Triality

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TRIALITY

The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings. But the octonions are the crazy old uncle nobody lets out of the attic: they are nonassociative. - John C. Baez.

Abstract. In 1925, Élie Cartan discovered what nowadays is called the triality principle. It asserts that for each special orthogonal operator $\gamma \in \text{SO}(8)$ there exist $\alpha, \beta \in \text{SO}(8)$ such that $\alpha(x)\beta(y) = \gamma(xy)$ for all $x, y \in \mathbb{R}^8$, where the products involved in this identity are given by the octonion algebra structure on $\mathbb{R}^8$. A proof of this is given and a recent application of the principle in the classification of 8 dimensional absolute valued algebras is reported on.

Thanks goes to my advisor Ernst Dieterich, for dedicating much more of his precious time than can or should be expected, to explain various concepts and theorems that I needed to understand the sources from which I’ve drawn the majority of the contents of this thesis.

1. Inverse Loops.

1.1. Basic structure.

Definition 1. An inverse loop, is a pair $(L, \cdot)$, $L$ being a set, and “·” a function $L \times L \to L$, $(x, y) \mapsto xy$, such that the following two conditions hold:

1. There is an identity element $1 \in L$: $1x = x1 = x$.
2. For all $x \in L$, there is a $y \in L$ such that $(xy)z = z(xy)$, for all $z \in L$.

Thus neither commutativity nor associativity of the operation is assumed. Condition 1 implies that $L \neq \emptyset$. Following convention, we shall often confuse $L$ the set with $(L, \cdot)$ - the structure on $L$.

Remark 2. The identity element is unique, by the standard proof: $1' = 1' \cdot 1 = 1$.

Proposition 3. The inverse is unique.

Proof. Assume $x \in L$ has inverses $y, z \in L$, i.e. $xy = 1 = yx$ and $xz = 1 = zx$. By condition 2 in definition 1, we have that:

$y = y1 = y(xy) = (yx)z = 1z = z$. \hfill $\Box$

Remark 4. Uniqueness being indicated, we will denote the inverse by $x^{-1}$. It follows from uniqueness that $(x^{-1})^{-1} = x$.

Proposition 5. The relation $(xy)z = 1$ is equivalent to $x(yz) = 1$.

Proof. From $xy = z$, $x, y, z \in L$, it follows on the one hand that:

$y = x^{-1}z \Leftrightarrow yz^{-1} = x \Leftrightarrow (yz^{-1})x = 1$,

and on the other hand that:

$z^{-1} = y^{-1}x^{-1} \Leftrightarrow z^{-1}x = y^{-1} \Leftrightarrow y(z^{-1}x) = 1$. \hfill $\Box$
Remark 6. We omit parenthesis, merely writing \(xyz=1\), when the above relation holds.

1.2. Isotopies.

Definition 7. An isotopy of an inverse loop \(L\), is a triple of invertible maps \(\alpha, \beta, \gamma\) \(L \rightarrow L\), such that

\[
\alpha(x) \cdot \beta(y) = \gamma(x \cdot y), \text{ for all } x, y \in L.
\]

We write \((\alpha, \beta | \gamma)\) for this isotopy relation. If \(xyz=1\) implies that \(\alpha(x)\beta(y)\gamma(z) = 1\), we write \((\alpha, \beta, \gamma)\).

Remark 8. We call \((\alpha, \beta | \gamma)\) the duplex form of an isotopy, and \((\alpha, \beta, \gamma)\) the triplex form of an isotopy.

Proposition 9. \((\alpha, \beta | \gamma) \Leftrightarrow (\alpha, \beta, \iota \gamma \iota)\), where \(\iota\) is the inverse map: \(x \mapsto x^{-1}\).

Proof. \(xyz=1 \Leftrightarrow xy = z^{-1} \Rightarrow \alpha(x)\beta(y) = \gamma(z^{-1}) \Leftrightarrow \alpha(x)\beta(y)(\gamma(z^{-1}))^{-1} = 1\), and we write \((\alpha, \beta, \iota \gamma \iota)\) for the isotopy in triplex form equivalent to the duplex form isotopy \((\alpha, \beta | \gamma)\).

\(\gamma(z^{-1})\) is well defined, since \(z^{-1} \in L\) and therefore \(\iota(\gamma(\iota)) : x \mapsto (\gamma(x^{-1}))^{-1}\) is also a bijection, by the uniqueness of the inverse.

Let me now state the first theorem of this section, so that the discussion leading to its proof will be clearer in purpose.

Theorem 10. Let \(I(L)\) be the set of isotopies on an inverse loop \(L\). Then \(I(L)\) is a group under component wise composition.

We have considered an isotopy to be a relation on triplets of invertible maps on an inverse loop \(L\). The theorem states that we can view isotopies as objects with a group structure. We shall use the same notation to denote the isotopy as an object as we use for denoting the relation on the triplets of maps, namely \((\alpha, \beta | \gamma)\).

Definition 11. Let \((\alpha, \beta | \gamma)\) be an isotopy. Then each of the invertible maps in the triple \(\alpha, \beta, \gamma\) are called monotopies. We denote the set of all monotopies on \(L\) by \(M(L) := \{\alpha | \alpha \text{ is a monotopy on } L\}\).

In other words, \(\alpha\) is a member of \(M(L)\) if and only if there exists invertible maps \(\beta\) and \(\gamma\) such that \((\alpha, \beta | \gamma)\) is an isotopy.

Remark 12. Analogously, a map \(\varphi\) is an endomorphism on a group \(G\), if and only if \((\varphi, \varphi | \varphi)\) is an isotopy on \(G\).

Definition 13. The product of two isotopies is their component wise composition:

\[
(\alpha, \beta | \gamma)(\alpha', \beta' | \gamma') := (\alpha(\alpha'), \beta(\beta') | \gamma(\gamma')).
\]

Lemma 14. The product of two isotopies is an isotopy.

Proof. Let \((\alpha, \beta | \gamma), (\alpha', \beta' | \gamma') \in I(L)\). Then for all \(x, y \in L\) we have that

\[
\alpha(\alpha'(x))\beta(\beta'(y)) = \gamma(\alpha'(x)\beta'(y)) = \gamma(\gamma'(xy)),
\]

where the last equality follows from \((\alpha', \beta' | \gamma')\) being an isotopy. Hence \((\alpha\alpha', \beta\beta' | \gamma\gamma') := (\alpha(\alpha'), \beta(\beta') | \gamma(\gamma'))\), is an isotopy.

Lemma 15. Composition of isotopies is associative.
Proof. Exercise. \hfill \Box

**Lemma 16.** The component wise inverse of an isotopy \( I \) is the inverse of \( I \) in \( I(L) \), i.e. \( (\alpha, \beta | \gamma)(\alpha^{-1}, \beta^{-1} | \gamma^{-1}) = (1_L, 1_L | 1_L) = 1_{I(L)} \).

