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Abstract

This PhD thesis consists of a summary and five papers which all deal with asymptotic problems on certain homogeneous spaces.

In Paper I we prove asymptotic equidistribution results for pieces of large closed horospheres in cofinite hyperbolic manifolds of arbitrary dimension. All our results are given with precise estimates on the rates of convergence to equidistribution.

Papers II and III are concerned with statistical problems on the space of n-dimensional lattices of covolume one. In Paper II we study the distribution of lengths of non-zero lattice vectors in a random lattice of large dimension. We prove that these lengths, when properly normalized, determine a stochastic process that, as the dimension n tends to infinity, converges weakly to a Poisson process on the positive real line with intensity 1/2. In Paper III we complement this result by proving that the asymptotic distribution of the angles between the shortest non-zero vectors in a random lattice is that of a family of independent Gaussians.

In Papers IV and V we investigate the value distribution of the Epstein zeta function along the real axis. In Paper IV we determine the asymptotic value distribution and moments of the Epstein zeta function to the right of the critical strip as the dimension of the underlying space of lattices tends to infinity. In Paper V we determine the asymptotic value distribution of the Epstein zeta function also in the critical strip. As a special case we deduce a result on the asymptotic value distribution of the height function for flat tori. Furthermore, applying our results we discuss a question posed by Sarnak and Strömbergsson as to whether there in large dimensions exist lattices for which the Epstein zeta function has no zeros on the positive real line.

Keywords: Analysis on homogeneous spaces, hyperbolic manifolds, spectral theory, equidistribution, the space of lattices, length statistics, Poisson process, moments, Epstein zeta function, value distribution, height function.

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Till mamma och pappa
List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.


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1. Introduction

1.1 Homogeneous spaces

In this thesis we study two special families of homogeneous spaces, namely hyperbolic \( n \)-space and the space of \( n \)-dimensional lattices. These spaces are special in the sense that they have more structure than is needed to be a homogeneous space. Nevertheless, let us start by giving the definition of a (general) homogeneous space.

Formally, a homogeneous space is a pair \((X,\rho)\) of a set \(X\) and a transitive action \(\rho\) on \(X\) by a group \(G\). Let us assume for the moment that \(\rho\) is a left action. When \(G\) and \(\rho\) are clear from the context, we will adopt the common practice and denote the homogeneous space simply by \(X\) and write the action of \(g \in G\) on \(X\) as \(x \mapsto g(x)\). If the set \(X\) has more structure it is natural to demand that the action \(\rho\) preserves this structure. In particular, if \(X\) is a smooth manifold we demand that the group \(G\) acts on \(X\) by diffeomorphisms.

Every homogeneous space can in a natural but non-canonical way be realized as a space of cosets as follows. Fix any \(x_0 \in X\) and let \(G_{x_0} = \{g \in G \mid g(x_0) = x_0\}\) be the stabilizer of \(x_0\). Then the set of left cosets \(G/G_{x_0}\) can be identified with \(X\) via the map \(\eta_{x_0} : G/G_{x_0} \to X\) given by \(gG_{x_0} \mapsto g(x_0)\).

We will be interested in the following special case. Let \(X\) be a Hausdorff topological space and \(G\) a Lie group acting transitively and continuously on \(X\). Here the last condition means that \((g, x) \mapsto g(x)\) is a continuous mapping of \(G \times X\) onto \(X\). Now, given \(x_0 \in X\) we find that \(G_{x_0}\) is a closed subgroup of \(G\) and hence [9, Thm. 4.2, p. 123] implies that \(X\) has a unique smooth structure such that \((g, x) \mapsto g(x)\) is a smooth mapping and \(\eta_{x_0}\) is a diffeomorphism.

Many common spaces can be realized as homogeneous spaces of the above type. To mention a few examples; the unit sphere \(S^{n-1} \subset \mathbb{R}^n\) is diffeomorphic to the homogeneous space \(SO(n)/SO(n-1)\), the standard \(n\)-dimensional torus \(\mathbb{T}^n\) is diffeomorphic to \(\mathbb{R}^n/\mathbb{Z}^n\) and the real projective space \(\mathbb{R}P^{n-1}\) is diffeomorphic to \(SO(n)/O(n-1)\) (where \(O(n-1)\) is considered as a subgroup of \(SO(n)\) in an appropriate way, see [30, Sec. 3.65]).
1.1.1 Hyperbolic $n$-space

The $n$-dimensional hyperbolic space is the unique $n$-dimensional simply connected Riemannian manifold with constant sectional curvature equal to $-1$. It is one of the standard examples of an $n$-dimensional space with non-Euclidean geometry. A commonly used model of hyperbolic $n$-space is the set

$$\mathcal{H}^n = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 - x_2^2 - \ldots - x_{n+1}^2 = 1, x_1 > 0 \right\}$$

equipped with the Riemannian metric induced from the pseudo-Riemannian metric

$$ds^2 = -dx_1^2 + dx_2^2 + \ldots + dx_{n+1}^2$$
on $\mathbb{R}^{n+1}$. The full group of orientation preserving isometries of $\mathcal{H}^n$ is given by the Lie group $\text{PSO}(1,n)$ (see e.g. [13, Sec. 3.2 (Cor. 3)]). It is now easy to verify that the stabilizer of $(1,0,\ldots,0) \in \mathcal{H}^n$ equals

$$\left\{ \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & & & A \end{pmatrix} \in \text{PSO}(1,n) : A \in \text{SO}(n) \right\},$$

which is clearly isomorphic to $\text{SO}(n)$. Hence we find that $\mathcal{H}^n$ is diffeomorphic to the homogeneous space $\text{PSO}(1,n)/\text{SO}(n)$.

We remark that there are several other important models of hyperbolic $n$-space. In Paper I we consider the upper half-space model $\mathbb{H}^n$ since it is in many ways convenient in connection with the problem we study. For a description of $\mathbb{H}^n$ see Section 2.1 and Paper I.

1.1.2 The space of $n$-dimensional lattices

By definition, an $n$-dimensional lattice $L$ is the $\mathbb{Z}$-span of a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^n$, i.e.

$$L = \left\{ \sum_{i=1}^{n} a_i \mathbf{v}_i \mid a_i \in \mathbb{Z}, 1 \leq i \leq n \right\}.$$The set of points

$$P = \left\{ \sum_{i=1}^{n} t_i \mathbf{v}_i \mid 0 \leq t_i < 1, 1 \leq i \leq n \right\}$$
is called a fundamental parallelogram (or a fundamental region) for the lattice $L$. Note that the set $P$ is naturally identified with the torus $\mathbb{R}^n/L$. The covolume of $L$ is defined as the $n$-dimensional Euclidean volume of
\[ P \] (or equivalently, the volume of \( \mathbb{R}^n / L \)). We let \( X_n \) denote the space of \( n \)-dimensional lattices of covolume 1.

