Filtered historical simulation and option pricing

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ABSTRACT

My thesis describes how Filtered Historical Simulation (FHS) and Least-Square Monte Carlo Method (LSM) can be used in connection with pricing of American options.
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1 INTRODUCTION

1.1 TIME SERIES CONCEPTS

1.1.1 STATIONARITY PROPERTY

Let \( \varepsilon = \{ \varepsilon_t \}_{t=0,1,2,\ldots} \) be a stochastic process. The process \( \varepsilon \) is said to be weakly stationary (or wide-sense stationary or covariance stationary) if:

(i) \( E(\varepsilon_t^2) < \infty \), for any \( t \)

(ii) \( E(\varepsilon_t) \) is independent of \( t \)

(iii) for each natural number \( k \) the quantity \( \text{Cov}(\varepsilon_t, \varepsilon_{t+k}) \) is independent of \( t \).

Since in this thesis we consider only this type of stationarity, we will drop the terms “weak” or “weakly” when referring to weakly stationary sequences.

1.1.2 GJR-GARCH(1,1)-MODEL

The GJR-GARCH was introduced by Glosten, Jagannathan and Runkle in 1993. Let \( p, q \) be non-negative integers. We say that a stationary stochastic process \( \{ y_t \}_{t=0,1,2,\ldots} \) is of the type GJR-GARCH\((p,q)\) if:

\[
\begin{align*}
  y_t &= \mu + \varepsilon_t, \\
  \varepsilon_t &= \sigma_t Z_t, \\
  \sigma_t^2 &= \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 + \sum_{k=1}^{q} \gamma_k I_{t-k} \varepsilon_{t-k}^2 \\
  Z_t &\sim N(0,1) \\
  E(\varepsilon_t \mid F_t) &= 0 \\
  \text{Var}(\varepsilon_t \mid F_t) &= \sigma_t^2
\end{align*}
\]

Where \( I_{t-1} \) are dummy variables defined by the formula:

\[
\begin{cases}
  I_t = 1, & \text{if } \varepsilon_t < 0 \\
  I_t = 0, & \text{if } \varepsilon_t \geq 0
\end{cases}
\]
It is assumed that the random variables $Z_t$ are independent and identically distributed with probability distribution $f$, mean 0 and variance 1. It is often assumed that $f$ is the Gaussian distribution, but it is crucial for FHS that it does not have to be Gaussian. The symbol $F_t$ denotes all information available at time $t$. The constant coefficients satisfy the inequalities $\mu, \alpha, \beta, \gamma \geq 0$ and $\omega > 0$.

If $\beta_1 = \cdots = \beta_p = 0$ and $\gamma_1 = \cdots = \gamma_q = 0$, the model reduces to the classic ARCH (q) model proposed by Engle in 1982.

If $\gamma_1 = \cdots = \gamma_q = 0$, the model reduces to the classic GARCH (p,q) model introduced by Bollerslev in 1986.

Note that

$$E[y_t] = \mu + E[e_t] = \mu + E[E[e_t | F_t]] = \mu$$

We will be particularly interested in the GJR-GARCH (1,1) model. In this case we have:

$$y_t = \mu + \epsilon_t,$$
$$\epsilon_t = \sigma_t Z_t,$$
$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma I_{t-1} \epsilon_{t-1}^2$$
$$Z_t \sim f(0,1) are IID$$

With some constants $\mu, \omega, \alpha, \beta, \gamma$. Let $\theta > 0$ denote the probability that $\epsilon_t < 0$.

Obviously for the normal distribution $\theta = \frac{1}{2}$. It has been shown in [4] that a GJR-GARCH(1,1) process is stationary if

$$\alpha + \beta + \frac{1}{2} \gamma < 1$$

In this case

$$\text{Var}[y_t] = \text{Var}[\epsilon_t] = \frac{\omega}{1-(\alpha + \beta + \theta \gamma)}$$
1.2 Financial Concepts

1.2.1 Options

Options is a type of derivative securities. An option is an agreement that gives the buyer the right to buy from, or sell to, the seller of the option a certain amount of an underlying asset (such as stock) at a predetermined price before or on a given date. To every type of options, there are two kinds in each: call options and put options.

- **Call** options is a contract that gives the owner the right (not the obligation) to buy a specified underlying asset at a predetermined price before or on a given date (expiration date).

- While **put** options is a contract that gives the owner the right (not the obligation) to sell a specified underlying asset at a predetermined price before or on a given date (expiration date).

The predetermined price of the underlying asset is also called exercise price or striking price, usually signed as \( K \). The expiration date, also called maturity date, is usually signed as \( T \).

The current price of the underlying asset at time \( t < T \), can be represented by \( S_t \).

The value at time \( t \) of an option corresponding to one share of the underlying asset, is often denoted by \( C_t \) for call options and by \( P_t \) for put options. The options value at expiration date is \( C = [0, S - K]^+ \) for call and \( P = [0, K - S]^+ \) for put respectively.

Options can be divided into several styles, European options, American options Barrier options, Exotic options, and so on. In this thesis, we will focus on the European options and American options.

- **European options**

European options can be exercised only on the expiration date \( T \).

- **American options**

American options can be exercised before or on the expiration date \( T \). Usually, the value of American options is larger than that of the European options. Meanwhile, the valuation for American options is more complicated, for the fact that the exercise date of American options is more flexible than that of the European options.
1.2.2 Least-Square Method for Pricing

The key of Francis A. Longstaff and Eduardo S. Schwartz (2001) is that by using least squares method, our target is to estimate American options’ conditional expected payoff under the assumption that the options holder choose continuation.

