Knotted Legendrian surfaces with few Reeb chords

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KNOTTED LEGENDRIAN SURFACES WITH FEW REEB CHORDS

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Abstract. For $g > 0$, we construct $g + 1$ Legendrian embeddings of a surface of genus $g$ into $J^1(\mathbb{R}^2) = \mathbb{R}^3$ which lie in pairwise distinct Legendrian isotopy classes and which all have $g + 1$ transverse Reeb chords ($g + 1$ is the conjecturally minimal number of chords). Furthermore, for $g$ of the $g + 1$ embeddings the Legendrian contact homology DGA does not admit any augmentation over $\mathbb{Z}_2$, and hence cannot be linearized. We also investigate these surfaces from the point of view of the theory of generating families. Finally, we consider Legendrian spheres and planes in $J^1(S^2)$ from a similar perspective.
1. Introduction.

We will consider contact manifolds of the form \( J^1(M) = T^*M \times \mathbb{R} \), where \( M \) is a 2-dimensional manifold, equipped with the contact form \( \alpha := dz + \theta \). Here \( \theta = -\sum_i p_i dq_i \) denotes the canonical (or Liouville) form on \( T^*M \) and \( z \) is the coordinate of the \( \mathbb{R} \)-factor.

An embedded surface \( L \subset J^1(M) \) is called Legendrian if \( L \) is everywhere tangent to the contact distribution \( \ker(\alpha) \). The Reeb vector field, which is defined by \( \iota_R d\alpha = 0, \alpha(R) = 1 \), here becomes \( R = \partial_z \). A Reeb chord on \( L \) is an integral curve of \( R \) having positive length and both endpoints on \( L \). When considering immersed Legendrian submanifolds, we say that self-intersections are zero-length Reeb chords.

We call the natural projections
\[
\Pi_F : J^1(M) \to M \times \mathbb{R},
\Pi_L : J^1(M) \to T^*M,
\]
the front projection and the Lagrangian projection, respectively. Legendrian submanifolds \( L \subset J^1(M) \) project to exact immersed Lagrangian manifolds \( \Pi_L(L) \) of the exact symplectic manifold \( (T^*M, d\theta) \), and Reeb chords of \( L \) correspond to self-intersections of the Lagrangian projection.

For a generic closed Legendrian submanifold \( L \subset J^1(M) \) there are only finitely many Reeb chords, each projecting to a transverse double-point of \( \Pi_L(L) \) under the Lagrangian projection. We call a Legendrian satisfying this property chord generic.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{standard_sphere}
\caption{The front projection of the standard sphere \( L_{std} \subset J^1(\mathbb{R}^2) \).}
\end{figure}

Let \( L_{std} \subset J^1(\mathbb{R}^2) = \mathbb{R}^5 \) denote the Legendrian sphere whose front projection is shown in Figure 1. Note that \( L_{std} \) only has one Reeb chord, and that up to isotopy it is the only known Legendrian sphere in \( J^1(\mathbb{R}^2) \) with this property.

In Section 4 we construct, for each \( g > 0 \), the Legendrian surfaces \( L_{g,k} \subset J^1(\mathbb{R}^2) = \mathbb{R}^5 \) of genus \( g \) by attaching \( k \) “knotted” and \( g - k \) “standard” Legendrian handles to \( L_{std} \), where \( k = 0, \ldots, g \). Each surface has \( g + 1 \) transverse Reeb chords, which according to a conjecture of Arnold is the minimal number of Reeb chords for a Legendrian surface in \( J^1(\mathbb{R}^2) \) of genus
This conjecture is only known to be true for \( g \leq 1 \). It follows from elementary properties of generic Lagrangian immersions when \( g = 0 \), and from Gromov’s theorem of non-existence of exact Lagrangian submanifolds in \( \mathbb{C}^n \) when \( g = 1 \).

We will study the Legendrian contact homology of \( L_{g,k} \). This theory associates a DGA (short for differential graded algebra) to a Legendrian submanifold. The DGA is then invariant up to homotopy equivalence under Legendrian isotopy. Legendrian contact homology was introduced by Eliashberg, Givental and Hofer in [EGH], and by Chekanov in [Che] for standard contact \( \mathbb{R}^3 \). We will also study the \( L_{g,k} \) in terms of generating families (See Definition 2.14). We show the following theorem.

**Theorem 1.1.** The \( g + 1 \) Legendrian surfaces \( L_{g,k} \subset J^1(\mathbb{R}^2) \) of genus \( g \), where \( k = 0, \ldots, g \), are pairwise non-Legendrian isotopic. Furthermore, \( L_{g,k} \) has a Legendrian contact homology DGA admitting an augmentation with coefficients in \( \mathbb{Z}_2 \) only if \( k = 0 \). Also, \( L_{g,k} \) admits a generating family only if \( k = 0 \).

**Remark 1.2.** There is a correspondence between generating families for a Legendrian knot in \( J^1(\mathbb{R}) \cong \mathbb{R}^3 \) and augmentations for its DGA with coefficients in \( \mathbb{Z}_2 \). See e.g. [FR]. It is not known whether a similar result holds in higher dimensions.

When \( k > 0 \), the DGA of \( L_{g,k} \) with coefficients in \( \mathbb{Z}_2 \) has 1 in the image of the boundary operator. Hence its homology vanishes, and it cannot be used to distinguish the different \( L_{g,k} \). Moreover, it follows that its DGA has no augmentation with coefficients in \( \mathbb{Z}_2 \).

To distinguish the different \( L_{g,k} \) we consider DGAs with coefficients in group ring \( \mathbb{Z}[H_1(L_{g,k};\mathbb{Z})] \) (one may also use coefficients in \( \mathbb{Z}_2[H_1(L_{g,k};\mathbb{Z})] \)). We will study the augmentation varieties of these DGAs. This is a Legendrian isotopy invariant introduced by L. Ng in [Ng].

In Section 5 we construct a Legendrian sphere \( L_{knot} \subset J^1(S^2) \) with one Reeb-chord which is not Legendrian isotopic to \( L_{std} \). However, according to Proposition 5.5, \( L_{knot} \) has a Lagrangian projection which is smoothly ambient isotopic to \( \Pi_L(L_{std}) \). Observe that the unit disk bundle \( D^*S^2 \) with its canonical symplectic form is symplectomorphic to a neighbourhood of the anti-diagonal in \( S^2 \times S^2 \). We show the following result.

**Theorem 1.3.** \( L_{std} \) and \( L_{knot} \) are not Legendrian isotopic. Furthermore, \( \Pi_L(L_{knot}) \subset D^*S^2 \subset S^2 \times S^2 \) cannot be mapped to \( \Pi_L(L_{std}) \subset D^*S^2 \) by a symplectomorphism of \( S^2 \times S^2 \) which is Hamiltonian isotopic to the identity.

The first result is proved by computing the Legendrian contact homology of the link \( L_{knot} \cup T_qS^2 \). The second result follows by relating \( \Pi_L(L_{knot}) \) to the non-displaceable Lagrangian tori treated in [FOOO].

In Section 6 we study the following Legendrian planes. Let \( F_0 := T^*_pS^2 \subset T^*S^2 \) be a Lagrangian fibre and let \( F_k \subset T^*S^2 \), where \( k \in \mathbb{Z} \), be the image of
$F_0$ under $k$ iterations of a Dehn twist (or its inverse) along the zero section. The plane $F_k$ coincides with $F_0$ outside of a compact set.

Since $H^1(F_k; \mathbb{R}) = 0$, $F_k$ is an exact embedded Lagrangian and hence we may lift $F_k$ to a Legendrian submanifold of $J^1S^2$ and a Lagrangian isotopy of $F_k$ induces a Legendrian isotopy of the lift. Moreover, since $F_k$ is a plane, a compactly supported Lagrangian isotopy may be lifted to a compactly supported Legendrian isotopy.

By computing the Legendrian contact homology of the Legendrian lift of the link $F_k \cup T^*qS^2$, we show the following.

**Theorem 1.4.** There is no compactly supported Legendrian isotopy taking $F_k$ to $F_l$ if $k \neq l$. Consequently, there is no compactly supported Lagrangian isotopy taking $F_k$ to $F_l$ if $k \neq l$. However, there are such compactly supported smooth isotopies if $k \equiv l \mod 2$.

The effect of Dehn twists on Floer Homology was studied by P. Seidel in [Sei], and our argument is a version of it.

### 2. Background.

In this section we recall the needed results and definitions. In particular, we give a review of Legendrian contact homology, linearizations and the augmentation variety. We also give a description of gradient flow trees, which will be used for computing the differentials of the DGAs. Finally, we briefly discuss the theory of generating families for Legendrian submanifolds.

#### 2.1. Legendrian contact homology.

We now recall the results in [EES1], [EES2] and [EES4] in order to define Legendrian contact homology for Legendrian submanifolds of $J^1(M)$ with coefficients in $\mathbb{Z}_2$ and $\mathbb{Z}$. For our purposes we will only need the cases $M = \mathbb{R}^2$ and $M = S^2$, respectively.

The Legendrian contact homology algebra is a DGA associated to a Legendrian submanifold $L$, assumed to be chord generic, which is generated by the Reeb chords of $L$. The differential counts pseudoholomorphic disks. The homotopy type, and even the stable isomorphism type (see below), of the DGA is then invariant under Legendrian isotopy. The most obvious consequence is that the homology of the complex, the so called Legendrian contact homology, is invariant under Legendrian isotopy.

**2.1.1. The algebra.** For a chord generic Legendrian submanifold $L \subset J^1(M)$ with the set $Q$ of Reeb chords, we consider the algebra $\mathcal{A}_\Lambda(L) = \Lambda(Q)$ freely generated over the ring $\Lambda$. We may always take $\Lambda = \mathbb{Z}_2$, but in the case when $L$ is spin we may also take $\Lambda = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{C}$. In the latter case, the differential depends on the choice of a spin structure on $L$ and orientations of capping operators. For details we refer to [EES4].