Note that \( \text{SL}(n, \mathbb{R}) \) acts transitively from the right on \( X_n \) by the mapping \( (L, g) \mapsto Lg \), where of course 
\[ Lg = \{ xg \mid x \in L \}. \]

The fact that \( \text{SL}(n, \mathbb{Z}) \) is the stabilizer of \( \mathbb{Z}^n \in X_n \) implies that the elements of \( X_n \) can be identified with the elements of the homogeneous space \( \text{SL}(n, \mathbb{Z}) \backslash \text{SL}(n, \mathbb{R}) \). Note in particular that this identification induces a smooth structure on \( X_n \).

As any locally compact (Hausdorff) topological group, \( \text{SL}(n, \mathbb{R}) \) is equipped with a right Haar measure (cf. [2, Chap. 9]). In fact, since \( \text{SL}(n, \mathbb{R}) \) is even a semisimple Lie group, any right Haar measure is also a left Haar measure and we can thus without ambiguity speak about a Haar measure on \( \text{SL}(n, \mathbb{R}) \) (cf. [11, Sec. VIII.2]). Recall that a right (left) Haar measure is a non-zero regular Borel measure that is invariant under right (left) translations. When such a measure exists it is unique up to multiplication with a positive constant. We can define a Haar measure on \( \text{SL}(n, \mathbb{R}) \) as the measure \( \nu_n \) on \( \text{SL}(n, \mathbb{R}) \) which satisfies
\[ d\nu_n(M') \frac{dt}{t} = (\det(x_{ij}))^{-n} dx_{11} dx_{12} \ldots dx_{nn}, \quad (1.1.1) \]
when parametrizing \( \text{GL}^+(n, \mathbb{R}) \) as \( (x_{ij})_{i,j=1}^n = t^{1/n} M' \), with \( M' \in \text{SL}(n, \mathbb{R}) \) and \( t > 0 \). Let us note that, in more concrete terms, the measure \( \nu_n \) is given by the following explicit formula, for any given Borel subset \( A \subset \text{SL}(n, \mathbb{R}) \):
\[ \nu_n(A) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \nu_n' \left( \left\{ M \in \text{GL}^+(n, \mathbb{R}) \mid 1 \leq \det M \leq 1 + \varepsilon, \pi(M) \in A \right\} \right), \]
where \( \nu_n' \) is the measure on \( \text{GL}^+(n, \mathbb{R}) \) given by (1.1.1) and \( \pi : \text{GL}^+(n, \mathbb{R}) \to \text{SL}(n, \mathbb{R}) \) is the natural projection given by \( \pi(M) = (\det M)^{-1/n} M \).

By restricting \( \nu_n \) to a fundamental region \( F_n \subset \text{SL}(n, \mathbb{R}) \) for \( \text{SL}(n, \mathbb{Z}) \backslash \text{SL}(n, \mathbb{R}) \) we can induce a measure \( \tilde{\mu}_n \) also on \( X_n \). Note in particular that this measure is invariant under right multiplication with elements in \( \text{SL}(n, \mathbb{R}) \). It is of great importance in this thesis that \( \tilde{\mu}_n \), or equivalently the restriction of \( \nu_n \) to \( F_n \), is a finite measure. For a proof of this result we refer to Raghunathan [12, Cor. 10.5]. However, for \( n \geq 2 \) we note that it is possible to give a more precise result. In fact, if we define \( \nu_n'' \) to be the (non-Haar) measure on \( \text{GL}^+(n, \mathbb{R}) \) given by
\[ d\nu_n''(M) = dx_{11} dx_{12} \ldots dx_{nn}, \]
it is straightforward to prove that for any measurable set $A \subset \text{SL}(n, \mathbb{R})$ we have the relation

$$
\nu_n(A) = n \cdot \nu''_n\left(\left\{ M \in \text{GL}^+(n, \mathbb{R}) \mid 0 < \det M \leq 1, \pi(M) \in A \right\}\right).
$$

(1.1.2)

Furthermore, if we let

$$
\mathcal{F} = \left\{ M \in \text{GL}^+(n, \mathbb{R}) \mid 0 < \det M \leq 1, \pi(M) \in F_n \right\}
$$

it follows from [20, Sec. XV] that

$$
\nu''_n(\mathcal{F}) = \frac{\zeta(2)\zeta(3) \cdots \zeta(n)}{n},
$$

where $\zeta(s)$ is the Riemann zeta function. Thus it follows from (1.1.2) that

$$
\tilde{\mu}_n(X_n) = \nu_n(F_n) = \zeta(2)\zeta(3) \cdots \zeta(n).
$$

Finally we normalize $\tilde{\mu}_n$ as

$$
\mu_n = \left(\zeta(2)\zeta(3) \cdots \zeta(n)\right)^{-1} \cdot \tilde{\mu}_n,
$$

in order to turn it into a probability measure on $X_n$. The probability space $(X_n, \mu_n)$ is the scene for the investigations presented in Papers II,III,IV and V.

1.2 Zeta functions

One of the most famous and important functions in all of mathematics is the Riemann zeta function. It is defined as the absolutely convergent sum

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
$$

(1.2.1)

for Re $s > 1$. It is well-known that $\zeta(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$ and that $s = 1$ is a simple pole with residue 1. In this connection we note that $\zeta(s)$ satisfies a functional equation. More precisely we have

$$
\xi(s) = \xi(1 - s),
$$

(1.2.2)

where

$$
\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).
$$
The relation between $\zeta(s)$ and number theory origins from the so called Euler product formula

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad (1.2.3)$$

which holds for $\text{Re } s > 1$. The product on the right hand side in (1.2.3) is taken over the set of all prime numbers. This remarkable formula relates the additive structure of the positive integers on the left hand side to the multiplicative structure of the integers and primes on the right hand side. It turns out that even though (1.2.3) only holds for $\text{Re } s > 1$, the behavior of $\zeta(s)$ in the critical strip $0 < \text{Re } s < 1$ has important implications in the theory of prime numbers. In particular there is a close connection (as we will see parts of below) between the complex numbers $s$ satisfying $\zeta(s) = 0$ and the prime counting function $\pi(x) = \#\{p \text{ prime } | p \leq x\}$.

