A weak boundary procedure for high order finite difference approximations of hyperbolic problems

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Abstract

We introduce a new weak boundary procedures for high order finite difference operators on summation-by-parts type applied to hyperbolic problems. The boundary procedure is applied in an extended domain where data is known. We show how to raise the order of accuracy for a diagonal norm based approximation and how to modify the spectrum of the resulting operator to get a faster convergence to steady-state. Furthermore, we also show how to construct better non-reflecting properties at the boundaries using the above procedure. Numerical results that corroborate the analysis are presented.

Keywords: Summation-by-parts, weak boundary conditions, penalty technique, high-order scheme, steady-state, non-reflecting boundary conditions.

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1. Introduction

High order finite difference methods provide an efficient approach for problems in computational science. The efficiency can be used either to increase the accuracy for a fixed number of mesh points or to reduce the computational cost for a given accuracy by reducing the number of mesh points [1]. The main drawback with high order finite difference methods is the complicated boundary treatment, required to get a stable method.

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Finite difference operators which satisfy summation-by-parts (SBP) property [2, 3, 4], are central difference operators in the interior domain augmented with special stencils near the domain boundaries. These SBP operators in combination with weak well-posed boundary conditions lead to energy stability [5, 6, 7, 8, 9]. One such boundary treatment is the simultaneous approximation term (SAT) method [10], which linearly combines the partial differential equation to be solved with well-posed boundary conditions [1, 5, 11, 12].

In this paper we will extend this technique by applying the boundary conditions in an extended domain. As an introduction, consider the continuous one-dimensional right going ($a > 0$) advection problem

$$u_t + au_x = 0, \quad 0 \leq x \leq 1, \quad t > 0, \tag{1}$$

with a boundary condition $u(0,t) = g_0(t)$ at $x = 0$ for well-posedness. The energy method applied to (1) yields the following continuous energy rate

$$\frac{d}{dt}||u||^2 = au(0,t)^2 - au(1,t)^2, \tag{2}$$

where $||u||^2 = \int_0^1 u^2 dx$. By letting $u(0,t) = g_0(t)$, well-posedness follows.

Let the approximative solution at grid point x_i be denoted u_i, and the discrete solution vector $\mathbf{u}^T = [u_0, u_1, \ldots, u_N]$. A finite difference approximation of (1) using an SBP operator with SAT treatment for the boundary conditions can be written as

$$\mathbf{u}_t + aP^{-1}Q\mathbf{u} = P^{-1}\alpha_{00}\,(u_0 - g_0)\,e_0, \tag{3}$$

where $P^{-1}Q$ approximates d/dx, P is diagonal and positive definite, $Q + Q^T = B = diag\,(-1, 0, ..., 0, 1)$, α_{00} is called the penalty coefficient and $e_0 = [1, 0, \ldots, 0]^T$. The discrete energy method on (3) gives

$$\frac{d}{dt}||\mathbf{u}||_P^2 = (a + 2\alpha_{00})\,u_0^2 - 2\alpha_{00}u_0 g_0 - au_N^2, \tag{4}$$

where $||\mathbf{u}||_P^2 = \mathbf{u}^T P\mathbf{u}$. For $\alpha_{00} \leq -(a/2)$, we have a bounded energy. Without boundary conditions ($\alpha_{00} = 0$), the rate (4) mimics (2) perfectly. For more details using this technique, see [4, 5, 9, 10, 13].

The SAT technique (weak imposition of boundary condition or penalty technique) is normally applied only at one grid point (as in the example above) [9, 13, 14, 15, 16, 17]. However, it is quite possible to impose the weak boundary conditions at multiple grid points. This has many advantages, and gives the designer of the scheme lots of flexibility and many options.

However, the flexibility comes at a price. For accuracy reasons, the solution must be *known* at those grid points. Examples where one knows the boundary data in an extended domain include external fluid dynamics and various forms of wave propagation problems close to far-field boundaries. In many cases, this drawback can be circumvented and boundary data in the extended domain can be derived using Taylor's series expansions and given data at the boundary.

The boundary procedure that we present in this paper has clear connections and many similarities to methods refereed to as fringe region, sponge layers and buffer layers [18, 19, 20, 21] but is more general.

To illustrate the procedure, we add on additional penalty terms in (3) at two new grid points close to the boundary $x = 0$ as

$$\begin{aligned}\mathbf{u}_t + aP^{-1}Q\mathbf{u} = P^{-1}\{&\alpha_{00}\,(u_0 - g_0)\,e_0 + \alpha_{01}\,(u_1 - g_1)\,e_0 \\ &+ \alpha_{10}\,(u_0 - g_0)\,e_1 + \alpha_{11}\,(u_1 - g_1)\,e_1 + \alpha_{12}\,(u_2 - g_2)\,e_1 \\ &+ \alpha_{21}\,(u_1 - g_1)\,e_2 + \alpha_{22}\,(u_2 - g_2)\,e_2\},\end{aligned} \tag{5}$$

where $e_1 = [0, 1, \ldots, 0]^T$ and $e_2 = [0, 0, 1, \ldots, 0]^T$. α_{ij}, $i, j = 0, 1, 2$ are the penalty coefficients and g_i, $i = 0, 1, 2$ are the boundary data. Equation (5) is as accurate as the original scheme (3) as long as $g_1\,(\Delta x, t)$ and $g_2\,(2\Delta x, t)$ are known. Let $g_i = 0$ and use the energy method. The result corresponding to (4) with $g_0 = 0$ becomes

$$\frac{d}{dt}||\mathbf{u}||_P^2 = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}^T \underbrace{\begin{pmatrix} a + 2\alpha_{00} & \alpha_{01} + \alpha_{10} & 0 \\ \alpha_{01} + \alpha_{10} & 2\alpha_{11} & \alpha_{12} + \alpha_{21} \\ 0 & \alpha_{12} + \alpha_{21} & 2\alpha_{22} \end{pmatrix}}_{R} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} - au_N^2. \tag{6}$$

For stability of (6), we need to choose α_{ij}, such that the matrix R is negative semi-definite. (The most obvious choice would be $R_{ii} \leq 0$ and $R_{ij} = 0$ for $i \neq j$.)

Once the stability requirements are fulfilled, a number of free parameters α_{ij} still exist. These can be used to

(i) increase the accuracy of the scheme,

(ii) increase the speed of convergence to steady-state,

(iii) design non-reflecting boundary conditions,

(iv) any combination of (i)-(iii).

The rest of the paper proceeds as follows. In section 2 we present a systematic procedure on how to generalize the multiple penalty technique to systems. Numerical experiments which show the treatment of (i)-(iii) are presented in section 3 for scalar problems and in section 4 for systems. The paper is concluded in section 5.

2. Multiple penalties for hyperbolic system of equations

The computational domain and the extended penalty regions in $[0, \epsilon_0]$ and $[\epsilon_1, 1]$ are shown in figure 1. For a certain mesh, there are p number of grid points in $[0, \epsilon_0]$ and q number of grid points in $[\epsilon_1, 1]$. We consider the

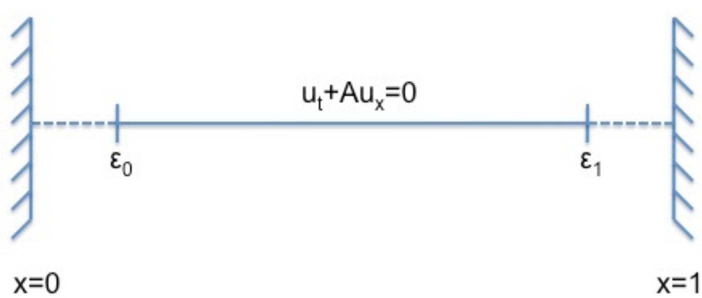


Figure 1: Illustration of multiple penalty domains $[0, \epsilon_0]$ and $[\epsilon_1, 1]$.

problem

$$\begin{aligned} u_t + Au_x &= 0, \quad x \in [0,1], \quad t > 0, \\ A^+ u(0,t) &= g_L(t), \\ A^- u(1,t) &= g_R(t), \end{aligned} \tag{7}$$

together with an initial condition which leads to a well-posed problem [1, 6, 14, 22, 23]. A is an $m \times m$ symmetric matrix that can be split according to the sign of its eigenvalues as $A = A^+ + A^-$, where $A^+ = X\Lambda^+X^T$ and $A^- = X\Lambda^-X^T$. Here Λ^+ and Λ^- contain the positive and negative (including zeros) eigenvalues respectively, and X is the corresponding eigenvector matrix of A.

The approximative solution at grid point x_i is $\mathbf{u}_i = [(u_0)_i, (u_1)_i, \ldots, (u_m)_i]$, for $i = 0, 1, \ldots, N$, and $\mathbf{u} = [\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_N]^T$. We consider $\mathbf{v}$ to be the known exact solution close to the boundaries and have a similar notation for $\mathbf{v}$ and the error $\mathbf{e} = \mathbf{u} - \mathbf{v}$.

The semi-discrete form of (7) using Kronecker products and weak boundary conditions is

$$\mathbf{u}_t + \left(P^{-1}Q \otimes A\right)\mathbf{u} = \left(P^{-1} \otimes I\right) R\left(\mathbf{u} - \mathbf{v}\right), \tag{8}$$

where R is a general penalty matrix. By inserting $\mathbf{v}$ in (8) we obtain

$$\mathbf{v}_t + \left(P^{-1}Q \otimes A\right)\mathbf{v} = \left(P^{-1} \otimes I\right) R\left(\mathbf{v} - \mathbf{v}\right) + Te, \tag{9}$$

where Te is the truncation error. By subtracting (9) from (8) we obtain,

$$\mathbf{e}_t + \left(P^{-1}Q \otimes A\right)\mathbf{e} = \left(P^{-1} \otimes I\right) R\,\mathbf{e} + Te. \tag{10}$$

We split

$$R = R_0 + R_{mul}, \tag{11}$$

where R_0 is the standard penalty term and R_{mul} an additional penalty term operating in the extended regions.