As to the least square method, we aim to regress $y$ on $x$ by a fitting curve function $E(y | x) = f(x)$ which is the conditional expected payoff. The fitting curve function contains $k$ basis functions, which is based on $x$, and $k$ corresponding coefficients $a_j, j = 1, 2, \cdots, k$. Suppose there are $N$ observed data set $(x_i, y_i), \cdots, (x_N, y_N)$, then the difference between the observed $y_i, i = 1, \cdots, N$ and the conditional expected payoff for each data set is denoted by $\hat{u}$, which is

$$\hat{u} = [y_i - f(x_i)], i = 1, \cdots, N.$$  

To find the best fitting curve $f(x)$ as well as the corresponding coefficients $a_j, j = 1, 2, \cdots, k$, the essence of the least square method is to minimize the sum of squared $\hat{u}$, that is:

$$\left\{a_1, a_2, \cdots, a_k\right\} = \arg \min_{a_1, a_2, \cdots, a_k} \sum \hat{u}^2 \quad \text{(1)}$$

In the procedure of the valuation of American options, we will use Least-Square Monte Carlo method, which is to estimate the conditional expected payoff in order to determine the optimal exercise strategy. The least-square method will help to value the American options in every time point. More details will be given in chapter 3 [9].

1.2.3 Risk-Neutral Valuation

- **P-measure**

P-measure, also called objective probability measure, is based on the “real” financial market and the related historical data.

- **Q-measure**
Q-measure is a probability measure equivalent to P allowing the use of the risk-free rate $r$ in valuation. This type of valuation is called the risk-neutral valuation. The measure Q is also referred to as the martingale measure equivalent to P [5].

Under Q-measure, the risk-neutral stock price process under the stochastic differential equation is:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

And the value of a unit of the risk-free asset satisfies:

$$dB(t) = rB(t)dt$$

Then the risk-neutral valuation formula is:

$$F(t, s) = e^{-(T-t)}E_t^Q[\chi]$$

Where the contingent claim $\chi$ is of the form $\chi = \Phi(S(T))$. The notation $E_t^Q$ means the Q-expectation, and the subscripts $t, s$ means that the solution $S$ in the stock price process is taken as $S(t) = s$.

1.2.4 PUT-CALL PARITY RELATIONSHIP

The relationship between put options and call options can be expressed by put-call parity contracts.

For European options there is a fixed relationship between the price of put options and call options with the same maturity date in an arbitrage free economy. By letting $q$ be the continuous dividend yield on stock price, and $P_E$ and $C_E$ be the European options premiums, i.e. prices, the put-call parity for European options is:

$$C_E - P_E = S_t e^{-qr} - Ke^{-rt}$$

By denoting $C_A$ and $P_A$ be the American options premiums for call and put options, the put-call parity condition for American options on stocks without dividends is:

$$C_E - K \leq C_A - P_A \leq S_t - Ke^{-rt}$$
1.2.5 S&P 100 INDEX OEX OPTIONS

The empirical analysis of FHS is usually based on real data from the market, such as S&P 100 (OEX) or S&P 500 (OEX), instead of individual stock options. The reason is that individual stocks usually need to pay discrete dividends, therefore, in order to avoid some expectation problems, S&P 100 (OEX) or S&P 500 (OEX) becomes a better choice.

“The Standard and Poor’s 100® Index is capitalization-weighted and provides a measure of overall large company performance because it comprises 100 blue chip stocks [6].” “A blue-chip stock is stock in a company with a national reputation for quality, reliability and the ability to operate profitably in good times and bad [7].”

OEX means American style exercise options, while XEO means European style exercise options. The OEX and XEO are established markets traded only at the Chicago Board Options Exchange (CBOE), which is the largest options market in the world, as well as one of the largest securities exchanges in the United States.

1.2.6 LIQUIDITY PROBLEM

Generally speaking, liquidity means the ability of money converting from an asset or security to cash. Liquidity is also known as marketability.

In terms of the real options market data, OTM (out-of-money) options hold higher liquidity, i.e. daily volumes of out-of-money options are usually several times as large as the volumes of in-the-money options.

1.2.7 LEVERAGE EFFECTS

Normally, the influence from shocks of positive and negative variance models should impose the same impact on future volatility. However, future volatility has been shown to be more influenced by past negative return shocks. This phenomenon was defined by Black as “leverage effects” in 1996 [8].
2 Motivation of FHS

2.1 Literature Review

Black and Scholes in 1973 proposed the option pricing model by let the volatility to be constant [11]. Merton in 1976 first explored mixed jump diffusion models when pricing option. Johnson and Shanno in 1987 studied the option pricing in the case that the instantaneous variance of the asset price follows stochastic process [8]. At the same time, Hull and White in 1987 found that the Black-Scholes models have the problems of overpricing in at-the-money option, underpricing in deep-in-the-money option and in deep-out-of-the-money option under the existing of stochastic volatility [8]. Heston, who in 1993 extended the model of Hull and White in 1987, derives a closed-form solution for an European call pricing under the existing of stochastic volatility [8].

Because of the consensus that variance of asset returns are changing through time, in the recent 20 years, the researchers of option pricing incorporate more about time series models into option pricing, GARCH model is a very preferable choice to model the time-variant variances. Duan in 1995, Heston and Nandi in 2000 among others take the assumption of Gaussian innovation as well as historical and pricing (i.e. risk-neutral) return dynamics into consideration when deriving pricing models based on GARCH-type stochastic volatility. The shortcoming of their research is that the conditional volatilities of historical data and pricing distributions are governed by the same parameters [11]. Christoffersen, Heston and Jacobs in 2006 obtained the pricing model by incorporating leverage effect, time-variant volatility, skewness, etc. and combined inverse Gaussian innovation with asymmetric GARCH-model [8].

2.2 **Review for Barone-Adesi, Engle, Mancini (2008)**

“A GARCH Option Pricing Model with Filtered Historical Simulation”, by Barone-Adesi, Engle, Mancini (2008), proposed a new method of pricing options based on asymmetric GARCH models with filtered historical of innovations. The new method which they proposed for pricing options is a nonparametric method.

They allow for different distributions for historical and pricing distributions in an incomplete market framework. These different distributions have different shapes, skewness, kurtosis and other features. Then the new method enhances the flexibility to fit market option prices.

The empirical analysis in this literature is based on S&P500 index options. The result of their empirical analysis indicated the flexible change of measure, the asymmetric GARCH volatility dynamic, and they also got the result that the nonparametric innovation distribution, which contains different features, lead to the accurate pricing performance of their model.