Studying the symmetry coming from the functional equation (1.2.2) it is easy to see that $\zeta(s)$ has zeros at the negative even integers $-2, -4, -6, \ldots$. These are the so called trivial zeros. The Riemann hypothesis asserts that all the remaining zeros of $\zeta(s)$ are located on the (critical) line $\text{Re } s = \frac{1}{2}$. If true the hypothesis would among other things imply that the counting function $\pi(x)$ is very well approximated by the logarithmic integral $\text{Li}(x) = \int_{2}^{x} \frac{dt}{\log t}$

in the sense that

$$\pi(x) = \text{Li}(x) + O(x^{1/2+\epsilon}) \quad (1.2.4)$$

would hold as $x \to \infty$ (cf. [10]). The estimate in (1.2.4) is in fact equivalent to the Riemann hypothesis. Despite numerous attempts and massive numerical evidence in its favor, the Riemann hypothesis has not yet been proved or disproved. It is widely considered as one of the most important unsolved problem in modern mathematics.

Functions defined in various contexts (possibly very different from prime number theory) that are in one way or another analogous to $\zeta(s)$ are called zeta functions or in some cases $L$-functions. One example is the Dedekind zeta function of a number field $k$, arising in algebraic number theory. Let $\mathcal{Q}$ denote the ring of integers in $k$. For each non-zero ideal $a$ of $\mathcal{Q}$, the norm of $a$, $N(a)$, is defined as the cardinality of the quotient ring $\mathcal{Q}/a$. Now the Dedekind zeta function of $k$ is defined for $\text{Re } s > 1$ by the series

$$\zeta_k(s) = \sum_{a \subseteq \mathcal{O}_k} \frac{1}{N(a)^s},$$
where the sum is over all non-zero ideals of \( \mathcal{O} \). Note that this formula reduces to (1.2.1) when \( k = \mathbb{Q} \). For each number field \( k \) the associated zeta function \( \zeta_k \) has an analytic continuation to \( \mathbb{C} \setminus \{1\} \) and it satisfies a functional equation similar to (1.2.2) relating \( \zeta_k(s) \) to \( \zeta_k(1-s) \). Furthermore, using the unique factorization of ideals (into prime ideals) in \( \mathcal{O} \) and the multiplicativity of the norm \( N \), one can for \( \text{Re}\, s > 1 \) write also \( \zeta_k \) as an Euler product

\[
\zeta_k(s) = \prod_{\mathfrak{p} \subseteq \mathcal{O}_k} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1},
\]

where the product is over all prime ideals of \( \mathcal{O} \). It follows from this formula that \( \zeta_k(s) \) also for general number fields \( k \) is closely related to the study of prime ideals in \( \mathcal{O} \). Finally we mention that there exists an extended Riemann hypothesis asserting that for every number field \( k \) all zeros of \( \zeta_k(s) \) with \( 0 < \text{Re}\, s < 1 \) are in fact on the line \( \text{Re}\, s = \frac{1}{2} \).

### 1.2.1 Epstein’s zeta function

Let \( n \in \mathbb{Z}^+ \) be given. For lattices \( L \in X_n \) and \( s \in \mathbb{C} \) with \( \text{Re}\, s > \frac{n}{2} \) we define the Epstein zeta function by

\[
E_n(L,s) = \sum_{m \in L}' |m|^{-2s},
\]

where \( ' \) denotes that the zero vector should be omitted. These functions (actually a slightly more general family of functions) were introduced by Epstein in the papers [3] and [4] in an attempt to find the most general form of a function satisfying a functional equation of the same type (1.2.2) as the Riemann zeta function. Exploring the relationship between \( E_n(L,s) \) and the lattice theta function, defined for \( \text{Im}\, z > 0 \) by

\[
\Theta(L,z) = \sum_{m \in L} e^{\pi i z|m|^2},
\]

Epstein proved that \( E_n(L,s) \) has an analytic continuation to \( \mathbb{C} \) except for a simple pole at \( s = \frac{n}{2} \) with residue \( \pi^{n/2} \Gamma(\frac{n}{2})^{-1} \). Furthermore \( E_n(L,s) \) satisfies the functional equation

\[
F_n(L,s) = F_n(L^*, \frac{n}{2} - s),
\]

where

\[
F_n(L,s) = \pi^{-s} \Gamma(s) E_n(L,s),
\]

and \( L^* \) is the dual lattice of \( L \).
As can be seen above, $E_n(L, s)$ is in many ways analogous to the Riemann zeta function. In particular we have the relation

$$\zeta(2s) = \frac{1}{2} E_1(\mathbb{Z}, s).$$

The Epstein zeta function is further related to the Dedekind zeta function via Hecke’s integral formula (cf. [7, pp. 198-207]). Note however that $E_n(L, s)$ in general cannot be written as an Euler product. We also mention that for all $n \geq 2$ there exist lattices $L \in X_n$ for which the Riemann hypothesis for $E_n(L, s)$ is known to fail in the sense that $E_n(L, s)$ has zeros in the critical strip $0 < \text{Re } s < \frac{n}{2}$ which are not on the line $\text{Re } s = \frac{n}{4}$ (cf. [27, Thm. 1]; see also [1], [22], [26] and [28]).

Finally we remark that the Epstein zeta function can alternatively be defined in terms of positive definite quadratic forms in $n$ variables. In this setting it is also possible to consider generalizations of the Epstein zeta function twisted with some character (cf. [23],[24]). It is a curious fact that such character sums are, in the case with binary quadratic forms, related to Gauss’ class number problem for imaginary quadratic fields. In particular Stark considers a particular instance of such a character sum in his proof [21] that there exist exactly nine imaginary quadratic fields of class-number one. For a historical account on the class number problem as well as some indications of its relation to the Epstein zeta function see [6].
2. Summary of the papers

In this chapter we give a short description of the content of each paper included in the thesis.

2.1 Paper I

In Paper I we study the asymptotic equidistribution of pieces of large closed horospheres in a cofinite hyperbolic manifold $M$ of arbitrary dimension. To present the problem, let us first consider the case when $M$ is two dimensional. Then the surface $M$ has a finite number of cusps, and to each cusp corresponds a one-parameter family of closed horocycle curves in $M$. In each family there exists a unique closed horocycle of any given length $\ell > 0$, and it is known that as $\ell \to \infty$, the closed horocycle becomes asymptotically equidistributed on $M$ with respect to the hyperbolic area measure. Investigations related to this fact have been carried out by a number of people over the years, including Selberg (unpublished), Zagier [31], Sarnak [16], Hejhal [8], Flaminio and Forni [5], and Strömbergsson [25].