We will also consider the algebra $\Lambda[H_1(L; \mathbb{Z})] \otimes_\Lambda \mathcal{A}_\Lambda(L)$ with coefficients in the group ring $\Lambda[H_1(L; \mathbb{Z})]$. 
2.1.2. The grading. For a Legendrian submanifold $L \subset J^1(M) = T^*M \times \mathbb{R}$ there is an induced Maslov class

$$\mu : H_1(L; \mathbb{Z}) \to \mathbb{Z},$$

which in our setting can be computed using the following formula. Let $L$ be front generic, and let $\eta$ be a closed curve on $L$ which intersects the singular set of the front transversely at cusp edges. Recall that $z$ is the coordinate of the $\mathbb{R}$-factor of $J^1(M) = T^*M \times \mathbb{R}$. Let $D(\eta)$ and $U(\eta)$ denote the number of cusp edges transversed by $\eta$ in the downward and upward direction relative the $z$-coordinate, respectively. In [EES2] it is proved that

$$\mu([\eta]) = D(\eta) - U(\eta). \tag{2.1}$$

We will only consider the case when the Maslov class vanishes. In this case the algebra $A_{\Lambda}(L)$ is graded as follows. For each generator, i.e. Reeb chord $c \in Q$, we fix a path $\gamma_c : I \to L$ with both ends on the Reeb chord such that $\gamma_c$ starts at the point with the higher $z$-coordinate. Again, we assume that $\gamma_c$ intersects the singularities of the front projection transversely at cusp edges. We call $\gamma_c$ a \textit{capping path} for $c$.

Let $f_u$ and $f_l$ be the local functions on $M$ defining the $z$-coordinates of the upper and lower sheets of $L$ near the endpoints of $c$, respectively. We define $h_c := f_u - f_l$. Let $p \in M$ be the projection of $c$ to $M$. Observe that the fact that $c$ is a transverse Reeb chord is equivalent to $h_c$ having a non-degenerate critical point at $p$.

We grade the generator $c$ by

$$|c| = \nu(\gamma_c) - 1,$$

where $\nu(\gamma_c)$ denotes the Conley-Zehnder index of $\gamma_c$. Here, the Conley-Zehnder index may be computed by

$$\nu(\gamma_c) = D(\gamma_c) - U(\gamma_c) + \text{index}_p(d^2 h_c), \tag{2.2}$$

where $D$ and $U$ are defined as above and $\text{index}_p(d^2 h_c)$ is the Morse index of $h_c$ at $p \in M$. See [EES2] for a general definition of the Conley-Zehnder index and a proof of the above formula.

If the Maslov class does not vanish, we must use coefficients in $\Lambda[H_1(L; \mathbb{Z})]$ to have a well defined grading over $\mathbb{Z}$. Elements $A \in H_1(L; \mathbb{Z})$ are then graded by

$$|A| = -\mu(A).$$

In our cases, since $\mu$ vanishes, the coefficients have zero grading.

In the case when $L$ has several connected components, Reeb chords between two different components are called \textit{mixed}, while Reeb chords between the same component are called \textit{pure}. Mixed Reeb chords can be graded in the following way. For each pair of components $L_0, L_1$, select points $p_0 \in L_0$ and $p_1 \in L_1$ both projecting to the same point on $M$, and such that neither lies on a singularity of the front projection. Let $c$ be a mixed Reeb chord starting on $L_0$ and ending on $L_1$. A capping path is then chosen as a path on $L_1$ starting at $c$ and ending on $p_1$, together with a path on $L_0$ starting
at \( p_0 \) and ending on \( c \). The grading of a mixed chord can then be defined as before, where the Conley-Zehnder index is computed as in Formula (2.2) for this (discontinuous) capping path.

Observe that the choice of points \( p_0 \) and \( p_1 \) may affect the grading of the mixed chords, hence this grading is not invariant under Legendrian isotopy in general. However, the difference in degree of two mixed chords between two fixed components is well defined.

2.1.3. The differential. Choose an almost complex structure \( J \) on \( T^*M \). We are interested in finite-energy pseudoholomorphic disks in \( T^*M \) having boundary on \( \Pi_L(L) \) and boundary punctures asymptotic to the double points of \( \Pi_L(L) \). A puncture of the disk will be called positive in case the positive boundary of the disk jumps to a sheet with higher \( z \)-coordinate at the Reeb chord, and will otherwise be called negative. We assume that \( J \) is chosen so that the solution spaces of pseudoholomorphic disks with one positive puncture are transversely cut out manifolds of the expected dimension. (See [EliSor3] for the existence of such almost complex structures.)

Since \( \Pi_L(L) \) is an exact immersed Lagrangian, one can easily show the following formula for the (symplectic) area of a disk \( D \subset T^*M \) with boundary on \( \Pi_L(L) \), having the positive punctures \( a_1, \ldots, a_n \) and the negative punctures \( b_1, \ldots, b_n \):

\[
\text{Area}(D) = \ell(a_1) + \ldots + \ell(a_n) - \ell(b_1) - \ldots - \ell(b_n),
\]

where \( \ell(c) \) denotes the action of a Reeb chord \( c \), which is defined by

\[
\ell(c) := \int_c \alpha > 0.
\]

One thus immediately concludes that a non-constant pseudoholomorphic disk with boundary on \( \Pi_L(L) \) must have at least one positive puncture.

Let \( \mathcal{M}(a; b_1, \ldots, b_n; A) \) denote the moduli space of pseudoholomorphic disks having boundary on \( L \), a positive puncture at \( a \in Q \) and negative punctures at \( b_i \in Q \) in the above order relative the orientation of the boundary. We moreover require that when closing up the boundary of the disk with the capping paths at the punctures (oriented appropriately), the cycle obtained is contained in the class \( A \in H_1(L; \mathbb{Z}) \). We define the differential on the generators by the formula

\[
\partial a = \sum_{\dim \mathcal{M} = 0} |\mathcal{M}(a, b_1, \ldots, b_n, A)| Ab_1 \cdots b_n.
\]

where \( |\mathcal{M}(a; b_1, \ldots, b_n, A)| \) is the algebraic number of elements in the compact zero-dimensional moduli space. The above count has to be performed modulo 2 unless the moduli spaces are coherently oriented. When \( L \) is spin, a coherent orientation can be given after making initial choices. If we are working with coefficients in \( A \) instead of \( \Lambda[H_1(L; \mathbb{Z})] \), we simply project the group ring coefficient \( A \) to 1 in the above formula.
For a generic almost complex structure $J$, the dimension of the above moduli space is given by
\[ \dim \mathcal{M}(a; b_1, ..., b_n, A) = |a| - |b_1| - ... - |b_n| + \mu(A) - 1, \]
and it follows that $\partial$ is a map of degree $-1$.

The differential defined on the generators is extended to arbitrary elements in the algebra by linearity and by the Leibniz rule
\[ \partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b). \]

Since $L$ is an exact Lagrangian immersion, no bubbling of disks without punctures can occur, and a standard argument from Floer theory shows that $\partial^2 = 0$. Observe that the sum occurring in the differential always is finite because of Formula (2.3) and since there are only finitely many Reeb chords.

2.1.4. Invariance under Legendrian isotopy. Let $\mathcal{A} = R\langle a_1, ..., a_m \rangle$ and $\mathcal{A}' = R\langle a'_1, ..., a'_m \rangle$ be free algebras over the ring $R$. An isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ of semi-free DGAs is tame if, after some identification of the generators of $\mathcal{A}$ and $\mathcal{A}'$, it can be written as a composition of elementary automorphisms, i.e. automorphisms defined on the generators of $\mathcal{A}$ by
\[ \varphi(a_i) = \begin{cases} a_i & i \neq j \\ Aa_j + b & i = j \end{cases} \]
for some fixed $j$, where $A \in R$ is invertible and $b$ is an element of the subalgebra generated by $\{a_i; \ i \neq j\}$.

The stabilization in degree $j$ of $(\mathcal{A}, \partial)$, denoted by $S_j(\mathcal{A}, \partial)$, is the following operation. Add two generators $a$ and $b$ with $|a| = j$ and $|b| = j - 1$ to the generators of $\mathcal{A} = R\langle a_1, ..., a_m \rangle$. The new differential $\partial'$ for the stabilization is defined to be $\partial$ on old generators, while $\partial'(a) = b$ and $\partial' b = 0$ for the new generators. It is a standard result (see [Che]) that $(\mathcal{A}, \partial)$ and $S_j(\mathcal{A}, \partial)$ are homotopy equivalent.

Theorem 2.1 ([EES4]). Let $L \subset J^1(M)$ be a Legendrian submanifold. The stable tame isomorphism class of its associated DGA $\Lambda[H_1(L; \mathbb{Z})] \otimes \Lambda_L(L)$ is preserved (after possibly shifting the degree of the mixed chords) under Legendrian isotopy and independent of the choice of a generic almost complex structure. Hence, the homology
\[ HC_\bullet(L; \Lambda[H_1(L; \mathbb{Z})]) := H_\bullet(\Lambda[H_1(L; \mathbb{Z})] \otimes \Lambda_L(L), \partial) \]
is invariant under Legendrian isotopy. In particular, the homology
\[ HC_\bullet(L; \mathbb{Z}_2) := H_\bullet(\Lambda_{\mathbb{Z}_2}(L), \partial) \]
with coefficients in $\mathbb{Z}_2$ is invariant under Legendrian isotopy.

Remark 2.2. Different choices of capping paths give tame isomorphic DGAs. Changing the capping path of a Reeb chord $c \in Q$ amounts to adding a representative of some class $\eta_c \in H_1(L; \mathbb{Z})$ to the old capping path. This gives the new DGA $(\Lambda[H_1(L; \mathbb{Z})] \otimes \Lambda \mathcal{A}_L, \varphi \partial \varphi^{-1})$, where
\[ \varphi : (\Lambda[H_1(L; \mathbb{Z})] \otimes \Lambda \mathcal{A}_L, \partial) \to (\Lambda[H_1(L; \mathbb{Z})] \otimes \Lambda \mathcal{A}_L, \varphi \partial \varphi^{-1}) \]
is the tame automorphism defined by mapping \( c \mapsto \eta_cc \), while acting by identity on the rest of the generators.

