be used in this hedging method.

**Black-Scholes Model**

$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$

where the drift, $\alpha$, and the volatility, $\sigma$, are constant.

**Dupire Model**

$$dS_t = \alpha(t) S_t dt + \sigma(S_t, t) S_t dW_t,$$

where $\sigma(S_t, t) = b(S_t, t)$ and

$$\frac{b^2(K, T) \partial^2 C}{2 \partial K^2} = \frac{\partial C}{\partial T}.$$  

This is an extension of the Black-Scholes model which takes the volatility smile into account.

**Merton’s Jump Diffusion Model**

$$S_t = S_0 \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\} \prod_{i=1}^{N(t)} Y_i,$$
where $\sigma^2$ is the instantaneous variance of the return, conditional on that the Poisson event does not occur. $\{Y_i\}_{i \geq 1}$ are log-normally distributed random variables and $N_t$ is a Poisson process with intensity $\lambda$.

The Black-Scholes model is a special case of this, when $\lambda = 0$. In this model one assumes the underlying follows a jump stochastic process by adding a compound Poisson process to a Brownian motion with drift. The jumps reflect abnormal vibrations in the price due to the arrival of important new information.

**Variance Gamma Model**

$$S_t = S_0 \exp\{\alpha t + X(\Gamma_t, \theta, \sigma, \nu)\},$$

where $X(t, \theta, \sigma, \nu) = \theta t + \sigma W_t$ and $\Gamma_t$ is a gamma process which satisfies $E(\Gamma_t) = t$, and $\text{Var}(\Gamma_t) = \nu t$.

This model evaluates the Wiener process at a random time change given by a gamma process. The Black-Scholes model is a special case of this, when $\Gamma_t = t$. This model accounts for kurtosis ($\nu$) and skewness ($\theta$) of the return distribution.

5.2. **General case.** We now discuss the general method for down-and-out barrier options. Suppose we have a down-and-out barrier option on the underlying $S$, with strike $K$, barrier $B$ and expiration at time $T$. The option has the reset times $t_1, t_2, \ldots, t_n$, where $t_n < T$.

The first step in the replication method is to replicate the final payoff ignoring the barrier, using vanilla options. If the final payoff is linear, this is easily done. If the payoff is non-linear, we can create an arbitrarily close replication by approximating with straight lines. The value of the portfolio consisting of this final payoff is denoted $\Pi_0(S, t)$.

Once we have replicated the final payoff, we want to make the replicating portfolio worthless below the barrier at times $t_1, t_2, \ldots, t_n$. To do this we work backward, looking at time $t_n$ next. We first want to make the value of the portfolio zero at the point $B$ at time $t_n$. We do this by shorting a digital put with notional $\Pi_0(B, t_n)$ struck at $B$ which expires at $t_n$. Denote this new portfolio, consisting of the final payoff replication and the digital put, $\Pi_0^1(S, t)$. We now have a portfolio which has zero value at $B$, but may not be zero on $[0, B)$. Partition $[0, B)$ into intervals $[0, x_1], [x_1, x_2], \ldots, [x_{k-1}, B)$, for some $k$. We approximate the value of our portfolio on each subinterval by a straight line; this we can replicate using vanilla options. To do this we work backward, first working on the interval $[x_{k-1}, B)$. We buy a put with strike $B$ expiring
at time $t_n$. To get the portfolio to be worthless at $x_{k-1}$, we choose the notional, $n$, so that

$$n(B - x_{k-1}) + \Pi_0^1(x_{k-1}, t_n) = 0$$

or equivalently

$$n = -\frac{\Pi_0^1(x_{k-1}, t_n)}{(B - x_{k-1})}.$$ 

Note that $\Pi_0^1(x_{k-1}, t_n)$ may be positive, in which case we will sell the put instead of buying it. The portfolio consisting of the final payoff, digital put and put option (denoted $\Pi_1^1(S, t)$) has zero value at both points $B$ and $x_{k-1}$ at time $t_n$ and is very small on the interval $(x_{k-1}, B)$. We continue moving backward. For each partition point $x_j$, $j = k - 1, \ldots, 2, 1$, we short a put struck at $x_j$ with notional