In [8], Hejhal asked to what exact degree of uniformity does this equidistribution hold? Specifically, given a subsegment of length $\ell_1 < \ell$ of a closed horocycle of length $\ell$, under what conditions on $\ell_1$ can we ensure that this subsegment becomes asymptotically equidistributed on $M$? Hejhal proved that this holds as long as we keep $\ell_1 \geq \ell^{c+\varepsilon}$ ($\varepsilon > 0$) as $\ell \to \infty$, where $c \geq \frac{2}{3}$ is a constant which only depends on the surface $M$. This was later improved in [25] to allowing $c = \frac{1}{2}$, independently of $M$. This constant is optimal. The equidistribution results both in [8] and [25] were obtained with explicit rates.

The purpose of Paper I is to generalize these equidistribution results to the case when $M$ is a hyperbolic manifold of arbitrary dimension $n + 1$. We realize $M$ as the quotient $M = \Gamma \backslash \mathbb{H}^{n+1}$, where $\mathbb{H}^{n+1}$ is the hyperbolic upper half space,

$$
\mathbb{H}^{n+1} = \left\{ P = (x, y) \mid x \in \mathbb{R}^n, y \in \mathbb{R}_{>0} \right\}
$$

with Riemannian metric $ds^2 = y^{-2}(dx_1^2 + \ldots + dx_n^2 + dy^2)$, and $\Gamma$ is a cofinite (but not cocompact) discrete subgroup of the group $G$ of orientation preserving isometries of $\mathbb{H}^{n+1}$. Without loss of generality we can assume that one of the cusps is placed at infinity. Then the fixator group
Γ_∞ ⊂ Γ contains a subgroup of finite index consisting of translations,

$$\Gamma_\infty' = \left\{ (x, y) \mapsto (x + \omega, y) \mid \omega \in \Lambda \right\}$$

where Λ is some lattice in \( \mathbb{R}^n \). Let \( \Omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \) be a basis of Λ. Now, for each \( y > 0 \) and any \( \alpha_i, \beta_i \in \mathbb{R} \) with \( \alpha_i < \beta_i \), we set

$$\mathcal{B} = \left\{ (u_1\omega_1 + \cdots + u_n\omega_n, y) \mid u_i \in [\alpha_i, \beta_i] \text{ for } i = 1, \ldots, n \right\}.$$

This is a box-shaped subset of a closed horosphere in \( M \). Our first main theorem says that as \( y \to 0 \), the box \( \mathcal{B} \) becomes asymptotically equidistributed in \( M \) with respect to the hyperbolic volume measure \( d\nu = y^{-n-1}dx_1 \cdots dx_n dy \), so long as we keep all \( \beta_i - \alpha_i \geq y^{1/2-\varepsilon} \).

**Theorem 1.** Let \( \Gamma \) be a cofinite discrete subgroup of \( G \) such that \( \Gamma \setminus \mathbb{H}^{n+1} \) has a cusp at infinity. Let \( \varepsilon > 0 \), and let \( f \) be a fixed continuous function of compact support on \( \Gamma \setminus \mathbb{H}^{n+1} \). Then

$$\frac{1}{(\beta_1 - \alpha_1) \cdots (\beta_n - \alpha_n)} \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_n}^{\beta_n} f(u_1\omega_1 + \cdots + u_n\omega_n, y) \, du_1 \cdots du_n \to \frac{1}{\nu(\Gamma \setminus \mathbb{H}^{n+1})} \int_{\Gamma \setminus \mathbb{H}^{n+1}} f(P) \, d\nu(P),$$

(2.1.1)

uniformly as \( y \to 0 \) so long as \( \beta_1 - \alpha_1, \ldots, \beta_n - \alpha_n \geq y^{1/2-\varepsilon} \).

Note that when \( n = 1 \) this specializes to the equidistribution result from [25] described above, since \( \ell \sim y^{-1} \) in this case. The exponent \( \frac{1}{2} \) in Theorem 1 is in fact the best possible in arbitrary dimension, if we restrict to considering boxes which have all side lengths comparable, as \( y \to 0 \).

We obtain all equidistribution results in Paper I with precise rates, provided that \( f \) is sufficiently smooth. In this direction we obtain the best results if instead of the box \( \mathcal{B} \) we use a smooth cutoff function in the closed horosphere. Thus let us fix \( \chi : \mathbb{R}^n \to \mathbb{R} \) to be a smooth function of compact support. Given \( \delta_1, \ldots, \delta_n \in (0, 1] \) and \( \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n \) we define

$$\chi_{\delta, \gamma}(u) = \chi\left(\frac{u_1 - \gamma_1}{\delta_1}, \ldots, \frac{u_n - \gamma_n}{\delta_n}\right),$$

and consider the horosphere integral

$$\frac{1}{\delta_1 \cdots \delta_n} \int_{\mathbb{R}^n} \chi_{\delta, \gamma}(u) f(u_1\omega_1 + \cdots + u_n\omega_n, y) \, du,$$

where \( du = du_1 \cdots du_n \). Note that if we relax the condition that \( \chi \) be smooth, and take \( \chi \) to be the characteristic function of the unit cube
with \([-\frac{1}{2}, \frac{1}{2}]^n\), then we get back the left hand side of (2.1.1) with \(\alpha_i = \gamma_i - \delta_i/2\) and \(\beta_i = \gamma_i + \delta_i/2\).