**Proof.** \( xy = \alpha(\alpha^{-1}(x))\beta(\beta^{-1}(y)) = \gamma(\alpha^{-1}(x)\beta^{-1}(y)) \), implies that \( \gamma^{-1}(xy) = \alpha^{-1}(x)\beta^{-1}(y) \), and so \( (\alpha^{-1}, \beta^{-1} | \gamma^{-1}) \) is indeed an isotopy.

That \( (\alpha, \beta | \gamma)(\alpha^{-1}, \beta^{-1} | \gamma^{-1}) = (1_L, 1_L | 1_L) \) holds is obvious, as is the last equality of the lemma. \hfill \Box

And now we can prove our theorem 10:

**Proof.** Lemma 14 establishes the closure of \( I(L) \) under the defined product, lemma 15 states that the product is associative, lemma 16 yields inverses and discloses the nature of the unit element. \hfill \Box

13. **Companions.** If \( \gamma \) is a monotopy, we have by definition that there exists \( \alpha, \beta \in M(L) \) such that \( xy = z \) implies that \( \alpha(x)\beta(y) = \gamma(z) \), and in particular that

\[
\alpha(z)\beta(1) = \gamma(z) \iff \alpha(z) = \gamma(z)(\beta(z))^{-1} = \gamma(z)b,
\]

for some \( b \in L \), and likewise \( \beta(z) = \alpha\gamma(z) \), for some \( a \in L \).

**Proposition 17.** \( \gamma \in M(L) \) if and only if there exists \( a, b \in L \), such that \( \gamma(xy) = (\gamma(x)b)(\alpha\gamma(y)) \).

**Proof.** We have already established " \( \Rightarrow \) " and " \( \Leftarrow \) " follows by taking \( \alpha = \gamma R_b \) and \( \beta = \gamma L_a \), where left and right multiplication by a fixed element defines the functions \( R_b : L \to L, x \mapsto xb \), and \( L_a : L \to L, x \mapsto ax \). \hfill \Box

**Definition 18.** A companion is one of either \( a \) or \( b \) in proposition 15.

So proposition 15 asserts that the existence of companions is a necessary and sufficient condition for a bijective function \( \gamma : L \to L \) to be a monotopy.

Note that a group endomorphism has \( a = 1 = b \) as companions, so that monotopies are a natural generalization of endomorphisms.

**Lemma 19.** \( B_a \) is well defined, i.e. \( B_a(x) := R_a(L_a(x)) = L_a(R_a(x)) \) is independent of the order of multiplication, and we can omit parenthesis: \( a(xa) = (ax)a =: axa \).

**Proof.** Let \( (\alpha, \beta | \gamma)^{-1} \) and \((\alpha\alpha, \gamma | \beta)\) be two isotopies on \( L \). Then

\[
(\alpha, \beta | \gamma)^{-1} \cdot (\alpha\alpha, \gamma | \beta) = (\alpha^{-1}\alpha\alpha, \beta^{-1} | \gamma^{-1} \beta) := h, \text{ is an isotopy}.
\]

Noting that \( \gamma^{-1} \beta = L_a \) and so \( \beta^{-1} \gamma = (L_a)^{-1} = L_{a^{-1}}, \) and so \( h = (\alpha^{-1}\alpha L_{a^{-1}} | L_a) \). Applying \( h \) to \( xa \) yields: \( (\alpha^{-1}\alpha\alpha(x), a^{-1}a = \alpha^{-1}\alpha\alpha(x) = a(x) \) and so \( a(xa), a^{-1}y = a(xy) \) and in particular \( a(xa), a^{-1} = ax, \) hence \( a(xa) = (ax)a \). \hfill \Box

**Lemma 20.** If \( (\alpha, \beta | \gamma) \) is an isotopy, then so are \((\alpha\alpha, \gamma | \beta), (\gamma \gamma, \alpha | \gamma \gamma), (\gamma \beta \gamma, \beta | \gamma \gamma), (\gamma, \gamma \beta | \alpha) \).

**Proof.** If \( xy = z \), then the following are equivalent expressions:

\[
xy = z \iff x = zy^{-1} \iff
\]
\[ z^{-1}x = y^{-1} \iff \]
\[ z^{-1} = y^{-1}x^{-1} \iff \]
\[ yz^{-1} = x^{-1} \iff \]
\[ y = x^{-1}z \]

It follows that if \((\alpha, \beta | \gamma)\) is an isotopy then these relations will also be equivalent:

\[ \alpha(x)\beta(y) = \gamma(z) \iff \]
\[ \gamma(x) = \alpha(z)\beta(y^{-1}) \iff \]
\[ \alpha(z^{-1})\beta(x) = \gamma(y^{-1}) \iff \]
\[ \gamma(z^{-1}) = \alpha(y^{-1})\beta(x^{-1}) \iff \]
\[ \alpha(y)\beta(z^{-1}) = \gamma(x^{-1}) \iff \]
\[ \gamma(y) = \alpha(x^{-1})\beta(z). \]

Again moving around the factors, restoring the order of \(x, y\) and \(z\), will yield the following equivalences:

\[ \alpha(x)\beta(y) = \gamma(z) \iff \]
\[ \gamma(x)\beta(y^{-1})^{-1} = \alpha(z) \iff \]
\[ \beta(x)\gamma(y^{-1})^{-1} = \alpha(z^{-1})^{-1} \iff \]
\[ \beta(x^{-1})^{-1}\alpha(y^{-1})^{-1} = \gamma(z^{-1})^{-1} \iff \]
\[ \gamma(x^{-1})^{-1}\alpha(y) = \beta(z^{-1})^{-1} \iff \]
\[ \alpha(x^{-1})^{-1}\gamma(y) = \beta(z). \]

Reading the duplex form of isotopy relations implied by the above set of equivalences, we get:

\((\alpha, \beta | \gamma) \iff \)
\((\gamma, \iota\beta\iota | \alpha) \iff \)
\((\beta, \iota\gamma\iota | \iota\alpha\iota) \iff \)
\((\iota\beta\iota, \iota\alpha\iota | \iota\gamma\iota) \iff \)
\((\iota\gamma\iota, \alpha | \iota\beta\iota) \iff \)
\((\iota\alpha\iota, \gamma | \beta). \)

If we transform the above set of equivalences to the triplex form and replace \(\gamma\) by \(\iota\gamma\iota\) - to make the symmetry more obvious - we get the following six isotopies which will come in handy later:

\((\alpha, \beta, \gamma) \iff \)
\((\iota\gamma\iota, \iota\beta\iota, \iota\alpha\iota) \iff \)
\((\beta, \gamma, \alpha) \iff \)
\((\iota\beta\iota, \iota\alpha\iota, \iota\gamma\iota) \iff \)
\((\gamma, \alpha, \beta) \iff \)
\((\iota\alpha\iota, \iota\gamma\iota, \iota\beta\iota). \)
Theorem 21. If \( a = \alpha(1) \) for some monotype \( \alpha \in M(L) \) we have that \( M_{a}(L) := \{L_{a}, L_{a}^{-1}, R_{a}, R_{a}^{-1}, B_{a}, B_{a}^{-1}\} \) are monotypes since \( \{L_{a}, R_{a} | B_{a}\}, \{B_{a}, L_{a}^{-1} | L_{a}\}, \{R_{a}, B_{a}^{-1} | R_{a}^{-1}\}, \{L_{a}^{-1}, R_{a} | B_{a}^{-1}\}, \{B_{a}^{-1}, L_{a} | L_{a}^{-1}\}, \{R_{a}^{-1}, B_{a} | R_{a}\} \) are isotopies.

