Stable Treatment of Discontinuities in the Numerical Pricing of Options with Dividends

Wei Hu
Abstract

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Black-Scholes is the widely used model of option pricing in practice. In this thesis, discontinuous dividends are involved into the Black-Scholes model. With some substitutions, the Black-Scholes equation can be transferred into another form. The main part of the thesis is about how to avoid the oscillations near the discontinuities using several oscillation-free methods. Furthermore, the error analysis and the result of the methods based on both the Black-Scholes equation and the transferred one are introduced.
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1 Introduction

In the financial market, options are the financial contracts that give the holder rights to buy or sell something based on the future price of the underlying assets [8]. There are many different kinds of underlying assets, in this thesis we are dealing with stock options. In Section 2.3.1, we will introduce some different kinds of stock options. More details of options can be found in [1].

In 1973, F. Black and M. Scholes introduced a mathematical model which can price the time-consistent option perfectly. This is the most important contribution to the option pricing area, more details about this will be demonstrated in Section 2.1. In this thesis, we also use the Black-Scholes equation to numerically approximate the value of the options. There are many methods which can compute the value of the options according to the stock price, such as Monte-Carlo methods, finite difference methods and finite element methods, also adaptive mesh refinement can be involved. More details about the methods can be found in [2]. Here, we will focus on the finite difference method with equidistant mesh.

In this thesis, we are dealing with two properties of the Black-Scholes model: dividend and discontinuity. There are two types of dividends: the continuous dividend and the discontinuous dividend, here we will focus on the latter one. When the discontinuous dividend is paid, the stock price will be changed immediately according to the dividend amount, so we need to handle this jump numerically. The treatment can be found later in Section 3. This is one type of discontinuity. Even more severe discontinuities will be dealt with later, especially with the barrier option.

The usual numerical approximation, which is the central space approximation, will cause some oscillations near the discontinuity. There are some methods that can avoid the oscillations, such as the adaptive mesh methods. The idea of finite difference methods based on adaptive mesh is that, near the discontinuity use more steps on the mesh, with sufficiently small step size the oscillation may be avoided. Also adaptive radial basis functions have been done, this can also solve the options with discontinuities smoothly [10].

In Section 4 we will introduce some oscillation-free methods to deal with the discontinu-
ities. In Section 5, we will analyze the accuracy of these methods for the Black-Scholes model. Finally, in Section 6 the numerical results will be presented using the oscillation-free methods to display how they work.

2 Analytical equation

2.1 Black-Scholes equation

The widely used model of option pricing in practice is the Black-Scholes model, which was introduced by F. Black and M. Scholes in 1973 [3]. In this paper, they derived a PDE which can price the option over a specific time interval. This PDE is called the Black-Scholes equation now:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0 \tag{1}
\]

Here \( S(t) \) is the stock price at time \( t \); \( V(S, t) \) is the option price of one underlying asset at time \( t \) and stock price \( S(t) \). And also with some parameters, \( r \) is the risk-free interest rate; \( \sigma \) is the volatility of stock returns; \( K \) is the strike price of the option; \( t \) is the time.

2.2 Transferred differential equation

Based on the Black-Scholes equation, we can use two substitutions to simplify the equation to a transferred equation:

\[
\begin{align*}
    x &= \log(S e^{r(T-t)}) \\
    C &= V e^{r(T-t)}
\end{align*}
\tag{2}
\]

\( T \) is the expiry time. With these substitutions above, the Black-Scholes equation will result in a different equation:

\[
\frac{\partial C}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 C}{\partial x^2} - \frac{\sigma^2}{2} \frac{\partial C}{\partial x} = 0 \tag{3}
\]

From this transferred PDE, we can notice that the risk-free interest rate \( r \) is removed. Also compared to the original Black-Scholes equation, the coefficients of the first derivative and the second derivative in stock price \( S \) are constant instead of functions of \( S \). Later on, we will see how these modifications affect the numerical approximation solution.
2.3 Boundary condition

There are many kinds of options in the finance market, such as European call/put options, American call/put options and Barrier options. In this article, we will compute the price of these options numerically later on. First of all, let us introduce some properties of them. Call options give the holder right to buy the underlying assets. On the contrary, put options give the holder right to sell the underlying assets. The pay-off of both European and American options are the same, which is \( \max(S - K, 0) \) for a call option and \( \max(K - S, 0) \) for a put option. For European options, the holder of them has the right to sell or buy the underlying assets, or to exercise the option at the expiry time \( T \) with a strike price \( K \). For American options, the holder of them can buy or sell the underlying asset at any time before the expiry time \( T \) with a strike price \( K \). This is called early exercise. With the Barrier options, the idea is that when the underlying crosses the barrier level, the value of the option will jump to a certain value or reduce to just a cash rebate, and also this rebate can be zero. There are four main types of barrier options: Up-and-out, Down-and-out, Up-and-in, Down-and-in.

2.3.1 Final time condition

In the option pricing problem, we know the pay-off when we exercise a European option at the expiry time \( T \), but we want to know the option price before the expiry time. Hence the Black-Scholes equation is solved backward in time. The Black-Scholes equation can price different kinds of options with different kinds of final conditions, which is at the expiry time \( T \) for the European style options. The final time conditions for some options follows.