Also, they obtained decreasing state-price densities per unit probability which validated their pricing model.

2.3 **Motivation of Filtered Historical Simulation (FHS)**

Simply speaking, filtered historical simulation (FHS) method is the procedure of sampling from the empirical innovation density to simulate the asset dynamics. The essence of FHS method is similar to that of the bootstrapping method. Bootstrapping method, introduced by Efron in 1979, it is a method used in cases where the value of the estimation does not give enough information [8].

Let $S_t, (t = 0, 1, 2, \cdots)$ be the equity index, then $r_t = \log \left(\frac{S_t}{S_{t-1}}\right)$ is the log return of equity index. To model $r_t$, Barone-Adesi, Engle, Mancini (2008) proposed a model which was based on GJR-GARCH(1,1) model. The FHS approach relies on sampling from the empirical density function $f$ of the scaled innovations $\{Z_t\}$ to simulate the future $\{Z_t\}$ as well as the asset dynamics. This is referred to as the historical simulation (FHS) method.
3 Least Square Monte Carlo (LSM)

3.1 Introduction of the LSM Approach

There are 3 major approaches to price American options: Lattice method, finite difference method and Monte-Carlo method.

Lattice method and finite difference method are efficient in computational step, but they are both difficult to be used in calculation when facing multiple assets or multiple state variables. When faced with multiple assets or multiple state variables, Monte-Carlo method is a good choice to calculate American options pricing because of its flexible and intuitive application in solving dimensional-problems.

For European options, holders will choose to exercise the option if it is in-the-money on the final exercise date. For American options, holders should compare the immediate exercise value with the expected cash flows if they continue to hold the options. Then by the result of comparison, holders will choose to exercise if immediate exercise is more profitable, or not to exercise if in other cases. Thus, we can know that the key to optimally exercising an American option is decided by the conditional expected value under the condition that option holders choose continuation. The LSM approach aims to use least squares to estimate the approximation of the conditional expectation function at every time periods by backwards approach.

3.2 Summary of LSM Method

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Assume there are $M$ periods before options expiration</td>
</tr>
<tr>
<td>2</td>
<td>Stock price paths matrix (mark out stock prices that are “in-the-money”)</td>
</tr>
<tr>
<td>3</td>
<td>Cash-flow matrix at time $T$</td>
</tr>
<tr>
<td>4</td>
<td>Regression of $X, Y$ at time $t_{M-1}$ matrix &amp; Optimal early exercise decision at time $t_{M-1}$.</td>
</tr>
<tr>
<td></td>
<td>Regression of $X, Y$ at time $t_{M-2}$ matrix &amp; Optimal early exercise decision at time $t_{M-2}$.</td>
</tr>
<tr>
<td></td>
<td>:</td>
</tr>
<tr>
<td></td>
<td>Regression of $X, Y$ at time $t_i$ matrix &amp; Optimal early exercise decision at time $t_i$</td>
</tr>
</tbody>
</table>
The LSM method to calculate American options price can be summarized into 5 steps. In the next section, a simple case will be studied and the pricing process will follow the 5 LSM method steps.

In the estimation process, we only use in-the-money paths since the exercise decision only happens on these paths where the option is in-the-money.

Different basis function can be used in the non-linear regression model. For example, one can use ordinary polynomials, (weighted) Laguerre polynomials, as well as Hermite, Legendre, Chebyshev, Gegenbauer, and Jacobi polynomials [8]. In the next section we use the (weighted) Laguerre polynomials to describe our regression procedure.

**Regression procedure Laguerre polynomials basis**

With the help of the (weighted) Laguerre polynomials, the conditional expectation can be represented as a linear function of the elements of the orthonormal basis. The basic functions under the (weighted) Laguerre polynomials are:

\[
L_0(X) = \exp(-X/2)
\]

\[
L_1(X) = \exp(-X/2)(1 - X)
\]

\[
L_2(X) = \exp(-X/2)(1 - 2X + X^2/2)
\]

\[
\vdots
\]

\[
L_n(X) = \exp(-X/2) \frac{e^x}{n!} \frac{d^n}{dX^n} (X^n e^{-X})
\]

Then the expected function based on the (weighted) Laguerre polynomials is:

\[
E[Y_{T=K} | X_{t=K-1}] = \sum_{j=0}^{\infty} a_j L_j(X)
\]

where \( a_j \) are constants coefficients and we need to calculate \( a_j \) by the in-the-money underlying asset price at time \( t_{k-1} \) and corresponding discounted cash flows received at time \( t_k \).

By Francis A.Longstaff and Eduardo S.Schwartz (2001), in the context of option pricing, there is no significant difference on the result when we choose more than three basis functions.
Thus with three Laguerre polynomials, it is sufficient to obtain the result of regression [9]. So for simplicity, we regress the expected function with \( L_0(X), \ L_1(X) \) and \( L_2(X) \), i.e.

\[
\begin{align*}
L_0(X) &= \exp(-X/2) \\
L_1(X) &= \exp(-X/2)(1 - X) \\
L_2(X) &= \exp(-X/2)(1 - 2X + X^2/2)
\end{align*}
\]

\[
E[Y_{T+K} \mid X_{T-K+1}] = \sum_{j=0}^{2} a_j L_j(X)
\]

(3)

In order to get the values of coefficients \( a_j \), we will use the least square method. Suppose there are \( N \) paths values for \( x \) at time \( t \), \( \{x_1, x_2, \ldots, x_N\} \), and also there are \( N \) paths values for \( y \) at time \( t + 1 \), \( \{y_1, y_2, \ldots, y_N\} \).

Under the least-square method, according to equation (1), we need to minimize the difference between the real and the predicted value in the model of equation 3.