Proof. \( L_{a} \) takes 1 to a, and is a monotypy, since \( \{L_{a}, R_{a} | B_{a}\} \) is an isotopy, by lemma 17. By lemma 20 we have the six isotopies, where we can write \( tL_{a}t = R_{a}^{-1} \), \( tR_{a}t = L_{a}^{-1} \) and \( tB_{a}t = B_{a}^{-1} \). By definition the members of \( M_{a}(L) \) are monotypies.

Theorem 22. The action of \( M(L) \) on \( L, M \times L \to L, \alpha:x \mapsto \alpha(x) \) is transitive if and only if \( (zx)(yz) = z(xy)z \) holds for all \( x, y \in L \). \( (zx)(yz) = z(xy)z \) is also known as the Moufang identity, and inverse loops in which it holds are called Moufang loops.

Proof. Assume \((zx)(yz) = z(xy)z, \text{ for all } x, y \in L. \) But

\[
(zx)(yz) = L_{x}(z)R_{x}(y) \quad \text{and} \quad (z(xy)z) = B_{x}(z)(y),
\]

hence \( (L_{x}, R_{x} | B_{x}) \) for all \( z \in L \), and so \( L_{x}, R_{x} \) and \( B_{x} \) are monotypies for all \( z \in L \).

So given \( x, y \in L \), we have that \( L_{y}x^{-1}(x) = (yx^{-1})x = y(x^{-1}x) = y \) and likewise for \( R \), and the action is transitive.

Conversely assume the action \( M \times L \to L \) is transitive, i.e. for all \( x, y \in L \) there exists \( \alpha \in M(L) \) such that \( \alpha(x) = y \), which is equivalent to there being only one orbit of the action on \( M(L) \) on \( L \).

In particular for all \( z \in L \) there exists \( \alpha \in M \) such that \( \alpha(1) = z \). By theorem 19 \( L_{x} \) is a monotypy, taking 1 to \( x, \) and we have the isotopy \( (L_{x}, R_{x} | B_{x}) \) for all \( z \in L \), and so \( L_{x}(z)R_{x}(x) = B_{x}(z) \), i.e. \( (zx)(yz) = z(xy)z \). \( \square \)

2. Composition Algebras, Absolute Valued Algebras and the Octonions.

In this section we will define absolute valued algebras and composition algebras, derive some basic consequences and describe some features of their intersection.

Definition 23. Let \( k \) be a field of characteristic not 2. A \( k \)-algebra is vector space \( A \) over \( k \) that is equipped with a \( k \)-bilinear multiplication \( A \times A \to A, (x,y) \mapsto xy \).

We restrict the choice of field to be able to define a quadratic form on the algebra.

Definition 24. A composition algebra \( A = (A, m) \) is a non-zero \( k \)-algebra endowed with a quadratic form \( m : A \to k \) that is multiplicative and non-degenerate, i.e. \( m(xy) = m(x)m(y) \), and \( <x, y> := \frac{1}{2}(m(x+y) - m(x) - m(y)) \) is non-degenerate: \( <x, y> = 0 \) for all \( y \in A \) if and only if \( x = 0 \).

We'll now deduce some consequences the composition law, \( m(xy) = m(x)m(y) \).

Proposition 25. \( <xy, xz> = m(x)<y, z> \quad \text{and} \quad <xx, yz> = <x, y > m(z). \) (The scaling laws).

Proof. Replacing \( y \) by \( y + z \) in the the composition law, gives:

\[
m(xy + xz) = 2 <xy, xz> + m(xy) + m(xz), \]

but

\[
m(xy + xz) = m(x(y + z)) = m(x)m(y + z) = m(x)(2 <y, z> + m(y) + m(z)),
\]

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so we we cancel the terms $m(xy)$ and $m(xz)$ using the composition law, divide by two and we have our equality. $<xz, yz> = <x, y > m(z)$ is proved in a similar way. \[\square\]

**Proposition 26.** $<xy, uz> = 2 <x, u > <y, z > - <xz, uy >$. *(The Exchange Law).*

**Proof.** Replacing $x$ by $x + u$ in $<xy, xz> = m(x) <y, z>$, gives

$< (x + u)y, (x + u)z > = < xy + uy, xz + uz > = < xy, xz > + < xy, uz >$

$+ < uy, xz > + < uy, uz >$, which by the scaling laws is

$(m(x) + 2 <x, u > + m(u)) <y, z >$.

We cancel terms and rearrange. \[\square\]

**Definition 27.** Conjugation of an element $x$, denoted $x^*$, is defined as $x^* := 2 <x, 1 > - x$.

**Proposition 28.** $<xy, z> = <y, x^*z>$, and $<xy, z> = <x, zy^*>$. *(The Braid Laws).*

**Proof.** Put $u = 1$ in the exchange law and we get

$2 <x, 1 > <y, z > - < xz, y > = < y, (2 <x, 1 > - x)z >$

The left hand side is equal to $<xy, z>$, and the right hand side is by definition equal to $<y, x^*z>$. The other case is similar. \[\square\]

**Proposition 29.** $x^{**} := (x^*)^* = x$. *(Biconjugation)*

**Proof.** Setting $y = 1$ and $z = t$ and applying the braid law twice we have:

$<x, t > = < x, 1 >$, $t > = < (x^*)^*, t > = < x^{**}, t >$, for all $t$. \[\square\]

**Proposition 30.** $(xy)^* = y^*x^*$. *(Product Conjugation)*

**Proof.** Repeated use of biconjugation gives that:

$< y^*x^*, t > = < x^*, y t > = < x^*t^*, y > = < t^*, xy > = < t^*, (xy)1 >$

$= < t^*(xy)^*, 1 > = < (xy)^*, t >$. \[\square\]

**Proposition 31.** Define $x^{-1} = x/m(x)$, for $x \neq 0$. Then $x^*(xy) = m(x)y = (yx)x^*$, or equivalently $x^{-1}(xy) = y = (yx)x^{-1}$

**Proof.** $< x^*(xy), t > = < xy, xt > = m(x) < y, t > = < m(x)y, t >$. \[\square\]

**Theorem 32.** The Moufang laws hold: $(xy)(zx) = x((yz)x)$ for all $x$, $y$, $z$ in a composition algebra.

**Proof.** We have the following equalities:

$< (xy)(zx), t > = < xy, t(x^*z^*) >$

$= 2 < x, t > < y, x^*z^* > = < x(x^*z^*), ty >$

$= 2 < x, t > < yz, x^* > = < yz, x^*z^* >$

$= 2 < yz, x^* > < x, t > = m(x) < z^*y^*, t >$

$= 2 < x, (yz)^* > < x, t > = m(x) < (yz)^*, t >$. 