**European call option**

\[
V(S, T) = \max(S - K, 0)
\]

The European call option gives the holder right to buy the underlying asset only at the expiry time \( T \) with a strike price \( K \).

**European put option**

\[
V(S, T) = \max(K - S, 0)
\]
The European put option gives the holder right to sell the underlying asset only at the expiry time $T$ with a strike price $K$. Compared to the European call option, the only difference is the pay-off at expiry time $T$, which is the final condition.

**Down and out digital barrier option**

$$V(S, T) = \begin{cases} 
0 & \text{if } S \leq B \text{(below the barrier)} \\
\text{payoff} & \text{if } S > B \text{(above the barrier)} 
\end{cases}$$

Digital option, also known as binary option, is an option whose pay-off is a fixed amount of asset or nothing at all. There are two main types: asset-or-nothing and cash-or-nothing. Here we will price the cash-or-nothing numerically, which pays a certain amount of cash at the expiry time or nothing [4]. And for the Down-and-out barrier option, when the stock price $S$ is below the barrier level $B$, the value of the option will reduce to the rebate immediately, which is nothing for the digital option. If the stock price $S$ remains above the barrier level $B$ until the expiry time $T$, the price of the option will be the pay-off.

**Up and out digital barrier option**

$$V(S, T) = \begin{cases} 
\text{payoff} & \text{if } S \leq B \text{(below the barrier)} \\
0 & \text{if } S > B \text{(above the barrier)} 
\end{cases}$$

Compared to the Down and out digital barrier option, the difference is that when the stock price $S$ is above the barrier level $B$, the value of option will reduce to the rebate immediately, which is zero. If the stock price $S$ remains under the barrier level $B$ at the expiry time $T$, the price of the option will be the pay-off.

**Up and out barrier call option**

$$V(S, T) = \begin{cases} 
\max(S - K, 0) & \text{if } S \leq B \text{(below the barrier)} \\
0 & \text{if } S > B \text{(above the barrier)} 
\end{cases}$$

This kind of option is a combination of a European call option and an up-and-out barrier option. When the stock price $S$ is above the barrier level $B(S < B)$, the value of option will reduce to the rebate immediately, which can be zero. If the stock price $S$ remains
under the barrier level \( B \) until the expiry time \( T \), the price of the option will be the European call option pay-off.

**Down and out barrier call option**

\[
V(S, T) = \begin{cases} 
0 & \text{if } S \leq B \text{(below the barrier)} \\
\max(S - K, 0) & \text{if } S > B \text{(above the barrier)}
\end{cases}
\]

This kind of option is a combination of a European call option and a down-and-out barrier option. When the stock price \( S \) is above the barrier level \( B \), the value of the option will reduce to the rebate immediately, which can be nothing. If the stock price \( S \) remains over the barrier level \( B \) until the expiry time \( T \), the price of the option will be the European call option pay-off.

### 3 Discontinuous dividend

The dividend is paid to the stock holder by the stock company. There are two kinds of dividends: continuous dividend and discontinuous dividend. In this thesis we only consider how to deal with the discontinuous dividend, where the dividend date and dividend amount are declared by the stock company in advance. As we know, when the dividend is paid, the stock price is expected to drop by the dividend amount on the ex-dividend date, this drop will also affect the price of the option.

Since the price of the option will not jump at one single date, how the discontinuous dividend affects the price of option can be modeled as:

\[
V(S_j, t^-_d) = V(S_j - D, t^+_d); \tag{4}
\]

where \( t_d \) is the declared dividend moment, \( t^-_d \) is the ex-dividend date, \( t^+_d \) is the date just after the dividend is paid and \( D \) is the declared dividend amount.

With this condition on the option price at dividend date, we can split the entire time interval into two parts \([0, t_d)\) and \([t_d, T]\):

- From expiry time \( T \) with the final condition \( V(S, T) \) according to different options (see Section 2.3.1) to \( t_d \), we can solve the PDE numerically as explained in
Section 4.1 iteratively backward in time. Finally we can get the value of the option \( V(S_j, t_d) \) at time \( t_d \).

- Then consider \( t_d^- \) as the expiry time during \([0, t_d)\). Now we already know the value of \( V(S_j, t_d^-) \) at time \( t_d \), according to (4) we take \( V(S_j - D, t_d^+) \) which is our final condition during \([0, t_d)\). Again we can solve the PDE numerically backward from \( t_d \) to 0, finally we can get \( V(S, 0) \), which is the price of option initially.

4 Numerical methods

4.1 Mesh definition

Here we define an equidistant mesh, \( M + 1 \) points in the space dimension and \( N + 1 \) points in the time dimension, \( \Delta t = T/N \) is the time step and \( \Delta S = S_{max}/M \) (Black-Scholes PDE) or \( \Delta x = (X_{max} - X_{min})/M \) (Transferred PDE) is the space step, here we define \( S_{max} = 4K \), \( X_{max} = 5 \), \( X_{min} = -5 \).