By multiplying (10) with $\mathbf{e}^T(P \otimes I_m)$ and adding its transpose, (let $Te = 0$, this does not influence stability), we obtain

$$\frac{d}{dt}||\mathbf{e}||_P^2 + \mathbf{e}^T\left[\left(Q + Q^T\right) \otimes A\right]\mathbf{e} = \mathbf{e}^T\left[R_0 + R_0^T + R_{mul} + R_{mul}^T\right]\mathbf{e}. \tag{12}$$

Let $R_{mul} = 0$, and choose

$$R_0 = \Sigma_0 \otimes A^+ + \Sigma_N \otimes A^-,\ \Sigma_0 = \begin{bmatrix} \alpha_{00} & & \\ & \mathbf{0} & \\ & & \alpha_{NN} \end{bmatrix},\ \Sigma_N = \begin{bmatrix} \beta_{00} & & \\ & \mathbf{0} & \\ & & \beta_{NN} \end{bmatrix}. \tag{13}$$

Since $P^{-1}Q$ is an SBP operator, the energy rate (12) is bounded if

$$\alpha_{00} \leq -\frac{1}{2} \quad \text{and} \quad \beta_{NN} \geq \frac{1}{2}, \quad \text{while} \quad \alpha_{NN} = \beta_{00} = 0. \tag{14}$$

As we will see below, the demands on R_{mul} will vary depending on the specific task we want it to do.

2.1. Increasing the accuracy of the scheme

If the difference operator $P^{-1}Q$ is a uniformly high order difference operator with same order of accuracy everywhere in the domain including the boundary region, then it is not an SBP operator. In this case we get additional terms in the symmetric part of Q,

$$Q + Q^T = B + \tilde{B}_s, \tag{15}$$

where $\tilde{B}_s$ is a symmetric matrix with non-zero blocks of fixed sizes at the upper-left and lower-right corners. The size of the blocks depend on the order of the difference operator under consideration.

Remark 1. *There are many variants of Q, and they all depend on which choice of P that we make. In our analysis, for clarity, we will restrict ourselves to using the standard diagonal SBP norm.*

In view of (11) and (15) we propose a penalty matrix R^{acc}_{mul} of the form

$$R^{acc}_{mul} = \Sigma^{acc} \otimes A, \quad \Sigma^{acc} = \begin{bmatrix} \Sigma^a_L & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Sigma^a_R \end{bmatrix}. \tag{16}$$

The energy rate (12) is bounded if

$$\mathbf{e}^T [R^{acc}_{mul} + R^{acc\,T}_{mul}]\mathbf{e} = \mathbf{e}^T \left(\tilde{\Sigma}^{acc} \otimes A\right) \mathbf{e} \leq 0,$$

where $\tilde{\Sigma}^{acc} = -\tilde{B}_s + \Sigma^{acc} + \Sigma^{acc\,T}$. By choosing $\tilde{\Sigma}^{acc} = 0$ we get

$$\frac{d}{dt}||\mathbf{e}||^2_P \leq 0,$$

and energy stability.

Remark 2. *For a uniformly 2nd, 4th and 6th order operators based on the standard diagonal SBP norm, weak boundary conditions must be imposed at one, four and six grid points respectively. If we choose $P = \Delta x I$ (I is the identity matrix) for our uniformly high order operator, it lowers the boundary data requirements. For 2nd, 4th and 6th order operators, the weak boundary conditions will be required at one, two, and three grid points respectively.*

In Appendices A and B we present uniformly 2nd, 4th and 6th order accurate difference operators based on SBP and identity norms respectively and also the corresponding Σ^{acc} matrices.

2.2. Increasing the rate of convergence to steady-state

To increase the rate of convergence to steady-state, we use multiple penalties for the incoming waves. In (11), we propose a penalty matrix R^{in}_{mul} of the form

$$R^{in}_{mul} = \Sigma^{in}_L \otimes A^+ + \Sigma^{in}_R \otimes A^-,\ \Sigma^{in}_L = \begin{bmatrix} \Sigma^i_L & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},\ \Sigma^{in}_R = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Sigma^i_R \end{bmatrix}, \tag{17}$$

where Σ^i_L covers the domain $[0,\, \epsilon_0]$ and Σ^i_R covers the domain $[\epsilon_1,\, 1]$ and are of the dimension p and q respectively. The energy rate (12) is bounded if

$$\mathbf{e}^T[R^{in}_{mul} + R^{in\,T}_{mul}]\mathbf{e} = \mathbf{e}_p{}^T\left(\tilde{\Sigma}^i_L \otimes A^+\right)\mathbf{e}_p + \mathbf{e}_q{}^T\left(\tilde{\Sigma}^i_R \otimes A^-\right)\mathbf{e}_q \leq 0.$$

Here $\tilde{\Sigma}^i_L = \Sigma^i_L + \Sigma^{i\,T}_L$, $\tilde{\Sigma}^i_R = \Sigma^i_R + \Sigma^{i\,T}_R$, $\mathbf{e}_p$ is the error vector in $[0, \epsilon_0]$, and $\mathbf{e}_q$ is the error vector in $[\epsilon_1, 1]$. For energy stability we need to choose Σ^i_L such that $\tilde{\Sigma}^i_L$ is negative semi-definite and Σ^i_R such that $\tilde{\Sigma}^i_R$ is positive semi-definite.

One can also modify waves passing out of the domain by choosing

$$R^{out}_{mul} = \Sigma^{out}_L \otimes A^- + \Sigma^{out}_R \otimes A^+,\ \Sigma^{out}_L = \begin{bmatrix} \Sigma^o_L & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},\ \Sigma^{out}_R = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Sigma^o_R \end{bmatrix}. \tag{18}$$

In (18), $\Sigma^{out}_{L,R}$ adds damping on the outgoing waves at the left and right boundary respectively. The energy rate (12) is bounded if

$$\mathbf{e}^T[R^{out}_{mul} + R^{out\,T}_{mul}]\mathbf{e} = \mathbf{e}_p{}^T\left(\tilde{\Sigma}^o_L \otimes A^-\right)\mathbf{e}_p + \mathbf{e}_q{}^T\left(\tilde{\Sigma}^o_R \otimes A^+\right)\mathbf{e}_q \leq 0,$$

where $\tilde{\Sigma}^o_L = \Sigma^o_L + \Sigma^{o\,T}_L$ and $\tilde{\Sigma}^o_R = \Sigma^o_R + \Sigma^{o\,T}_R$. Hence we need to choose $\tilde{\Sigma}^o_L$ to be positive semi-definite and $\tilde{\Sigma}^o_R$ negative semi-definite.

2.3. Changing the wave speed

For the construction of boundary closures which change the error propagation, we propose R^{wave}_{mul} of the form

$$R^{wave}_{mul} = \Sigma^{wave}_+ \otimes A^+ + \Sigma^{wave}_- \otimes A^-. \tag{19}$$

We partition Q as

$$Q = \begin{bmatrix} Q_L & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Q_R \end{bmatrix}$$

and $\Sigma^{wave}_{+,-}$ as

$$\Sigma^{wave}_{+} = \begin{bmatrix} \alpha_L^+ Q_L & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha_R^+ Q_R \end{bmatrix}, \Sigma^{wave}_{-} = \begin{bmatrix} \alpha_L^- Q_L & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha_R^- Q_R \end{bmatrix}.$$

Here Q_L and Q_R are square matrices of size p and q respectively and α_L, α_R are scalars. The relation (10) is modified to

$$\mathbf{e}_t + \left(P^{-1}\tilde{Q}_+ \otimes A^+\right)\mathbf{e} + \left(P^{-1}\tilde{Q}_- \otimes A^-\right)\mathbf{e} = \left(P^{-1} \otimes I\right) R_0 \mathbf{e}, \tag{20}$$

where $\tilde{Q}_{+,-} = Q - \Sigma^{wave}_{+,-}$ is given by

$$\tilde{Q}_{+,-} = \begin{bmatrix} \left(1-\alpha_L^{+,-}\right) Q_L & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \left(1-\alpha_R^{+,-}\right) Q_R \end{bmatrix}.$$

The modified difference operator $\tilde{Q}_{+,-}$ will serve as a building block for the construction of non-reflecting boundary conditions. By choosing $\alpha_L^{+,-}$ and $\alpha_R^{+,-}$ properly, we can control the wave speed of the errors in the extended regions.

The energy rate (12) in view of (20) becomes

$$\frac{d}{dt}||\mathbf{e}||_P^2 = \tilde{\alpha}_{00}\mathbf{e}_0^T A^+ \mathbf{e}_0 + \tilde{\alpha}_{NN}\mathbf{e}_N^T A^+ \mathbf{e}_N + \tilde{\beta}_{00}\mathbf{e}_0^T A^- \mathbf{e}_0 + \tilde{\beta}_{NN}\mathbf{e}_N^T A^- \mathbf{e}_N,$$

where for stability we require

$$\begin{aligned} \tilde{\alpha}_{00} &= \left(1-\alpha_L^+\right) + 2\alpha_{00} \le 0, \quad \tilde{\alpha}_{NN} = -\left(1-\alpha_R^+\right) + 2\alpha_{NN} \le 0, \\ \tilde{\beta}_{00} &= \left(1-\alpha_L^-\right) + 2\beta_{00} \ge 0, \quad \tilde{\beta}_{NN} = -\left(1-\alpha_R^-\right) + 2\beta_{NN} \ge 0. \end{aligned} \tag{21}$$

Note that $\mathbf{e}_0$ and $\mathbf{e}_N$ are the elements of the error vector $\mathbf{e}$. For $\alpha_L^{+,-} = \alpha_R^{+,-} = 0$, (21) reduce to the standard penalty limits in (14).