So \((xy)(zx) = 2 < x, (yz)^* > x - m(x)(yz)^*\) is a function of \(x\) and \(yz\) only. We can therefore replace \(y\) and \(z\) by any two other elements with the same product, so deduce that:
\[
(xy)(zx) = (x(yz))1x = (x(yz))x, \quad \text{and} \quad (xy)(zx) = x1((yz)x) = x((yz)x).
\]

**Corollary 33. Bimultiplication is well defined in a composition algebra:**

**Proof.** \((xy)x = (xy)(1x) = (x1)(yx) = x(yx)\).

**Remark 34.** Thus we have proved that unital composition algebras are special cases of Moufang Loops.

**Definition 35.** An absolute valued algebra \(A\), is a non-zero real algebra and a multiplicative norm \(\| \cdot \|: A \to \mathbb{R}_{\geq 0}\) i.e. for all \(x, y \in A\), \(\| xy \| = \| x \| \| y \|\), where the last juxtaposition is multiplication in \(\mathbb{R}\). We also say that the norm respects the multiplication on \(A\).

**Proposition 36.** \(\| xy \| = \| x \| \| y \| \) implies that \(A\) has no zero dividers.

**Proof.** For otherwise there would be non-zero \(x\) and \(y\) in \(A\) such that \(0 = \| xy \| = \| x \| \| y \| \neq 0\).

**Proposition 37.** If the dimension of \(A\) is finite, then \(A\) contains no zero dividers if and only if \(A\) is a division algebra, i.e. \(L_a\) and \(R_a\) are bijective for all \(a \neq 0\).

**Proof.** The lack of zero dividers in \(A\) implies that \(L_a\) and \(R_a\) are injective and linear algebra tells us that linear operators on finite vector spaces are surjective if they are injective.

**Theorem 38.** If \(A\) is a finite dimensional absolute valued algebra then \(A\) is a composition algebra.

**Proof.** We need a quadratic form that is multiplicative and non-degenerate. \(m : A \to \mathbb{R}\). Our candidate is \(m(x) := \| x \|^2\). It is multiplicative since \(m(xy) = \| xy \|^2 = \| x \| \| y \| \| x \|^2 = m(x)m(y)\). It is non-degenerate since if \(< x, y > = 0\) for all \(y \in A\) then in particular \(0 = \| xy \| = \| x \|^2\) which implies that \(x = 0\).

However it is not in general a quadratic form since the inner product it induces \(< x, y > := \frac{1}{4}(m(x + y) - m(x) - m(y))\) need not be bilinear. However, Urbanik and Wright [7] proved that if \(A\) is unital then \(A\) is finite dimensional and then the norm induces a quadratic form [1].

### 2.1. The octonions algebra.

**Definition 39.** The real octonion algebra \(\mathbb{O}\) is the vectorspace \(\mathbb{R}^8\) with a bilinear multiplication \(\mathbb{O} \times \mathbb{O} : (x, y) \mapsto xy\), given by subjecting the basis vectors \((1, i_0, i_1, i_2, i_3, i_4, i_5, i_6) := (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8)\) to the following relations:

\[
i_n^2 = -1 \quad \text{and} \quad i_{n+1}i_{n+2} = -i_{n+1}\]

where the subscripts run modulo 7.

Each octonion \(o \in \mathbb{O}\) is thus on the form:
\[ a = x_\infty + x_0 i_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 + x_4 i_4 + x_5 i_5 + x_6 i_6, \quad x_i \in \mathbb{R}. \]

The standard norm on \( \mathbb{R}^8 \) makes the octonion algebra an 8 dimensional absolute valued algebra.

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**Proposition 40.** Every element \( \alpha \) of \( O(n) \) that fixes a \( k \)-dimensional subspace can be written as a product of at most \( n - k \) reflections.

**Proof.** Take a vector \( v \) not fixed by \( \alpha \), i.e. \( \alpha(v) = w \neq v \). The reflection in \( v - w \), call it \( \sigma \), restores \( w \) to \( v \) and fixes any vector \( u \) in \( \mathcal{U} \), the subspace fixed by \( \alpha \). To see this first note that since \( \alpha \in O(n) \) we have that \( \| v \| = \| \alpha(v) \| = \| w \| \). Hence the parallelogram spanned by \( v \) and \( w \) is a rhombus and so the diagonals \( v + w \) and \( v - w \) intersect at right angles, indeed:

\[ < v + w, v - w > = < v, v > - < v, w > + < w, v > - < w, w > = 0. \]

We can thus make a unique orthogonal decomposition of \( v \) and \( w \), namely:

\[ v = \frac{1}{2}(v + w) + \frac{1}{2}(v - w) \quad \text{and} \quad w = \frac{1}{2}(v + w) - \frac{1}{2}(v - w). \]

The reflection \( \sigma \) in \( v - w \) will fix \( v + w \) since it's orthogonal to \( v - w \) and \( \sigma(v - w) = -(v - w) \).

It follows that \( \sigma(w) = v \).

If \( u \) is fixed by \( \alpha \) then \( < u, v > = < \alpha(u), \alpha(v) > = < u, w > \) which implies that \( < u, v - w > = 0 \) and \( u \) is thus orthogonal to \( v - w \) and hence fixed by the reflection so we have that \( \alpha(u) = u \) implies \( \sigma(u) = u \). But we also have that \( \sigma \alpha(v) = \sigma(w) = v \) and so \( \sigma \alpha(x) = x \), for all \( x \) in \( \mathcal{U} + \mathbb{R}v \). But \( \mathcal{U} \subseteq \mathcal{U} + \mathbb{R}v \) since \( v \) was not in \( \mathcal{U} \), and so we proceed by induction.