4.2 Time approximation

In the Black-Scholes equation (1), there is only a first derivative in time. As we discussed in Section 2.3.1, we price the option from the expiry time to the beginning, which means that we solve the equation backwards in time. The first derivative in time can be approximated as:

\[
\frac{\partial V}{\partial t}(S_j, t_n) \approx \frac{V_j^n - V_j^{n-1}}{\Delta t} \tag{5}
\]

4.3 Space approximation

We observe from the Black-Scholes equation that we need to approximate second and first derivatives in space.

4.3.1 Boundary condition

Since at the beginning and the end of space(which means \( S = 0 \) and \( S = S_{max} \)), the shape of the option price \( V \) with respect to the stock price \( S \) can be considered to be
flat. In this thesis, we deal with the boundary condition as

\[ \frac{\partial^2 V}{\partial S^2} = 0 \quad (6) \]

Also there are other approaches to deal with the boundary conditions, such as\[4\]

\[ V(0, t) = 0, \]
\[ V(S_{max}, t) = \text{payoff} \]

### 4.3.2 Approximation for second derivative in space

The second derivative is approximated by a central difference. This approximation is second order accurate.

\[ \frac{\partial^2 V}{\partial S^2}(S_j, t_n) \approx \frac{V_{j-1}^n - 2V_j^n + V_{j+1}^n}{\Delta S^2} \quad (7) \]

Now we turn to the boundary condition for space equation. By (6), we can get

\[ \frac{\partial^2 V}{\partial S^2} = 0 \]
\[ \Rightarrow \frac{V_{j-1}^n - 2V_j^n + V_{j+1}^n}{\Delta S^2} = 0 \]
\[ \Rightarrow V_{j-1}^n - 2V_j^n + V_{j+1}^n = 0 \]

At the left hand side, we start at \( j = 2 \) since \( V_0 \) is not defined. Then the boundary condition at \( j = 1 \) can be formulated as:

\[ V_1^n = 2V_2^n - V_3^n \quad (8) \]

At the right hand side, we have \( j = M \) since \( V_{M+2} \) is not defined. Then the boundary condition at \( j = M + 1 \) can be formulated as:

\[ V_{M+1}^n = 2V_M^n - V_{M-1}^n \quad (9) \]

### 4.3.3 Approximation for first derivative in space

There are many different kinds of numerical approximations for the first derivative in space, and this is the part which may cause oscillations near the discontinuities. This will be demonstrated in the numerical results later on.
Central difference

\[ \frac{\partial V}{\partial S}(S_j, t_n) \approx \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S} \]

This is the common approximation for the first derivative in space we usually use. It is second order accurate but this method will cause oscillations near the discontinuity. Later in Section 4.7, we can see this in Figure 1.

Upwind difference

In the usual case which is forward in time, the upwind method is

\[ \frac{a\partial V}{\partial S}(S_j, t_n) \approx \begin{cases} 
  a \frac{V_{j+1}^n - V_j^n}{\Delta S} & \text{if } a > 0 \\
  a \frac{V_{j+1}^n - V_j^n}{\Delta S} & \text{if } a < 0 
\end{cases} \]

In our case we iterate backward in time, so the upwind method should be

\[ \frac{a\partial V}{\partial S}(S_j, t_n) \approx \begin{cases} 
  a \frac{V_{j+1}^n - V_j^n}{\Delta S} & \text{if } a > 0 \\
  a \frac{V_{j+1}^n - V_j^n}{\Delta S} & \text{if } a < 0 
\end{cases} \]

Upwind difference with flux limiters

First, we introduce the color equation from [5], which is

\[ V_t + u(S)V_S = 0 \] (10)

Stepping backward in time, this kind of equation can be approximated using the flux function:

\[ \frac{V_j^n - V_{j-1}^{n-1}}{\Delta t} + \frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta S} = 0 \] (11)

where \( F_{j+\frac{1}{2}}^n \) is the numerical flux approximation at \( S = S_{j+\frac{1}{2}} \) and \( F_{j-\frac{1}{2}}^n \) is the numerical flux approximation at \( S = S_{j-\frac{1}{2}} \):

\[ F_{j+\frac{1}{2}}^n = \begin{cases} 
  u(S_j)V_{j+1}^n - \frac{1}{2} u(S_j)(1 - u(S_j)\frac{\Delta t}{\Delta S})\delta_{j+1}^n & \text{if } u(S) > 0 \\
  u(S_j)V_j^n - \frac{1}{2} |u(S_j)|(1 - |u(S_j)|\frac{\Delta t}{\Delta S})\delta_j^n & \text{if } u(S) < 0
\end{cases} \] (12)

\[ F_{j-\frac{1}{2}}^n = \begin{cases} 
  u(S_j)V_j^n - \frac{1}{2} u(S_{j-1})(1 - u(S_{j-1})\frac{\Delta t}{\Delta S})\delta_j^n & \text{if } u(S) > 0 \\
  u(S_j)V_{j-1}^n - \frac{1}{2} |u(S_{j-1})|(1 - |u(S_{j-1})|\frac{\Delta t}{\Delta S})\delta_{j-1}^n & \text{if } u(S) < 0
\end{cases} \] (13)