The precise rate of equidistribution which we obtain depends on the spectrum of the Laplace-Beltrami operator \(\Delta\) on \(M\). If there are small eigenvalues \(0 < \lambda < n^2/4\) of \(-\Delta\) on \(M\), then we take \(\sigma_1 \in (\frac{n}{2}, n)\) so that \(\lambda = \sigma_1(n - \sigma_1)\) is the smallest non-zero eigenvalue; otherwise, if there are no small eigenvalues, we set \(\sigma_1 = \frac{n}{2}\).

\[\text{Theorem 2.} \quad \text{Let } \Gamma \text{ be a cofinite discrete subgroup of } G \text{ such that } \Gamma \backslash \mathbb{H}^{n+1} \text{ has a cusp at infinity. Let } \varepsilon > 0, \text{ let } f \text{ be a fixed } C^\infty \text{-function on } \Gamma \backslash \mathbb{H}^{n+1} \text{ which is } \Gamma \text{-invariant and of compact support on } \Gamma \backslash \mathbb{H}^{n+1}, \text{ and let } \chi : \mathbb{R}^n \to \mathbb{R} \text{ be a smooth function of compact support. Then, uniformly over all } \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n \text{ and all } \delta_1, \ldots, \delta_n \text{ satisfying } \sqrt{\gamma} \leq \delta_1, \ldots, \delta_n \leq 1,\]

\[
\frac{1}{\delta_1 \cdots \delta_n} \int_{\mathbb{R}^n} \chi_{\delta, \gamma}(u)f(u_1\omega_1 + \cdots + u_n\omega_n, y) \, du \\
= \frac{\langle \chi \rangle}{\nu(\Gamma \backslash \mathbb{H}^{n+1})} \int_{\Gamma \backslash \mathbb{H}^{n+1}} f(P) \, d\nu(P) + O\left(\frac{y^{-\varepsilon} (y/\delta_{\min}^2)^{n-\sigma_1}}{n-\sigma_1}\right), \tag{2.1.2}
\]

where \(\sigma_1 \in (\frac{n}{2}, n)\) is as above, \(\langle \chi \rangle = \int_{\mathbb{R}^n} \chi(x) \, dx\), \(\delta_{\min} = \min_{i \in \{1, \ldots, n\}} \delta_i\), and the implied constant depends on \(\Gamma, f, \chi, \varepsilon\) and \(\Omega\).

Our method of proof of these results is to use the spectral decomposition of the Laplace-Beltrami operator on \(M\), involving the theory of Eisenstein series. To obtain our results we develop several bounds on sums over the Fourier coefficients of the individual eigenfunctions of \(M\), which are also of independent interest. In particular we prove a precise Rankin-Selberg type bound and bounds on coefficient sums twisted with an additive character.

2.2 Paper II

In Paper II we study the distribution of lengths of non-zero lattice vectors in a random lattice of large dimension. Given a lattice \(L \in X_n\), we order its non-zero vectors by increasing lengths as \(\pm \mathbf{v}_1, \pm \mathbf{v}_2, \pm \mathbf{v}_3, \ldots,\) set \(\ell_j = |\mathbf{v}_j|\) (thus \(0 < \ell_1 \leq \ell_2 \leq \ell_3 \leq \ldots\)), and define

\[\mathcal{V}_j = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \ell_j^n,\]

so that \(\mathcal{V}_j\) is the volume of an \(n\)-dimensional ball of radius \(\ell_j\). We also let, for \(t \geq 0,\)

\[\tilde{N}_t(L) = \#\{j \mid \mathcal{V}_j \leq t\}.
\]
The first result concerning the statistics of the sequence \(\{\ell_j\}_{j=1}^\infty\) is due to Rogers. In [15] he shows that, for fixed \(t\), \(\tilde{N}_t(L)\) has a distribution which converges weakly to the Poisson distribution with mean \(\frac{t}{2}\) as \(n \to \infty\). Clearly Rogers’ result determines the limit distribution of each fixed \(\ell_j\) as \(n \to \infty\). In particular the volume \(V_1\) has an exponential distribution with expectation value 2 as \(n \to \infty\). In [19] Schmidt presents a more direct study of the distribution of \(\ell_1\) using another method resulting in better error bounds as \(n \to \infty\).

The results of Rogers and Schmidt suggest very naturally that it should be possible to understand not only the distribution of each fixed \(\ell_j\) but also the joint distribution of the entire sequence \(\{\ell_j\}_{j=1}^\infty\) for a random lattice \(L \in X_n\) as \(n \to \infty\). The purpose of Paper II is to point out that this is possible using Rogers’ methods. Our main theorem states that, as \(n \to \infty\), the volumes \(\{V_j\}_{j=1}^\infty\) behave like the points of a Poisson process on the positive real line with intensity \(\frac{1}{2}\).

**Theorem 3.** Let \(\{N(t), t \geq 0\}\) be a Poisson process on the positive real line with intensity \(\frac{1}{2}\). Then the stochastic process \(\{\tilde{N}_t(\cdot), t \geq 0\}\) converges weakly to \(\{N(t), t \geq 0\}\) as \(n \to \infty\).

There is a close conceptual relation between Theorem 3 and the main result (Theorem 1.1) by VanderKam in [29] (cf. also Sarnak [17]), which states that for a generic fixed lattice \(L \in X_n\), the local spacing statistics of the infinite sequence \(\{V_j\}_{j=1}^\infty\) exhibit Poissonian behavior (to a larger extent the larger the dimension). VanderKam’s theorem is of particular interest in the context of the Berry-Tabor conjecture (cf. [29, Cor. 1.2]). In this direction, let us note that Theorem 3 can be viewed as a kind of “low-eigenvalue-high-dimension” analog of the Berry-Tabor conjecture for flat tori. Indeed, Theorem 3 can be reformulated as follows.

**Theorem 4.** For a random flat torus \(\mathbb{R}^n/L\) with \(L \in X_n\) the distribution of the non-zero eigenvalues (normalized to have mean-spacing one) converges to the distribution of the points of a Poisson process on the positive real line with intensity one as \(n \to \infty\).

### 2.3 Paper III

In this paper we continue the study in Paper II of statistical properties of the non-zero lattice vectors \(\pm v_1, \pm v_2, \pm v_3, \ldots\) in a random lattice of large dimension. The purpose of Paper III is to determine the distribution of lengths and relative positions of the vectors \(\pm v_1, \ldots, \pm v_N\) for a random lattice \(L \in X_n\), for \(N\) fixed and \(n \to \infty\).