If \( \dim(\mathcal{U}) = n \) then \( \alpha = \prod_{i=0}^{n} \sigma_i \). Assume that if \( \alpha \) fixes \( \mathcal{U} \) pointwise and that \( \dim(\mathcal{U}) = k \geq 1 \) then \( \alpha = \sigma_1 \ldots \sigma_l \), \( l \leq n - k \). Now if \( \alpha \) fixes \( \mathcal{T} \) where \( \dim(\mathcal{T}) = k - 1 \), then \( \sigma_{k+1} \alpha \) fixes the subspace \( \mathcal{T} + \mathbb{R}v \) and \( \dim(\mathcal{T} + \mathbb{R}v) = \dim(\mathcal{T}) + 1 = k \), and by assumption we must have that \( \sigma_{k+1} \alpha = \sigma_i \alpha \sigma_1 \ldots \sigma_l \), where \( \sigma_i = \sigma_1 \) and \( l + 1 = n - k + 1 = n - (k - 1) \).

\[ \square \]

**Theorem 41.** If \( \alpha \in SO(8) \), then \( \alpha \) is a product of an even number of reflections.

**Proof.** Reflections \( \alpha \) have \( \det(\alpha) = -1 \), \( SO(n) \) is the subgroup of maps \( \beta \) with \( \det(\beta) = 1 \) and determinants are multiplicative, i.e. \( \det(\alpha \beta) = \det(\alpha) \det(\beta) \). It follows from proposition 40 that all elements of \( SO(n) \) must be the product of an even number of reflections.

The octonions are strongly non-associative in the sense that

**Proposition 42.** If \( x(ry) = (xr)y \) for all octonions \( x, y \) and \( r \), then \( r \) is real.

**Proof.** \((i_{n+1}i_{n+2})(i_{n+1}i_{n+2}) = i_{n+3}i_{n+2} = i_{n+5} \) and \( i_{n+1}(i_{n+1}i_{n+2}) = i_{n+1}i_{n+2} = i_{n+1}i_{n+2} = i_{n+3} \). Thus for \( r \) to associate with all octonions \( r \) must have all imaginary coefficients equal to zero so \( r \) is real.

\[ \square \]

**Proposition 43.** If \( a, b \) is any pair of companions for the monotopy \( \gamma \) then any other pair has the form \( ar, r^{-1}b \), where \( r \) is real.
TRIALITY

Proof. The condition that A, B should be a second pair of companions is the identity 
\( (\gamma(x)a)(b(y)) = (\gamma(x)A)(B\gamma(y)) \). If we write \( \gamma(x) = X \) and \( \gamma(y) = Y \) then the identity becomes \( (Xa)(bY) = (XA)(BY) \) for all X, Y. We can choose r such that 
\( (R_r \) is transitive) \( A = ar \) and since \( \gamma \) is bijective, there is x and y such that 
\( X = a^{-1}, Y = 1 \), which gives \( b = rB \), so \( B = r^{-1}b \) and the identity is now that 
\( (Xa)(bY) = (X(ar))(r^{-1}bY) \), for all X, Y. Set \( Y = (r^{-1}b)^{-1} = b^{-1}r \), shows that 
\( (Xa)r = X(ar) \), and setting \( X = (ar)^{-1} = r^{-1}a^{-1} \) shows that \( r^{-1}(bY) = (r^{-1}b)Y \) and we get the identity \( (Xa)(bY) = ((Xa)r)((r^{-1}b)Y) \), for all X, Y. Finally set 
\( Xa = x \) and \( bY = rz \) and we get the identity on the desired form \( x(rz) = (xr)z \), which by proposition 42 implies that \( r \) is real.

\( \square \)

**Theorem 44.** Let A be a real division algebra with \( 2 \leq \dim(A) < \infty \). Let x and y be elements in \( A \setminus \{0\} \). Then \( \det(L_x) \) and \( \det(L_y) \) have the same sign.

**Proof.** Consider the sign map: \( \mathbb{R} \setminus \{0\} \rightarrow \mathbb{C}_2 \), sign(x) = \( \frac{1}{2} \), as having values in the cyclic group \( \mathbb{C}_2 = \{1, -1\} \). We define the generalized sign map 
\( s : GL_2(A) \rightarrow \mathbb{C}_2 \), \( s(x) = \text{sign}(\det(x)) \, \det(x) \),
which is defined for every finite-dimensional real vector space A. The maps:
\( L : A \setminus \{0\} \rightarrow GL_2(A), a \mapsto L_a \) and \( R : A \setminus \{0\} \rightarrow GL_2(A), a \mapsto R_a \)
yield maps \( l \) and \( r \) when composed with the generalized sign map:
\( l : A \setminus \{0\} \rightarrow \mathbb{C}_2, l(a) = s(L_a) \) and \( r : A \setminus \{0\} \rightarrow \mathbb{C}_2, r(a) = s(R_a) \).

If A is either \( \mathbb{C} \), \( \mathbb{H} \) or \( \mathbb{O} \) then both \( l \) and \( r \) are constant. For we equip \( A \setminus \{0\} \) with the standard Euclidean topology of \( \mathbb{R}^{\dim(A)} \) and \( \mathbb{C}_2 \) with the discrete topology. \( A \setminus \{0\} \) is connected, since the dimension is \( \geq 2 \), and \( l \) and \( r \) are continuous - since they are the composition of continuous functions - hence their image is connected and they must therefore be constant.

\( \square \)

**Proposition 45.** Let A be an absolute valued algebra. If \( \|x\| = 1 \), then \( L_x \) is an orthogonal map.

**Proof.** We have the equality:
\( <v,v> = \|v\|^2 = \|x\|^2 \|v\|^2 = \|xv\|^2 = \|L_xv\|^2 = <L_xv,L_xv> \).

\( \square \)

**Corollary 46.** \( \det(L_x) = 1 \) for all unit octonions.

**Proof.** \( \det(L_1) = 1 \), and if \( \|x\| = 1 \) then \( x \) is in \( O(8) \), and so it must be in \( SO(8) \).

\( \square \)

**Proposition 47.** \( B_x \) is a scalar multiple of \( \text{ref}(1)\text{ref}(x) \), where \( \text{ref}(x) \) is the reflection in the vector \( x \), i.e. in the hyperplane of which \( x \) is orthogonal.

**Proof.** In theorem 32 we derived that \( B_x(yz) = 2 <x,(yz)^* > x - m(x)(yz)^* \). Comparing this with the formula for the reflection: \( \text{ref}(x) : t \mapsto t - 2 <x,t > x <x,x> \), we get that \( - <x,x> \text{ref}_x((yz)^*) = - <x,x> (yz)^* + 2 <x,(yz)^*> x = (xy)(zx) = B_x(yz) \). So \( <x,x> \text{ref}_x(yz) = B_x(yz) \). This is so since for real \( r \), \( rz^* = a - b_iq_0 - ... - b_iq_6 \) which implies that \( -rz^* = -a + b_iq_0 + ... + b_iq_6 \) which implies that \( \text{ref}_1(-rz^*) = a + b_iq_0 + ... + b_iq_6 = rz \). Of course real numbers commutes with reflections.