Here \( \delta_j^n \) is the flux limiter. There are many kinds of limiters. They will be introduced later on in this section. Then, insert (12) and (13) into (11), and compare the result with
We can get the flux approximation of the first derivative in space (Later on we will use \( \hat{F}_{j+\frac{1}{2}} \) and \( \hat{F}_{j-\frac{1}{2}} \) as the correction item instead of \( F_{j+\frac{1}{2}} \) and \( F_{j-\frac{1}{2}} \) introduced before):

\[
\frac{\partial V}{\partial S}(S_j, t_n) \approx \begin{cases} 
\frac{u(S_j)}{\Delta S} \frac{V_{j+1} - V_j}{\Delta S} & \text{if } u(S) > 0 \\
\frac{u(S_j)}{\Delta S} \frac{V_j - V_{j-1}}{\Delta S} & \text{if } u(S) < 0 
\end{cases}
\]

where

\[
\tilde{F}_{j+\frac{1}{2}} = \begin{cases} 
\frac{1}{2} u(S_j)(1 - \frac{\Delta t}{\Delta S} \Delta(\delta_{j+1})) & \text{if } u(S) > 0 \\
\frac{1}{2} | u(S_j) | (1 - \frac{\Delta t}{\Delta S} \Delta(\delta_j)) & \text{if } u(S) < 0 
\end{cases}
\]

\[
\tilde{F}_{j-\frac{1}{2}} = \begin{cases} 
\frac{1}{2} u(S_{j-1})(1 - \frac{\Delta t}{\Delta S} \Delta(\delta_j)) & \text{if } u(S) > 0 \\
\frac{1}{2} | u(S_{j-1}) | (1 - \frac{\Delta t}{\Delta S} \Delta(\delta_{j-1})) & \text{if } u(S) < 0 
\end{cases}
\]

\[
\delta_j \approx \Delta S \frac{\partial u(S_j)}{\partial S}
\]

Different kinds of flux limiters

Now, different kinds of flux limiters will be introduced: The flux limiters are usually used in fluid dynamics. They avoid the oscillations near the discontinuity in stock problems. Here we introduce them in the Black-Scholes PDE to price the options with discontinuities.

Firstly, we will describe the linear methods.

**Fromm limiter** This is a centered approximation of the first derivative over the \( j \)th grid cell. With this limiter in (11) the Fromm method is obtained.

\[
\delta_j = \frac{V_{j+1} - V_{j-1}}{2}
\]

**Beam-Warming limiter** This is an upwind approximation of the first derivative over the \( j \)th grid cell. Introducing this type of limiter in (11) we can get the Beam-Warming method.

\[
\delta_j = V_j - V_{j-1}
\]
**Lax-Wendroff limiter** Downwind approximation of the first derivative is used here. This kind of limiter will result in the Lax-Wendroff method.

$$\delta_j^n = V_{j+1}^n - V_j^n$$

Then, some high-resolution limiters will be introduced.

**Minmod limiter**

$$\delta_j^n = \minmod(V_j^n - V_{j-1}^n, V_{j+1}^n - V_j^n);$$

where

$$\minmod(a, b) = \begin{cases} 
a & \text{if } |a| < |b| \text{ and } ab > 0 \\
b & \text{if } |a| > |b| \text{ and } ab > 0 \\
0 & \text{if } ab < 0\end{cases}$$

Here, the minmod method compares the Beam-Warming limiter and the Lax-Wendroff limiter, then choose the one which is smaller in magnitude. Furthermore, if these two have different signs, then $\delta_j^n = 0$.

**Superbee limiter**

$$\delta_j^n = \maxmod(\delta_j^{n(1)}, \delta_j^{n(2)})$$

where

$$\delta_j^{n(1)} = \minmod((V_{j+1}^n - V_j^n), 2(V_j^n - V_{j-1}^n)),$$

$$\delta_j^{n(2)} = \minmod(2(V_{j+1}^n - V_j^n), (V_j^n - V_{j-1}^n)),$$

$$\maxmod(a, b) = \begin{cases} 
a & \text{if } |a| > |b| \text{ and } ab > 0 \\
b & \text{if } |a| < |b| \text{ and } ab > 0 \\
0 & \text{if } ab < 0\end{cases}$$

Here, the minmod method compares the Lax-Wendroff limiter with twice of the Beam-Warming limiter and chooses the smallest one in magnitude, also twice the Lax-wendroff limiter with the Beam-Warming limiter and chooses the smallest one in magnitude. Finally choose the one which is larger in magnitude from the previous results. Furthermore, if these two results have the difference signs, then $\delta_j^n = 0$. 

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MC limiter

\[ \delta_j^n = \text{minmod}\left( \frac{V_{j+1}^n - V_{j-1}^n}{2}, 2(V_j^n - V_{j-1}^n), 2(V_{j+1}^n - V_j^n) \right) \]

This is the monotonous central-difference limiter, proposed by van Leer [7]. It compares the central difference of the Fromm limiter with twice the Beam-warming limiter and the Lax-Wendroff limiter, then choose the minimum one in magnitude.