Therefore the estimation of \( \{a_1, a_2, a_3\} \) can be expressed as:

\[
\{\hat{a}_1, \hat{a}_2, \hat{a}_3\} = \arg \min_{a_1, a_2, a_3} \sum \hat{u}^2
\]

where \( \hat{u} = \|y - A(x)a\| \), with

\[
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_N
\end{bmatrix} = \begin{bmatrix}
L_0(x_1) & L_1(x_1) & L_2(x_1) \\
L_0(x_2) & L_1(x_2) & L_2(x_2) \\
\vdots & \vdots & \vdots \\
L_0(x_N) & L_1(x_N) & L_2(x_N)
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\]

(4)
by trying different possible values for $\hat{a}$ again and again. When the value of $u^2$ is minimized, we can get the appropriate values for $\hat{a}$.

### 3.3 A Simple Example of LSM

In this section we will study a simple example which proposed by Francis A. Longstaff and Eduardo S. Schwartz (2001). For simplicity we will use here regression based on the polynomials $1, x, x^2$.

In the example, there exists an American put options, and assume the information (strike price, dividend, riskless rate) can be obtained from the financial market.

**Strike price** $K$ is 1.10.

**Dividend** $d$ is 0, i.e. this is an non-dividend paying stock of American put options.

**Riskless rate** $r$ is 6%, i.e. we can discount it back to time $t$ by $\exp(-0.06)$.

Now with the procedures in section 2.2, we can see how the algorithm of Least-Square method works.

1. **Assume there are $M$ periods before options expiration** $0 = t_0 < t_1 < t_2 < \cdots < t_M = T$

   In this example, $0 = t_0 < t_1 < t_2 < t_3 = T$, which means, our American put options is exercisable at a strike price of 1.10 at any of the 3 periods.

2. **Stock price paths matrix (mark out stock prices that are “in-the-money”)**

   Table 3-1 Stock price paths

<table>
<thead>
<tr>
<th>Path</th>
<th>t=0</th>
<th>t=1</th>
<th>t=2</th>
<th>t=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>1.09</td>
<td>1.08</td>
<td>1.34</td>
</tr>
<tr>
<td>2</td>
<td>1.00</td>
<td>1.16</td>
<td>1.26</td>
<td>1.54</td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>1.22</td>
<td>1.07</td>
<td>1.03</td>
</tr>
<tr>
<td>4</td>
<td>1.00</td>
<td>0.93</td>
<td>0.97</td>
<td>0.92</td>
</tr>
<tr>
<td>5</td>
<td>1.00</td>
<td>1.11</td>
<td>1.56</td>
<td>1.52</td>
</tr>
<tr>
<td>6</td>
<td>1.00</td>
<td>0.76</td>
<td>0.77</td>
<td>0.90</td>
</tr>
<tr>
<td>7</td>
<td>1.00</td>
<td>0.92</td>
<td>0.84</td>
<td>1.01</td>
</tr>
</tbody>
</table>
We know that if current stock price is lower than the strike price in put options, i.e. when $S_i < K = 1.10$, our American put options in this example is in-the-money. Table 3-1 summaries the options prices under each periods and each paths. The boldface and italic items marked in Table 3-1 are in-the-money options while the others are in out-of-money state.

3. Cash-flow matrix at time T (T=3)

We now focus on time $t = T = 3$, the last period. With 4 items in in-the-money state, we can calculate options value at time $t = T = 3$, as shown in Table 3-2, the cash-flow matrix. At time $T$, the pay-off is the same as for the corresponding European option.

Table 3-2 Cash-flow matrix at time t=T=3

<table>
<thead>
<tr>
<th>Path</th>
<th>t=1</th>
<th>t=2</th>
<th>t=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>---</td>
<td>---</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>---</td>
<td>---</td>
<td>0.00</td>
</tr>
<tr>
<td>3</td>
<td>---</td>
<td>---</td>
<td>1.10-1.03=0.07</td>
</tr>
<tr>
<td>4</td>
<td>---</td>
<td>---</td>
<td>1.10-0.92=0.18</td>
</tr>
<tr>
<td>5</td>
<td>---</td>
<td>---</td>
<td>0.00</td>
</tr>
<tr>
<td>6</td>
<td>---</td>
<td>---</td>
<td>1.10-0.90=0.20</td>
</tr>
<tr>
<td>7</td>
<td>---</td>
<td>---</td>
<td>1.10-1.01=0.09</td>
</tr>
<tr>
<td>8</td>
<td>---</td>
<td>---</td>
<td>0.00</td>
</tr>
</tbody>
</table>

4. Regression of X, Y at time $t_{M-1}$ matrix & Optimal early exercise decision at time $t_{M-1}$

Regression of X, Y at time $t_{M-2}$ matrix & Optimal early exercise decision at time $t_{M-2}$

...:

Regression of X, Y at time $t_i$ matrix & Optimal early exercise decision at time $t_i$
In this step, we will do the recursive process of matrix of regression of $X_{t_i}$ and $Y_{t=3}$ at time $t_i$ and the corresponding optimal early exercise decision at time $t_i$, for $i = 1, \cdots, M - 1$, i.e. $i = 1, 2$ in our case.

- For time $t=2$

For the American options, however, before the $M$th period, option holder should decide whether to exercise our American put options immediately, or continue to hold the options to the next period until the final date. Now for time $t=2$, we check if the option holder should exercise our put options immediately, or continue to hold the options to time $t=3$ period.

Let $X_{t=2}$ denote the stock prices at time $t=2$, and $Y_{t=3}$ denote the corresponding discounted cash flows that will receive at time $t=3$ (As shown in Table 3-2) if option holder choose to continue options’ life. As shown in Table 3-1, there are 5 paths that are in-the-money at time $t=2$. They are, the 1st, 3rd, 4th, 6th, and 7th.

Note that, 0.94176 is the discount factor given by riskless rate 6%, i.e. $\exp(-0.06)=0.94176$.