\( \square \)

**Lemma 48.** The operations \( \text{ref}(1)\text{ref}(a) \) and \( \text{ref}(a)\text{ref}(1) \) are bimultiplications by unit octonions.
Proof. Reflections are naturally unaffected if we rescale \( a \) to have norm 1. But by proposition we have that the first one is nothing but bimultiplication with \( a \): \( \text{ref}(\text{ref}(a)) = B_a \). The operations are obviously inverses of each other, \( \text{ref}(a)\text{ref}(1) \circ \text{ref}(1)\text{ref}(a) = 1 \), and the inverse of \( B_a \) is of course given by \( (B_a)^{-1} = B_{a^{-1}} \), and so the second operation is also bimultiplication by a unit octonion.

\[ \Box \]

**Definition 49.** \((\alpha, \beta \mid \gamma)\) is called a special orthogonal isotopy if \( \alpha, \beta, \gamma \) are all in \( SO(8) \).

**Proposition 50.** The special orthogonal isotopies form a subgroup of the group of all isotopies.

**Proof.** \( SO(8) \) is a group, so the component wise composition of isotopies is closed in the set of special orthogonal isotopies.

**Remark 51.** The group of special orthogonal isotopies is also called the spin group \( \text{Spin}_8 \).

**Theorem 52.** If \( \gamma \) is any element in \( SO(8) \) then there exists \( \alpha, \beta \) in \( SO(8) \) for which \((\alpha, \beta \mid \gamma)\) is an isotopy which implies that \( \gamma \) is a monotoropy. Moreover, \( \alpha \) and \( \beta \) are uniquely determined by \( \gamma \) up to sign, the only other pair being \(-\alpha \) and \(-\beta\).

**Proof.** We can write \( \gamma \) as the product of an even number of reflections,

\[
\gamma = \text{ref}(a_1)\text{ref}(b_1)\text{ref}(a_2)\text{ref}(b_2)\ldots\text{ref}(a_{2n})\text{ref}(b_{2n}).
\]

But

\[
\text{ref}(a_i)\text{ref}(b_i) = \text{ref}(1)\text{ref}(1)\text{ref}(b_i)
\]

and the left hand is the product of two bimultiplications of unit octonions, so \( \gamma \) is the product of \( 4n \) unit bimultiplications, \( B_{c_1}B_{c_2}\ldots B_{c_{4n}} \). But then

\[
(\alpha, \beta \mid \gamma) = (L_{c_1}L_{c_2}\ldots L_{c_{4n}}, R_{c_1}R_{c_2}\ldots R_{c_{4n}} \mid B_{c_1}B_{c_2}\ldots B_{c_{4n}})
\]

is the desired isotopy, since \( c_1c_2\ldots c_{4n} \) are unit octonions \( \alpha \) and \( \beta \) are in \( SO(8) \).

Companions being unique up to scalar multiplication \( \alpha, \beta \) are unique up to sign, as the only non-trivial scalar that preserves \( SO(8) \) membership is \(-1\), for \( \det(\lambda x) = \lambda^n\det(x) \) and \(-L_{\alpha} = L_{-\alpha} \), so if \( L_{\alpha} \) is in \( SO(8) \) then \(-L_{\alpha} \) is in \( SO(8) \). Therefore we also have the isotopy \((-\alpha, -\beta \mid \gamma)\), where \(-\alpha = -L_{c_1}L_{c_2}\ldots L_{c_{4n}} = L_{-1}L_{c_1}L_{c_2}\ldots L_{c_{4n}}\).

\[ \Box \]

**Remark 53.** The theorem states that \( \text{Spin}_8 \) forms a 2-to-1 cover of \( SO(8) \), a 2-to-1 cover meaning a surjective homomorphism where the pre-image of each element in the codomain is a set containing exactly two elements.

**Lemma 54.** Conjugate elements in \( SO(8) \) fix subspaces of the same dimension.

**Proof.** Given \( \alpha, \beta \) in \( SO(n) \) such that \( \alpha = \gamma\beta\gamma^{-1} \) for some \( \gamma \) in \( SO(n) \), consider

\[
\text{Fix}(\alpha) := \{ x \in \mathbb{R}^n \mid \alpha(x) = x \} = \ker(\alpha - 1_n),
\]

where the last equality, apart from being obvious, shows that \( \text{Fix}(\alpha) \) is a subspace. Let \( \text{Fix}(\alpha) = \{ x \in \mathbb{R}^n \mid \beta(x) = x \} \) and our goal is to show that \( \dim(\text{Fix}(\alpha)) = \dim(\text{Fix}(\beta)) \).
But $x \in \text{Fix}(\alpha)$ gives that $\alpha(x) = x$ which is equivalent to $\gamma \beta \gamma^{-1}(x) = x$, if and only if $\beta \gamma^{-1}(x) = \gamma^{-1}(x)$, if and only if $\gamma^{-1}(x) \in \text{Fix}(\beta)$. Let $\gamma^{-1}(x) = y$ and the expression is that $y \in \text{Fix}(\beta)$ if and only if $\gamma(x) \in \text{Fix}(\alpha)$, thus $\text{Fix}(\alpha) = \gamma(\text{Fix}(\beta))$. But $\gamma$ is a bijection, therefore $\dim(\text{Fix}(\alpha)) = \dim(\text{Fix}(\beta))$. \hfill $\Box$

**Lemma 55.** $B_{i_0}$ fixes a space of dimension $\geq 6$. $L_{i_0}$ is fix-point free.

**Proof.** We study the multiplication table:

\[
\begin{align*}
i_0i_0 &= -1, \\
i_0i_1 &= -i_0, \\
i_0i_2i_0 &= i_3i_0 = i_1, \\
i_0i_3i_0 &= i_0i_1 = i_3, \\
i_0i_4i_0 &= i_5i_0 = i_4, \\
i_0i_5i_0 &= i_0i_4 = i_5, \\
i_0i_6i_0 &= i_0i_2 = i_6.
\end{align*}
\]

and the first statement of the theorem follows. For the second statement observe that $L_{i_0}(x) = i_0x$. So $x \in \text{Fix}(L_{i_0})$ implies that $i_0x = x = lx$ if and only if $(i_0 - 1)x = 0$ but $i_0 - 1 \neq 0$ and $\mathbb{O}$ has no zero-divisors. This implies that $x = 0$, therefore $\dim(\text{Fix}(L_{i_0})) = 0$. \hfill $\Box$

**Theorem 56.** The spin group $\text{Spin}_8$ has an outer automorphism of order 3, namely $(\alpha, \beta, \gamma) \mapsto (\beta, \gamma, \alpha)$.

**Proof.** By lemma 20 the image of an isotopy is an isotopy, and it obviously respects composition, so this is an automorphism and it obviously has order 3. To see that it is an outer automorphism (i.e. not a conjugation $x \mapsto y^{-1}xy$) consider the isotopy $(L_{i_0}, R_{i_0}, B_{i_0})$ which is sent to $(R_{i_0}, B_{i_0}, L_{i_0})$. The two are not conjugates since $L_{i_0}$ has no fixpoint and $B_{i_0}$ fixes a 6-space. \hfill $\Box$

**Remark 57.** $\text{SO}(8)$ does not have this triality because $\gamma$ determines $\alpha$ and $\beta$ only up to sign.