In conclusion, the methods with all these limiters will have second-order accuracy for constant coefficients of the first derivative in the smooth region. But for the linear limiters, oscillations may appear near the discontinuities. The high-resolution limiters can avoid the oscillations near the discontinuities.

4.4 Numerical approximation for the Black-Scholes PDE

Since in the Black-Scholes equation \( rS > 0 \), the first derivative will be approximated as follows:

\[ rS \frac{\partial V}{\partial S} \approx rS_j \frac{V_{j+1}^n - V_j^n}{\Delta S} - \frac{\tilde{F}_{j+\frac{1}{2}}^n - \tilde{F}_{j-\frac{1}{2}}^n}{\Delta S} \]

where

\[ \tilde{F}_{j+\frac{1}{2}}^n = \frac{1}{2} rS_j (1 - \frac{\Delta t}{\Delta S} rS_j) \delta_{j+1}^n, \]

\[ \tilde{F}_{j-\frac{1}{2}}^n = \frac{1}{2} rS_{j-1} (1 - \frac{\Delta t}{\Delta S} rS_{j-1}) \delta_j^n \]

4.4.1 Explicit method

Here we combine (5), (7) and (14) to numerically approximate the Black-Scholes equation:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

\[
\Rightarrow \frac{V_j^n - V_{j-1}^{n-1}}{\Delta t} + \frac{1}{2} \sigma^2 S_j^2 \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta S^2} + rS_j \frac{V_{j+1}^n - V_j^n}{\Delta S} + \frac{\tilde{F}_{j+\frac{1}{2}}^n - \tilde{F}_{j-\frac{1}{2}}^n}{\Delta S} - rV_j^n = 0
\]

for \( j = 2, \cdots, M \).

According to the spatial boundary condition, which is explained in (8) and (9), this
method can be formulated as

\[
\begin{pmatrix}
V_2 \\
V_3 \\
\vdots \\
V_{M-1} \\
V_M
\end{pmatrix}
^{(n-1)} =
\begin{pmatrix}
b_2 + 2a_2 & c_2 - a_2 & 0 \\
a_3 & b_3 & c_3 & \ldots \\
& a_{M-1} & b_{M-1} & c_{M-1} & \ldots \\
0 & & a_M - c_M & b_M + 2c_M
\end{pmatrix}
\begin{pmatrix}
V_2 \\
V_3 \\
\vdots \\
V_{M-1} \\
V_M
\end{pmatrix}
^{(n)}
\]

where

\[
a_j = \frac{1}{2} \sigma_j^2 S_j^2 \Delta t \Delta S^2,
\]

\[
b_j = 1 - \sigma_j^2 S_j^2 \Delta t \frac{\Delta t}{\Delta S} rS_j - rdt,
\]

\[
c_j = \frac{1}{2} \sigma_j^2 S_j^2 \Delta t \frac{\Delta t}{\Delta S} rS_j,
\]

\[
d_j = -\frac{\Delta t}{\Delta S} (\tilde{F}_{j+\frac{1}{2}} - \tilde{F}_{j-\frac{1}{2}}).
\]

4.4.2 Implicit method

Here we use a modification of the Crank-Nicolson method with flux limiters, but notice that there is a difference in approximating the first derivative in space.

In the Crank-Nicolson method with flux limiters we approximate

\[
rS \frac{\partial V}{\partial S} \approx \frac{1}{2} (rS_j V^{n+1}_{j+1} - V^n_j) - \frac{\tilde{F}^{n+1}_{j+\frac{1}{2}} - \tilde{F}^{n-1}_{j-\frac{1}{2}}}{\Delta S} + rS_j \frac{V^{n+1}_{j+1} - V^{n-1}_{j-1}}{\Delta S} - \frac{\tilde{F}^{n+1}_{j+\frac{1}{2}} - \tilde{F}^{n-1}_{j-\frac{1}{2}}}{\Delta S}
\]

From equation (15) and (16), we can see that if we want to compute \( \tilde{F}^{n-1}_{j+\frac{1}{2}} \) and \( \tilde{F}^{n-1}_{j-\frac{1}{2}} \), we need to compute \( \delta^{n-1}_{j+1} \) and \( \delta^{n-1}_{j-1} \). From Section 4.3.3 with different kinds of limiters, we need to know the value of \( V^{n-1} \) in Minmod, Superbee and MC limiters. Thus we can not compute the value of the limiter to \( rS_j \frac{V^{n+1}_{j+1} - V^{n-1}_{j-1}}{\Delta S} \), but without this limiter to this term, this method is not second order accurate any more because of the missing correction.
term. Notice that, if we use Fromm, Beam-Warming and Lax-Wendrof limiters, we can add the correction items to the left-hand side matrix instead of computing them. Hence, we remove the term \( rS_j \frac{V_{j+1}^n - V_j^{n-1}}{\Delta S} \) and use the approximation mentioned in (14).