$$n = -\frac{\Pi_{k-j}^1(x_{j-1}, t_n)}{(x_j - x_{j-1})}, \; j = k, \ldots, 1$$

all expiring at $t_n$. This creates a replicating portfolio, $\Pi_1^1(S, t)$, with value close to zero on $[0, B]$ at time $t_n$. Once this is complete, we move back one time step to $t_{n-1}$ and repeat the same process starting with the portfolio $\Pi_1$ instead of $\Pi_0$. We continue moving back in time until we have a replicating portfolio with the correct final payoff and zero value on

$$[0, B] \times t_1 \cup [0, B] \times t_2 \cup \cdots \cup [0, B] \times t_n.$$ 

Once we have created the replicating portfolio, we have no additional trading until the first time the underlying lies below the barrier at one of the reset times. At this time, we liquidate the portfolio at zero cost. If the underlying does not lie below the barrier at any of the reset times, we hold the portfolio until the final time $T$ and get the desired payoff.

**Remark** The replicating portfolio is independent of the initial spot. This means that we can get the value of the option as a function of spot by revaluing the replicating portfolio. Letting $\Phi$ denote the price of our discrete barrier option we have

$$\Pi(S, t) \approx \Phi(S, t) \text{ for all } S, \; \text{and } t \leq T,$$

where we have equality in the limit as we (appropriately) increase the number of options in our replicating portfolio. This means that the delta, gamma and theta of the option are equal to the delta, gamma and theta of the replicating portfolio. To see that the deltas are equal we calculate

$$\frac{\partial \Pi}{\partial S}(S, t) = \lim_{h \to 0} \frac{\Pi(S + h, t) - \Pi(S - h, t)}{2h} \approx \lim_{h \to 0} \frac{\Phi(S + h, t) - \Phi(S - h, t)}{2h} = \frac{\partial \Phi}{\partial S}(S, t).$$
We do not necessarily have equality for Greeks with respect to parameters other than the underlying and time as they generally affect the composition of the replicating portfolio. For instance, let $\sigma_1$ and $\sigma_2$ denote two different volatilities and let $\Pi^1$ be the replicating portfolio constructed with volatility $\sigma_1$. Then

$$\Pi^1(S, t; \sigma_1) \approx \Phi(S, t; \sigma_1)$$

for all $S$, and $t \leq T$ but this does not imply

$$\Pi^1(S, t; \sigma_2) \approx \Phi(S, t; \sigma_2).$$

**Remark** The method presented above can be generalized to down-and-out discrete barrier options, up-and-out discrete barrier options and discrete double barrier options. It is generalized to up-and-out options by replacing puts by calls. In this case we would begin in the same manner by replicating the final payoff at $T$, assuming the option has not knocked out. In the next step we add more options to make the portfolio worthless on $[B, \infty) \times t_i$ for all $i$ (or, on $[B, N] \times t_i$ for some reasonably large $N$). We would partition the interval

$$[B, N] = [B, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, N]$$

and begin first working on the interval $[B, x_1]$, then move forward to $[x_1, x_2]$ and so on. Double barrier options are replicated by treating the up-barrier and down-barrier separately then combining the portfolios. This same method also applies when the boundary is not constant, for example, if $B = B(t_i)$.

This method can also be used to hedge and price in options since the relations

$$C(S, K) = DIC(S; K, B) + DOC(S; K, B)$$

$$= UOC(S; K, B) + UIC(S; K, B)$$

hold even for discrete barrier options. To hedge a down-and-in (up-and-in) option we enter the hedge implied by the equations in (25). When or if the underlying lies below (above) the barrier at a reset time, we liquidate the portfolio and use the profits to buy the desired final payoff. Otherwise, the hedging portfolio expires worthless.
6. Case study

In this section we compare the price we get for a specific option using the hedging method presented in Section 5 with its analytical price. The option we will consider is called an Auto Call. This particular option was sold to Swedbank’s private banking customers in 2009 and many similar products are regularly sold. We can describe this option as an up-and-out discrete barrier option with rebates. This option has a reset every year for five years \( t = 1, 2, 3, 4, 5 \). Let \( S_0 \) denote the initial spot price, this value also makes the barrier. If the underlying lies above the initial spot, \( S_0 \), at the first reset time, i.e. after one year, then the option knocks out and we get the rebate

\[(1 + 0.15) \cdot \text{investment} \]