2.4. Other possible combinations of penalty terms

We noted above that R_0 in (13) is sufficient to make the SBP scheme stable. All other additions of R_{mul} (R^{acc}_{mul}, R^{in}_{mul}, R^{out}_{mul}, R^{wave}_{mul}) are specifically constructed for the tasks we want them to perform.

However, it is of course possible to combine these additional matrices. For effects on the convergence rate to steady-state, we can use (17) and (18) as

$$R_{mul} = R^{in}_{mul} + R^{out}_{mul}. \tag{22}$$

Another useful combination of (17), (18) and (19) is given by

$$R_{mul} = R^{wave}_{mul} + R^{in}_{mul} + R^{out}_{mul}. \tag{23}$$

By using (23) we can control the wave speed of errors in the penalty regions and damp the reflections at the same time.

If $P^{-1}Q$ is a uniformly high order operator instead of an SBP operator, we can use (16) for stability, while adding any one of the combinations (22) and (23).

3. Applications to scalar problems

3.1. Higher order accuracy

We consider the problem (1) and the corresponding semi-discrete formulation (8), (15) and penalty matrix (16). As an exact solution we use $u(x,t) = \sin(2\pi(x-t))$ and compute the solution at $t = 1$. We take $P^{-1}Q$ as a uniformly fourth and sixth order accurate difference operator based on the diagonal SBP norm as defined in (A.1, A.3, A.7) respectively. The results compared with the standard SBP schemes are shown in tables 1 and 2.

Remark 3. *Uniformly fourth and sixth order difference operators based on identity norm given by (B.1, B.4, B.7) were also verified for the same accuracy.*

Remark 4. *In some cases, the boundary data in the extended regions can be derived using a Taylor's series expansion and given boundary data, see Appendix C.*

Table 1: L_2-error and convergence rates (q), for 3rd order SBP and uniformly 4th order schemes.

Points	3rd order SBP scheme		uniformly 4th order scheme	
	l_2-error	q	l_2-error	q
21	$7.19e-03$	–	$7.44e-04$	–
41	$9.16e-04$	2.97	$5.00e-05$	3.89
81	$1.17e-04$	2.97	$3.23e-06$	3.95
161	$1.48e-05$	2.98	$2.05e-07$	3.98
321	$1.87e-06$	2.99	$1.29e-08$	3.99

Table 2: L_2-error and convergence rates (q), for 4th order SBP and uniformly 6th order schemes.

Points	4th order SBP scheme		uniformly 6th order scheme	
	l_2-error	q	l_2-error	q
21	$8.11e-03$	–	$7.18e-06$	–
41	$7.61e-04$	3.41	$1.96e-07$	5.19
81	$5.29e-05$	3.84	$3.76e-09$	5.70
161	$3.41e-06$	3.95	$6.41e-11$	5.87
321	$2.16e-07$	3.98	$1.04e-12$	5.94

3.2. Steady-state computations

Consider the advection problem (1) with initial data

$$u(x,0) = 1 + e^{-100(x-0.5)^2}.$$

Equations (10), (13) and (17) lead to (let $Te = 0$)

$$\mathbf{e}_t + aP^{-1}Q\mathbf{e} = aP^{-1}\left(\Sigma_0 + \Sigma_L^{in}\right)\mathbf{e}. \tag{24}$$

We choose the penalty matrices Σ_0 and Σ_L^{in} in (24) in such a way that eigenvalues shift away from the imaginary axis, see figure 2. The best effect is obtained for coarse meshes.

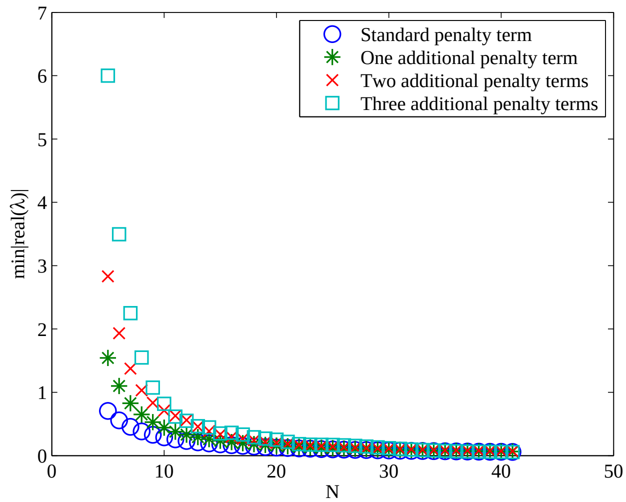


Figure 2: The effect of adding penalty terms on more grid points to the spectrum. The $\min |real(\lambda)|$ is shown for various N.

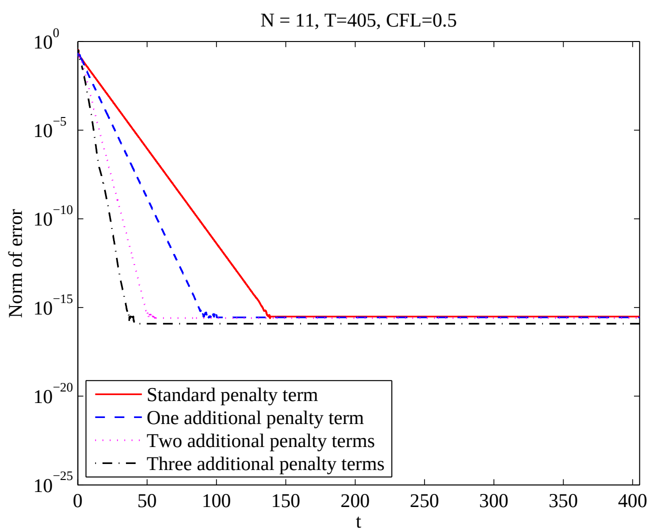


Figure 3: The effect of adding penalty terms on more grid points and corresponding convergence rates to steady-states for $N = 11$.

In the calculations below we plot the l_2-norm of $\mathbf{e} = \mathbf{u} - \mathbf{v}$ as a function of time. In figure 3 for $N = 11$ we see the effect of adding penalty on more grid points. It results in an increased convergence rate to steady-state.

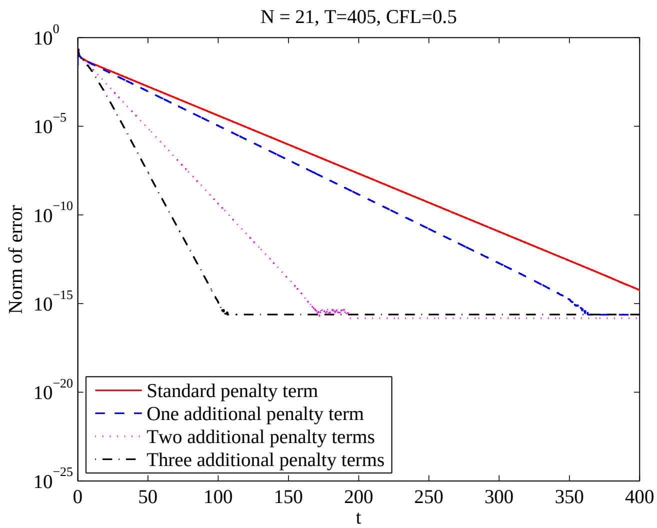


Figure 4: The effect of adding penalty terms on more grid points and corresponding convergence rates to steady-states for $N = 21$.

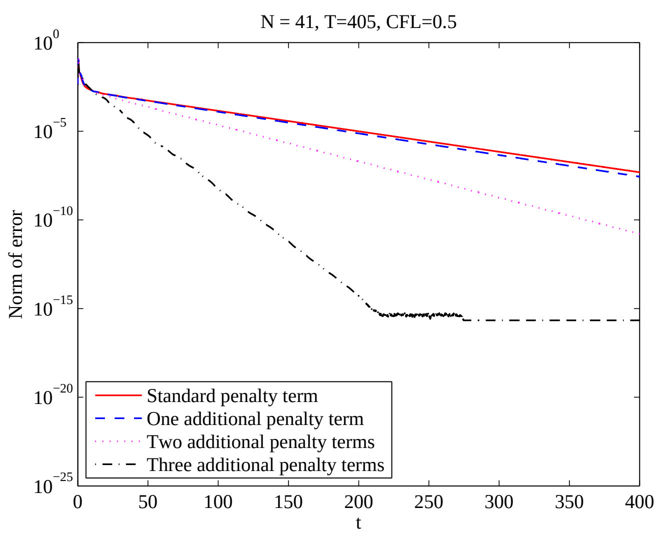


Figure 5: The effect of adding penalty terms on more grid points and corresponding convergence rates to steady-states for $N = 41$.

Next, we refined the mesh. The convergence rate decrease for a fixed number of multiple penalties and can be seen in figure 4 for $N = 21$ and in figure 5 for $N = 41$.

It can also be seen that the convergence rate remains approximately constant if one adds on penalty terms as the mesh is refined, see figure 6. Note that all the results are consistent with figure 2.

Next we consider (10), (13) and (22), so that (24) is modified to

$$\mathbf{e}_t + aP^{-1}Q\mathbf{e} = aP^{-1}\left(\Sigma_0 + \Sigma_L^{in} + \Sigma_R^{out}\right)\mathbf{e}. \tag{25}$$

The last additional term in (25) damps the outgoing waves close to the outflow boundary. There are many ways to choose the elements of the matrix Σ_R^{out} in (25). For simplicity we consider a constant diagonal matrix of the form $\Sigma_R^{out} = c_0\, diag(1, \dots, 1)$ in $[\epsilon_1, 1]$, where c_0 is a tuning parameter.

The convergence rates to steady-state for $N = 11, 21$ are compared with the corresponding results from the standard penalty and one additional penalty as shown in figures 7 and 8. As can be seen, the convergence rate to steady-state is improved considerably.