With this mixed explicit-implicit method, the Black-Scholes equation can be approximated as:

\[
\frac{V_j^n - V_j^{n-1}}{\Delta t} + \frac{1}{4} \sigma^2 S_j^2 \left( \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta S^2} + \frac{V_j^{n-1} - 2V_{j-1}^{n-1} + V_{j-2}^{n-1}}{\Delta S^2} \right) + rS_j \frac{V_{j+1}^n - V_j^n}{\Delta S} - \frac{1}{2} \sigma^2 S_j^2 \left( \frac{V_{j+1}^n - 2V_{j-1}^n + V_{j-2}^{n-1}}{\Delta S} \right) = 0
\]

This method can be formulated as a linear system:

\[
\begin{pmatrix}
\hat{b}_2 + 2\hat{a}_2 & \hat{c}_2 - \hat{a}_2 & 0 \\
\hat{a}_3 & \hat{b}_3 & \hat{c}_3 \\
& \ddots & \ddots & \ddots \\
\hat{a}_{M-1} & \hat{b}_{M-1} & \hat{c}_{M-1} & 0 \\
0 & \hat{a}_M - \hat{c}_M & \hat{b}_M + 2\hat{c}_M & \hat{a}_M \\
\end{pmatrix}
\begin{pmatrix}
V_2 \\
V_3 \\
\vdots \\
V_{M-1} \\
V_M \\
\end{pmatrix}
= 
\begin{pmatrix}
\hat{b}_2 + 2\hat{a}_2 & c_2 - a_2 & 0 \\
a_3 & b_3 & c_3 \\
& \ddots & \ddots & \ddots \\
a_{M-1} & b_{M-1} & c_{M-1} & 0 \\
0 & a_M - c_M & b_M + 2c_M & \end{pmatrix}
\begin{pmatrix}
V_2^{(n-1)} \\
V_3^{(n-1)} \\
\vdots \\
V_{M-1}^{(n-1)} \\
V_M^{(n-1)} \\
\end{pmatrix}
+ 
\begin{pmatrix}
d_1 \\
d_2 \\
\vdots \\
d_{M-1} \\
d_M \\
\end{pmatrix}
\]

where

\[
\hat{a}_j = \frac{1}{4} \sigma^2 S_j^2 \frac{\Delta t}{\Delta S^2}, \\
\hat{b}_j = \frac{1}{2} \sigma^2 S_j^2 \frac{\Delta t}{\Delta S^2} + \frac{1}{2} r \Delta t, \\
\hat{c}_j = \frac{1}{4} \sigma^2 S_j^2 \frac{\Delta t}{\Delta S^2}, \\
a_j = \frac{1}{4} \sigma^2 S_j^2 \frac{\Delta t}{\Delta S^2}, \\
b_j = 1 - \frac{1}{2} \sigma^2 S_j^2 \frac{\Delta t}{\Delta S^2} - rS_j \frac{\Delta t}{\Delta S} - \frac{1}{2} r \Delta t, \\
c_j = \frac{1}{4} \sigma^2 S_j^2 \frac{\Delta t}{\Delta S^2}, \\
d_j = \frac{\Delta t}{\Delta S} (\tilde{F}_{j+\frac{1}{2}}^n - \tilde{F}_{j-\frac{1}{2}}^n).
\]
4.4.3 Splitting method

This idea in the splitting method is to split the space operator of the Black-Scholes equation into three parts, which means split one time step into three small steps. This was proposed by Gilbert Strang in 1968 [9]

First step: Firstly, we approximate the first derivative part explicitly based on half time step.

\[ V^{n-\frac{1}{3}} = V^n + \frac{\Delta t}{2} rS \frac{\partial V}{\partial S} \]

\[ \Rightarrow V^{n-\frac{1}{3}}_j = V^n_j + \frac{\Delta t}{2} rS_j \frac{V_{j+1}^n - V_j^n}{\Delta S} - \frac{1}{2} \frac{\Delta t}{\Delta S} (\tilde{F}^n_{j+\frac{1}{2}} - \tilde{F}^n_{j-\frac{1}{2}}) \]

Second step: Then, we approximate the second derivative part implicitly based on one time step.

\[ V^{n-\frac{2}{3}} = V^{n-\frac{1}{3}} + \frac{\Delta t}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r\Delta tV \]

\[ \Rightarrow V^{n-\frac{2}{3}}_j = V^{n-\frac{1}{3}}_j + \frac{1}{2} \frac{\Delta t}{\Delta S^2} (\frac{V_{j+1}^{n-\frac{1}{3}} - 2V_j^{n-\frac{1}{3}} + V_{j-1}^{n-\frac{1}{3}}}{\Delta S^2} - \frac{V_{j+1}^{n-\frac{1}{3}} - 2V_j^{n-\frac{1}{3}} + V_{j-1}^{n-\frac{1}{3}}}{\Delta S^2}) \]

\[ - \frac{1}{2} r\Delta t (V^{n-\frac{1}{3}}_j + V_j^{n-\frac{2}{3}}) \]

Last step: Finally, we approximate the first derivative part explicitly again based on half time setp.