If the underlying lies below the initial spot the option continues for another year. If the underlying lies above the initial spot after \( k \) years, \( k < 5 \), we get the rebate

\[(1 + k \cdot 0.15) \cdot \text{investment} \]

and the option expires. If the underlying lies below the initial spot the option continues. If the option has not knocked out at the previous reset times, then after the fifth year we get the final payoff

\[
\begin{cases}
(1 + 5 \cdot 0.15) \cdot \text{investment} & \text{if } S_5 \geq S_0 \\
1 \cdot \text{investment} & \text{if } B \leq S_5 < S_0 \\
\frac{S_5}{S_0} \cdot \text{investment} & \text{if } S_5 < B
\end{cases}
\]

where \( B \) is a predetermined lower barrier. In Figure 13 we see the final payoff when \( S_0 = 100 \) and \( B = 65 \). This contract is an up-and-out contract as opposed to our previous example which was a down-and-out contract, so we go through the hedge explicitly. We begin by replicating the payoff at maturity, given that the barrier has not been crossed at a previous reset time. For simplicity, say we invest 1. For an investment amount different from 1 we multiply the price we derive

![Figure 13. Payoff at time \( T = 5 \) of the Auto Call, \( S_0 = 100 \), \( B = 65 \).](image-url)
by the investment amount. To replicate the final payoff we first buy \( \frac{1}{S_0} \) units of the underlying. We then short \( \frac{1}{S_0} \) call options with strike \( B \) expiring at final time, \( T = 5 \). This will give us the payoff in Figure 14 at final time. To get the desired payoff we also buy \( \left(1 - \frac{1}{S_0}B\right) \) digital calls with strike \( B \). We need another digital call so that the portfolio is worth the rebate \((1 + 5 \cdot 0.15) = 1.75\) if \( S_T > S_0 \). This is accomplished by buying \((1.75 - 1)\) digital calls with strike \( S_0 \). We have now matched the payoff in Figure 13. The portfolio consisting of the underlying, call and digital calls we denote by \( \Pi_0(S, t) \). The next step is to move back one time step to time \( t = 4 \). We now want to create a portfolio \( \Pi_1(S, t) \) such that

\[
\Pi_1(S, 4) = 1 + 4 \cdot 0.15 = 1.6 \quad \text{when} \quad S \geq S_0,
\]

\[
\Pi_1(S, 4) = \Pi_0(S, 4) \quad \text{when} \quad S < S_0
\]

and

\[
\Pi_1(S, t) = \Pi_0(S, t) \quad \text{for} \quad t > 4.
\]

We start with the first condition and buy a digital call with notional \( 1.6 - \Pi_0(S_0, 4) \) struck at \( S_0 \) expiring at \( t = 4 \). Adding this option to \( \Pi_0 \) we get the portfolio \( \Pi_1^0 \). Next, partition \([S_0, N]\) for some reasonably large \( N \). For example, if \( S_0 = 100 \) we might choose \( N = 200 \). This gives the partition

\[
[S_0, N] = [S_0, x_1] \cup [x_1, x_2] \cup [x_2, x_3] \cup \cdots \cup [x_{n-1}, N].
\]

We buy or sell call options with strike \( S_0, x_1, x_2, \ldots, x_{n-1}, \) all expiring at \( t = 4 \) so that the replicating portfolio is worth 1.6 at each of the points, \( x_1, x_2, \ldots, x_{n-1}, N \) and arbitrarily close to 1.6 on each of the intervals \([S_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, N]\). To do this we first sell \( n \) call options with strike \( S_0 \). We choose \( n \) such that

\[
n(- (x_1 - S_0)) + \Pi_1^0(x_1, 4) = 1.6
\]
or

\[ n = -\frac{1.6 - \Pi_0^1(x_1, 4)}{x_1 - S_0}. \]

Adding this option to \( \Pi_0^1 \) we get the portfolio \( \Pi_1^1 \) which is worth 1.6 (the rebate) when \( S = S_0 \) and \( S = x_1 \). We continue in this manner, selling calls with strike \( x_i \) all expiring at \( t = 4 \) with notional

\[ n = -\frac{1.6 - \Pi_1^1(x_{i+1}, 4)}{x_{i+1} - x_i} \]