**Remark 2.3.** The choice of spin structure on \( L \) induces the following isomorphism of the DGAs involved. Let \( s_0 \) and \( s_1 \) be two spin structures, and let \( s_i \) induce the DGA \( ([H_1(L;\mathbb{Z})] \otimes A, \partial_{s_i}) \). Then there is an isomorphism of DGAs (considered as \( \mathbb{Z} \)-algebras)
\[
\varphi : (\mathbb{Z}[H_1(L;\mathbb{Z})] \otimes A, \partial_{s_0}) \to (\mathbb{Z}[H_1(L;\mathbb{Z})] \otimes A, \partial_{s_1})
\]
defined by
\[
\varphi(A) = (-1)^{d(s_0,s_1)(A)} A
\]
for \( A \in H_1(L;\mathbb{Z}) \), while acting by identity on all generators coming from Reeb chords. Here \( d(s_0,s_1) \in H^1(L;\mathbb{Z}_2) \) is the difference cochain for the two spin structures.

2.1.5. **Linearizations and augmentations.** Linearized contact homology was introduced in [Che]. This is a stable isomorphism invariant of a DGA and hence a Legendrian isotopy invariant.

Let \( A\Lambda = \bigoplus_{i=0}^{\infty} A^i\Lambda \) be the module decomposition with respect to word-length. Decompose \( \partial = \partial_0 + \partial_1 \) accordingly and note that if \( \partial_0 = 0 \) on generators, it follows that \( (\partial_1)^2 = 0 \). We will call a DGA satisfying \( \partial_0 = 0 \) good, and call the homology \( H_\bullet(A^1, \partial_1) \) its linearized contact homology.

An augmentation of \( (A\Lambda, \partial) \) is a unital DGA morphism
\[
\epsilon : (A\Lambda, \partial) \to (\Lambda, 0).
\]
It induces a tame automorphism \( \Phi^\epsilon \) defined on the generators by \( c \mapsto c + \epsilon(c) \). \( \Phi^\epsilon \) conjugates \( \partial \) to
\[
\partial^\epsilon := \Phi^\epsilon \partial (\Phi^\epsilon)^{-1} = \Phi^\epsilon \partial,
\]
where \( (A\Lambda, \partial^\epsilon) \) can be seen to be good. We denote the induced linearized contact homology by
\[
LCH_\bullet(L;\Lambda, \epsilon) := H_\bullet\left(A(L)^1, (\partial^\epsilon)_1\right).
\]

**Theorem 2.4 ([Che] 5.1).** Let \( (A, \partial) \) be a DGA. The set of isomorphism classes of the graded vector spaces
\[
H_\bullet\left(A^1, (\partial^\epsilon)_1\right)
\]
for all augmentations \( \epsilon \) is invariant under stable tame isomorphism. Hence, for the DGA of a Legendrian submanifold, this set is a Legendrian isotopy invariant.

2.1.6. **The augmentation variety.** The augmentation variety was introduced in [Ng]. Here we let \( \mathbb{F} \) be an algebraically closed field. The maximal ideal spectrum
\[
\text{Sp}(\mathbb{F}[H_1(L;\mathbb{Z})]) \simeq (\mathbb{F}^*)^{\text{rank}H_1(L;\mathbb{Z})}
\]
can be identified with the set of unital algebra morphisms
\[
\rho : \mathbb{F}[H_1(L;\mathbb{Z})] \to \mathbb{F}.
\]
Extending \( \rho \) by identity on the generators induces a unital DGA chain map

\[ \rho : (\mathbb{F}[H_1(L; \mathbb{Z})] \otimes \mathcal{A}, \partial) \to (\mathcal{A}, \partial^\rho := \rho \partial). \]

**Definition 2.5.** Let \((\mathbb{F}[H_1(L; \mathbb{Z})] \otimes \mathcal{A}_F, \partial)\) be a DGA with coefficients in the group ring. Its *augmentation variety* is the subvariety

\[ \text{AugVar}(\mathbb{F}[H_1(L; \mathbb{Z})] \otimes \mathcal{A}_F, \partial) \subset \text{Sp}(\mathbb{F}[H_1(L; \mathbb{Z})]) \]

defined as the Zariski closure of the set of points \( \rho \in \text{Sp}(\mathbb{F}[H_1(L; \mathbb{Z})]) \) for which the chain complex \((\mathcal{A}_F, \partial^\rho)\) has an augmentation.

This construction can be seen to be a contravariant functor from the category of finitely generated semi-free DGAs with coefficients in the group-ring \(\mathbb{F}[H_1(L; \mathbb{Z})]\) to the category of algebraic subvarieties of \(\text{Sp}(\mathbb{F}[H_1(L; \mathbb{Z})])\). A unital DGA morphism will induce an inclusion of the respective subvarieties.

In the following we suppose that \(H_1(L; \mathbb{Z})\) is a free \(\mathbb{Z}\)-module and that the coefficient ring \(\mathbb{F}[H_1(L; \mathbb{Z})]\) consists of elements in degree zero only (i.e. that the Maslov class vanishes).

**Lemma 2.6.** Let \(\rho : \mathbb{F}[H_1(L; \mathbb{Z})] \to \mathbb{F}\) be a unital algebra map, and let \(\Phi : (\mathbb{F}[H_1(L; \mathbb{Z})] \otimes \mathcal{A}_F, \partial) \to (\mathbb{F}[H_1(L; \mathbb{Z})] \otimes \mathcal{B}_F, \partial_B)\) be a unital DGA morphism. The existence of an augmentation of \((\mathcal{B}_F, \partial_B^\rho)\) implies the existence of an augmentation of \((\mathcal{A}_F, \partial_A^\rho)\).

**Proof.** Augmentations pull back with unital DGA morphisms. The proposition follows from the fact that the induced map

\[ \rho \Phi : (\mathcal{A}_F, \partial_A^\rho) \to (\mathcal{B}_F, \partial_B^\rho) \]

is a unital DGA morphism:

\[ \rho \Phi \partial_A^\rho = \rho \Phi \rho \partial_A = \rho \Phi \partial_A = \rho \partial_B \Phi = \rho \partial_B \rho \Phi = \partial_B^\rho \rho \Phi. \]

In particular, since stable tame isomorphic DGAs are chain homotopic, the following proposition follows.

**Corollary 2.7.** The isomorphism class of the augmentation variety of a DGA associated to a Legendrian submanifold is invariant under Legendrian isotopy.

We will now give an alternative description of the augmentation variety which is more suitable for computations.

Let \(\mathcal{A}\) be a semi-free DGA and let \(\mathcal{N} \subset \mathcal{A}\) be the ideal generated by the generators of negative degree. This is a subcomplex. Consider the quotient complex \((\mathcal{A}_+, \partial^+) := \mathcal{A}/\mathcal{N} \).
Since tame isomorphisms of $\mathcal{A}$ preserve $\mathcal{N}$, they induce tame isomorphisms of $A_+$. Stabilizations of $\mathcal{A}$ in degrees different from zero also induce stabilizations of $A_+$. However, stabilization of $A_+$ in degree zero yields

$$(S_0(\mathcal{A}))_+ = A_+ \ast \langle a \rangle,$$

where “$\ast$” denotes the free product, and where $\partial^+ a = 0$ and $|a| = 0$.

An augmentation of $\mathcal{A}$ has $\mathcal{N}$ in its kernel, and so induces an augmentation of $A_+$. Conversely, an augmentation of $A_+$ pulls back to an augmentation of $\mathcal{A}$ via the quotient projection.

Augmentations $A_+ \to \mathbb{F}$ correspond bijectively to unital algebra maps $H_*(A_+;\partial^+) \to \mathbb{F}$ vanishing in degrees different from zero. Since $A_+$ has no generators in negative degrees, augmentations simply correspond to unital algebra maps

$$\alpha : H_0(A_+;\partial^+) \to \mathbb{F}.$$

Let $C(B)$ denote the algebra $B$ quoted by the commutator ideal. Consider the affine algebra

$$C(H_0(A_+;\partial^+))/\sqrt{0},$$

where $\sqrt{0}$ denotes the nilradical. Its maximal ideal spectrum

$$\text{TotAug}(A_+;\partial^+) := \text{Spec}(H_0(A_+;\partial^+))/\sqrt{0}$$

will be called the total augmentation variety.

Unital algebra maps $\alpha : H_0(A_+;\partial^+) \to \mathbb{F}$ factor through the algebra $\mathcal{O}_{\text{TotAug}}(A_+;\partial^+)$ of regular functions of the total augmentation variety. Hence we have found a correspondence between augmentations of $(A_+;\partial^+)$ and points in the total augmentation variety.

**Proposition 2.8.** The isomorphism class of the total augmentation variety $\text{TotAug}(A_+;\partial^+)$ is invariant under stable tame isomorphism of $(\mathcal{A},\partial)$, up to stabilization by taking the cartesian product with an affine plane $\mathbb{F}^n$.

**Proof.** By the above reasoning, the isomorphism class of $A_+$ is invariant under stable tame isomorphism of $\mathcal{A}$, up to stabilization by the free product with a finitely generated free algebra, i.e.

$$A_+ \ast \langle b_1, \ldots, b_n \rangle,$$

such that $|b_i| = 0$ and $\partial_+ b_i = 0$. This gives the result. \qed

Let $(A)_0$ denote the homogeneous degree zero component of the DGA $\mathcal{A}$. Then the following holds.