\[ V^{n-1} = V^{n-\frac{2}{3}} + \frac{\Delta t}{2} rS \frac{\partial V}{\partial S} \]

\[ \Rightarrow V^{n-1}_j = V^{n-\frac{2}{3}}_j + \frac{\Delta t}{2} rS_j \frac{V_{j+1}^{n-\frac{2}{3}} - V_j^{n-\frac{2}{3}}}{\Delta S} - \frac{1}{2} \frac{\Delta t}{\Delta S} (\tilde{F}^n_{j+\frac{1}{2}} - \tilde{F}^n_{j-\frac{1}{2}}) \]

4.5 Numerical approximation for the transferred PDE

In the transferred differential equation (3), the coefficient of the first derivative is \(-\frac{\sigma^2}{2} < 0\), which is a negative constant. Then the first derivative will be approximated as follows:

\[ -\frac{\sigma^2}{2} \frac{\partial V}{\partial x} \approx -\frac{\sigma^2}{2} \frac{V^n_j - V^n_{j-1}}{\Delta x} - \frac{\tilde{F}^n_{j+\frac{1}{2}} - \tilde{F}^n_{j-\frac{1}{2}}}{\Delta x} \] (17)

where

\[ \tilde{F}^n_{j+\frac{1}{2}} = \frac{1}{4} \sigma^2 (1 - \frac{1}{2} \frac{\Delta t}{\Delta x}) \delta^n_j, \]

\[ \tilde{F}^n_{j-\frac{1}{2}} = \frac{1}{4} \sigma^2 (1 - \frac{1}{2} \frac{\Delta t}{\Delta x}) \delta^n_{j-1} \]
4.5.1 Explicit method

Here we combine (5), (7) and (17) to numerically approximate the transferred differential equation:

\[
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \left( \frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x} \right) = 0
\]

\[
\Rightarrow \frac{V^n_j - V^{n-1}_j}{\Delta t} + \frac{1}{2} \sigma^2 \left( \frac{V^n_{j+1} - 2V^n_j + V^n_{j-1}}{\Delta x^2} - \frac{V^{n-1}_j - V^{n-1}_{j-1}}{\Delta x} \right) + \frac{\tilde{F}^{n}_{j+\frac{1}{2}} - \tilde{F}^{n}_{j-\frac{1}{2}}}{\Delta x} = 0
\]

Notice that, for a smooth solution, the method with the limiters in Section 4.3.3 has the second-order accuracy \[9\].

4.5.2 Implicit method

For the same reason as explained in Section 4.4.2, we approximate the first derivative in (17) explicitly in time and the second derivative implicitly to arrive at:

\[
\frac{V^n_j - V^{n-1}_j}{\Delta t} + \frac{1}{4} \sigma^2 \left( \frac{V^n_{j+1} - 2V^n_j + V^n_{j-1}}{\Delta x^2} + \frac{V^{n-1}_j - 2V^{n-1}_{j-1} + V^{n-1}_{j-1}}{\Delta x^2} \right) - \frac{1}{2} \sigma^2 \frac{V^n_j - V^{n-1}_j}{\Delta x} - \frac{\tilde{F}^{n}_{j+\frac{1}{2}} - \tilde{F}^{n}_{j-\frac{1}{2}}}{\Delta x} = 0
\]

4.5.3 Splitting method

Split the transferred equation into three parts as in Section 4.4.3.

First step:

\[
V^{n-\frac{1}{2}} = V^n - \frac{\Delta t}{2} \frac{\sigma^2 \partial V}{\partial x}
\]

\[
\Rightarrow V^{n-\frac{1}{2}} = V^n_j - \frac{\Delta t}{2} \frac{1}{\sigma^2} \frac{V^n_{j+1} - V^n_{j-1}}{\Delta x} - \frac{1}{2} \Delta t \frac{1}{\Delta x S} \left( \tilde{F}^{n}_{j+\frac{1}{2}} - \tilde{F}^{n}_{j-\frac{1}{2}} \right)
\]

Second step:

\[
V^{n-\frac{3}{2}} = V^{n-\frac{1}{2}} + \frac{\Delta t}{2} \frac{\sigma^2 \partial^2 V^n}{\partial x^2}
\]

\[
\Rightarrow V^{n-\frac{3}{2}} = V^{n-\frac{1}{2}}_j + \frac{1}{2} \Delta t \frac{1}{\sigma^2} \left( \frac{V^{n-\frac{3}{2}}_{j+1} - 2V^{n-\frac{3}{2}}_j + V^{n-\frac{3}{2}}_{j-1}}{\Delta x^2} + \frac{V^{n-\frac{3}{2}}_{j+\frac{1}{2}} - 2V^{n-\frac{3}{2}}_{j-\frac{1}{2}} + V^{n-\frac{3}{2}}_{j-\frac{3}{2}}}{\Delta x^2} \right)
\]