(if \( n < 0 \) we buy instead of sell). Doing this for all points \( S_0, x_1, \ldots, x_{n-1}, \) we get the portfolio \( \Pi_1 \). Because all of the added contracts are calls with strikes above \( S_0 \) and all expire at \( t = 4 \), this portfolio has the value

\[ \Pi_1(S, 4) = 1.6 \quad \text{when} \quad S \in \{S_0, x_1, \ldots, N\} \]

\[ \Pi_1(S, 4) \approx 1.6 \quad \text{when} \quad S \geq S_0 \]

\[ \Pi_1(S, 4) = \Pi_0(S, 4) \quad \text{when} \quad S < S_0 \]

and

\[ \Pi_1(S, t) = \Pi_0(S, t) \quad \text{for} \quad t > 4. \]

Next we move back one time step and repeat this process starting with \( \Pi_1 \) instead of \( \Pi_0 \). We continue moving back through all of the reset times. When we have done this we have a replicating portfolio. This replicating portfolio is set up at initial time and liquidated when or if \( S_t \geq S_0 \). The proceeds from liquidating the portfolio match the given rebate.

To summarize, at initial time we set up the portfolio found in Table 16. The table shows that this replicating portfolio will consist of \( 4 + 4 \times \) (number of partitions) contracts. This hedging method is an improvement over delta hedging in that the hedge is set up at initial time and needs to be rebalanced at most once, when the option knocks out. In the previous section we mentioned that this hedge is both delta and gamma neutral. This method does have the disadvantage that to get an accurate hedge, we must enter quite a lot of contracts.

6.1. Example. In this section we examine how accurately this hedging method prices the Auto Call. We compare the price of the Auto Call computed analytically in a Black-Scholes setting with the price computed by our hedging method. Let \( S_0 = 100 \), volatility be 20\%, risk free rate be 2.5\%, dividend yield be 0 and let the lower barrier be \( B = 65 \). The analytical price of the option is 1.0727, we compare this to the values in Table 17. We see that even with a small number of partitions we have quite an accurate price.

We also examine the price of our replicating portfolio under a Merton jump diffusion model. Assume that \( S = 100 \), the lower barrier is 60,
<table>
<thead>
<tr>
<th>notional</th>
<th>option</th>
<th>strike</th>
<th>expiry</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{S_0} )</td>
<td>underlying</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( -\frac{1}{S_0} )</td>
<td>call</td>
<td>( B )</td>
<td>( T )</td>
</tr>
<tr>
<td>( 1 - \frac{1}{S_0}B )</td>
<td>digital call</td>
<td>( B )</td>
<td>( T )</td>
</tr>
<tr>
<td>( 1.75 - 1 )</td>
<td>digital call</td>
<td>( S_0 )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

\[
1.6 - \Pi_0(S_0, 4) = \frac{1.6 - \Pi_0(x_1, 4)}{x_1 - S_0} \quad \text{call} \quad S_0 \quad 4
\]

\[
\vdots
\]

\[
\frac{1.6 - \Pi_0^4(N, 4)}{N - x_{n-1}} \quad \text{call} \quad x_{n-1} \quad 4
\]

\[
1.45 - \Pi_1(S_0, 3) = \frac{1.45 - \Pi_1^3(x_1, 3)}{x_1 - S_0} \quad \text{call} \quad S_0 \quad 3
\]

\[
\vdots
\]

\[
\frac{1.45 - \Pi_1^3(N, 3)}{N - x_{n-1}} \quad \text{call} \quad x_{n-1} \quad 3
\]

\[
1.3 - \Pi_2(S_0, 2) = \frac{1.3 - \Pi_2^2(x_1, 2)}{x_1 - S_0} \quad \text{call} \quad S_0 \quad 2
\]

\[
\vdots
\]

\[
\frac{1.3 - \Pi_2^2(N, 2)}{N - x_{n-1}} \quad \text{call} \quad x_{n-1} \quad 2
\]

\[
1.15 - \Pi_3(S_0, 1) = \frac{1.15 - \Pi_3^1(x_1, 1)}{x_1 - S_0} \quad \text{call} \quad S_0 \quad 1
\]

\[
\vdots
\]

\[
\frac{1.15 - \Pi_3^1(N, 1)}{N - x_{n-1}} \quad \text{call} \quad x_{n-1} \quad 1
\]