We remark that, since the vectors \(\{v_j\}_{j=1}^\infty\) are determined only up to sign, the angles between them are a priori not well-defined. We avoid this ambiguity by introducing a "symmetrized" angle measure \(\varphi\), taking values in the interval \([0, \pi/2]\). To be more specific, we denote the Euclidean
angle between the vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ by $\phi(\mathbf{x}_1, \mathbf{x}_2)$ and define

$$
\varphi(\mathbf{x}_1, \mathbf{x}_2) = \begin{cases} 
\phi(\mathbf{x}_1, \mathbf{x}_2) & \text{if } \phi(\mathbf{x}_1, \mathbf{x}_2) \in [0, \frac{\pi}{2}], \\
\pi - \phi(\mathbf{x}_1, \mathbf{x}_2) & \text{otherwise}.
\end{cases}
$$

Given $L \in X_n$ and $i, j \in \mathbb{Z}_{\geq 1}$, we let $\varphi_{ij} = \varphi(v_i, v_j)$. Using Rogers’ mean value formula [14] we show that for a random lattice $L \in X_n$ the angles $\{\varphi_{ij}\}_{i<j}$ accumulate to $\frac{\pi}{2}$, as $n \to \infty$, with a rate at least comparable with $n^{-\frac{1}{2}}$. This fact suggests that it is natural to study the normalized variables

$$
\tilde{\varphi}_{ij} = \sqrt{n} \left( \frac{\pi}{2} - \varphi_{ij} \right) = \sqrt{n} \left( \frac{\pi}{2} - \varphi(v_i, v_j) \right).
$$

The main result in Paper III is the following theorem, where we use the term positive Gaussian variable to denote a random variable $\Phi$ satisfying $\Phi = |X|$ for a random variable $X \in \mathcal{N}(0, 1)$.

**Theorem 5.** Let $N \in \mathbb{Z}_{\geq 2}$. The joint distribution of $\mathcal{V}_1, \ldots, \mathcal{V}_N$ and $\tilde{\varphi}_{ij}$, $1 \leq i < j \leq N$, converges, as $n \to \infty$, to the joint distribution of the first $N$ points of a Poisson process on the positive real line with intensity $\frac{1}{2}$ and a collection of $\left( \begin{array}{c} N \\ 2 \end{array} \right)$ independent positive Gaussian variables (which are also independent of the first $N$ variables).

Paper III concludes with a short discussion of the limit distribution of any fixed number of successive minima of a random lattice $L \in X_n$ as $n \to \infty$. We show that, for each $N \in \mathbb{Z}_{\geq 1}$, the first $N$ successive minima (suitably normalized) has the same limit distribution as the $N$-tuple $(\mathcal{V}_1, \ldots, \mathcal{V}_N)$ as $n \to \infty$.

**2.4 Paper IV**

In Paper IV we study the asymptotic value distribution of the the Epstein zeta function $E_n(L, s)$ for $s > \frac{n}{2}$ and a random lattice $L \in X_n$ of large dimension $n$. This paper is mainly motivated by [18, Sec. 6], where Sarnak and Strömbergsson study the height function for flat tori in large dimensions. For the flat torus $\mathbb{R}^n/L$, with $L \in X_n$, the height function is given by

$$
h_n(\mathbb{R}^n/L) = 2 \log 2\pi + \frac{\partial}{\partial s} E_n(L^*, s)|_{s=0}.
$$

Sarnak and Strömbergsson show that $h_n$ concentrates at the value $\log 4\pi - \gamma + 1$ as $n \to \infty$. Interpreted in terms of the Epstein zeta function this result states that if $\varepsilon > 0$ is fixed then

$$
\text{Prob}_{\mu_n} \left\{ L \in X_n \left| \left| \frac{\partial}{\partial s} E_n(L, s)|_{s=0} - (1 - \gamma - \log \pi) \right| < \varepsilon \right\} \to 1 \quad (2.4.1)
$$

\[1\] Here $\gamma$ is Euler’s constant.
as \( n \to \infty \). Note that (2.4.1) together with \( E_n(L,0) = -1 \) \((\forall L \in X_n)\) give a fairly precise description of the behavior of \( E_n(L,s) \) at the point \( s = 0 \) for most \( L \in X_n \) when \( n \) is large.

The results in Paper IV give information about the value distribution of \( E_n(L,s) \) for \( s > \frac{n}{2} \) with large \( n \). The main result is that the value distribution, as \( n \to \infty \), of the (suitably normalized) Epstein zeta function can be completely described in terms of the points of a Poisson process on the positive real line.

**Theorem 6.** Let \( V_n \) denote the volume of the \( n \)-dimensional unit ball. Let \( \mathcal{P} \) be a Poisson process on the positive real line with intensity \( \frac{1}{2} \) and let \( T_1, T_2, T_3, \ldots \) denote the points of \( \mathcal{P} \) ordered so that \( 0 < T_1 < T_2 < T_3 < \cdots \). Then, for fixed \( c > \frac{1}{2} \), the distribution of the random variable \( V_n^{-2c}E_n(\cdot, cn) \) converges to the distribution of \( 2 \sum_{j=1}^{\infty} T_j^{-2c} \) as \( n \to \infty \). In fact, for any \( m \geq 1 \) and fixed \( \frac{1}{2} < c_1 < \cdots < c_m \), the distribution of the random vector

\[
\left( V_n^{-2c_1}E_n(\cdot, c_1 n), \ldots, V_n^{-2c_m}E_n(\cdot, c_m n) \right)
\]

converges to the distribution of

\[
\left( 2 \sum_{j=1}^{\infty} T_j^{-2c_1}, \ldots, 2 \sum_{j=1}^{\infty} T_j^{-2c_m} \right)
\]

as \( n \to \infty \).

We can strengthen the result in Theorem 6 by showing that we do not only have convergence in distribution but also convergence in moments after an explicit and tractable truncation. (A truncation is necessary already in order for the moments of \( E_n(\cdot, s) \) to exist.)

**Theorem 7.** Let \( c > \frac{1}{2} \) and \( \delta > 0 \) be fixed. Let \( I(\cdot) \) be the indicator function and let \( E_{n,\delta}(L,s) \) be the truncation of \( E_n(L,s) \) that discards the contribution to \( E_n(L,s) \) from all lattice vectors in \( L \) belonging to the \( n \)-ball of volume \( \delta \) centered at the origin. Then every moment of \( V_n^{-2c}E_{n,\delta}(\cdot, cn) \) converges to the corresponding moment of \( 2 \sum_{j=1}^{\infty} I(T_j > \delta)T_j^{-2c} \) as \( n \to \infty \). Furthermore, for any \( m \geq 1 \) and fixed \( \frac{1}{2} < c_1 < \cdots < c_m \), the joint moments of the random vector

\[
\left( V_n^{-2c_1}E_{n,\delta}(\cdot, c_1 n), \ldots, V_n^{-2c_m}E_{n,\delta}(\cdot, c_m n) \right)
\]

converge to the corresponding joint moments of

\[
\left( 2 \sum_{j=1}^{\infty} I(T_j > \delta)T_j^{-2c_1}, \ldots, 2 \sum_{j=1}^{\infty} I(T_j > \delta)T_j^{-2c_m} \right)
\]

as \( n \to \infty \).
Paper IV concludes with a discussion of the random function $c \mapsto V_n^{-2c}E_n(\cdot, cn)$ for $c \in [A, B]$ with $\frac{1}{2} < A < B$. Using the fact that $c \mapsto V_n^{-2c}E_n(L, cn)$ is convex for each fixed lattice $L \in X_n$, we prove a result analogous to Theorem 6 determining the asymptotic value distribution of this $C([A, B])$-valued random function.