4. **Triality and an Isomorphism Conditions for Absolute Valued Algebras.**

We start by stating a classic result by Albert [3].

**Theorem 58.** Let $A_f$ denote the category of finite dimensional absolute valued algebras. Let $O_1(V) \subset O(V)$ be the set of orthogonal maps fixing the identity $1_V$ of the vector space $V$. Let $A$ be either $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. Then $\{A_{f,g} \mid f, g \in O_1(A)\}$, is dense in $A_f$, i.e. for every $A$ in $A_f$ there is $f$ and $g$ in $O_1$ such that $A_{f,g} \simeq A$.

**Definition 59.** If $A$ is in $\{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $f, g$ in $O_1(A)$ then $A_{f,g} = A$ as a vector space and multiplication is defined as $x \circ y = f(x)g(y)$, where the juxtaposition is multiplication in $A$ and the norm is the norm of $A$.

**Proposition 60.** If $f$ and $g$ are in $O(A)$ then $A_{f,g}$ is an absolute valued algebra.

**Proof.** The vectorspace is left intact and all that needs to be proved is that the norm respects the multiplication. This is established by the following calculation:

\[
\| x \circ y \| = \| f(x)g(y) \| = \| f(x) \cdot g(y) \| = \| x \| \cdot \| y \|.
\]
**Definition 61.** Let $A$ be an absolute valued algebra with $2 \leq \dim(A) \leq \infty$, then the double sign of $A$ is $(l(A), r(A)) \in C_2 \times C_2$, where $l(A) := l(a)$ for some $a \in A \setminus \{0\}$.

**Remark 62.** $l(A) := l(a)$, $a \in A \setminus \{0\}$ is well defined, since $l(a)$ has been shown to be constant.

**Theorem 63.** If two finite dimensional absolute valued algebras have different double signs they are not isomorphic.

**Proof.** We will show that $\varphi : A \cong B \Rightarrow (l(A), r(A)) = (l(B), r(B))$.

But $l(A) := l(a) = \text{sign}(\det(L_a))$ for some $a \in A \setminus \{0\}$. Choose some such $a$ and set $b = \varphi(a)$. Then

$$L_b(\varphi(x)) = by = \varphi(a)\varphi(x) = \varphi(ax) = \varphi L_a(x), \ x = \varphi^{-1}(y) \in A.$$ 

So $L_b \varphi = \varphi L_a$. Therefore chose an arbitrary basis for $A$: $a = (a_1, ..., a_n)$ and let $b = \varphi(a) = (\varphi(a_1), ..., \varphi(a_n))$ be a basis for $B$. Then $\text{mat}(L_b)_{a} = \text{mat}(L_b)_{\varphi(a)}$. It follows in particular that $\text{sign}(\det(\text{mat}(L_b)_{a})) = \text{sign}(\det(\text{mat}(L_b)_{\varphi(a)}))$. 

**Proposition 64.** Let $\psi$ be an isomorphism of finite dimensional absolute valued algebras $A$ and $B$. Then $\psi$ is a proper orthogonal map.

**Proof.** For linear maps $f$ we have that $\langle v, v' \rangle = \langle f(v), f(v') \rangle$ if and only if $\|v\| = \|f(v)\|$. But isomorphisms of finite dimensional absolute valued algebras respect the norm [2], i.e. $\|v\| = \|\psi(v)\|$ and for finite dimensional absolute valued algebras

$$\langle v, w \rangle := \frac{1}{2}(\|v + w\| - \|v\| - w\|),$$

defines a scalar product [1]. Thus if $\psi$ is an isomorphism, $\psi$ is orthogonal.

But if $A$ and $B$ are isomorphic they have the same double sign $(l(A), r(A)) = (l(B), r(B))$ and to preserve that, $\psi$ must be proper orthogonal.

We will now prove two lemmas, masquerading as propositions, before proving our final theorem.

**Proposition 65.** Let $\psi : A \rightarrow B$ be an isomorphism of finite dimensional absolute valued algebras and let $f, g : A \rightarrow A$ be two linear isometries. Then

$$\psi f \psi^{-1}, \psi g \psi^{-1} : B \rightarrow B,$$

are linear isometries and

$$\psi : A_{f, g} \rightarrow B_{\psi f \psi^{-1}, \psi g \psi^{-1}},$$

is an isomorphism.

In particular we have that:

1. $\psi : A_{f, g} \rightarrow B$ is an isomorphism if and only if $\psi : A \rightarrow B_{\psi f^{-1} \psi^{-1}, \psi g^{-1} \psi^{-1}}$ is an isomorphism, where $A_{f, g}$ is unital with unity $\psi^{-1}(1)$.

2. If this unity is fixed by $f$ and $g$ then $\psi f^{-1} \psi^{-1}$ and $\psi g^{-1} \psi^{-1}$ fix $1$.

**Proof.** By Rodríguez [6] we have that any isomorphism of finite dimensional absolute valued algebras is linearly isometric. So $\psi f \psi^{-1}$ and $\psi g \psi^{-1}$ are linear isometries. That they are from $B$ to $B$ is clear.

That $A_{f, g} \cong B_{\psi f^{-1} \psi^{-1}, \psi g^{-1} \psi^{-1}}$ is clear since $A \cong B$ and we can therefore consider the multiplication in $B$ to be of the form

$$x \circ y = \psi 1_A \psi^{-1}(x) \psi 1_A \psi^{-1}(y).$$
Then clearly $A_{f,g} \simeq B_{\psi_f \psi^{-1}, \psi_g \psi^{-1}}$, since

$$\psi(x \circ y) = \psi(f(x)g(y)) = \psi(f(x))\psi(g(y)) = x \circ y,$$

where “$\circ$” is multiplication in $A_{f,g}$, “$\circ$” is multiplication in $B_{\psi_f \psi^{-1}, \psi_g \psi^{-1}}$ and the second equality holds since $\psi$ is an isomorphism on $A$.

(1) follows by taking $A = A_{f,g}$, $f = f^{-1}$ and $g = g^{-1}$ in the main statement of the theorem.

(2) is obvious. \qed

**Proposition 66.** Let $f, g$ be two linear isometries of an Euclidean space $\mathbb{A}$. Then $\mathbb{A}_{f,g}$ is isomorphic to $\mathbb{A}$ if and only if $f = R_a$ and $g = L_b$ for suitable norm-one $a$, $b$ of $\mathbb{A}$. If moreover the linear isometries $f, g$ fix 1, then $f = g = 1_\mathbb{A}$.