**Table 16.** Auto Call hedging portfolio.

<table>
<thead>
<tr>
<th>number of partitions</th>
<th>price</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.0795</td>
<td>0.0068</td>
</tr>
<tr>
<td>5</td>
<td>1.0770</td>
<td>0.0044</td>
</tr>
<tr>
<td>10</td>
<td>1.0737</td>
<td>0.0011</td>
</tr>
<tr>
<td>20</td>
<td>1.0729</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

**Table 17.** Replicated Auto Call price using Black-Scholes prices compared with the analytical price 1.0727.

Risk free rate is 2%, dividend yield zero, volatility 7.42% and the rebate at year \( k \) is \((1 + 0.1k)\). We also assume that the jump intensity, \( \lambda \), is
3.9481, expected value of the jumps is $-0.0729$ and standard deviation of the jumps is 0.0908. We compare the replicating prices in Table 18. The price does not change significantly when the number of partitions is increased, implying that our price is quite accurate even with a small number of partitions.

<table>
<thead>
<tr>
<th>number of partitions</th>
<th>price</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.0082</td>
</tr>
<tr>
<td>5</td>
<td>1.0067</td>
</tr>
<tr>
<td>10</td>
<td>1.0047</td>
</tr>
</tbody>
</table>

Table 18. Replicated Auto Call price under Merton’s jump diffusion model.

As mentioned in Section 5, the delta and gamma of the replicating portfolio should equal the delta of the Auto Call option. In Table 19 we compute the value of the (approximated) deltas for the Auto Call. Although to use this method we assume that volatility is deterministic,

<table>
<thead>
<tr>
<th>underlying</th>
<th>option delta</th>
<th>replicating portfolio delta</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>0.0050</td>
<td>0.0048</td>
</tr>
<tr>
<td>90</td>
<td>0.0045</td>
<td>0.0042</td>
</tr>
<tr>
<td>100</td>
<td>0.0040</td>
<td>0.0037</td>
</tr>
<tr>
<td>120</td>
<td>0.0031</td>
<td>0.0028</td>
</tr>
<tr>
<td>130</td>
<td>0.0029</td>
<td>0.0023</td>
</tr>
</tbody>
</table>

Table 19. The delta of the Auto Call analytical price compared with the delta of the replicating portfolio at initial time.

we are still interested in how the portfolio is affected by a change in volatility. In Table 20 we examine the vega of the portfolio. It is clear that the vega of the option does not match the vega of the replicating portfolio. This implies that the portfolio will need to be rebalanced when there is a change in the volatility. It is also interesting to note that the vega of the replicating portfolio is higher than that of the analytical price.
<table>
<thead>
<tr>
<th>volatility</th>
<th>option vega</th>
<th>replicating portfolio vega</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>-0.4925</td>
<td>-2.0410</td>
</tr>
<tr>
<td>20%</td>
<td>-0.74</td>
<td>-2.75</td>
</tr>
<tr>
<td>30%</td>
<td>-0.70</td>
<td>-2.33</td>
</tr>
<tr>
<td>40%</td>
<td>-0.6573</td>
<td>-1.5836</td>
</tr>
</tbody>
</table>

**Table 20.** The vega of the Auto Call analytical price compared with the vega of the replicating portfolio at initial time.
APPENDIX A

We verify that the price we derived for a down-and-in call using the replicating portfolio matches the traditional Black-Scholes price (when there are no carrying costs). We show the case when the barrier lies above the strike. The traditional Black-Scholes price given in (13) is

\[
DIC(S; K, B) = \left( \frac{B}{S} \right) \frac{2\tilde{\sigma}}{\sigma^2} \left[ C \left( \frac{B^2}{S}; B \right) + (B - K)DC \left( \frac{B^2}{S}; B \right) \right] \\
- (B - K)DC(S; B) + C(S; K) - C(S; B) \tag{26}
\]

where \( DC(S; B) \) denotes the price of a digital call on \( S \) with strike \( B \).