**Proposition 2.9.** The augmentation variety of the semi-free DGA $(\mathbb{F}[H_1(L;\mathbb{Z})] \otimes \mathcal{A},\partial)$ is the closure of the image of the total augmentation variety

$$\text{TotAug}(\mathbb{F}[H_1(L;\mathbb{Z})] \otimes A_+;\partial^+) \subset \text{Spec}(\mathbb{F}[H_1(L;\mathbb{Z})]) \times \text{Spec}(C(A_+)_0)$$

under the canonical projection

$$\text{Spec}(\mathbb{F}[H_1(L;\mathbb{Z})]) \times \text{Spec}(C(A_+)_0) \to \text{Spec}(\mathbb{F}[H_1(L;\mathbb{Z})]).$$
Proof. A point \((\rho, a) \in \text{Sp} \mathbb{F}[H_1(L; \mathbb{Z})] \times \text{Sp}(\mathcal{C}(\mathcal{A}_+)_0)\), where \(\rho\) corresponds to a unital algebra morphism
\[
\rho : \mathbb{F}[H_1(L, \mathbb{Z})] \rightarrow \mathbb{F},
\]
is contained in \(\text{TotAug}(\mathbb{F}[H_1(L; \mathbb{Z})] \otimes \mathcal{A}_+, \partial_+)\) if and only if
\[
a \in \text{TotAug}(\mathcal{A}_+, \partial^\rho_+).
\]
Hence, \(\rho \in \text{Sp} \mathbb{F}[H_1(L; \mathbb{Z})]\) being in the image of the projection
\[
\text{TotAug}(\mathbb{F}[H_1(L; \mathbb{Z})] \otimes \mathcal{A}_+, \partial_+) \rightarrow \text{Sp} \mathbb{F}[H_1(L; \mathbb{Z})],
\]
implies that \(\text{TotAug}(\mathcal{A}_+, \partial^\rho_+)\) is non-empty, i.e. that \((\mathcal{A}_+, \partial^\rho_+)\) admits an augmentation. By the above reasoning, this is equivalent to \((\mathcal{A}, \partial^\rho)\) admitting an augmentation. \(\square\)

2.2. Flow trees. Our computations of the differentials of the Legendrian contact homology DGAs relies on the technique of gradient flow trees developed in [Ekh]. We restrict ourselves to the case \(\dim M = 2\).

Definition 2.10. Given a metric \(g\) on \(M\), a flow tree on \(L\) is a finite tree \(\Gamma\) immersed by \(f : \Gamma \rightarrow M\), together with extra data, such that:

(a) On the interior of an edge \(e_i\), \(f\) is an injective parametrization of a flow line of
\[
-\nabla (h^\alpha_i - h^\beta_i),
\]
where \(h^\alpha_i\) and \(h^\beta_i\) are two local functions on \(M\), each defining the \(z\)-coordinate of a sheet of \(L\). To the flow line corresponding to \(e_i\) we associate its two 1-jet lifts \(\phi_i^\alpha, \phi_i^\beta\), parameterized by
\[
\phi_i^\alpha(t) = (dh_i^\alpha(e_i(t)), h_i^\alpha(e_i(t))) \in L \subset J^1(M) = T^* M \times \mathbb{R},
\]
\[
\phi_i^\beta(t) = \left( dh_i^\beta(e_i(t)), h_i^\beta(e_i(t)) \right) \in L \subset J^1(M) = T^* M \times \mathbb{R}
\]
and oriented by \(-\nabla (h^\alpha_i - h^\beta_i)\) and \(-\nabla (h^\beta_i - h^\alpha_i)\), respectively.

(b) For every vertex \(n\) we fix a cyclic ordering of the edges \(\{e_i\}\). We denote the unique 1-jet lift of the \(i\):th edge which is oriented towards (away from) the vertex \(n\) by \(\phi_i^{in,n}(\phi_i^{out,n})\).

(c) Consider the curves on \(L \subset J^1(M)\) given by the oriented 1-jet lifts of the flow lines. Give the curves a cyclic order by declaring that for every vertex \(n\) and edge \(i\), the curve \(\phi_i^{in,n}\) is succeeded by \(\phi_{i+1}^{out,n}\). We require that the Lagrangian projections of the oriented 1-jet lifts in this order forms a closed curve on \(\Pi_L(L) \subset T^*(M)\).

If the 1-jet lifts \(\phi_i^{in,n}\) and \(\phi_{i+1}^{out,n}\) have different values at the vertex \(n\), we call this a puncture at the vertex. The puncture is called positive if the oriented curve jumps from a lower to a higher sheet relative the \(z\)-coordinate, and is otherwise called negative.

We will also define a partial flow tree as above, but weakening condition (c) by allowing 1-valent vertices \(n\) for which \(\Pi_L \circ \phi^{in,n}\) and \(\Pi_L \circ \phi^{out,n}\) differ at \(n\). We call such vertices special punctures.
As for punctured holomorphic disks with boundary on \( \Pi_L(L) \), an area argument gives the following result.

**Lemma 2.11** (Lemma 2.13 [Ekh]). Every (partial) flow tree has at least one positive (possibly special) puncture.

See definitions 3.4 and 3.5 in [Ekh] for the notion of dimension of a gradient flow tree.

**Proposition 2.12** (Proposition 3.14 [Ekh]). For a generic perturbation of \( L \) and \( g \), all flow trees with at most one positive puncture is a transversely cut out manifold of the expected dimension.

Lemma 3.7 in [Ekh] implies that a generic, rigid and transversely cut out gradient flow tree has no vertices of valence higher than three, and that each vertex is one of the six types depicted in Figure 2 of which the vertices (\( P_1 \)) and (\( P_2 \)) are the possible punctures. (Note that there exists both positive and negative punctures of type (\( P_1 \)) and (\( P_2 \)).) The only picture in Figure 2 which is not self explanatory is the one for (S). Observe that the flow line at an (S)-vertex is tangent to the projection of the cusp edge. We refer to Remark 3.8 in [Ekh] for details.

Because of the following result, we may use rigid flow trees to compute the Legendrian Contact Homology.

**Theorem 2.13** (Theorem 1.1 [Ekh]). For a generic perturbation of \( L \) and the metric \( g \) on \( M \), there is a regular almost complex structure \( J \) on \( T^*M \), such that there is a bijective correspondence between rigid \( J \)-holomorphic disks with one positive puncture having boundary on a perturbation \( \tilde{L} \) of \( L \), and rigid flow trees on \( L \) with one positive puncture.
<table>
<thead>
<tr>
<th>Vertex</th>
<th>Lagrangian projection</th>
<th>Front projection</th>
<th>Flow tree</th>
</tr>
</thead>
<tbody>
<tr>
<td>((P_1))</td>
<td><img src="image" alt="Lagrangian projection" /></td>
<td><img src="image" alt="Front projection" /></td>
<td><img src="image" alt="Flow tree" /></td>
</tr>
<tr>
<td>((P_2))</td>
<td><img src="image" alt="Lagrangian projection" /></td>
<td><img src="image" alt="Front projection" /></td>
<td><img src="image" alt="Flow tree" /></td>
</tr>
<tr>
<td>((E))</td>
<td><img src="image" alt="Lagrangian projection" /></td>
<td><img src="image" alt="Front projection" /></td>
<td><img src="image" alt="Flow tree" /></td>
</tr>
<tr>
<td>((S))</td>
<td><img src="image" alt="Lagrangian projection" /></td>
<td><img src="image" alt="Front projection" /></td>
<td><img src="image" alt="Flow tree" /></td>
</tr>
<tr>
<td>((Y_0))</td>
<td><img src="image" alt="Lagrangian projection" /></td>
<td><img src="image" alt="Front projection" /></td>
<td><img src="image" alt="Flow tree" /></td>
</tr>
<tr>
<td>((Y_1))</td>
<td><img src="image" alt="Lagrangian projection" /></td>
<td><img src="image" alt="Front projection" /></td>
<td><img src="image" alt="Flow tree" /></td>
</tr>
</tbody>
</table>

**Figure 2.** \((P_1)\) and \((P_2)\) depict the punctures in the generic case. \((E)\) and \((S)\) depict the vertices corresponding to an *end* and a *switch*, respectively, while \((Y_0)\) and \((Y_1)\) describes the generic 3-valent vertices.
2.3. Generating families.

**Definition 2.14.** A generating family for a Legendrian submanifold $L \subset J^1(M)$ is a function $F : M \times E \to \mathbb{R}$, where $E$ is a smooth manifold, such that

$$L = \left\{ (q, p, z) \in J^1(M); \quad 0 = \frac{\partial}{\partial w_i} F(q, w), \quad p^i = \frac{\partial}{\partial q^i} F(q, w), \quad z = F(q, w) \right\}. $$

We think of $F$ as a family of functions $F_m : E \to \mathbb{R}$ parameterized by $M$, and require this family to be versal. Since we are considering the case $\dim M = 2$, versality implies that critical points of $F_m$ are isolated, non-degenerate outside a set of codimension 1 of $M$, possibly of $A_2$-type (birth/death type) above a codimension 1 subvariety of $M$ and possibly of $A_3$-type above isolated points of $M$.

We are interested in the case when $E$ is either a closed manifold or of the form $E = \mathbb{R}^N$. In the latter case we require that $F_m$ is linear and non-zero outside of a compact set for each $m \in M$.

In both cases, the Morse homology of a function $F_m$ in the family is well defined for generic data. In the case when $E$ is a closed manifold the Morse homology is equal to $H_{\bullet}(E; \mathbb{Z}_2)$, while it vanishes in the case $E = \mathbb{R}^N$.

The set of generating families for $L$ is invariant under Legendrian isotopy, up to stabilization of $E$ by a factor $\mathbb{R}^M$ and adding a non-degenerate quadratic form on $\mathbb{R}^M$ to $F_m$. (See [YVC].)

In the case $M = \mathbb{R}$ we have the following result. Consider the function

$$W : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, \quad W(m, x, y) = F(m, x) - F(m, y). $$

For sufficiently small $\delta > 0$ we consider the graded vector space

$$\mathcal{G}H_{\bullet}(F) = H_{\bullet+N+1}(W \geq \delta, W = \delta; \mathbb{Z}_2). $$

In $J^1(\mathbb{R})$ there are connections between generating families and augmentations. For example, we have the following result.