2.5 Paper V

In Paper V we study the asymptotic value distribution of the Epstein zeta function $E_n(L, s)$ for $0 < s < \frac{n}{2}$ and a random lattice $L \in X_n$ of large dimension $n$. More precisely, we give information on the value distribution of $E_n(L, s)$ for $\frac{n}{4} < s < \frac{n}{2}$ with large $n$; using (1.2.6) it is then easy to infer results also for the interval $0 < s < \frac{n}{4}$. In order to state our first main theorem we introduce some notation. We consider a Poisson process $\mathcal{P} = \{\hat{N}(V), V \geq 0\}$ on the positive real line with constant intensity $\frac{1}{2}$, and let $T_1, T_2, T_3, \ldots$ denote the points of the process ordered in such a way that $0 < T_1 < T_2 < T_3 < \ldots$. We let $N(V) = 2\hat{N}(V)$ and define, for all $V \geq 0$,

$$R(V) = N(V) - V. \quad (2.5.1)$$

Finally we let $V_n$ denote the volume of the unit ball in $\mathbb{R}^n$.

**Theorem 8.** Let $\frac{1}{4} < c_1 < c_2 < \frac{1}{2}$. For each $n \in \mathbb{Z}_{\geq 1}$ consider

$$c \mapsto V_n^{-2c}E_n(\cdot, cn)$$

as a random function in $C([c_1, c_2])$. Then the distribution of this random function converges to the distribution of

$$c \mapsto \int_0^\infty V^{-2c} dR(V)$$

as $n \to \infty$.

A crucial ingredient in the proof of Theorem 8 is our result Theorem 3 (in Paper II) on the distribution of lengths of lattice vectors in a random lattice $L \in X_n$. In view of that result the following definitions are natural. Given $L \in X_n$ and $V \geq 0$ we let $N_n(V)$ denote the number of non-zero lattice points of $L$ in the closed $n$-ball of volume $V$ centered at the origin, and define

$$R_n(V) = N_n(V) - V.$$

A second crucial ingredient in our proof of Theorem 8 is the following bound on $R_n(V)$.
Theorem 9. For all \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that for all \( n \geq 3 \) and \( C \geq 1 \) we have
\[
\Pr_{\mu_n}\left\{ L \in X_n \mid |R_n(V)| \leq C_\varepsilon (CV)^{\frac{1}{2}} (\log V)^{\frac{3}{2} + \varepsilon}, \quad \forall V \geq 10 \right\} \geq 1 - C^{-1}.
\]

Theorem 9 is interesting not only for being an important technical part of the proof of Theorem 8, but also for its connection with the famous circle problem generalized to dimension \( n \) and general ellipsoids. Given \( V > 0 \), \( n \geq 2 \) and \( L \in X_n \) the problem asks for the number \( N(V) = 1 + N_n(V) \) of lattice points of \( L \) in the closed \( n \)-ball of volume \( V \) centered at the origin. It is well-known that \( N(V) \) is asymptotic to the volume \( V \) of this ball. Hence \( 1 + R_n(V) \) equals the remainder term in this asymptotic relation, and Theorem 9 implies that this remainder is \( \ll V^{\frac{1}{2}} (\log V)^{\frac{3}{2} + \varepsilon} \) as \( V \to \infty \), for almost every \( L \in X_n \). As far as we are aware, this fact has not been pointed out previously in the literature.

In the last section of Paper V we extend the result in Theorem 8 to the case \( c_2 = \frac{1}{2} \). In order for this to make sense we have to subtract the singular part of \( V_n^{-2c} E_n(\cdot, cn) \) from both the random functions appearing in Theorem 8. As an application we prove the following theorem, strengthening a result [18, Thm. 3] by Sarnak and Strömbergsson on the asymptotic value distribution of the height function \( h_n \).

**Theorem 10.** The random variable
\[
n\left( h_n(L) - \left( \log(4\pi) - \gamma + 1 \right) \right) + \log n
\]
converges in distribution to
\[
2 \lim_{c \to \frac{1}{2}} \left( \int_0^\infty V^{-2c} dR(V) + \frac{1}{1 - 2c} \right) - \log \pi - 1
\]
as \( n \to \infty \).

Another application of Theorem 8 (and its generalization allowing \( c_2 = \frac{1}{2} \)) concerns a question posed by Sarnak and Strömbergsson in [18]. They note that if there exists a lattice \( L_0 \in X_n \) satisfying \( E_n(L, s) \geq E_n(L_0, s) \) for all \( 0 < s < \frac{n}{2} \) and all \( L \in X_n \) then \( E_n(L_0, s) < 0 \) for \( 0 < s < \frac{n}{2} \). Hence, for such a lattice \( L_0, E_n(L_0, s) \) has no zeros in \((0, \infty)\). Sarnak and Strömbergsson point out that it is an interesting question as to whether or not lattices \( L \in X_n \) with \( E_n(L, s) \neq 0, \forall s > 0 \), do exist for all \( n \) (or all large \( n \)). In this direction we prove the following result.

**Corollary 11.** For any fixed \( \frac{1}{4} < c_1 < c_2 \leq \frac{1}{2} \), the limit
\[
\lim_{n \to \infty} \Pr_{\mu_n}\left\{ L \in X_n \mid E_n(s, L) < 0 \text{ for all } s \in [c_1 n, c_2 n] \setminus \left\{ \frac{1}{2} n \right\} \right\}
\]
exists, and equals

\[ f(c_1, c_2) := \text{Prob}\left\{ \int_0^\infty V^{-2c} dR(V) < 0 \text{ for all } c \in [c_1, c_2] \setminus \{\frac{1}{2}\} \right\}. \]

Moreover, for all \( \frac{1}{4} < c_1 < c_2 \leq \frac{1}{2} \) the probability \( f(c_1, c_2) \) satisfies \( 0 < f(c_1, c_2) < 1 \).