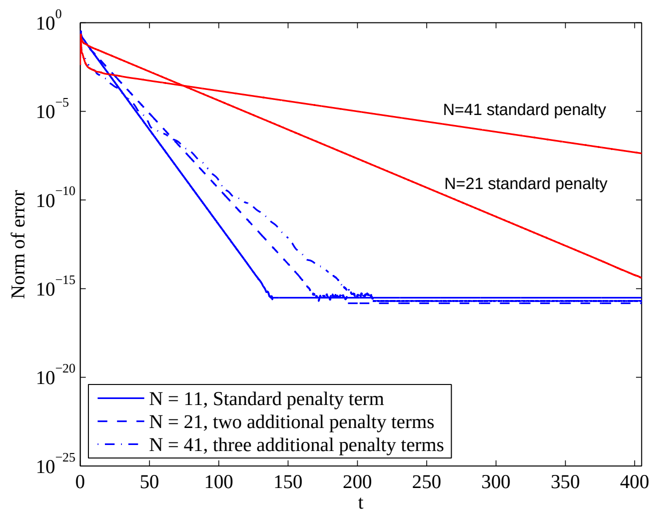


Figure 6: The effect of adding multiple penalty on mesh refinement.

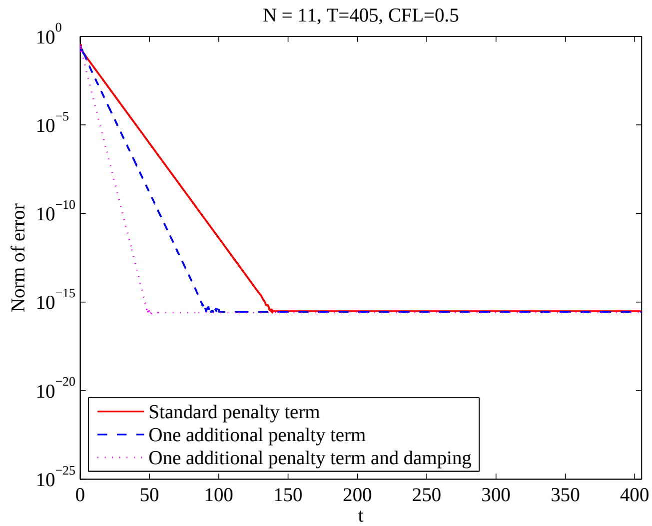


Figure 7: Convergence rate to steady-state using multiple penalty and additional damping on the outgoing waves given by (25) for $N = 11$.

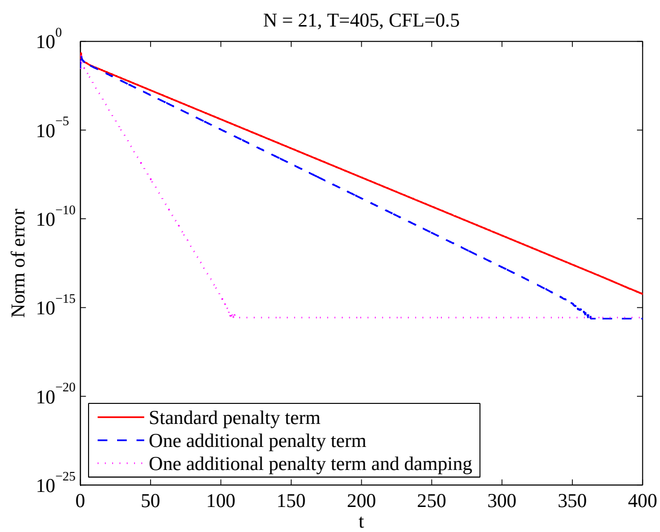


Figure 8: Convergence rate to steady-state using multiple penalty and additional damping on the outgoing waves given by (25) for $N = 21$.

3.3. Changing wave speeds

We consider the formulation (20) applied to the scalar problem (1)

$$\mathbf{e}_t + aP^{-1}\tilde{Q}\mathbf{e} = aP^{-1}\Sigma_0\mathbf{e}, \tag{26}$$

where we have ignored the subscript '+' on $\tilde{Q}$, α_L and α_R. If we choose $\alpha_R = -1$, the wave speed of the error is doubled in $[\epsilon_1, 1]$. By choosing $\alpha_L = 2$, we get a reversed error wave speed of $-a$ in $[0, \epsilon_0]$.

In the experiments below we use the exact solution and initial data,

$$u(x,t) = f(x - a\,t), \quad u(x,0) = f(x) = e^{-100(x-0.5)^2}.$$

We denote the error wave speed in $[\epsilon_1, 1]$ by a_R. The numerical solution moves with the wave speed a in the whole domain $[0,1]$. However, in figure 9 it can be seen that the error moves with $a_R = 2a$ in the region $[\epsilon_1, 1]$. We

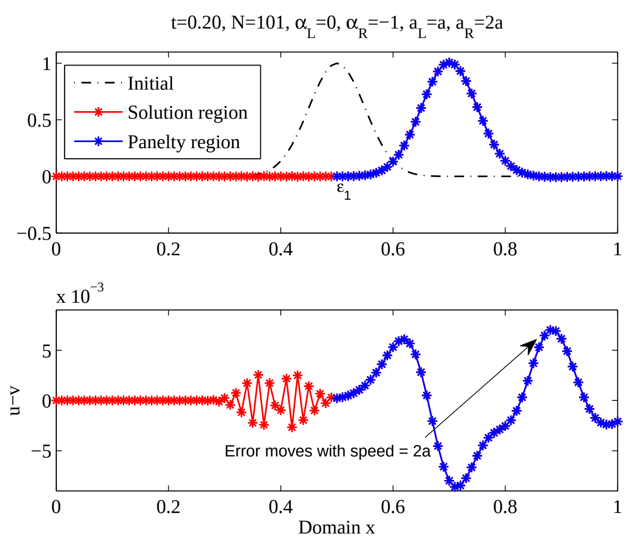


Figure 9: Error propagation with different speeds, $a_L = a$, $a_R = 2a$.

will use this technique below in the construction of non-reflecting boundary conditions.

4. Applications to system of equations

Consider the system of equations

$$u_t + Au_x = 0, \quad x \in [0,1], \tag{27}$$

where u and $A = X\Lambda X^T$ are

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The characteristics variables are $w = X^T u$, where $w_1 = \frac{1}{\sqrt{2}}(u_1 + u_2)$ and $w_2 = \frac{1}{\sqrt{2}}(u_1 - u_2)$. A general well-posed set of boundary conditions is given by

$$w_1(0,t) = \alpha w_2(0,t) + g_L(t), \quad w_2(1,t) = \beta w_1(1,t) + g_R(t). \tag{28}$$

In (28), we specify the ingoing characteristics variables in term of the outgoing ones and data, where α, β are constants.

We start by computing exact boundary data. For the initial data $u_1(x,0) = f_1(x)$, $u_2(x,0) = f_2(x)$, and using $w = X^T u$ we get $w_1(x,0) = \frac{1}{\sqrt{2}}(f_1(x) + f_2(x))$ and $w_2(x,0) = \frac{1}{\sqrt{2}}(f_1(x) - f_2(x))$.

The exact solution for the Cauchy problem in characteristics variables is given by

$$w_1(x,t) = \frac{1}{\sqrt{2}}(f_1(x-t) + f_2(x-t)), \; w_2(x,t) = \frac{1}{\sqrt{2}}(f_1(x+t) - f_2(x+t)),$$

and the exact solution in original variables using $u = Xw$ is given by

$$\begin{aligned} u_1(x,t) &= \frac{1}{2}[f_1(x-t) + f_2(x-t) + f_1(x+t) - f_2(x+t)], \\ u_2(x,t) &= \frac{1}{2}[f_1(x-t) + f_2(x-t) - f_1(x+t) + f_2(x+t)]. \end{aligned} \tag{29}$$

From (29) we can get $u(0,t) = g(0,t)$ and $u(1,t) = g(1,t)$ and apply the boundary conditions in the form

$$\begin{aligned} A^+ u(0,t) &= A^+ g(0,t) = g_L(t), \\ A^- u(1,t) &= A^- g(1,t) = g_R(t). \end{aligned}$$

As initial data we choose

$$u_1(x,0) = f_1(x) = e^{-200(x-0.5)^2}, \quad u_2(x,0) = f_2(x) = 0. \tag{30}$$

4.1. Higher order accuracy

The raised accuracy of the uniformly high order schemes for the system (27) is shown in table 3. We have used exactly the same technique as in section 3.1.

Table 3: Convergence rates for uniformly 4th and 6th order schemes.

Points	4th order scheme		6th order scheme	
	l_2-error	q	l_2-error	q
21	$4.10e-04$	–	$6.60e-06$	–
41	$2.64e-05$	3.95	$1.18e-07$	5.80
81	$1.66e-06$	3.99	$2.02e-09$	5.87
161	$1.03e-07$	4.00	$3.29e-13$	5.93
321	$6.48e-09$	4.00	$5.26e-13$	5.96

4.2. Steady-state computations

Equation (10) along with (13) and (22) becomes (let $Te = 0$)

$$\mathbf{e}_t + \left(P^{-1}Q \otimes A\right)\mathbf{e} = \left(P^{-1} \otimes I\right)\left[R_0 + R_{mul}^{in} + R_{mul}^{out}\right]\mathbf{e}. \tag{31}$$

We consider Σ_L^k, Σ_R^k, $k = [in, out]$ to be constant diagonal matrices with correct signs in (31). The comparison of results is done with the corresponding second order standard SBP scheme. For $N = 11, 21, 41$, the results are shown in figure 10, and indicates a considerably gain with the new approach.