**Theorem 2.15** (5.3 in [FR]). Let $F$ be a generic generating family for a Legendrian knot $L \subset J^1(\mathbb{R})$. Then there exists an augmentation $\epsilon$ of the DGA of $L$ which satisfies

$$\mathcal{G}H_{\bullet}(F) \simeq LCH(L; \mathbb{Z}_2, \epsilon). $$

3. The front cone.

In this section we describe the behavior of gradient flow trees on a particular Legendrian cylinder in $J^1(\mathbb{R}^2)$. We need this when investigating our Legendrian surfaces, since some of them coincide with this cylinder above open subsets of $M$ in the bundle $J^1(M) \to M$. 
3.1. **The front cone and its front generic perturbation.** We are interested in the Lagrangian cylinder embedded by

\[ S^1 \times \mathbb{R} \to T^*\mathbb{R}^2 \cong \mathbb{C}^2, \ (\theta, r) \mapsto (r + i)(\cos \theta, \sin \theta). \]

The Legendrian lift of this cylinder is not front generic, since its front projection is given by the double cone

\[ (\theta, r) \mapsto (r \cos \theta, r \sin \theta, r) \]

in which all of \( S^1 \times \{0\} \) is mapped to a point. We call this Legendrian manifold the *front cone*. We perturb it to become front generic as shown in Figure 3.

![Figure 3](image_url)

**Figure 3.** A generic perturbation of the front cone. The top picture is depicts the projection of the four cusp edges and the four swallow tail singularities, the bottom picture depicts five slices of the front projection.
3.2. The gradient flow trees near the front cone. The gradient flow outside the region in \( \mathbb{R}^2 \) bounded by the projection of the four cusp edges behaves like the gradient flow of the unperturbed cone, i.e \(-\nabla (h_u - h_l)\) points inwards to the center, where \( h_u \) is the \( z \)-coordinate of the upper sheet and \( h_l \) is the \( z \)-coordinate of the lower sheet.

We will now examine the behavior of flow trees on a Legendrian submanifold \( L \subset J^1(M) \) which has a front cone above some subset \( U \subset M \). More precisely, we assume that after some diffeomorphism of \( U \subset M \), the part of \( L \) above \( U \) coincides with the part of the front cone above an open disk centered at the origin. Moreover, we assume that the perturbation making the front cone generic has been performed in a much smaller disk.

**Lemma 3.1.** Let \( L \subset J^1(M) \) be a Legendrian submanifold which has a front cone above \( U \subset M \), and let \( T \) be a flow tree on \( L \) with one positive puncture. Removing the part of \( T \) contained inside \( U \) then creates a connected partial flow tree.

**Proof.** Consider the partial trees obtained by removing the part of the tree contained inside \( U \). Completing each edge ending at \( \partial U \) by continuing it into \( U \) in either of the two ways shown in Figure 4, we create a (possibly disconnected) flow tree without adding any additional punctures. Each component must however have a positive puncture by Lemma 2.11, and hence there can only be one component. \( \square \)

**Figure 4.** The possibilities for a rigid gradient flow tree with one positive puncture passing through the lower cusp edge. Inside the region bounded by the cusp edges, \( T_1 \) is a gradient flow of the height difference of sheet \( A \) and \( B \). For \( T_2 \), its the top left edge corresponds to the height difference of sheet \( A \) and \( D \), while the top right edge corresponds to the height difference of sheet \( B \) and \( C \).

**Proposition 3.2.** Let \( L \subset J^1(M) \) be a Legendrian submanifold which has a front cone above \( U \subset M \), and let \( T \) be a rigid flow tree on \( L \) with one positive puncture. Every edge \( e \) of \( T \) which intersects \( \partial U \) is then contained in a partial tree either like \( T_1 \) or \( T_2 \) shown Figure 4.
Proof. By the previous lemma, each edge \( e \) is connected to a partial tree contained entirely inside \( U \). By the dimension formula, and since the flow tree is rigid, this partial tree cannot have any \((Y_0)\)-vertices. Furthermore, we may exclude the existence of \((S)\)-vertices (switches) by examining the behavior of the gradient flows along the cusp edges.

An edge of a generic flow tree which intersects \( \partial U \) will do so away from the projections of the swallow tail singularities. At the cusp edge, the flow tree may either have an \((Y_1)\)-vertex (consider \( T_2 \) in Figure 4) or continue straight through (consider \( T_1 \) in Figure 4). In any case, the only possibility is that these edges end at \((E)\)-vertices at the cusp edges. \( \square \)

3.3. Generating families for the front cone.

Proposition 3.3. A generating family for the front cone has the property that the Morse homology of a generic function in the family is isomorphic to \( H^{•+i}(S^1; \mathbb{Z}_2) \). In particular, the front cone has no generating family with fibre \( R^N \) being linear at infinity. However, it has a generating family with fibre \( S^1 \).

Proof. The dashed diagonal in Figure 3 corresponds to points where the \( z \)-coordinate of sheet \( B \) is equal to that of sheet \( D \). Above (below) the dashed diagonal the \( z \)-coordinate of sheet \( B \) is greater (less) than that of sheet \( D \). Similarly, the diagonal orthogonal to it corresponds to points where the \( z \)-coordinate of sheet \( A \) is equal to that of sheet \( C \). Above (below) that diagonal, the \( z \)-coordinate of sheet \( C \) is greater (less) than that of sheet \( A \).

At the intersections of the two diagonals (at the center of the front cone) the \( z \)-coordinates of sheets \( A \) and \( C \) coincide, as well as those of sheets \( B \) and \( C \).

We let \( A, B, C \) and \( D \) denote the critical points of the function \( F_m \) in the family, where each critical point has been named after the sheet to which it corresponds. Since the pairs \( AB, AD, CB, CD \) all cancel at birth/death singularities at the cusp edges, as seen in Figure 3 we conclude that

\[
\text{index}(A) = \text{index}(C) = i + 1, \quad \text{index}(B) = \text{index}(D) = i,
\]

for some \( i \geq 0 \). Suppose that \( B \) is not in the image of \( \partial A \). Since \( A \) and \( B \) cancel at a birth/death singularity at the top cusp edge, there must have been a handle slide moment either from \( A \) to \( C \) or from \( D \) to \( B \) somewhere in the domain bounded by the top cusp edge and the diagonals. However, for \( m \) in this domain, \( F_m(A) < F_m(C) \) and \( F_m(D) < F_m(B) \), so there can be no such handle slide. By contradiction, we have shown that \( B \) is in the image of \( \partial A \).

Continuing in this manner, one can show that we must have the complex \( \partial(A) = B + D = \partial(C) \) at the center. Since its homology does not vanish, \( L \) has no generating family with fibre \( E = R^N \) being linear at infinity. Assuming that \( E \) is closed, we conclude that we must have \( E = S^1 \).

We give the following description of a generating family with fibre \( S^1 \) for the perturbed front cone. Consider the plane with coordinates \( q = (q^1, q^2) \),
and an ellipse parameterized by $\gamma(\theta), \theta \in S^1$. It can be shown that

$$F(q, \theta) := \|\gamma(\theta) - q\|$$

is such a generating family for $q$ in the domain bounded by the ellipse, given that its semi-axes satisfy $b < a < \sqrt{2}b$. The projection of the four cusp edges here correspond to points in the plane being the envelope of inward normals of the ellipse, where each normal has length equal to the curvature radius at its starting point.

\[ \square \]

**Remark 3.4.** The front cone also appears twice in the conormal lift of the unknot in $\mathbb{R}^3$ (see [EE]). More precisely, let $f : S^1 \rightarrow \mathbb{R}^3, f(\theta) = (\cos \theta, \sin \theta, 0)$ be the unknot. Identify $S^2$ with the unit sphere in $\mathbb{R}^3$. The conormal lift of $f$ is the Legendrian torus in $J_1^1(S^2)$ defined by the generating family $F : S^2 \times S^1 \rightarrow \mathbb{R}, F(q, w) = \langle q, f(w) \rangle$.

It can be seen to have front cones above the north and south pole.

4. **Knotted Legendrian surfaces in $J^1(\mathbb{R}^2)$ of genus $g > 0$.**

4.1. **The standard Legendrian handle.** Consider the Legendrian cylinder with one saddle-type Reeb chord whose front projection is shown in Figure 5. We will call this the standard Legendrian handle. As described in [EES2], we may attach the handle to a Legendrian surface by gluing its ends to cusp edges in the front projection. This amounts to performing “Legendrian surgery” in the case when the cusp edges are located on the same connected component, and “Legendrian connected sum” when the cusp edges are on different connected components.

4.2. **Two Legendrian tori in $J^1(\mathbb{R}^2)$.** Consider the tori $T_{std}$ and $T_{knot}$ whose fronts are depicted in Figure 6 and Figure 7 respectively. For the rotational-symmetric front there is an $S^1$-family of Reeb chords. After perturbation by adding a Morse function defined on $S^1$, having exactly two critical points, to the function defining the $z$-coordinate of one of the sheets, we obtain two non-degenerate Reeb chords from the circle of Reeb chords:

![Figure 5. A Legendrian handle with one saddle-type Reeb chord.](image-url)
c_M of maximum type and c_m of saddle type. Using Formula (2.1), after making the front cone contained in \( T_{knot} \) front generic as in Section 3, one sees that the Maslov class vanishes for both tori. Using the h-principle for Legendrian immersions one concludes that \( T_{std} \) and \( T_{knot} \) are regularly homotopic through Legendrian immersions.

Remark 4.1. An ambient isotopy of \( \mathbb{C}^2 \) inducing a Lagrangian regular homotopy between \( \Pi_L T_{std} \) and \( \Pi_L T_{knot} \) can be constructed as follows. Observe that the rotational-symmetric \( \Pi_L T_{std} \) and \( \Pi_L T_{knot} \) can be seen as the exactly immersed Lagrangian counterparts of the Clifford- and the Chekanov torus, respectively. In the Lefschetz fibration \( \mathbb{C}^2 \to \mathbb{C} \) given by \( (z, w) \mapsto z^2 + w^2 \), with the origin as the only critical value, we get a representation of these tori as the vanishing cycle fibred over a figure eight curve in the base. More precisely, \( \Pi_L(T_{knot}) \) is a figure eight curve in the base encircling the origin, while \( \Pi_L(T_{std}) \) is a figure eight curve not encircling the origin. From this picture it is easily seen that \( \Pi_L(T_{knot}) \) and \( \Pi_L(T_{std}) \) are ambient isotopic through immersed Lagrangians. Namely, we may disjoin the circle in the fibre from the vanishing cycle, and then pass the curve in the base through the critical value.