**Table 3-3 Data for regression at time $t=2$**

<table>
<thead>
<tr>
<th>Path</th>
<th>$Y_{t=3}$</th>
<th>$X_{t=2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>corresponding discounted cash flows received at time $t=3$</td>
<td>stock prices at time $t=2$</td>
</tr>
<tr>
<td>1</td>
<td>0.00*0.94176</td>
<td>1.08</td>
</tr>
<tr>
<td>2</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>3</td>
<td>0.07*0.94176</td>
<td>1.07</td>
</tr>
<tr>
<td>4</td>
<td>0.18*0.94176</td>
<td>0.97</td>
</tr>
<tr>
<td>5</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>6</td>
<td>0.20*0.94176</td>
<td>0.77</td>
</tr>
<tr>
<td>7</td>
<td>0.09*0.94176</td>
<td>0.84</td>
</tr>
<tr>
<td>8</td>
<td>---</td>
<td>---</td>
</tr>
</tbody>
</table>

With the regression method, we can estimate conditional expectation function by regressing $Y_{t=3}$ on a constant, $X_{t=2}$ and $X_{t=2}^2$ for the five paths. The regression function for the conditional expectation is $E[Y_{t=3} | X_{t=2}] = -1.070 + 2.983X_{t=2} - 1.813X_{t=2}^2$. And the result of conditional expectation for the five paths is shown in Table 3-4. We can see from Table
3-4 that, by comparing continuing the put options’ life until \( t=3 \) with immediately exercising value at \( t=2 \), option holder should optimal choose to exercise the options at \( t=2 \) at the 4\(^{th}\), 6\(^{th}\) and 7\(^{th}\) paths.

Table 3-4 Optimal early exercise decision at time \( t=2 \)

| Path | Exercise \((1.10 - X_{t=2})^+\) | Continuation \(E[Y_{t=3} | X_{t=2}] = -1.070 + 2.983X_{t=2} - 1.813X_{t=2}^2\) |
|------|-------------------------------|---------------------------------|
| 1    | 0.02                          | 0.0369                          |
| 2    | ---                           | ---                             |
| 3    | 0.03                          | 0.461                           |
| 4    | 0.13                          | 0.1176                          |
| 5    | ---                           | ---                             |
| 6    | 0.33                          | 0.1520                          |
| 7    | 0.26                          | 0.1565                          |
| 8    | ---                           | ---                             |

Therefore, we get the cash-flows matrix at \( t=2 \) as in Table 3-5

Table 3-5 Cash-flows matrix at time \( t=2 \)

<table>
<thead>
<tr>
<th>Path</th>
<th>( t=1 )</th>
<th>( t=2 )</th>
<th>( t=3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>---</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>---</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>3</td>
<td>---</td>
<td>0.00</td>
<td>( 0.07 )</td>
</tr>
<tr>
<td>4</td>
<td>---</td>
<td>( 0.13 )</td>
<td>0.00</td>
</tr>
<tr>
<td>5</td>
<td>---</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>6</td>
<td>---</td>
<td>( 0.33 )</td>
<td>0.00</td>
</tr>
<tr>
<td>7</td>
<td>---</td>
<td>( 0.26 )</td>
<td>0.00</td>
</tr>
<tr>
<td>8</td>
<td>---</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

- For time \( t=1 \)

There are 5 paths are in in-the-money state, which are the 1\(^{st}\), 4\(^{th}\), 6\(^{th}\), 7\(^{th}\) and 8\(^{th}\) paths as marked in Table 3-1. To calculate \( Y_{t=2} \) we discount to time \( t = 1 \) the cash flows payable at time \( t = 2 \).

Table 3-6 Data for regression at time \( t=1 \)

<table>
<thead>
<tr>
<th>Path</th>
<th>( Y_{t=2} )</th>
<th>( X_{t=1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>corresponding discounted cash flows</td>
<td>stock prices at time ( t=1 )</td>
</tr>
</tbody>
</table>
FHS and Option Pricing

received at time t=2

| Path | Exercise (if option is under in-the-money state) | Compare | Continuation $E[Y_{t=3} | X_{t=1}] = (1.10 - X_{t=1})^+ 2.038 - 3.335X_{t=1} + 1.356X_{t=1}^2$ |
|------|--------------------------------------------------|---------|-----------------------------------------------|
| 1    | 0.01                                             | <       | 0.0139                                        |
| 2    | ---                                              | ---     | ---                                           |
| 3    | ---                                              | ---     | ---                                           |
| 4    | **0.17**                                         | >       | **0.1092**                                     |
| 5    | ---                                              | ---     | ---                                           |
| 6    | **0.34**                                         | >       | **0.2866**                                     |
| 7    | **0.18**                                         | >       | **0.1175**                                     |
| 8    | **0.22**                                         | >       | **0.1533**                                     |

Similar to time t=2, we estimate conditional expectation function by regressing $Y_{t=3}$ on a constant, $X_{t=1}$ and $X_{t=1}^2$ for the five paths. The regression function for the conditional expectation is $E[Y_{t=3} | X_{t=1}] = 2.038 - 3.335X_{t=1} + 1.356X_{t=1}^2$. And the result of conditional expectation for the five paths is shown in Table 3-7. By comparing continuing the put options’ life until later time with the immediate exercise value at time t=1, option holder will optimal choose to exercise the options at time t=1 at the 4th, 6th, 7th and 8th paths.

Table 3-7 Optimal early exercise decision at time t=1

5. Then we can get the matrix of stopping rule as the outcome

All in all, we will get the conclusion from above analysis that the stopping rule of our American put options in this example is shown in Table 3-8. The items “1” in the matrix means option holder will choose to exercise immediately while items “0” means not to exercise and to continue the put options until expiration. The corresponding cash flow matrix in each period and each path is shown in Table 3-9.
The above tables reflect the cash flows generated by this American put option along each of the considered paths. By suitably discounting these cash flows and taking the average (over the number of all paths) we arrive at the price of this option at time $t = 0$:

$$\frac{(0.17 + 0.34 + 0.18 + 0.22) \times 0.94176 + 0.07 \times 0.94176^3}{8} = 0.11443371$$

In real life applications one would like to simulate many thousands of paths to make the result more realistic. The simulation of the paths is done with the help of Monte-Carlo methods.
4 Mathematically FHS

4.1 Asset Price Dynamics

4.1.1 Under P-measure

Under the objective probability measure $P$, the GJR-GARCH(1,1) model for log-returns

$$r_i = \log\left(\frac{S_i}{S_{i-1}}\right)$$

of the equity index $S_i$, is described by the following equations:

$$
\begin{align*}
    r_i &= \mu + \epsilon_i, \\
    \epsilon_i &= \sigma_i Z_i, \\
    Z_i &\sim f(0,1) \text{ are IID} \\
    \sigma_i^2 &= \omega + \alpha \epsilon_{i-1}^2 + \beta \sigma_{i-1}^2 + \gamma I_{i-1} \epsilon_{i-1}^2
\end{align*}
$$

where $f$ denotes the yet unknown probability distribution to be derived from the empirical data. We will use historical data from time $t-n+1$ to time $t$ (where $n > 0$ is an integer) and QMLE to calibrate the model. By calculating the scaled innovations $Z_i$, we will be able to determine the empirical probability distribution $f(0,1)$.