To compare with the analytic convergence rate, we write (31) as

$$\mathbf{e}_t + M\mathbf{e} = 0$$

and compute the minimum real eigenvalue λ_r of the matrix M. The analytic rate of convergence to steady-state is directly proportional to $e^{-\lambda_r t}$, where $\lambda_r = \max_i Re(\lambda_i)$. From the analytic rate of convergence, we can compute the time it takes to reach the steady-state by $t_{ss} \approx 15\ln(10)/\lambda_r$ (we considered the norm of error $\approx 10^{-15}$ to indicate the steady-state). Comparison of the different times to reach steady-state is shown in table 4. As can be seen, the gain is considerable.

4.3. Changing wave speeds

We consider the formulation (20) for the problem (27) and the initial conditions (30). Since the problem (27) is symmetric, we consider the penalty

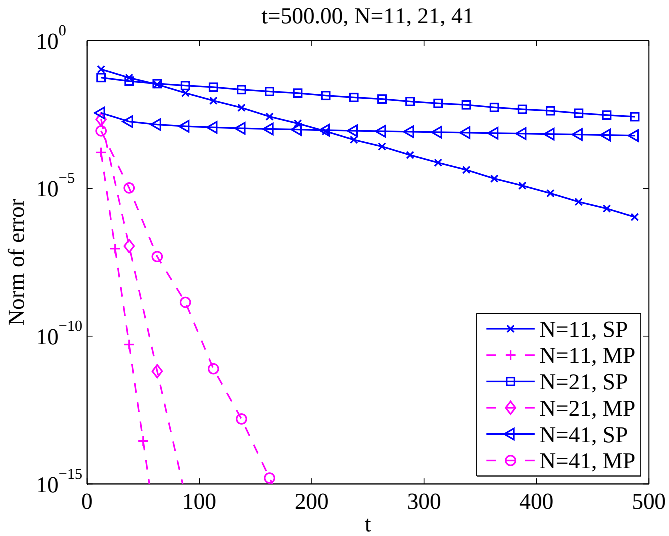


Figure 10: Rate of convergence to steady-state using standard and multiple penalty terms, $N = 11, 21, 41$. SP: Standard Penalty, MP: Multiple Penalty.

Table 4: Convergence rates to steady-state using standard and multiple penalty.

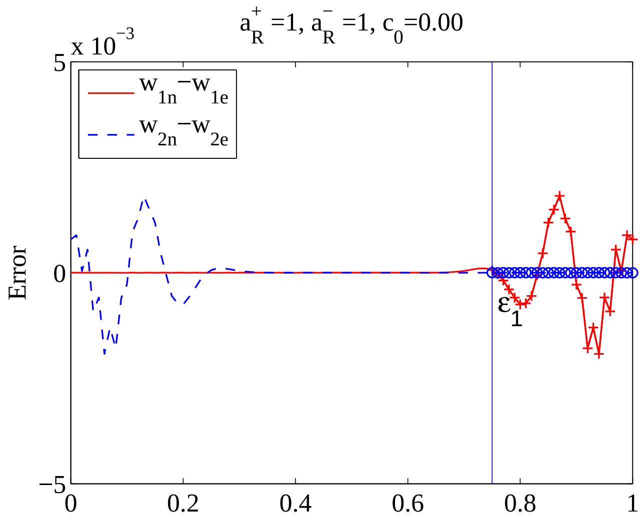
Points	Standard penalty term		Multiple penalty term		
	λ_r	$t^1_{ss} \approx$	λ_r	$t^2_{ss} \approx$	t^1_{ss}/t^2_{ss}
11	0.0242	1427.2	0.6015	57.4	24
21	0.0061	55662.1	0.3899	88.5	63
41	0.0015	23025.9	0.1772	194.9	118

region $[\epsilon_1, 1] = [0.75, 1]$ only. As long as the stability limits given by (21) are satisfied, we can freely choose the coefficients $\alpha_R^{+,-}$.

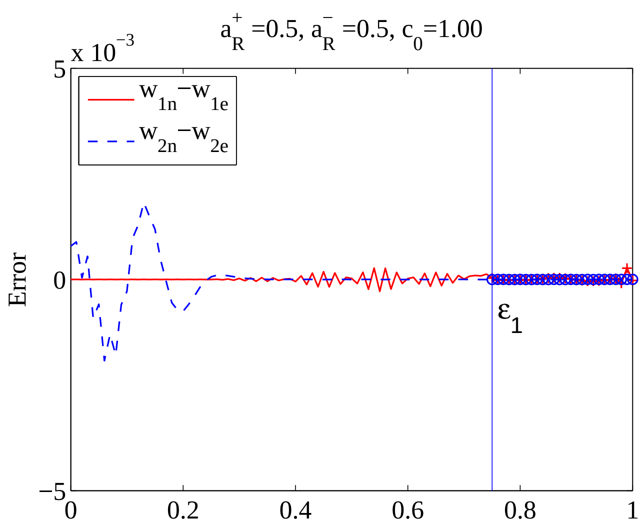
We choose the parameters α_R^+ and α_R^- in order to change the error wave speeds and damp the errors in the penalty region. We use,

$$\mathbf{e}_t + \left(P^{-1}\tilde{Q}_+ \otimes A^+\right)\mathbf{e} + \left(P^{-1}\tilde{Q}_- \otimes A^-\right)\mathbf{e} = \left(P^{-1} \otimes I\right)[R_0 + R_{mul}^{in} + R_{mul}^{out}]\mathbf{e}. \tag{32}$$

When $\alpha_R^{+,-} \neq 0$, we will call it the Modified Error Scheme (MES). We consider the following parameter combinations:



(a) SBP



(b) SBP + error speed change + damping

Figure 11: Non-reflecting properties for the 4th order SBP scheme. The error wave speed is modified and we have added damping in the penalty region. The penalty region is $[0.75, 1]$, $N = 101$ and $t = 0.4$.

(i) Standard SBP scheme ($\alpha_R^{+,-} = 0$, $R_{mul}^{in} = R_{mul}^{out} = 0$),

(ii) Standard SBP scheme with damping ($\alpha_R^{+,-} = 0$, $R_{mul}^{in} \neq 0, R_{mul}^{out} \neq 0$),

(iii) Modified errors scheme ($\alpha_R^{+,-} \neq 0$, $R_{mul}^{in} = R_{mul}^{out} = 0$) and

(iv) Modified error scheme with damping ($\alpha_R^{+,-} \neq 0$, $R_{mul}^{in} \neq 0, R_{mul}^{out} \neq 0$).

With $\alpha_R^{+,-} \neq 0$, the error wave speed in $[\epsilon_1, 1]$ is $a_R^{+,-} = (1 - \alpha_R^{+,-})$. In figure 11(a) the errors are shown using the standard 4th order SBP scheme. Adding damping reduces the error amplitude and modifying the error wave speed makes the errors move faster or slower. By combining both the previous settings properly we obtained reduced reflections as can be seen in figure 11(b).

Remark 5. *This procedure can also be seen as a way to increase the rate of convergence to steady-state, because we are aiming to reduce the reflection from the boundary.*

As another example of the flexibility of this method we will damp the outgoing disturbances and at the same time send in important signals. We consider the following initial data,

$$u_1(x, 0) = u_2(x, 0) = e^{-200(x-0.5)^2}.$$

For $t > 0$ we send in a sine wave from the boundary $x = 1$, which propagates left into the domain. To damp the outgoing signals we consider α_R^+ and R_{mul}^{out} to be non-zero, while α_R^- and R_{mul}^{in} which act on the ingoing signals are zero.

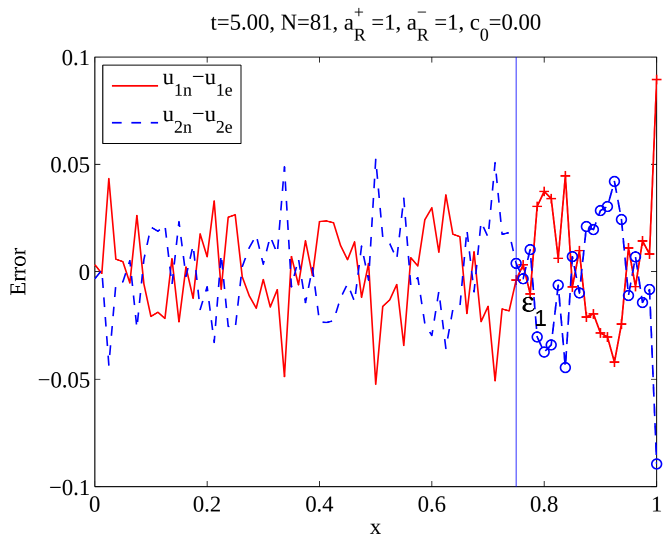
By adjusting the error wave speed and damping strength we obtain a more accurate solution. The results compared with the standard SBP scheme in terms of errors at $t = 5.0$ are shown in figure 12. The l_2-norm of the errors for a short as well as long time are shown in figure 13.

5. Conclusions

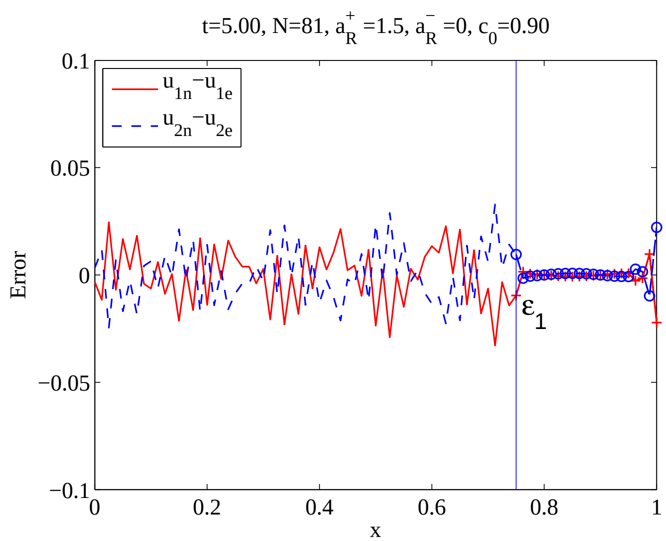
We have introduced a new weak boundary procedure by applying penalty terms in extended domains. The technique can be used if one knows the exact solution in parts of the domain.