4.3. The Legendrian genus-\( g \) surfaces \( L_{g,k} \), with \( 0 \leq k \leq g \). We construct the Legendrian surface \( L_{g,k} \) of genus \( g \) by taking the Legendrian direct sum of the standard sphere \( L_{std} \) and \( g - k \) copies of \( T_{std} \) (or equivalently, attaching \( g - k \) standard handles to \( L_{std} \)) and \( k \) copies of \( T_{knot} \). In all cases, we attach one edge of the handle to the unique cusp edge of the sphere, and
the other to the outer cusp edge of the tori. The attached knotted tori will be called \textit{knotted handles}.

It is clear that we may cancel the Reeb chord of each standard handle connecting the sphere and the torus, which is a saddle-type Reeb chord, with the maximum-type Reeb chord $c_M$ on the corresponding torus. We have thus created a Legendrian surface having one Reeb chord $c$ coming from the sphere and Reeb chords $c_i$, for $i = 1, \ldots, g$, coming from the saddle-type Reeb chords on the attached tori. Such a representative is shown in Figure 8.

![Figure 8](image)

\textbf{Figure 8.} The flow of $-\nabla (f_u - f_l)$, where $f_u$ and $f_l$ are $z$-coordinates of the upper and lower sheet, respectively, of the surface $L_{2,1} \simeq L_{\text{std}} \# T_{\text{knot}} \# T_{\text{std}}$.

The surfaces $L_{g,0}$ will be called the \textit{standard Legendrian surface of genus $g$} and they were studied in [EES2]. Observe that $T_{\text{std}}$ is Legendrian isotopic to $L_{1,0}$ and $T_{\text{knot}}$ is Legendrian isotopic to $L_{1,1}$.

\textbf{Remark 4.2.} Both $L_{1,0}$ and $L_{1,1}$ have exactly two transverse Reeb chords. This is the minimal number of transverse Reeb chords for a Legendrian torus in $J^1(\mathbb{R}^2)$, as follows by the adjunction formula

$$-\chi(L) + 2W(\Pi_C(L)) = 0,$$

where $W$ denotes the Whitney self-intersection index. When $L$ is a torus we get that $W(\Pi_C(L_{1,0})) = 0$, which together with the fact that there are no exact Lagrangian submanifolds in $\mathbb{C}^2$ implies the statement.

More generally, $L_{g,k}$ has $g + 1$ transverse Reeb chords. By a conjecture of Arnold, $g + 1$ is the minimal number of transverse Reeb chords for a closed Legendrian submanifold in $J^1(\mathbb{R}^2)$ of genus $g$.

\textbf{4.4. Computation of $HC_\bullet(L_{g,k}; \mathbb{Z})$.} We first choose the following basis \{$\mu_i, \lambda_i$\} for $H_1(L_{g,k}; \mathbb{Z})$. We let $\mu_i \in H_1(L_{g,k}; \mathbb{Z})$ be the class which is represented by (a perturbation of) the outer cusp edge on the $i$:th torus in the decomposition

$$L_{g,k} \simeq L_{\text{std}} \# T_{\text{knot}} \# \ldots \# T_{\text{knot}} \# T_{\text{std}} \# \ldots \# T_{\text{std}}.$$
If the $i$:th torus is standard, let $\gamma_1$ and $\gamma_2$ denote the 1-jet lifts of the flow trees on that torus corresponding to $D_1$ and $D_2$ shown in Figure 6 respectively. We let $\lambda_i$ be represented by the cycle $\gamma_1 - \gamma_2$.

If the $i$:th torus is knotted, $D_1$ depicted in Figure 7 is a partial flow tree ending near a front cone. We choose to extend it with the partial gradient flow tree $T_1$ shown in Figure 4 at the perturbed front cone. Again, denoting the 1-jet lifts of $D_1 \cup T_1$ and $D_2$ by $\gamma_1$ and $\gamma_2$, respectively, we let $\lambda_i$ be represented by the cycle $\gamma_1 - \gamma_2$.

Lemma 4.3. $L_{g,k}$ has vanishing Maslov class, and the generators of $\mathbb{Z}[H_1(L_{g,k};\mathbb{Z})] \otimes \mathcal{A}(L)$ can, after an appropriate choice of capping paths, spin structure and orientations for the capping operator, be made to satisfy

$$|c| = 2, \quad \partial c = 0,$$

$$|c_i| = 1, \quad \partial c_i = 1 + \lambda_i$$

if the $i$:th handle is standard, and

$$|c| = 1, \quad \partial c = 1 + \lambda_i + \mu_i \lambda_i$$

if the $i$:th handle is knotted.

Proof. Using Formula (2.1) one easily checks that the Maslov class of $L_{g,k}$ vanishes, since each closed curve on $L_{g,k}$ must traverse the cusp edges in upward and downward direction an equal number of times.

We choose the capping path for $c$ lying on $L_{std}$ (avoiding the handles) and for $c_i$ we take the 1-jet lift of $D_2$. Using Formula (2.2), we compute

$$|c| = 2, \quad |c_i| = 1.$$

After choosing a suitable spin structure on $L$ and orienting the capping operators, we get the following differential on the Reeb chords. For $c_i$ coming from a standard torus, we compute

$$\partial c_i = 1 + \lambda_i,$$

where the two terms come from the gradient flow trees $D_2$ and $D_1$, respectively. For a Reeb chord $c_i$ coming from a knotted torus we get

$$\partial c_i = 1 + \lambda_i + \mu_i \lambda_i$$

where the first term comes from the gradient flow tree $D_2$, and the last two terms come from the partial flow tree $D_1$ approaching the front cone of the $i$:th torus. By Proposition 3.2, the edge $D_1$ can be completed in exactly two ways to become a rigid flow tree with one positive puncture.

For the Reeb chord $c$ coming from the sphere we get, because of the degrees of the generators, that $\partial c$ is a linear combination of the $c_i$. The relation $\partial^2 = 0$, together with the fact that the $\partial c_i$ form a linearly independent set, implies that

$$\partial c = 0.$$
Remark 4.4. By changing the spin structure on \( L_{g,k} \), we may give each term different from 1 in the differential of a given generator an arbitrary sign.

Remark 4.5. For the DGA with \( \mathbb{Z}_2 \)-coefficients, observe that 1 is in the image of \( \partial \) for \( L_{g,k} \) when \( k > 0 \). Hence \( HC_\bullet(L_{g,k};\mathbb{Z}_2) = 0 \) in these cases.

Since the Maslov class vanishes, we may consider the invariant given by the augmentation variety in Section 2.1.6.

Proposition 4.6. The augmentation variety for \( L_{g,k} \) over \( \mathbb{C} \) is isomorphic to
\[
(C \setminus \{0\})^{g-k} \times (C \setminus \{1,0\})^k.
\]

Proof. In this case, since there are no generators in negative degrees, the discussion in Section 2.1.6 shows that the augmentation variety is given by
\[
Sp\left(F\left[H_1(L_{g,k};\mathbb{Z})\right]/\sqrt{\text{im}\partial}\right).
\]
After making the identification \( F\left[H_1(L_{g,k};\mathbb{Z})\right] \simeq F\left[Z_{\lambda_i}, Z_{\mu_i}, Z^{-1}_{\lambda_i}, Z^{-1}_{\mu_i}\right] \), the augmentation variety thus becomes
\[
\left\{ Z_{\lambda_i} + 1 = 0, \ k < i \leq g, \ Z_{\lambda_j}(1 + Z_{\mu_j}) = 1, \ 1 \leq j \leq k \right\} \subset (\mathbb{C}^*)^{2k},
\]
after some appropriate ordering of the handles. \( \square \)

Proof of Theorem 1.1. The first part follows immediately from the above proposition, together with Corollary 2.7.

Since \( L_{g,k} \) contains \( k > 0 \) front cones, Proposition 3.3 gives that there does not exist any linear at infinity generating family of the form \( F : \mathbb{R}^2 \times \mathbb{R}^N \to \mathbb{R} \) for \( L_{g,k} \) when \( k > 0 \).

It can easily be checked that \( L_{g,0} \) have linear at infinity generating families with fibre \( \mathbb{R} \), since both the standard Legendrian handle and the standard Legendrian sphere \( L_{\text{std}} \) have such generating families. \( \square \)

Remark 4.7. Theorem 1.1 also implies that the Lagrangian projections \( \Pi_L L_{g,k} \) and \( \Pi_L L_{g,l} \) never are Hamiltonian isotopic when \( k \neq l \), even when their actions coincide.

5. A knotted Legendrian sphere in \( J^1(S^2) \) which is smoothly ambient isotopic to the unknot.

Consider the front over \( S^2 \) given in Figure 9. It represents a Legendrian link \( L_{\text{knot}} \cup F \), where \( L_{\text{knot}} \) is a sphere with one maximum type Reeb chord \( c \), having a rotational symmetric front, and where \( F \) is the Legendrian lift of a perturbed fibre \( T^*_q S^2 \).

Remark 5.1. The Lagrangian projection \( \Pi_L(L_{\text{knot}}) \) of the sphere may be seen as the Lagrangian sphere with one transversal double point obtained by performing a Lagrangian surgery to the non-displaceable Lagrangian torus discovered in [AF], exchanging a handle for a transverse double point. More precisely, the torus in \( T^* S^2 \) is given by the image of the geodesic flow of a
fibre of $U^*S^2$. We may lift a neighbourhood of the fibre in the torus to a Legendrian submanifold in $J^1(S^2)$. The front projection of this lift looks like the front cone described in Section 3. $\Pi_L(L_{\text{knot}})$ is obtained from the torus by replacing the Lagrangian projection of the front cone with the Lagrangian projection of the two-sheeted front consisting of the graphs of the functions

$$f_1(x_1, x_2) = x_1^2 + x_2^2,$$

$$f_2(x_1, x_2) = -x_1^2 - x_2^2,$$

respectively. This produces the Reeb chord $c$ as shown at the south pole in Figure 9.