The GJR-GARCH (1, 1) model under P-measure is based on the historical data. With the historical return data, under P-measure, we aim to estimate $\mu$ and coefficient $\theta = \{\omega, \beta, \alpha, \gamma\}$ by QMLE method, and we also aim to obtain a series scaled return innovation $\{Z_i\}$ in order to do the simulation as well as the forecasting and prediction. The specific procedure will be presented in the following section.
4.1.2  **UNDER Q-MEASURE**

To calibrate the model under the Q-measure we will use the empirical probability distribution $f(0,1)$ established in the previous step. The GJR-GARCH(1,1) model under Q-measure is:

$$
\begin{align*}
    r_i &= \mu^* + \varepsilon_i, \\
    \varepsilon_i &= \sigma_i Z_i, \\
    Z_i &\sim f(0,1) \text{ are IID}, \\
    \sigma_i^2 &= \omega^* + \alpha^* \varepsilon_{i-1}^2 + \beta^* \sigma_{i-1}^2 + \gamma^* I_{t-i} \varepsilon_{t-i}^2
\end{align*}
$$

We are going to use this model to simulate the future values of the equity index $S_t$ (where $i \in \{t+1, t+2, t+\tau\}$ ). Since the value of $r_i$ for this period are unknown yet, we will use option prices known at time $t$, with expiry date $t+\tau$, to calibrate the model with respect to the Q measure.

Here the notion $\theta^* = \{\mu^*, \omega^*, \beta^*, \alpha^*, \gamma^*\}$ is a new set of pricing GARCH parameters obtained from the market prices. We will discuss it in the next section.

4.2  **FHS ALGORITHM**

4.2.1  **SUMMARY OF FHS ALGORITHM**

There are mainly 6 steps in the method of FHS. They are:

1. Calibrate GJR-GARCH(1,1) model in order to get $\hat{\theta} = \{\hat{\mu}, \hat{\omega}, \hat{\beta}, \hat{\alpha}, \hat{\gamma}\}$ by using historical returns $r_i$ with QMLE method.

2. Calculate the empirical innovations $\hat{Z}$ by using $\hat{\theta}$ and the historical returns.

3. Generate $N$ trajectories of the underlying price process by using a hypothetical $\theta^*$ and a series of empirical innovations which are chosen at random without replacement from $\hat{Z}$. 
4. Get hypothetical options prices for both European options and American options implied by the simulate trajectories which we have got in step 3.

5. Compare the hypothetical option prices with actual prices obtained from market data.

6. Calibrate the model under Q by choosing the best $\theta^*$.

The details of the 6 steps will be presented in the next section.

4.2.2 FHS ALGORITHM IN DETAIL

1. Calibrate GJR-GARCH(1,1) model in order to get $\hat{\theta} = \{\hat{\mu}, \hat{\omega}, \hat{\beta}, \hat{\alpha}, \hat{\gamma}\}$ by using historical returns $r_t$ with QMLE method. (Under P-measure)

The aim in the first step is to use $n$ historical return data at time $t$ to estimate the mean of returns $\mu$ and parameters $\theta = \{\omega, \beta, \alpha, \gamma\}$ of the GJR-GARCH (1,1) model under P measure (objective probability) by QMLE (Quasi-Maximum Likelihood Estimator) method. This will generate a series of historical innovations $Z_i$.

Suppose there are $n$ historical log-returns of the underlying asset at time $t$. And $n$ observed historical return data at time $t$ are:

$$\begin{align*}
\{r_{1-n+t} = \log \frac{S_{1-n+t}}{S_{0-n+t}}, r_{2-n+t} = \log \frac{S_{2-n+t}}{S_{1-n+t}}, \ldots, r_t = \log \frac{S_t}{S_{t-1}} \}\n\Rightarrow \{r_i\}_{i=1-n+t, 2-n+t, \ldots, t} = r_{1-n+t}, r_{2-n+t}, \ldots, r_t
\end{align*}$$

GJR-GARCH(1,1) from equation (1) leads to the following equations

$$\begin{align*}
\varepsilon_i &= r_i - \mu \\
Z_i &= \frac{\varepsilon_i}{\sigma_i} , \text{ where} \\
\sigma_i^2 &= \omega + \beta \sigma_{i-1}^2 + (\alpha + \gamma I_{i-1}) \varepsilon_{i-1}^2
\end{align*}$$

As to the estimation of parameters $\theta = \{\mu, \omega, \beta, \alpha, \gamma\}$, we use QMLE (Quasi-Maximum Likelihood Estimator) method. And the log quasi-likelihood function based on $\varepsilon_i$ is given as follows [10]:
\[
L(\theta) = \frac{1}{n-1} \sum_{i=1}^{n-t} l_i(\theta),
\]
where
\[
l_i(\theta) = -\frac{1}{2} \left( \log \sigma_i^2 + \frac{\varepsilon_i^2}{\sigma_i^2} \right)
\]

We can get the estimation of \( \theta = \{\mu, \omega, \beta, \alpha, \gamma\} \) by letting:
\[
\theta = \arg \max_{\theta} L(\theta)
\]

We will be assuming that \( \alpha + \beta + \frac{1}{2} \gamma < 1 \) to guarantee stationarity conditions.