By using this technique we have shown that one can:

(i) raise the accuracy,

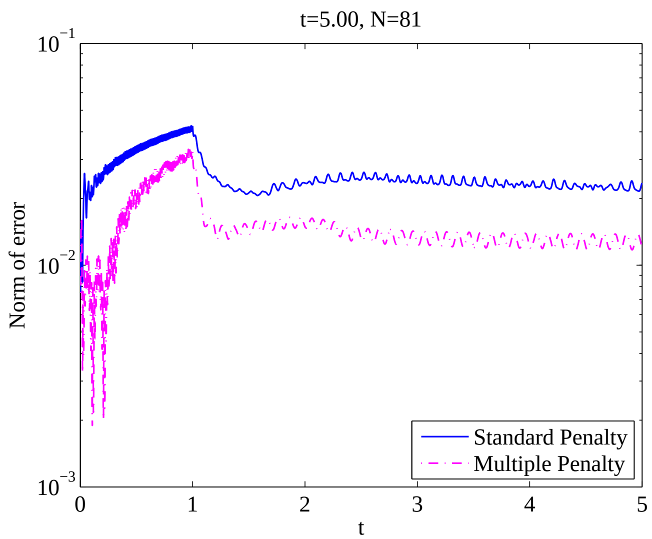


(a) SBP

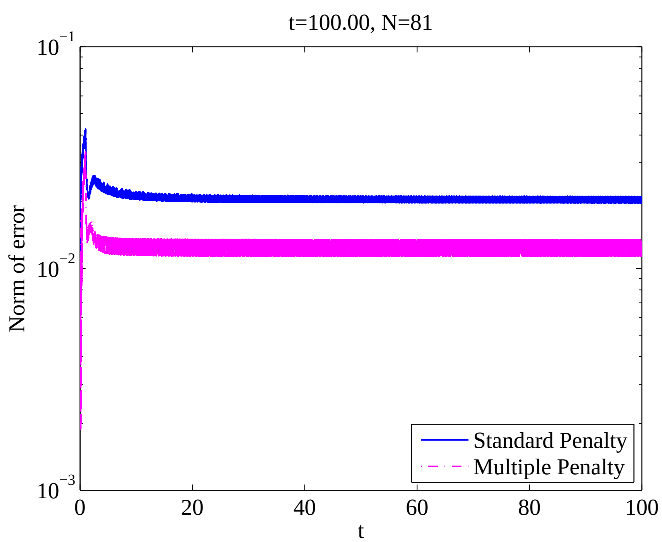


(b) SBP + error speed change + damping

Figure 12: The outgoing disturbance is damped while the ingoing wave is left unchanged. The penalty region is $[0.75, 1]$, $N = 81$ and $t = 5.0$.



(a) SBP



(b) SBP + error speed change + damping

Figure 13: Short and long time behavior of l_2-norm of errors using standard and multiple penalties.

(ii) increase the rate of convergence to steady-state,

(iii) use it to design non-reflecting boundary conditions,

(iv) use any combination of (i)-(iii).

The technique is easy to use and simple to program. The extension of this method to more general problems involving dissipative terms will be investigated in future work.

Appendix A. Uniformly high order operators based on SBP norms

For a uniformly second order accurate difference operator $P^{-1}Q$, the matrix Q and the second order SBP norm P are given as follows

$$Q = \begin{bmatrix} -\frac{3}{4} & 1 & -\frac{1}{4} & \cdots & & & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & & & \cdots & \frac{1}{4} & -1 & \frac{3}{4} \end{bmatrix}, P = \Delta x \begin{bmatrix} \frac{1}{2} & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & \frac{1}{2} \end{bmatrix}. \tag{A.1}$$

The matrix $\tilde{B}_s$ which raises the accuracy and the penalty matrix Σ^{acc} which maintain stability by making $\tilde{\Sigma}^{acc} = 0$ are given by

$$\tilde{B}_s = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \cdots & & & 0 \\ \frac{1}{2} & 0 & & & & & \\ -\frac{1}{4} & & & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & & & \frac{1}{4} \\ & & & & & 0 & -\frac{1}{2} \\ 0 & & \cdots & & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \Sigma^{acc} = \begin{bmatrix} -\frac{1}{4} & & & 0 \\ \frac{1}{2} & & & \\ -\frac{1}{4} & & & \\ \vdots & & \ddots & \vdots \\ & & & \frac{1}{4} \\ & & & -\frac{1}{2} \\ 0 & & & \frac{1}{4} \end{bmatrix}. \tag{A.2}$$

For a uniformly fourth order accurate difference operator on all grid points, we have

$$
P^{-1}Q = \frac{1}{\Delta x}\begin{bmatrix}
-\frac{25}{12} & 4 & -3 & \frac{4}{3} & -\frac{1}{4} & \cdots & & & & & 0 \\
-\frac{1}{4} & -\frac{5}{6} & \frac{3}{2} & -\frac{1}{2} & \frac{1}{12} & & & & & & \\
\frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & & & & & & \\
\vdots & & & & & \ddots & & & & & \vdots \\
 & & & & & & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} \\
 & & & & & & -\frac{1}{12} & \frac{1}{2} & -\frac{3}{2} & \frac{5}{6} & \frac{1}{4} \\
0 & & & & & \cdots & \frac{1}{4} & -\frac{4}{3} & 3 & -4 & \frac{25}{12}
\end{bmatrix}, \tag{A.3}
$$

$$
P = \Delta x \ \mathrm{diag}(\ \frac{17}{48} \quad \frac{59}{48} \quad \frac{43}{48} \quad \frac{49}{48} \quad 1 \quad \ldots \quad 1 \quad \frac{49}{48} \quad \frac{43}{48} \quad \frac{59}{48} \quad \frac{17}{48} \ )
$$

where P is the SBP norm. We obtain Q as

$$
Q = \begin{bmatrix}
-\frac{425}{576} & \frac{68}{48} & -\frac{51}{48} & \frac{17}{36} & -\frac{17}{192} & \cdots & & & & & 0 \\
-\frac{59}{192} & -\frac{295}{288} & \frac{59}{32} & -\frac{59}{96} & \frac{59}{576} & & & & & & \\
\frac{43}{576} & -\frac{43}{72} & 0 & \frac{43}{72} & -\frac{43}{576} & & & & & & \\
0 & \frac{49}{576} & -\frac{49}{72} & 0 & \frac{49}{72} & -\frac{49}{576} & & & & & \\
0 & 0 & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & & & & \\
\vdots & & & & & \ddots & & & & & \vdots \\
 & & & & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & 0 & 0 \\
 & & & & & \frac{49}{576} & -\frac{49}{72} & 0 & \frac{49}{72} & -\frac{49}{576} & 0 \\
 & & & & & & \frac{43}{576} & -\frac{43}{72} & 0 & \frac{43}{72} & -\frac{43}{576} \\
 & & & & & & -\frac{59}{576} & \frac{59}{96} & -\frac{59}{32} & \frac{295}{288} & \frac{59}{192} \\
0 & & & & & \cdots & \frac{17}{192} & -\frac{17}{36} & \frac{51}{48} & -\frac{68}{48} & \frac{425}{576}
\end{bmatrix}. \tag{A.4}
$$

The matrix $\tilde{B}_s$ which raises the accuracy is

$$\tilde{B}_s = \begin{bmatrix}
-\frac{137}{288} & \frac{71}{64} & -\frac{569}{576} & \frac{17}{36} & -\frac{17}{192} & 0 & \cdots & & & & & 0 \\
\frac{71}{64} & -\frac{295}{144} & \frac{359}{288} & -\frac{305}{576} & \frac{59}{576} & 0 & & & & & & \\
-\frac{569}{576} & \frac{359}{288} & 0 & -\frac{1}{12} & \frac{5}{576} & 0 & & & & & & \\
\frac{17}{36} & -\frac{305}{576} & -\frac{1}{12} & 0 & \frac{1}{72} & -\frac{1}{576} & & & & & & \\
-\frac{17}{192} & \frac{59}{576} & \frac{5}{576} & \frac{1}{72} & 0 & 0 & & & & & & \\
0 & 0 & 0 & -\frac{1}{576} & 0 & 0 & & & & & & \\
\vdots & & & & & \ddots & & & \vdots & & & \\
 & & & & & & 0 & 0 & \frac{1}{576} & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & -\frac{1}{72} & -\frac{5}{576} & -\frac{59}{576} & \frac{17}{192} \\
 & & & & & & \frac{1}{576} & -\frac{1}{72} & 0 & \frac{1}{12} & \frac{305}{576} & -\frac{17}{36} \\
 & & & & & & 0 & -\frac{5}{576} & \frac{1}{12} & 0 & -\frac{359}{288} & \frac{569}{576} \\
 & & & & & & 0 & -\frac{59}{576} & \frac{305}{576} & -\frac{359}{288} & \frac{295}{144} & -\frac{71}{64} \\
0 & & & & & \cdots & 0 & \frac{17}{192} & -\frac{17}{36} & \frac{569}{576} & -\frac{71}{64} & \frac{137}{288}
\end{bmatrix}. \tag{A.5}$$

The penalty matrix Σ^{acc} which maintain stability by making $\tilde{\Sigma}^{acc} = 0$ is given by

$$\Sigma^{acc} = \begin{bmatrix}
-\frac{137}{576} & 0 & & & & & & & 0 \\
\frac{71}{64} & -\frac{295}{288} & & & & & & & \\
-\frac{569}{576} & \frac{359}{288} & 0 & & & & & & \\
\frac{17}{36} & -\frac{305}{576} & -\frac{1}{12} & 0 & & & & & \\
-\frac{17}{192} & \frac{59}{576} & \frac{5}{576} & \frac{1}{72} & & & & & \\
0 & 0 & 0 & -\frac{1}{576} & 0 & & & & \\
\vdots & & & & & \ddots & & & \vdots \\
 & & & & & \frac{1}{576} & 0 & 0 & 0 \\
 & & & & & -\frac{1}{72} & -\frac{5}{576} & -\frac{59}{576} & \frac{17}{192} \\
 & & & & & & \frac{1}{12} & \frac{305}{576} & -\frac{17}{36} \\
 & & & & & & & -\frac{359}{288} & \frac{569}{576} \\
 & & & & & & & \frac{295}{288} & -\frac{71}{64} \\
0 & & & & & & & 0 & \frac{137}{576}
\end{bmatrix}. \tag{A.6}$$