As before, we get

\[
\frac{2\tilde{\sigma}}{\sigma^2} = -1
\]

so equation (13) becomes

\[
DIC(S; K, B) = \frac{S}{B} \left[ C \left( \frac{B^2}{S}; B \right) + (B - K)DC \left( \frac{B^2}{S}; B \right) \right] \\
- (B - K)DC(S; B) + C(S; K) - C(S; B) \\
= (B - K) \left( \frac{S}{B} DC \left( \frac{B^2}{S}; B \right) - DC(S; B) \right) \\
+ \frac{S}{B} C \left( \frac{B^2}{S}; B \right) + C(S; K) - C(S; B) \tag{27}
\]

Using the put-call parity we can write

\[
C(S; K) = S - K + P(S; K)
\]

and

\[
C(S; B) = S - B + P(S; B).
\]

We also note that we can rewrite \( DC \left( \frac{B^2}{S}; B \right) \) as

\[
DC \left( \frac{B^2}{S}; B \right) = E^Q \left[ 1_{B^2 > B} \right] = Q \left( \frac{B^2}{S} > B \right) \\
= Q(B > S) = E^Q \left[ 1_{S < B} \right] = DP(S; B)
\]

and \( DC(S; B) \) as

\[
DC(S; B) = E^Q \left[ 1_{S > B} \right] = Q(S > B) \\
= 1 - Q(S \leq B) = 1 - DP(S; B).
\]
We now rewrite equation (27) as
\[
DIC(S; K, B) = (B - K) \left( \frac{S}{B} DC \left( \frac{B^2}{S}; B \right) - DC(S; B) \right) \\
+ \frac{S}{B} C \left( \frac{B^2}{S}; B \right) + C(S; K) - C(S; B)
\]
\[
= (B - K) \left( \frac{S}{B} DP(S; B) - 1 + DP(S; B) \right) + \frac{S}{B} C \left( \frac{B^2}{S}; B \right) \\
+ (S - K + P(S; K)) - (S - B + P(S; B))
\]
\[
= (B - K) \left( \frac{S}{B} DP(S; B) - 1 + DP(S; B) \right) \\
+ \frac{S}{B} C \left( \frac{B^2}{S}; B \right) + (B - K) + P(S; K) - P(S; B)
\]
\[
= (B - K) \left( \frac{S}{B} DP(S; B) + DP(S; B) \right) + \frac{S}{B} C \left( \frac{B^2}{S}; B \right) \\
+ P(S; K) - P(S; B). 
\]

Using the homogeneity of the call price we can write
\[
\frac{S}{B} C \left( \frac{B^2}{S}; B \right) = C(B; S) = E^Q [(B - S)^+] = P(S; B)
\]
so (28) becomes
\[
DIC(S; K, B) = (B - K) \left( \frac{S}{B} DP(S; B) + DP(S; B) \right) 
\]
\[
= (B - K) \left( \frac{S}{B} DP(S; B) + DP(S; B) \right) + \frac{S}{B} C \left( \frac{B^2}{S}; B \right) \\
+ P(S; K) - P(S; B)
\]
\[
= (B - K) \left( \frac{S}{B} DP(S; B) + DP(S; B) \right) + P(S; K). 
\]