Given a potential \( \theta = \{\mu, \omega, \beta, \alpha, \gamma\} \) we calculate the corresponding residuals \( \varepsilon_i \) by the following algorithm:

**ASSUMED INITIAL VALUES FOR** \( i = 1-n+t \):
\[
\varepsilon_{1-n+t} = 0, \quad \sigma_{1-n+t}^2 = \frac{\omega}{1-\alpha - \beta - \frac{1}{2} \gamma}
\]

**STEP UP FOR** \( i = 2-n+t, \cdots, t \):
\[
\text{eq}(4) \quad r_i \rightarrow \varepsilon_i
\]
\[
\varepsilon_{i-1}, \sigma_{i-1}^2 \xrightarrow{\text{eq}(5)} \varepsilon_i^2, \sigma_i^2
\]

We use the log quasi-likelihood function which is described in equation (6) to find an optimal \( \theta = \{\mu, \omega, \beta, \alpha, \gamma\} \), which we will denote by \( \hat{\theta} = \left\{ \hat{\mu}, \hat{\omega}, \hat{\beta}, \hat{\alpha}, \hat{\gamma} \right\} \). Then we use \( \hat{\theta} \) to calculate the empirical innovations.

\[
\hat{\theta} = \left\{ \hat{\mu}, \hat{\omega}, \hat{\beta}, \hat{\alpha}, \hat{\gamma} \right\}
\]

2. **Calculate the empirical innovations** \( \hat{Z} \) by using \( \hat{\theta} \) and the historical returns

The outcome in step 1 is \( \hat{\theta} = \left\{ \hat{\mu}, \hat{\omega}, \hat{\beta}, \hat{\alpha}, \hat{\gamma} \right\} \), we will use it to calculate the empirical
innovations $\hat{Z}$.

**Algorithm:**

ASSUMED INITIAL VALUES FOR $i = 1 - n + t$:

$$\hat{\varepsilon}_{1-n+t} = 0, \quad \sigma_{1-n+t} = \frac{\omega}{1 - \alpha - \beta - \frac{1}{2} \gamma}$$

INITIAL RESIDUAL FOR $i = 1 - n + t$:

$$\hat{Z}_{1-n+t} = \frac{\hat{\varepsilon}_{1-n+t}}{\sigma_{1-n+t}}$$

STEP UP FOR $i = 2 - n + t, \ldots, t$:

$$\hat{r}_i \to \hat{\varepsilon}_i$$

$$\hat{\varepsilon}_{i-1}, \sigma_{i-1} \to \sigma_i$$

$$\hat{Z}_i = \frac{\hat{\varepsilon}_i}{\sigma_i}$$

RESULT:

$$\left\{\hat{Z}_{1-n+t}, \hat{Z}_{2-n+t}, \ldots, \hat{Z}_i\right\} \quad (7)$$

This series $\left\{\hat{Z}_{1-n+t}, \hat{Z}_{2-n+t}, \ldots, \hat{Z}_i\right\}$ are the empirical scaled innovations which we want to use to perform simulation in the next step.

3. Generate $N$ trajectories of the underlying price process by using a hypothetical $\theta^*$ and a series of empirical innovations which are chosen at random without replacement from $\hat{Z}$. 

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In this step we will generate \( N \) trajectories \( r_i \), where \( i = t + 1, \cdots , t + \tau \), for time \( t + \tau \). Equation (2) is poor to be used to price options directly, because it is the asset return model which is specified under the historical measure \( P \). So we need to change the measure from \( P \) to \( Q \) by calibrating a new set of pricing GARCH parameter \( \theta^* \). The notation ‘#’ below indicates simulated values.

There are two ways to choose the initial volatility and the hypothetical pricing GARCH parameter \( \theta^* \).

- One way is proposed by Barone-Adesi, Engle, and Mancini (2008). In this way, the initial volatility for the generation process is chosen under the \( P \)-measure. That is, at time \( j = t + 1, \sigma_{j-1}^2 = \sigma_j^2 = \sigma_p^2 \) which we have got in step 2. And the hypothetical pricing GARCH parameters are given by \( \theta^* = \{\mu^*, \omega^*, \beta^*, \alpha^*, \gamma^* \} \).

- The other method is proposed by Chueh-Yung Tsao and Wei-Yu Hung (2009), they modified and extended the FHS method of Barone-Adesi, Engle, and Mancini (2008). In their paper, they proposed a modified model, in which the initial value of the conditional volatility \( \sigma_i \) is regarded as unknown, denoted by \( \sigma_Q \) and should be determined from market data together with the coefficient of the GJR-GARCH model under the \( Q \)-measure.

Then the initial value for volatility is denoted by \( \sigma^* = \sigma_Q \), which will be a component of \( \theta^* \) and has to be estimated through optimization, similarly to other components. Therefore the hypothetical \( \theta^* \) is given by \( \theta^* = \{\sigma_Q, \mu^*, \omega^*, \beta^*, \alpha^*, \gamma^* \} \).

Below is the detail and algorithm in this step.

We can simulate \( \sigma_j^* \) and \( r_j^* \) by the following equation for time \( t + 1 \) to \( t + \tau \):

\[
\begin{align*}
\sigma_j^2 &= \omega^* + \beta^* \sigma_{j-1}^2 + (\alpha^* + \gamma^* I_{j-1}) e_{j-1}^2 \quad \text{(8)} \\
e_{j-1}^2 &= \sigma_{i+1}^2 Z_i^2 \\
\text{i.e. } e_j^* &= \sigma^2 Z_j^* \quad \text{(9)} \\
r_j^* &= \log \left( \frac{S_j}{S_{j-1}} \right) = \mu^* + e_j^* \quad \text{(10)}
\end{align*}
\]
Where \( j = t + 1, t + 2, \ldots, t + \tau \), which means we need to simulate the series \( \sigma_j^\# \), for \( j = t + 1, \cdots, t + \tau \) by using the GARCH pricing model and empirical innovation. That is, by choosing at random (with replacement) a series of \( Z_{t+1}^\#, Z_{t+2}^\#, \ldots, Z_{t+\tau}^\# \) from the series \( \left\{ \hat{Z}_{1-\tau}, \hat{Z}_{2-\tau}, \ldots, \hat{Z}_{t} \right\} \) which have been obtained in step 2.