For a uniformly sixth order accurate difference operator on all grid points, we have

$$P^{-1}Q = \frac{1}{\Delta x}\begin{bmatrix} -\frac{147}{60} & 6 & -\frac{15}{2} & \frac{20}{3} & -\frac{15}{4} & \frac{6}{5} & -\frac{1}{6} & \cdots & & & & & & & 0 \\ -\frac{1}{6} & -\frac{77}{60} & \frac{5}{2} & -\frac{5}{3} & \frac{5}{6} & -\frac{1}{4} & \frac{1}{30} & & & & & & & & \\ \frac{1}{30} & -\frac{2}{5} & -\frac{7}{12} & \frac{4}{3} & -\frac{1}{2} & \frac{2}{15} & -\frac{1}{60} & & & & & & & & \\ -\frac{1}{60} & \frac{9}{60} & -\frac{45}{60} & 0 & \frac{45}{60} & -\frac{9}{60} & \frac{1}{60} & & & & & & & & \\ \vdots & & & & & & & \ddots & & & & & & & \vdots \\ & & & & & & & & -\frac{1}{60} & \frac{9}{60} & -\frac{45}{60} & 0 & \frac{45}{60} & -\frac{9}{60} & \frac{1}{60} \\ & & & & & & & & \frac{1}{60} & -\frac{2}{15} & \frac{1}{2} & -\frac{4}{3} & \frac{7}{12} & \frac{2}{5} & -\frac{1}{30} \\ & & & & & & & & -\frac{1}{30} & \frac{1}{4} & -\frac{5}{6} & \frac{5}{3} & -\frac{5}{2} & \frac{77}{60} & \frac{1}{6} \\ 0 & & & & & & \cdots & & \frac{1}{6} & -\frac{6}{5} & \frac{15}{4} & -\frac{20}{3} & \frac{15}{2} & -6 & \frac{147}{60} \end{bmatrix} \tag{A.7}$$

$$P = \Delta x \,\mathrm{diag}\left(\frac{13649}{43200} \quad \frac{12013}{8640} \quad \frac{2711}{4320} \quad \frac{5359}{4320} \quad \frac{7877}{8640} \quad \frac{43801}{43200} \quad 1 \quad \cdots \quad 1 \quad \frac{43801}{43200} \quad \frac{7877}{8640} \quad \frac{5359}{4320} \quad \frac{2711}{4320} \quad \frac{12013}{8640} \quad \frac{13649}{43200} \right)$$

where P is the corresponding SBP norm. We can get Q at left boundary as

$$Q = \begin{bmatrix} -\frac{668801}{864000} & \frac{13649}{7200} & -\frac{13649}{5760} & \frac{13649}{6480} & -\frac{13649}{11520} & \frac{13649}{36000} & -\frac{13649}{259200} & 0 & 0 & \cdots & 0 \\ -\frac{12013}{51840} & -\frac{925001}{518400} & \frac{12013}{3456} & -\frac{12013}{5184} & \frac{12013}{10368} & -\frac{12013}{34560} & \frac{12013}{259200} & 0 & 0 & & \\ \frac{2711}{129600} & -\frac{2711}{10800} & -\frac{18977}{51840} & \frac{2711}{3240} & -\frac{2711}{8640} & \frac{2711}{32400} & -\frac{2711}{259200} & 0 & 0 & & \\ -\frac{5359}{259200} & \frac{5359}{28800} & -\frac{5359}{5760} & 0 & \frac{5359}{5760} & -\frac{5359}{28800} & \frac{5359}{259200} & 0 & 0 & & \\ 0 & -\frac{7877}{518400} & \frac{7877}{57600} & -\frac{7877}{11520} & 0 & \frac{7877}{11520} & -\frac{7877}{57600} & \frac{7877}{518400} & 0 & & \\ 0 & 0 & -\frac{43801}{5292000} & \frac{43801}{288000} & -\frac{43801}{57600} & 0 & \frac{43801}{57600} & -\frac{43801}{288000} & \frac{43801}{5292000} & & \\ 0 & 0 & 0 & -\frac{1}{60} & \frac{3}{20} & -\frac{3}{4} & 0 & \frac{3}{4} & \frac{3}{20} & \frac{1}{60} & \\ \vdots & & & & & & & \ddots & & & \vdots \end{bmatrix}. \tag{A.8}$$

The matrix $\tilde{B}_s$ which raises the accuracy at left boundary is

$$\tilde{B}_s = \begin{bmatrix}
-\frac{236801}{432000} & \frac{431299}{259200} & -\frac{608783}{259200} & \frac{540601}{259200} & -\frac{13649}{11520} & \frac{13649}{36000} & -\frac{13649}{259200} & \cdots & & 0 \\
\frac{431299}{259200} & -\frac{925001}{259200} & \frac{92879}{28800} & -\frac{552419}{259200} & \frac{197591}{172800} & -\frac{12013}{34560} & \frac{12013}{259200} & & & \\
-\frac{608783}{259200} & \frac{92879}{28800} & -\frac{18977}{25920} & -\frac{971}{10368} & -\frac{30589}{172800} & \frac{19231}{288000} & -\frac{2711}{259200} & & & \\
\frac{540601}{259200} & -\frac{552419}{259200} & -\frac{971}{10368} & 0 & \frac{947}{3840} & -\frac{3263}{96000} & \frac{1039}{259200} & & & \\
-\frac{13649}{11520} & \frac{197591}{172800} & -\frac{30589}{172800} & \frac{947}{3840} & 0 & -\frac{23}{300} & \frac{763}{57600} & -\frac{463}{314573} & & \\
\frac{13649}{36000} & -\frac{12013}{34560} & \frac{19231}{288000} & -\frac{3263}{96000} & -\frac{23}{300} & 0 & \frac{601}{57600} & -\frac{601}{288000} & \frac{117}{504599} & \\
-\frac{13649}{259200} & \frac{12013}{259200} & -\frac{2711}{259200} & \frac{1039}{259200} & \frac{763}{57600} & \frac{601}{57600} & 0 & & & \\
0 & 0 & 0 & 0 & -\frac{463}{314573} & -\frac{601}{288000} & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & \frac{117}{504599} & 0 & 0 & 0 & \\
\vdots & & & & & \ddots & & & & \vdots
\end{bmatrix}. \tag{A.9}$$

The penalty matrix Σ^{acc} which maintain stability by making $\tilde{\Sigma}^{acc} = 0$ at left boundary is given by

$$\Sigma^{acc} = \begin{bmatrix}
-\frac{236801}{864000} & & & & & & & & & \\
\frac{431299}{259200} & -\frac{925001}{518400} & & & & & & & & \\
-\frac{608783}{259200} & \frac{92879}{28800} & -\frac{18977}{51840} & & & & & & & \\
\frac{540601}{259200} & -\frac{552419}{259200} & -\frac{971}{10368} & 0 & & & & & & \\
-\frac{13649}{11520} & \frac{197591}{172800} & -\frac{30589}{172800} & \frac{947}{3840} & 0 & & & & & \\
\frac{13649}{36000} & -\frac{12013}{34560} & \frac{19231}{288000} & -\frac{3263}{96000} & -\frac{23}{300} & 0 & & & & \\
-\frac{13649}{259200} & \frac{12013}{259200} & -\frac{2711}{259200} & \frac{1039}{259200} & \frac{763}{57600} & \frac{601}{57600} & 0 & & & \\
0 & 0 & 0 & 0 & -\frac{463}{314573} & -\frac{601}{288000} & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & \frac{117}{504599} & 0 & 0 & 0 & \\
\vdots & & & & & \ddots & & & & \vdots
\end{bmatrix}. \tag{A.10}$$

Appendix B. Uniformly high order operators based on identity norm

For a uniformly second order accurate difference operator on all grid points, we use an identity norm $P = \Delta x I$ and obtain

$$Q = \begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} & \cdots & & & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & & & \cdots & \frac{1}{2} & -2 & \frac{3}{2} \end{bmatrix}, \tag{B.1}$$

the matrix $\tilde{B}_s$ becomes

$$\tilde{B}_s = \begin{bmatrix} -2 & \frac{3}{2} & -\frac{1}{2} & \cdots & & & 0 \\ \frac{3}{2} & 0 & & & & & \\ -\frac{1}{2} & & & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & & & \frac{1}{2} \\ & & & & & 0 & -\frac{3}{2} \\ 0 & & \cdots & & \frac{1}{2} & -\frac{3}{2} & 2 \end{bmatrix}. \tag{B.2}$$

The penalty matrix Σ^{acc} such that $\tilde{\Sigma}^{acc} = 0$ is given by

$$\Sigma^{acc} = \begin{bmatrix} -1 & & 0 \\ \frac{3}{2} & & \\ -\frac{1}{2} & & \\ \vdots & \ddots & \vdots \\ & & \frac{1}{2} \\ & & -\frac{3}{2} \\ 0 & & 1 \end{bmatrix} \tag{B.3}$$