**Proof.** Let $\psi : \mathbb{A} \to \mathbb{A}_{f,g}$ be an isomorphism. Then for every $x \in \mathbb{A}$ we have

$$\psi(x) = \psi(x \cdot 1) = f(\psi(x))g(\psi(1)) \quad \text{and} \quad \psi(x) = \psi(1 \cdot x) = f(\psi(1))g(\psi(x)).$$

This shows that $f = R_a$ and $g = L_b$ where $a = (g(\psi(1)))^{-1}$ and $b = (f(\psi(1)))^{-1}$. If $f$ and $g$ fix 1 then $a = R_a(1) = f(1) = 1 = g(1) = L_b(1) = b$.

For the converse observe that $\mathbb{A}_{f,g}$ is unital with unity $b^{-1}a^{-1}$:

$$x(b^{-1}a^{-1}) = xa(b^{-1}a^{-1}) = xa(a^{-1}) = x,$$

and

$$(b^{-1}a^{-1})x = ((b^{-1}a^{-1})a)bz = (b^{-1})bz = x,$$

and by Albert the only unital finite dimensional algebras are the classic ones. \qed

**Theorem 67.** All finite dimensional absolute valued algebras are isomorphic to one of the form $\mathbb{A}_{f,g}$ where $f$, $g$ are two linear isometries fixing 1 and $\mathbb{A}$ is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. Except in the case of $\mathbb{A} = \mathbb{R}$ these algebras fall into four disjoint double-sign classes that corresponds to the following cases:

1. $f, g \in SO(n)$, $(1, 1) \in C_2 \times C_2$,
2. $f \in SO(n)$, $g \in O(n) - SO(n)$, $(1, -1) \in C_2 \times C_2$,
3. $g \in SO(n)$, $f \in O(n) - SO(n)$, $(-1, 1) \in C_2 \times C_2$,
4. $f, g \in O(n) - SO(n)$, $(-1, -1) \in C_2 \times C_2$.

**Theorem 68.** The full subcategory $\mathcal{O} = \{\mathbb{O}_{f,g} \mid f, g \in O_1(\mathbb{O})\}$ of $\mathbb{A}_8$, the category of 8 dimensional absolute valued algebras, is dense, and is a grupoid.

If $\psi : \mathbb{O}_{f,g} \to \mathbb{O}_{f',g'}$ is an isomorphism in $\mathcal{O}$, then $\psi \in SO(8)$. Conversely, let $\mathbb{O}_{f,g}$ and $\mathbb{O}_{f',g'}$ be objects in $\mathcal{O}$ and let $\psi \in SO(8)$ with triality correspondents $\psi_1$, $\psi_2 \in SO(8)$. Then $\psi : \mathbb{O}_{f,g} \to \mathbb{O}_{f',g'}$ is an isomorphism in $\mathcal{O}$ if and only if

$$f' = R_{\psi_2(1)^{-1}} \psi f \psi^{-1} \quad \text{and} \quad g' = L_{\psi_1(1)^{-1}} \psi g \psi^{-1}.$$ 

**Proof.** It has already been established that $\psi$ is in $SO(8)$, $\mathcal{O}$ is dense by Albert’s theorem. It is a grupoid since all morphisms are isomorphisms (excluding the 0 morphisms from the category). This is so because morphisms in $A_8$ respect the multiplication. So if $f : A \to B$ is a morphism between two objects in $A_8$ then injectivity implies surjectivity, since $A$ and $B$ are vector spaces with the same finite dimension. To show that $f$ is injective let $x \in ker(f)$. Then for all $y$ in $A$ we have that $f(xy) = f(x)f(y) = 0f(y) = 0$. If $x \neq 0$ then $L_x$ has been shown to be bijective, thus every $z \in A$ is on the form $z = xy$ for some $y \in A$, this implies that $f = 0$, which we had already excluded. Thus $x \in ker(f)$ implies that $x = 0$. 


Assume that $\psi : \mathcal{O}_{f,g} \to \mathcal{O}_{f',g'}$ is an isomorphism. Take $\mathcal{O}_{f,g} = A$ and $\mathcal{O}_{f',g'} = B$ in proposition 66. It follows that $\psi : A \to \mathcal{O}_{f'\psi f^{-1},g\psi g^{-1}}$ is also an isomorphism. By proposition 67 we have that

$$(f'\psi f^{-1}, g\psi g^{-1}) = (R_{a}, L_{b}),$$

where

$$(a, b) = ((g\psi g^{-1}\psi(1))^{-1}, (f'\psi f^{-1}\psi(1))^{-1}) = ((g\psi g^{-1}(1))^{-1}, (f'\psi f^{-1}(1))^{-1}),$$

which implies that

$$(f', g') = (R_{a}\psi f\psi^{-1}, L_{b}\psi g\psi^{-1}).$$

But we also have that

$$\psi(xy) = \psi(x) \circ \psi(y) = f'\psi f^{-1}\psi^{-1}\psi(x)g'\psi g^{-1}\psi^{-1}(y) = f'\psi f^{-1}(x)g'\psi g^{-1}(y),$$

for all $x, y \in \mathcal{O}$. So $(f'\psi f^{-1}, g\psi g^{-1}) = (\psi_{1}, \psi_{2})$ both of which are clearly in $SO(8)$. And the first equivalence statement follows.

Conversely, assume that $\psi \in SO(8)$ with $\psi_{1}, \psi_{2} \in SO(8)$, the triality correspondents, let $f$ and $g$ be given and set $(f', g') = (R_{\psi_{2}(1)\psi f\psi^{-1}, L_{\psi_{1}(1)\psi g\psi^{-1}}}$).

But we have that $\psi(y) = \psi_{1}(1)\psi_{2}(y)$ if and only if $(L_{\psi_{1}(1))^{-1}\psi(y)} = \psi_{2}(y)$ if and only if $L_{\psi_{1}(1)}^{-1}\psi = \psi_{2}$, and likewise we have that $R_{\psi_{2}(1)\psi f\psi^{-1} = \psi_{1}}$. But

$$\psi(x \circ y) = \psi(f(x)g(y)) = \psi_{1}(f(x))\psi_{2}(g(y)) = R_{\psi_{2}(1)\psi f\psi^{-1}}(f(x))L_{\psi_{1}(1)\psi g\psi^{-1}}(g(y),$$

and

$$\psi(x) \circ \psi(y) = f'\psi(x)g'\psi(y) = R_{\psi_{2}(1)\psi f\psi^{-1}}(f(x)L_{\psi_{1}(1)\psi g\psi^{-1}}(y) =$$

$$= R_{\psi_{2}(1)\psi f\psi^{-1}}(f(x)L_{\psi_{1}(1)\psi g\psi^{-1}}(y).$$

So $\psi(x \circ y) = \psi(x) \circ \psi(y)$ and $\psi$ is an algebra morphism. It is bijective on the vectorspace and is therefore an algebra isomorphism.

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