Let’s look next at \( \left( \frac{S}{B} DP(S; B) + DP(S; B) \right) \). We rewrite this as
\[
\frac{S}{B} DP(S; B) + DP(S; B) = 2DP(S; B) - \left( 1 - \frac{S}{B} \right) DP(S; B)
\]
\[
= 2DP(S; B) - \frac{1}{B}(B - S)DP(S; B)
\]
and further
\[
(B - S)DP(S; B) = E^Q [(B - S)_{1[S<B]}] = E^Q [(B - S)^+] = P(S; B)
\]
so we have
\[ \frac{S}{B} DP(S; B) + DP(S; B) = 2DP(S; B) - \frac{1}{B} P(S; B). \]
Plugging this into equation (30) we get
\[ DIC(S; K, B) = (B - K) \left( 2DP(S; B) - \frac{1}{B} P(S; B) \right) + P(S; K) \]
which is the same as the value we have in (16) using the method in Carr et al. [5].
We derive the risk neutral value for the discrete barrier down-and-out call. The value of the down-and-out call is
\[
\Phi = e^{-r(T-t)} E^Q \left[ (S_T - K)^+ \cdot 1_{[S_{t_1} > B]} \right].
\]
We calculate
\[
\Phi = e^{-r(T-t)} E^Q \left[ (S_T - K) \cdot 1_{[S_T > K, \ S_{t_1} > B]} \right] - Ke^{-r(T-t)} E^Q \left[ 1_{[S_T > K, \ S_{t_1} > B]} \right]
\]
\[
= E^Q \left[ \frac{S_T}{S_T} S_T \cdot 1_{[S_T > K, \ S_{t_1} > B]} \right] - Ke^{-r(T-t)} Q(S_T > K, \ S_{t_1} > B)
\]
\[
= SQ^S(S_T > K, \ S_{t_1} > B) - Ke^{-r(T-t)} Q^T(S_T > K, \ S_{t_1} > B)
\]
\[
= SN_2(Y, 0, \Sigma_y) - Ke^{-r(T-t)} N_2(X, 0, \Sigma_x),
\]
where \(N_2(Y, 0, \Sigma_y)\) is the bivariate cumulative normal distribution of \(Y\) with mean zero and covariance matrix \(\Sigma_y\). Here we have used \(Q^S\) and \(Q^T\) to denote the probability when \(S\) and \(B\) are used as numeraire, respectively. We have
\[
Y = (y_1, y_2), \quad \text{where} \quad y_1 = \log \left( \frac{S_T}{B} \right) + (r + \frac{\sigma^2}{2})(t_1 - t), \quad y_2 = \log \left( \frac{S_T}{K} \right) + (r + \frac{\sigma^2}{2})(T - t)
\]
\[
X = (x_1, x_2) = (y_1 - \sigma(t_1 - t), y_2 - \sigma(T - t))
\]
and
\[
\Sigma_y = \Sigma_x = \begin{pmatrix}
    t_1 - t & t_1 - t \\
    t_1 - t & T - t
\end{pmatrix}.
\]
Appendix C. MATLAB Code

Calculating the price of the Auto Call option

function [price, notional] = replicationautocallprice(investment, ...
    upperBarrier, lowerBarrier, underlying, t, volatility, rate, yield, ...
    rebate, x)
% REPLICATIONAUTOCALLPRICE This function computes the price of the
% Auto Call option using a replication method.

% Arguments:
%-----------
% % investment = Initial investment.
% % upperBarrier = Barrier (usually the value of the underlying at initial
% % time).
% % lowerBarrier =
% % underlying = Current value of the underlying.
% % t = Current time.
% % volatility =
% % rate = Interest rate.
% % yield = Dividend yield.
% % rebate = A vector of knock out rebates.
% % x = [S0 x1 x2 x3... N], a 1xN vector of the partitionpoints.
% %
% % Returns:
%-----------
% % price = Value of the replicating portfolio.
% % notional = The quantity of digital calls and calls used in the
% % replicating portfolio.
% %
% % Example:
%-----------
% % investment = 1;
% % underlying = [0 : 0.1 : 200]; %100;
% % upperBarrier = 100;
% % rate = 0.025;
% % volatility = 0.8;
% % lowerBarrier = 0.65*upperBarrier;
% % yield = 0;
% % t = 0;
% % rebate = [1.15*investment, 1.3*investment, 1.45*investment, ...
% % 1.6*investment, 1.75*investment];
% % x = [upperBarrier : 25: upperBarrier + 100];
% %
% % Test:
%-----------
% test_replicationautocall
% Compute where to start the recursion in replicationautocall.

if t == 0
    expiry = 1;
    [price, notional] = replicationautocall(investment, upperBarrier, ... 
    lowerBarrier, underlying, t, expiry, volatility, rate, yield, ... 
    rebate, x);
elseif (t == 5)
    expiry = 5;
    [price, notional] = replicationautocall(investment, upperBarrier, ... 
    lowerBarrier, underlying, t, expiry, volatility, rate, yield, ... 
    rebate, x);
elseif (floor(t)-t == 0) && (underlying >= upperBarrier)
    price = rebate(t);
elseif (floor(t)-t == 0) && (underlying < upperBarrier)
    expiry = t+1;
    [price, notional] = replicationautocall(investment, upperBarrier, ... 
    lowerBarrier, underlying, t, expiry, volatility, rate, yield, ... 
    rebate, x);
else
    expiry = ceil(t);
    [price, notional] = replicationautocall(investment, upperBarrier, ... 
    lowerBarrier, underlying, t, expiry, volatility, rate, yield, ... 
    rebate, x);
end
function [price, notional] = replicationautocall(investment, ...
    upperBarrier, lowerBarrier, underlying, t, expiry, volatility, ...
    rate, yield, rebate, x)
% REPLICATIONAUTOCALL This function computes the value of the Auto Call
% option using a replication method, using recursion. The starting point
% must be given (indicated by the expiry, see replicationautocallprice).