For a uniformly fourth order accurate difference operator based on identity norm P, we have

$$Q = \begin{bmatrix} -\frac{25}{12} & 4 & -3 & \frac{4}{3} & -\frac{1}{4} & \cdots & & & & & 0 \\ -\frac{1}{4} & -\frac{5}{6} & \frac{3}{2} & -\frac{1}{2} & \frac{1}{12} & & & & & & \\ \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & & & & & & \\ \vdots & & & & & \ddots & & & & & \vdots \\ & & & & & & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} \\ & & & & & & -\frac{1}{12} & \frac{1}{2} & -\frac{3}{2} & \frac{5}{6} & \frac{1}{4} \\ 0 & & & & & \cdots & \frac{1}{4} & -\frac{4}{3} & 3 & -4 & \frac{25}{12} \end{bmatrix}, \quad \text{(B.4)}$$

the matrix $\tilde{B}_s$ becomes

$$\tilde{B}_s = \begin{bmatrix} -\frac{19}{6} & \frac{15}{4} & -\frac{35}{12} & \frac{4}{3} & -\frac{1}{4} & \cdots & 0 \\ \frac{15}{4} & -\frac{5}{3} & \frac{5}{6} & -\frac{5}{12} & \frac{1}{12} & & \\ -\frac{35}{12} & \frac{5}{6} & & & & & \\ \frac{4}{3} & -\frac{5}{12} & & & & & \\ -\frac{1}{4} & \frac{1}{12} & & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & & -\frac{1}{12} & \frac{1}{4} \\ & & & & & \frac{5}{12} & -\frac{4}{3} \\ & & & & & -\frac{5}{6} & \frac{35}{12} \\ & & -\frac{1}{12} & \frac{5}{12} & -\frac{5}{6} & \frac{5}{3} & -\frac{15}{4} \\ 0 & \cdots & \frac{1}{4} & -\frac{4}{3} & \frac{35}{12} & -\frac{15}{4} & \frac{19}{6} \end{bmatrix}. \quad \text{(B.5)}$$

The penalty matrix Σ^{acc} such that $\tilde{\Sigma}^{acc} = 0$ is given by

$$\Sigma^{acc} = \begin{bmatrix} -\frac{19}{12} & 0 & & & 0 \\ \frac{15}{4} & -\frac{5}{6} & & & \\ -\frac{35}{12} & \frac{5}{6} & & & \\ \frac{4}{3} & -\frac{5}{12} & & & \\ -\frac{1}{4} & \frac{1}{12} & & & \\ \vdots & & \ddots & & \vdots \\ & & & -\frac{1}{12} & \frac{1}{4} \\ & & & \frac{5}{12} & -\frac{4}{3} \\ & & & -\frac{5}{6} & \frac{35}{12} \\ & & & \frac{5}{6} & -\frac{15}{4} \\ 0 & & & 0 & \frac{19}{12} \end{bmatrix} \quad \text{(B.6)}$$

For a uniformly sixth order accurate difference operator based on identity norm P, we have

$$Q = \begin{bmatrix}
-\frac{147}{60} & 6 & -\frac{15}{2} & \frac{20}{3} & -\frac{15}{4} & \frac{6}{5} & -\frac{1}{6} & \cdots & & & & & & & 0 \\
-\frac{1}{6} & -\frac{77}{60} & \frac{5}{2} & -\frac{5}{3} & \frac{5}{6} & -\frac{1}{4} & \frac{1}{30} & & & & & & & & \\
\frac{1}{30} & -\frac{2}{5} & -\frac{7}{12} & \frac{4}{3} & -\frac{1}{2} & \frac{2}{15} & -\frac{1}{60} & & & & & & & & \\
-\frac{1}{60} & \frac{9}{60} & -\frac{45}{60} & 0 & \frac{45}{60} & -\frac{9}{60} & \frac{1}{60} & & & & & & & & \\
\vdots & & & & & & & \ddots & & & & & & & \vdots \\
& & & & & & & & -\frac{1}{60} & \frac{9}{60} & -\frac{45}{60} & 0 & \frac{45}{60} & -\frac{9}{60} & \frac{1}{60} \\
& & & & & & & & \frac{1}{60} & -\frac{2}{15} & \frac{1}{2} & -\frac{4}{3} & \frac{7}{12} & \frac{2}{5} & -\frac{1}{30} \\
& & & & & & & & -\frac{1}{30} & \frac{1}{4} & -\frac{5}{6} & \frac{5}{3} & -\frac{5}{2} & \frac{77}{60} & \frac{1}{6} \\
0 & & & & & & & \cdots & \frac{1}{6} & -\frac{6}{5} & \frac{15}{4} & -\frac{20}{3} & \frac{15}{2} & -6 & \frac{147}{60}
\end{bmatrix}, \tag{B.7}$$

the matrix $\tilde{B}_s$ becomes

$$\tilde{B}_s = \begin{bmatrix}
-\frac{117}{30} & \frac{35}{6} & -\frac{224}{30} & \frac{399}{60} & -\frac{15}{4} & \frac{6}{5} & -\frac{1}{6} & \cdots & & & & & & & 0 \\
\frac{35}{6} & -\frac{77}{30} & \frac{21}{10} & -\frac{91}{60} & \frac{49}{60} & -\frac{1}{4} & \frac{1}{30} & & & & & & & & \\
-\frac{224}{30} & \frac{21}{10} & -\frac{7}{6} & \frac{35}{60} & -\frac{21}{60} & \frac{7}{60} & -\frac{1}{60} & & & & & & & & \\
\frac{399}{60} & -\frac{91}{60} & \frac{35}{60} & & & & & & & & & & & & \\
-\frac{15}{4} & \frac{49}{60} & -\frac{21}{60} & & & & & & & & & & & & \\
\frac{6}{5} & -\frac{1}{4} & \frac{7}{60} & & & & & & & & & & & & \\
-\frac{1}{6} & \frac{1}{30} & -\frac{1}{60} & & & & & & & & & & & & \\
\vdots & & & & & & & \ddots & & & & & & & \vdots \\
& & & & & & & & & & & & \frac{1}{60} & -\frac{1}{30} & \frac{1}{6} \\
& & & & & & & & & & & & -\frac{7}{60} & \frac{1}{4} & -\frac{6}{5} \\
& & & & & & & & & & & & \frac{21}{60} & -\frac{49}{60} & \frac{15}{4} \\
& & & & & & & & & & & & -\frac{35}{60} & \frac{91}{60} & -\frac{399}{60} \\
& & & & & & & & \frac{1}{60} & -\frac{7}{60} & \frac{21}{60} & -\frac{35}{60} & \frac{7}{6} & -\frac{21}{10} & \frac{224}{30} \\
& & & & & & & & -\frac{1}{30} & \frac{1}{4} & -\frac{49}{60} & \frac{91}{60} & -\frac{21}{10} & \frac{77}{30} & -\frac{35}{6} \\
0 & & & & & & & \cdots & \frac{1}{6} & -\frac{6}{5} & \frac{15}{4} & -\frac{399}{60} & \frac{224}{30} & -\frac{35}{6} & \frac{117}{30}
\end{bmatrix}. \tag{B.8}$$

The penalty matrix Σ^{acc} such that $\tilde{\Sigma}^{acc} = 0$ is given by

$$
\Sigma^{acc} = \begin{bmatrix}
-\frac{117}{60} & 0 & 0 & \cdots & & & 0 \\
\frac{35}{6} & -\frac{77}{60} & 0 & & & & \\
-\frac{224}{30} & \frac{21}{10} & -\frac{7}{12} & & & & \\
\frac{399}{60} & -\frac{91}{60} & \frac{35}{60} & & & & \\
-\frac{15}{4} & \frac{49}{60} & -\frac{21}{60} & & & & \\
\frac{6}{5} & -\frac{1}{4} & \frac{7}{60} & & & & \\
-\frac{1}{6} & \frac{1}{30} & -\frac{1}{60} & & & & \\
\vdots & & & \ddots & & & \vdots \\
& & & & \frac{1}{60} & -\frac{1}{30} & \frac{1}{6} \\
& & & & -\frac{7}{60} & \frac{1}{4} & -\frac{6}{5} \\
& & & & \frac{21}{60} & -\frac{49}{60} & \frac{15}{4} \\
& & & & -\frac{35}{60} & \frac{91}{60} & -\frac{399}{60} \\
& & & & \frac{7}{12} & -\frac{21}{10} & \frac{224}{30} \\
& & & & 0 & \frac{77}{60} & -\frac{35}{6} \\
0 & & & \cdots & 0 & 0 & \frac{117}{60}
\end{bmatrix} \tag{B.9}
$$

Appendix C. Numerical boundary conditions

Here we describe that how boundary data at more inner points can be generated from a single boundary point data using the Taylor's series expansions. Consider the problem (1), where the boundary condition is given at one point x_0,

$$u_0 = u(x_0, t) = g(x_0, t). \tag{C.1}$$

We generate boundary data numerically at grid points x_1 and x_2 by

$$
\begin{aligned}
u_1 &= u(x_1, t) = u_0 + (u_x)_0 h + (u_{xx})_0 \frac{h^2}{2!} + \ldots + O(h^{p+1}), \\
u_2 &= u(x_2, t) = u_0 + (u_x)_0 (2h) + (u_{xx})_0 \frac{(2h)^2}{2!} + \ldots + O(h^{p+1}).
\end{aligned} \tag{C.2}
$$

Here p is the order of the scheme being used in computation. We compute

$$
\begin{aligned}
(u_x)_0 &= -(u_t)_0 = -g', \\
(u_{xx})_0 &= -(u_{xt})_0 = (u_{tt})_0 = g'', \\
(u_{xxx})_0 &= (u_{ttx})_0 = (u_{xtt})_0 = -(u_{ttt})_0 = -g''',
\end{aligned} \tag{C.3}
$$

where $'$ denotes the derivative with respect to t. It can be extended to nth order derivatives as

$$(\frac{\partial^n u}{\partial x^n})_0 = (-1)^n \frac{\partial^n g}{\partial t^n}. \tag{C.4}$$

Using (C.3) and (C.4), we can obtain data in (C.2) to the required accuracy.

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