**Remark 5.2.** One can also obtain $L_{\text{knot}}$ by the following construction, which involves a Dehn twist. Consider the standard sphere $L_{\text{std}}$ shown in Figure 11 and suppose that $\Pi_L(L_{\text{std}}) \subset T^*D^2 \subset T^*\mathbb{R}^2$. We may symplectically embed $T^*D^2 \subset T^*S^2$. Perturb one of the sheets so that it coincides with a fibre of $T^*S^2$ in a neighbourhood of the double point. Removing a neighbourhood of this sheet and replacing it with its image under the square of a Dehn-twists along the zero section (see Section 6), such that the Dehn twist has support in a small enough neighbourhood, yields $\Pi_L L_{\text{knot}}$.

Since $L_{\text{knot}}$ has a front cone above the north pole, Proposition 3.3 implies that the fibre of a generating family must be $S^1$. However, the following holds.

**Proposition 5.3.** $L_{\text{knot}}$ has no generating family

$$F : S^2 \times S^1 \to \mathbb{R}.$$

**Proof.** Suppose there is such a generating family. Then

$$\Pi_L L_{\text{knot}} = dF \cap (T^*S^2 \times S^1),$$

where $dF$ is considered as a section in $T^*S^2 \times T^*S^1$. Hence

$$\Pi_L L_{\text{knot}} \cap S^2 = dF \cap (S^2 \times S^1),$$
which by topological reasons must consist of at least four point when the
intersection is transversal. However, one sees that $L_{knot}$ intersects the zero
section transversely in only two points, which leads to a contradiction. □

Remark 5.4. $L_{knot}$ can be seen to have a generating family defined on an
$S^1$-bundle over $S^2$ of Euler number 1.

Proposition 5.5. $\Pi_L(L_{knot})$ and $\Pi_L(L_{std})$ are smoothly ambient isotopic.

Proof. Consider $L_{knot}$ given by the rotation symmetric front in Figure 9. We assume that the Reeb chord $c$ is above the south pole and that the front
cone is above the north pole. We endow $S^2$ with the round metric.

The goal is to produce a filling of $L_{knot}$ by an embedded $S^1$-family of
disks in $T^*S^2$ with corners at $c$. More precisely, we want a map

$$\varphi : S^1 \times D^2 \to T^*S^2,$$

such that

- $\varphi$ is a diffeomorphism on the complement of $S^1 \times \{1\}$.
- $\varphi^{-1}(c) = S^1 \times \{1\}$.
- $\varphi_{|S^1 \times \partial D^2}$ is a foliation of $\Pi_L(L_{knot})$ by embedded paths starting and
  ending at the double point.
- On a neighbourhood $U \supset S^1 \times \{1\}$, $\varphi_{|(\{\theta\} \times D^2) \cap U}$ maps into the plane
  given by $(s, t) \mapsto t\dot{\gamma}_{\theta}(s) \in T_{\gamma_{\theta}(s)}S^2 \simeq T^*_{\gamma_{\theta}(s)}S^2$ (identified using the
  round metric), where $\gamma_{\theta}$ is a geodesic on $S^2$ starting at the south
  pole with angle $\theta$.

The existence of such a filling by disks will prove the claim, since an
isotopy then may be taken as a contraction of $\Pi_L(L_{knot})$ within the disks to
a standard sphere contained in the neighbourhood of $S^1 \times \{1\}$. To that end, observe that such a neighbourhood contains a standard sphere intersecting
each plane $(s, t) \mapsto t\dot{\gamma}_{\theta}(s) \in T_{\gamma_{\theta}(s)}S^2$ in a figure eight curve, with the double
point coinciding with that of $L_{knot}$.

We begin by considering the embedded $S^1$-family of disks with boundary on $\Pi_L(L_{knot})$ and two corners at $c$, where each disk is contained in the annulus

$$\{t\dot{\gamma}_{\theta}(s) \in T_{\gamma_{\theta}(s)}S^2 \simeq T^*_{\gamma_{\theta}(s)}S^2; \quad t \in \mathbb{R}, \quad 0 \leq s < 2\pi\},$$

and where $\gamma_{\theta}$ is the geodesic described above. (In some complex structure,
these disks may be considered as pseudoholomorphic disks with two positive
punctures at $c$, or alternatively, gradient flow trees on $L_{knot}$ with two positive
punctures.)

Let $R_{\theta} : \mathbb{C}^2 \to \mathbb{C}^2$ be the complexified rotation of $\mathbb{R}^2$ by angle $\theta$. We can take a chart near the north pole of $T^*S^2$ which is symplectomorphic to a
neighbourhood of the origin in $\mathbb{C}^2 \simeq T^*\mathbb{R}^2$ such that:

- The image of $L_{knot}$ in the chart is invariant under $R_{\theta}$.
- The disk in the above family corresponding to the geodesic $\gamma_{\theta}$ is
  contained in the plane $R_{\theta}(z, 0)$. 


Figure 10. A neighbourhood of the north pole of $T^*S^2$ identified as a subset of $\mathbb{C}^2$ where $L_{\text{knot}}$ and $T$ are invariant under $R_\theta$. The curves $(R_\theta D) \cap T$ can be completed to a foliation of $T$ by closed curves as shown on the left.

The image of $L_{\text{knot}}$ in such a chart is shown in Figure [10]

The torus $T$ invariant under $R_\theta$ shown Figure [10] is symplectomorphic to a Clifford torus $S^1 \times S^1 \subset \mathbb{C}^2$. It can be parameterized by $f : \mathbb{R}/\pi \mathbb{Z} \times \mathbb{R}/\pi \mathbb{Z} \to \mathbb{C}^2$, $\varphi, \theta \mapsto f(\varphi, \theta) = \gamma(\varphi + \theta)(\cos(\varphi - \theta), \sin(\varphi - \theta))$, where $\gamma : \mathbb{R}/2\pi \mathbb{Z} \to \mathbb{C}$ is a parametrization of $T \setminus (\mathbb{C} \times \{0\})$ satisfying $\gamma(t + \pi) = -\gamma(t)$.

We now cut each disk in the above family along $T$, splitting the disk corresponding to the geodesic $\gamma_0$ into the pieces $R_\theta D$, $-R_\theta D$ and $R_\theta D'$, where $R_\theta \partial D' \subset T$.

Each curve $(R_\theta D) \cap T$ can be extended to a unique leaf in a foliation of $T$ by closed curves, as shown in Figure [10] This foliation of $T$ extends to a filling of $T$ by an embedded $S^1$-family of disks disjoint from $\{R_\theta D\}$.

To see this, one can argue as follows. The foliation of $T$ is isotopic to a foliation where all leaves are of the form $\theta \equiv C$. Using this isotopy, we may create a family of annuli with one boundary component being a leaf in the foliation of $T$, and the other boundary component being the curve $\epsilon e^{i(\varphi + C)}(\cos(\varphi - C), \sin(\varphi - C))$ for some $C$ and $\epsilon > 0$. The latter curve is a leaf of a foliation on the torus parameterized by $\epsilon e^{i(\varphi + \theta)}(\cos(\varphi - \theta), \sin(\varphi - \theta))$. The smaller torus, which is depicted by $T'$ in Figure [10] is clearly isotopic to $T$ by an isotopy preserving each $R_\theta D'$. Finally, each leaf in the foliation on $T'$ is bounded by a disk contained in the plane $\left(\frac{\epsilon}{2} e^{i2C}, -iz - \frac{\epsilon}{2i} e^{i2C}\right)$.

We will now compute the linearized Legendrian contact homology for the link $L_{\text{knot}} \cup F$ with coefficients in $\mathbb{Z}_2$. Observe that the DGA of each component is good since, as we shall see, the differentials vanish. Observe that even though $F$ is not compact, the Legendrian contact homology is still well defined under Legendrian isotopy of the component $L_{\text{knot}}$. 

□
Lemma 5.6. The differential vanishes on all generators of the DGA for the link $L_{knot} \cup F$, and hence

$$LCH_\bullet(L_{knot} \cup F; \mathbb{Z}_2) \simeq \mathbb{Z}_2 c \oplus \mathbb{Z}_2 a \oplus \mathbb{Z}_2 b,$$

where we have chosen the unique isomorphism class of its linearized homology.

Proof. Since both components have zero Maslov number, we may consider DGAs graded over $\mathbb{Z}$.

Formula (2.2) gives $|c| = 2$. We may assume that $\ell(c) < \min\{\ell(a), \ell(b)\}$. Thus, by the area formula (2.3), we immediately compute $\partial(c) = 0$ for the only pure Reeb chord. Hence there is an unique augmentation of the DGA for each component, namely the trivial one.

For the Reeb chords $a$ and $b$, we grade them as in the proof of Lemma 6.3, i.e. $|a| = 2, |b| = 1$. The same proof carries over to give $\partial a = \partial b = 0$. □

Remark 5.7. The above lemma shows that the subspace of $LCH_\bullet(L_{knot} \cup F; \mathbb{Z}_2)$ spanned by the mixed chords is isomorphic to $H_{\bullet+1}(S^1; \mathbb{Z}_2)$. This graded vector space is, in turn, isomorphic to the Morse homology of a generic function in the generating family considered above (with shifted degrees).

Corollary 5.8. Every isotopy class of $L_{knot}$ has a Reeb chord intersecting $F$.

Proof. If $L_{knot}$ and $F$ could be unlinked, i.e. if $L_{knot}$ could be isotoped so that the link carries no mixed Reeb chords, then we would have

$$LCH_\bullet(L_{knot} \cup F; \mathbb{Z}_2) \simeq LCH_\bullet(L_{knot}; \mathbb{Z}_2) \oplus LCH_\bullet(F; \mathbb{Z}_2) \simeq \mathbb{Z}_2 c,$$

where we again have chosen the unique (trivial) augmentations. This leads to a contradiction. □

Proof of Theorem 1.3. Suppose that $L_{knot}$ is Legendrian isotopic to $L_{std}$. It would then be possible to unlink $L_{knot}$ and $F$, contradicting the previous corollary.