In particular, for any given \( \varepsilon > 0 \) the probability that

\[ E_n(s, L) < 0 \text{ for all } s \in (0, (\frac{1}{4} - \varepsilon)n] \cup [(\frac{1}{4} + \varepsilon)n, \frac{1}{2}n) \]

holds tends to a positive limit as \( n \to \infty \)!
Summary in Swedish

Den här avhandlingen behandlar olika asymptotiska problem på två speciella familjer av homogena rum, närmare bestämt de \( n \)-dimensionella hyperboliska rummen och rummen av \( n \)-dimensionella gitter. För de senare rummen inför vi beteckningen \( X_n \).

I Artikel I studerar vi den asymptotiska likafördelningen av delar av slutna horosfärer i en icke-kompakt hyperbolisk mångfald \( M \) av ändlig volym och godtycklig dimension \( n + 1 \). Vi beskriver \( M \) som en kvot av typen \( M = \Gamma \setminus \mathbb{H}^{n+1} \), där

\[
\mathbb{H}^{n+1} = \left\{ (x, y) \mid x \in \mathbb{R}^n, y \in \mathbb{R}_{>0} \right\}
\]

betecknar övre halvrumsmodellen för det \((n+1)\)-dimensionella hyperboliska rummet och \( \Gamma \) är en diskret delgrupp till gruppen \( G \) av orienteringsbevarande isometrier av \( \mathbb{H}^{n+1} \). Vi kan utan inskränkning anta att \( \Gamma \) har en spets i oändligheten. Då innehåller stabilisatorn \( \Gamma_\infty \subset \Gamma \) en delgrupp av ändligt index bestående av translationer,

\[
\Gamma'_\infty = \left\{ (x, y) \mapsto (x + \omega, y) \mid \omega \in \Lambda \right\}
\]

där \( \Lambda \) är något gitter i \( \mathbb{R}^n \). Låt \( \Omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \) vara en bas i \( \Lambda \) och sätt, för varje \( y > 0 \) och alla \( \alpha_i, \beta_i \in \mathbb{R} \) sådana att \( \alpha_i < \beta_i \),

\[
\mathcal{B} = \left\{ (u_1 \omega_1 + \cdots + u_n \omega_n, y) \mid u_i \in [\alpha_i, \beta_i] \text{ för } i = 1, \ldots, n \right\}.
\]

Ett av huvudresultaten i Artikel I säger att då \( y \to 0 \) blir mängden \( \mathcal{B} \) asymptotiskt likafördelad i \( M \) med avseende på det hyperboliska volymsmåttet, så länge vi håller \( \beta_i - \alpha_i \geq y^{1/2-\epsilon} \) för alla \( 1 \leq i \leq n \).

I Artikel II studerar vi fördelningen av längder av nollskilda gittervektorer i ett slumpvis valt gitter av hög dimension. Givet ett gitter \( L \in X_n \), ordnar vi dess nollskilda vektorer efter växande längd som \( \pm v_1, \pm v_2, \pm v_3, \ldots \) och låter \( \ell_j = |v_j| \) (alla så gäller \( 0 < \ell_1 \leq \ell_2 \leq \ell_3 \leq \ldots \)). Dessutom definierar vi

\[
\mathcal{V}_j = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} \ell_j^n,
\]
så att \( V_j \) betecknar volymen av en \( n \)-dimensionell boll med radie \( \ell_j \). Inspirerade av resultat av Rogers och Schmidt visar vi att då \( n \to \infty \) konvergerar följen \( \{V_j\}_{j=1}^{\infty} \) i fördelning mot punkterna i en Poissonprocess på \( \mathbb{R}^+ \) med konstant intensitet \( \frac{1}{2} \).

I Artikel III fortsätter vi att studera statistiska egenskaper hos följen \( \pm v_1, \pm v_2, \pm v_3, \ldots \) av nollskilda vektorer i ett slumpvis valt gitter av hög dimension. Bland annat visar vi att vinklarna mellan de kortaste nollskilda vektorerna i ett slumpvis valt gitter \( L \in X_n \) konvergerar i fördelning mot en familj av oberoende Gaussiska variabler då \( n \to \infty \).

I Artikel IV och Artikel V studerar vi den asymptotiska värdefördelningen hos Epsteins zeta funktion som för \( L \in X_n \) och \( \text{Re } s > \frac{n}{2} \) definieras genom

\[
E_n(L, s) = \sum_{m \in L} |m|^{-2s},
\]

där ′ betecknar att vi endast summerar över nollskilda gittervektorer. Huvudresultatet i Artikel IV säger att om vi fixerar \( c > \frac{1}{2} \) så kan den asymptotiska värdefördelningen då \( n \to \infty \) hos \( E_n(\cdot, cn) \) (lämpligt normaliserad) fullständigt beskrivas i termer av en Poissonprocess på \( \mathbb{R}^+ \) med konstant intensitet \( \frac{1}{2} \).

I Artikel V studerar vi den asymptotiska värdefördelningen hos \( E_n(L, s) \) för \( 0 < \text{Re } s < \frac{n}{2} \) och ett slumpvis valt gitter \( L \in X_n \). Givet en Poissonprocess \( \mathcal{P} = \{\hat{N}(V), V \geq 0\} \) på \( \mathbb{R}^+ \) med konstant intensitet \( \frac{1}{2} \) låter vi \( T_1, T_2, T_3, \ldots \) beteckna processens punkter ordnade så att \( 0 < T_1 < T_2 < T_3 < \ldots \). Vi sätter \( N(V) = 2\hat{N}(V) \) och definierar, för alla \( V \geq 0 \),

\[
R(V) = N(V) - V.
\]

Dessutom låter vi \( V_n \) beteckna volymen av enhetsbollen i \( \mathbb{R}^n \). Ett av huvudresultaten i Artikel V säger att om vi för \( \frac{1}{4} < c_1 < c_2 < \frac{1}{2} \) betraktar

\[
c \mapsto V_n^{-2c} E_n(\cdot, cn)
\]

som en stokastisk variabel med värden i \( C([c_1, c_2]) \) så konvergerar den i fördelning mot

\[
c \mapsto \int_0^\infty V^{-2c} dR(V)
\]

då \( n \to \infty \). Detta resultat tillämpas sedan på flera problem av statistisk natur på rummet \( X_n \). En viktig del av beviset av vårt huvudresultat är en uppskattning som visar sig vara av intresse även i samband med Gauss cirkelproblem generaliserat till godtycklig dimension och generella ellipsoider.
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