% Arguments:
%investment = Initial investment.
%upperBarrier = Barrier (usually the value of the underlying at initial
%    time).
%lowerBarrier =
%underlying = Current value of the underlying.
%t = Current time.
%expiry =
%volatility =
%rate = Interest rate.
%yield = Dividend yield.
%rebate = A vector of knock out rebates.
%x = [S0 x1 x2 x3... N], a 1xN vector of the partition points.
%
% Reutrns:
%----------
%price = Value of the replicating portfolio.
%notional = The quantity of digital calls and calls used in the
%    replicating portfolio.
%
% Author: Julie Loucks
% Date: 2010-04-13

notional = zeros(1,length(x));
if expiry > 4

    stock = (investment / upperBarrier) .* underlying;
    strike = lowerBarrier;
    call = (investment / upperBarrier) * blsprice(underlying, strike, ...
        rate, expiry-t, volatility, yield);
    digitalCalls065S0 = (1-(1/upperBarrier)*lowerBarrier) * investment * ...
        digitalcall(underlying, strike, expiry-t, volatility, rate, yield);
    digitalCallsS0 = (rebate(5) - investment) * digitalcall(underlying, ...
        upperBarrier, expiry-t, volatility, rate, yield);
    price = stock - call + digitalCalls065S0 + digitalCallsS0;

end
else

% Set up the digital call.
notional(1) = rebate(expiry) - replicationautocall(investment, ...
upperBarrier, lowerBarrier, upperBarrier, expiry, expiry + 1, ...
vvolatility, rate, yield, rebate, x);
digitalCallS0 = notional(1) * digitalcall(underlying, upperBarrier, ...
expiry-t, volatility, rate, yield);

% Compute the value of the call options- a matrix where each row is the
% value of an option. We compute one call option for each strike x(i),
% i=1:length(x)-1.
calls = zeros(length(x)-1,length(underlying));
for index = 1 : length(x) - 1
    strike = x(index);
    notional(index + 1) = -(sum(notional(2:index+1)) ...
        * (x(index+1) - x(index)) + (replicationautocall(investment, ...
        upperBarrier, lowerBarrier, x(index+1), expiry, expiry + 1, ...
        volatility, rate, yield, rebate, x) ...
        - replicationautocall(investment, upperBarrier, ...
        lowerBarrier, x(index), expiry, expiry + 1, volatility, ...
        rate, yield, rebate, x)) / (x(index+1) - x(index));
    calls(index,:) = notional(index + 1) * blsprice(underlying, ...
        strike, rate, expiry - t, volatility, yield);
end

price = replicationautocall(investment, upperBarrier, lowerBarrier, ...
underlying, t, expiry+1, volatility, rate, yield, rebate, x) ...
+ digitalCallS0 + sum(calls);
end
function price = digitalcall(underlying, strike, T, volatility, rate, ... yield)

% DIGITALCALL computes the Black-Scholes price of a digital call option.

% Arguments:
% ----------
% underlying = Current value of the underlying.
% strike =
% T = Time to maturity.
% volatility =
% rate = Risk free rate.
% yield = Dividend yield.
%
% Author: Julie Loucks
% Date: 2010-04-13
%
%---------------------------------------

price = zeros(1,length(underlying));
if (T == 0)
    for i = 1 : length(underlying)
        if underlying(i) >= strike
            price(i) = 1;
        end
    end
else
    d2 = (log(underlying / strike) + (rate - yield - 0.5 * ... (volatility)^2)*(T)) / (volatility * sqrt(T));
    price = exp(-rate*T) * normcdf(d2);
end
References