We now show that $\Pi_L(L_{knot}) \subset D^* S^2 \subset S^2 \times S^2$, where $D^* S^2$ denotes the unit disk bundle, cannot be Hamiltonian isotoped to $\Pi_L(L_{std})$. We use the fact that the torus considered in [AF], and fibre-wise rescalings of it, are non-displaceable in $S^2 \times S^2$, as is shown in [FOOO]. After Hamiltonian isotopy, such a torus can be placed in an arbitrarily small neighbourhood of $\Pi_L(L_{knot})$.

A Hamiltonian isotopy of $S^2 \times S^2$ mapping $\Pi_L(L_{knot})$ to a small Darboux chart, would do the same with a non-displaceable torus sufficiently close to it, which leads to a contradiction. □

Consider the Legendrian embedding of a plane $F_2 \subset J^1S^2$ whose front projection is shown in Figure 11. By abuse of notation, we will sometimes use $F_2$ to denote its Lagrangian projection in $T^*S^2$, which is an embedding.

![Figure 11](image_url)

**Figure 11.** The front projection of a Legendrian lift of $F_2 \cup F$ drawn over $S^2$, together with a partial flow tree $D$. $F$ is the Legendrian lift of a generic perturbation of $T^*_qS^2$.

**Remark 6.1.** $F_2$ can be seen as the (Legendrian lift of the) image of a Lagrangian fibre $F_0 := T^*_pS^2$ under the square of a symplectic Dehn twist along the zero section (consider [Sei] for a treatment of the symplectic Dehn twist). The square of the Dehn twist can be described by the time-$2\pi$ map of the Hamiltonian flow induced by the Hamiltonian

$$H(q,p) = \varphi(\|p\|)\|p\|,$$

where we have used the round metric, and where $\varphi$ is a non-decreasing function satisfying $\varphi(t) = 0$ when $|t|$ is small and $\varphi(t) = 1$ when $t > 0$ is large. Outside of a compact set, this flow corresponds to the Reeb flow on the contact boundary $U^*S^2 = \partial(D^*S^2)$ extended to a flow on the symplectization $\mathbb{R} \times U^*S^2 \cong T^*S^2 \setminus 0$ independently of the $\mathbb{R}$-factor.

We now compute the linearized contact homology $LCH(F_{2k} \cup F; \mathbb{Z}_2)$, where $F_m$ denotes the image of $F_0$ under the time-$m\pi$ Hamiltonian flow induced by $H$ above, i.e. the image of $F_0$ after $2k$ symplectic Dehn twists along the zero section, and $F$ is the Legendrian lift of a Hamiltonian perturbation of $T^*_qS^2$ shown in Figure 11.

Since there are no pure Reeb chords, the DGA of the link is good, and since the Maslov class vanishes for both $F$ and $F_{2k}$, we may grade the DGA over $\mathbb{Z}$. Even though the Legendrian surfaces involved are non-compact, the Legendrian contact homology is well defined since $F_{2k}$ and $F$ are disjoint outside of a compact set, and it is invariant under compactly supported...
Legendrian isotopies of $F_{2k}$. Observe that since $H^1(\mathbb{R}^2; \mathbb{R}) = 0$, and since $F_{2k}$ is a plane, any compactly supported Lagrangian isotopy of $F_{2k}$ lifts to a compactly supported Legendrian isotopy.

The Legendrian link consisting of the lift of $F_{2k} \cup F$, where $F$ is translated far enough in the Reeb direction, has the mixed Reeb-chords $a_1, b_1, \ldots, a_k, b_k$, where the Reeb chords have been labelled such that

$$\ell(b_1) < \ell(a_1) < \ell(b_2) < \ell(a_2) < \ldots < \ell(b_k) < \ell(a_k).$$

For $F_2 \cup F$ in Figure 11 we have $b_1 = b$ and $a_1 = a$.

**Remark 6.2.** The linearized complex for the DGA of the link is the Floer complex $CF_* (F_{2k}, F; \mathbb{Z}_2)$.

**Lemma 6.3.** For a Legendrian lift of $F_{2k} \cup F$, where the lift of $F$ has been translated so that all Reeb chords start on $F_{2k}$ and end on $F$, the differential of the corresponding DGA vanishes. Consequently,

$$LCH_* (F_{2k} \cup F; \mathbb{Z}_2) \cong \bigoplus_{i=1}^k \mathbb{Z}_2 a_i \oplus \mathbb{Z}_2 b_i$$

where the grading is given by

$$|b_i| = 2i - 1, \quad |a_i| = 2i,$$

and where we have chosen the unique isomorphism class of its linearized homology.

**Proof.** We choose capping paths for each Reeb chord $c$ as follows: We fix a point $p \in F$ close to some Reeb chord endpoint. By $p' \in F_{2k}$ we denote the point on the highest sheet of $F_{2k}$ whose projection to $S^2 \subset J^1(S^2)$ coincides with that of $p$. A capping path for $c$ will be a path on $F$ starting on the endpoint of $c$ and ending at $p$, followed by a path on $F_{2k}$ starting on $p'$ and ending on the starting point of $c$. Using formula (2.2) one computes

$$|b_i| = 2(i - 1) + 2 - 1, \quad |a_i| = (2i - 1) + 2 - 1,$$

since the Reeb-chords all are of maximum type, and since one has to pass $2(i - 1)$ (respectively $2i - 1$) front cones in downward direction to go from $p'$ to $b_i$ (respectively from $p'$ to $a_i$). After perturbing each front cone as in Section 3 to making it front generic, we see that traversing a front cone in downward (upward) direction amounts to traversing one cusp edge in downward (upward) direction.

By comparing indices, we immediately get that

$$\partial_{b_i} b_i = m_{i-1} a_{i-1}, \quad \partial_{a_i} a_i = n_i b_i, \quad m_i, n_i \in \mathbb{Z}_2,$$

and that $\partial b_1 = 0$. We want to show that $m_i = n_i = 0$ for all $i$. We show the case $\partial a = 0$ for $F_2$ and note that the general case is analogous.

Consider the front projection of the Legendrian lift of the link shown in Figure 11. We will compute $\partial a$ by counting rigid flow trees. We are interested in flow trees having a positive puncture at $a$ and a negative puncture at $b$. 
Observe that since the puncture at \( b \) is negative with a maximum-type Reeb chord, it must be of type \((P_2)\). This vertex is 2-valent, with one of the edges connected to it being a flow line for the height difference of the lowest sheet of \( F_2 \) and a sheet of \( F \), while the other edge is living on \( F_2 \).

Suppose that the first edge does not originate directly from a positive \((P_1)\) puncture at \( a \). Thus, the edge has to end in an \((Y_0)\)-vertex. However, there can be no such vertex on this edge, since this would contradict the rigidity of the flow tree.

The other edge adjacent to the \((P_2)\) vertex is a flow line living on \( F_2 \) approaching the front cone. Hence, we are in the situation depicted by the partial front tree \( D \) in Figure 11. Applying Proposition 3.2, we conclude that this edge can be completed to a rigid flow tree with one positive puncture in exactly two ways. We have thus computed

\[ \partial a = 0. \]

\[ \square \]

**Proposition 6.4.** \( F_{2k} \) is smoothly isotopic to \( F_0 \) by an isotopy having compact support.

**Proof.** This follows from the fact that the square of a Dehn twist is smoothly isotopic to the identity by a compactly supported isotopy. \[ \square \]

**Proof of Theorem 1.4.** Suppose that \( F_{2k} \) and \( F_{2l} \) are Legendrian isotopic by an isotopy having compact support, where \( k, l \geq 0 \). After translating \( F \) far enough in the \( z \)-direction, we get that \( F_{2k} \cup F \) is Legendrian isotopic to \( F_{2l} \cup F \) by a compactly supported isotopy. Hence,

\[ LCH_\bullet(F_{2k} \cup F; \mathbb{Z}_2) \simeq LCH_\bullet(F_{2l} \cup F; \mathbb{Z}_2) \]

and we get that \( k = l \) by the previous lemma.

After applying a Dehn twist, we likewise conclude that if \( F_{2k+1} \) and \( F_{2l+1} \), where \( k, l \geq 0 \), are Legendrian isotopic by an isotopy having compact support then \( k = l \). Observe that \( F_k \) and \( F_l \) cannot be isotopic when \( k \not\equiv l \pmod{2} \) because of topological reasons.

Similarly, one may define \( F_k \) for \( k < 0 \) by applying \( k \) inverses of Dehn twists. After applying \( \text{min}(|k|, |l|) \) Dehn twists to \( F_k \) and \( F_l \), the above result gives that \( F_k \) and \( F_l \) with \( k, l \in \mathbb{Z} \) are Legendrian isotopic by a compactly supported isotopy if and only if \( k = l \).

The above proposition similarly gives that \( F_k \) and \( F_l \) are smoothly isotopic by an isotopy having compact support if and only if \( k \equiv l \pmod{2} \). \[ \square \]

The above computations are closely related to the result in [Sei], where the existence of the following exact triangle for Floer homology is proved:

\[ \begin{array}{ccc} HF(L_0, L_1) & \longrightarrow & HF(L_0, \tau L_1) \\ \downarrow & & \downarrow \\ HF(L_0, L) \otimes HF(L, L_1). \end{array} \]
Here $L_0, L_1$ are closed exact Lagrangians in a Liouville domain, $L$ is a Lagrangian sphere and $\tau$ is the Dehn twist along $L$ for some choice of an embedding of $L$. The map $\$\$ is given by the pair of pants coproduct composed with the isomorphism $HF(\tau_L L, \tau_L L_1) \simeq HF(L, L_1)$, while $\ltimes$ is given by the pair of pants product.

REFERENCES