**Algorithm:**

**HYPOTHETICAL PARAMETERS:**

<table>
<thead>
<tr>
<th>[Barone-Adesi et al.]</th>
<th>( \theta^* = { \mu^<em>, \omega^</em>, \beta^<em>, \alpha^</em>, \gamma^* } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Tsao &amp; Hung]</td>
<td>( \theta^* = { \sigma_0^<em>, \mu^</em>, \omega^<em>, \beta^</em>, \alpha^<em>, \gamma^</em> } )</td>
</tr>
</tbody>
</table>

**INITIAL VALUES:**

<table>
<thead>
<tr>
<th>[Barone-Adesi et al.]</th>
<th>( \epsilon_t^# = \hat{\epsilon}_t, \quad \sigma_t^# = \sigma_p = \hat{\sigma}_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Tsao &amp; Hung]</td>
<td>( \epsilon_t^# = \hat{\epsilon}_t, \quad \sigma_t^# = \sigma_u )</td>
</tr>
</tbody>
</table>

**SELECTION:**

Based on \( \left\{ \hat{Z}_{1-\tau}, \hat{Z}_{2-\tau}, \ldots, \hat{Z}_{t} \right\} \) we have got in step 2, we generate a series of innovations for time \( j = t + 1, \cdots, t + \tau \):

\[
\left\{ Z_{t+1}^\#, Z_{t+2}^\#, \ldots, Z_{t+\tau}^\# \right\}
\]

**STEP UP FOR \( j = t + 1, \cdots, t + \tau \)**

\[
\begin{align*}
\sigma_{j-1}^\# \epsilon_{j-1}^\# & \xrightarrow{eq.(8)} \sigma_j^\# \\
\sigma_j^\#, Z_j^\# & \xrightarrow{eq.(9)} \epsilon_j^\# \\
\mu^*, \epsilon_j^\# & \xrightarrow{eq.(10)} r_j^\#
\end{align*}
\]

**RESULT:**
\[
\{r^\#, r^\#_{t+1}, \ldots, r^\#_{t+\tau}\}
\]
\[
\Leftrightarrow
\{r^\#_j\}, \quad j = t+1, \ldots, t+\tau
\]

With the series \( \{r^\#_{t+1}, r^\#_{t+2}, \ldots, r^\#_{t+\tau}\} \), we can calculate the simulated value of the equity index \( S_i \) for any \( i = t+1, \ldots, t+\tau \) by the following formula:

\[
S_i / S_t = \exp\left[ i\mu^* + \sum_{j=t+1}^{t+\tau} \sigma^\# Z_j \right]
\]

And if the number of simulation is \( N \), for each period, we get \( N \) simulated returns:

\[
\{r^\#_1\}, \quad \{r^\#_2\}, \ldots, \{r^\#_N\} \quad (11)
\]

The \( N \) simulated returns will be used in the next step for calculating the hypothetical options price.

\section*{4. Get hypothetical options prices for both European options and American options by the simulate trajectories which we got in step 3.}

By using the \( N \) simulated returns, we can calculate the hypothetical price through direct risk-neutral valuation for European options or through the Least-Square Method for American options.

\begin{itemize}
\item \textbf{European options hypothetical price}
\end{itemize}

For European options, we can calculate the hypothetical price by the following equation, with the help of the \( N \) simulated returns \( \{r^\#_1\}, \ldots, \{r^\#_N\} \) in equation (11):

European put options hypothetical price:

\[
P_E = \frac{1}{N} \left[ \sum_{j=1}^{N} e^{-rT_j} \max\left(X - S^\#_{j,T}, 0\right) \right]
\]

European call options hypothetical price:

\[
C_E = \frac{1}{N} \left[ \sum_{j=1}^{N} e^{-rT_j} \max\left(S^\#_{j,T} - X, 0\right) \right]
\]
**American options hypothetical price**

In chapter 2 the LSM approach has been studied. We can get the matrix of stopping rule by LSM method and the matrix can tell us the best exercise point \( T_j^* \) along the \( n \)th path, with the assumption that the total numbers of the simulated paths is \( N \).

American put options hypothetical price:

\[
P_d = \frac{1}{N} \left[ \sum_{j=1}^{N} e^{-r f (T_j^*-t)} \max(X - S_{j,x_j^*}, 0) \right]
\]

American call options hypothetical price:

\[
C_d = \frac{1}{N} \left[ \sum_{j=1}^{N} e^{-r f (T_j^*-t)} \max(S_{j,x_j^*}^a - X, 0) \right]
\]

where \( r f \) is the corresponding risk-free rate at the best exercise point \( T_j^* \) for the \( n \)th path, and \( S_{j,x_j^*}^a \) is the corresponding asset price at the best exercise point \( T_j^* \) for the \( n \)th path.

**5. Comparison between hypothetical option prices and actual prices obtained from market data.**

In this step we compare the hypothetical option prices and actual prices obtained from market data. The above simulation procedure is applied to a \( N_j \) options written on the considered equity index, with strike prices \( K_j \) and expiry dates \( T_j \). Let \( e(K_j, T_j) \) denote the square of the difference between the market price and the simulated price of such an option. The difference in pricing is calculated as the mean square error

\[
MSE = \frac{1}{N} \sum_{j=1}^{N} e(K_j, T_j)
\]

**6. Calibration is achieved by choosing an optimal \( \theta^* \).**
By varying GARCH parameters $\theta^*$, and repeating step 3 and step 4 again and again, we aim to find a best $\theta^* = \{\omega^*, \beta^*, \alpha^*, \gamma^*\}$ that will lead the MSE to a smallest level or below a threshold.

Once the model is calibrated under the Q-measure, it can be used to price options different from those used in the calibration procedure. Of course these options have to have the same underlying asset. The valuation process is similar to calculations described above in Steps 3 and 4, but with the optimal $\theta^*$.
REFERENCE

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