Last step:

\[
V^{n-1} = V^{n-\frac{3}{2}} - \frac{\Delta t}{2} \frac{1}{\sigma^2} \frac{\partial V^{n-\frac{3}{2}}}{\partial x}
\]
\[ V_{n-1}^j = V_{n-1}^{j-\frac{3}{2}} - \Delta t \frac{1}{2} \sigma^2 \frac{V_{n-1}^{j-\frac{3}{2}} - V_{n-1}^{j-\frac{3}{2}}}{\Delta x} - \frac{1}{2} \Delta x \left( \frac{F_{n-1}^{j+\frac{1}{2}} - F_{n-1}^{j-\frac{1}{2}}}{\Delta x} \right) \]

### 4.6 The discontinuous dividend

We have introduced the method how to deal with discontinuous dividend in Section 3, but there is another problem with the dividend. When the dividend is paid, the stock price \( S \) will be reduced by a known value. However, in the numerical approximation, we are using the grid points in the mesh. If \( S \) reduces to a value which is not exactly on the grid point, we need to introduce some approximation in this case.

#### 4.6.1 European option

In the case of a European call option, if \( S \) takes a value between the grid points, we can use linear interpolation. For example, let \( \hat{S} \) be the stock price before the dividend is paid, and \( \hat{S}_j \) is between two grid points \( S_j \) and \( S_{j+1} \). Then we can use these two points \( S_j \) and \( S_{j+1} \) to approximate at \( \hat{S}_j \). This solution can be used in some other kinds of options such as American put option.

#### 4.6.2 Down and out digital barrier

In this case, it is more complicated than the European call option, because there is a barrier and when the stock price is smaller than that barrier the option price is 0. So the problem will be focused on the first point near the barrier \( B \) when the dividend is paid and the stock price changes to a value away from a grid point. Here three points will be used, the price option \( V_0 \) at the barrier, which is not at a grid point, and the first two points next to the barrier \( V_1 \) and \( V_2 \).

Firstly we use the Taylor expansion around \( V_1 \):

\[
V_0 \approx V_1 - \Delta S_0 \frac{\partial V_1}{\partial S} + \frac{\Delta S_0^2}{2} \frac{\partial^2 V_1}{\partial S^2} - \frac{\Delta S_0^3}{6} \frac{\partial^3 V_1}{\partial S^3} \tag{18}
\]

\[
V_2 \approx V_1 + \Delta S \frac{\partial V_1}{\partial S} + \frac{\Delta S^2}{2} \frac{\partial^2 V_1}{\partial S^2} + \frac{\Delta S^3}{6} \frac{\partial^3 V_1}{\partial S^3} \tag{19}
\]
where $\Delta S_0$ is the space step between $V_0$ and $V_1$ and $\Delta S$ is the space between $V_1$ and $V_2$.

And also we can approximate $\frac{\partial V_1}{\partial S}$ and $\frac{\partial^2 V_1}{\partial S^2}$ using $V_0, V_1$ and $V_2$:

$$\frac{\partial V_1}{\partial S} \approx a_0 V_0 + a_1 V_1 + a_2 V_2$$ (20)

$$\frac{\partial^2 V_1}{\partial S^2} \approx a_0 V_0 + a_{11} V_1 + a_{22} V_2$$ (21)

Now put (18) and (19) into (20), we can get:

$$\frac{\partial V_1}{\partial S} \approx (a_0 + a_1 + a_2) V_1 + (a_2 \Delta S - a_0 \Delta S_0) \frac{\partial V_1}{\partial S} + \left( \frac{a_0 \Delta S_0^2}{2} - \frac{a_2 \Delta S}{2} \right) \frac{\partial V_1}{\partial S^2}$$ (22)

Comparing the coefficients of (22)

$$\begin{cases} 
  a_1 + a_2 + a_3 = 0 \\
  a_2 \Delta S - a_0 \Delta S_0 = 1 \\
  a_0 \Delta S_0^2 + a_2 \Delta S^2 
\end{cases}$$ (23)

Finally, $\frac{\partial V_1}{\partial S}$ can be approximated as following

$$\frac{\partial V_1}{\partial S} \approx -\frac{\Delta S_0}{\Delta S(\Delta S + \Delta S_0)} V_0 + \frac{\Delta S_0 - \Delta S}{\Delta S \Delta S_0} V_1$$ (24)

Again, put (18) and (19) into (21), and compare the coefficients

$$\frac{\partial^2 V_1}{\partial S^2} \approx \frac{2}{\Delta S(\Delta S + \Delta S_0)} V_0 - \frac{2}{\Delta S \Delta S_0} V_1$$ (25)

Hence, when the stock price reduces to a value which is not on the defined mesh, we can use (24) and (25) to approximate the PDE on the first point near the barrier.

### 4.6.3 Up and out digital barrier option

This case is quite similar to the Down and out digital barrier. The only difference is that the first value of $S$, where $V$ is not zero, is less than $S_0$, not larger than $S_0$.

### 4.7 Stability and oscillation

First of all, let us have a look at the result of the normal numerical approximation of Down and out digital barrier option with one discontinuous dividend with a central difference approximation of the first derivative and Euler backward in time. Here we use a
small volatility $\sigma = 0.025$ to display the oscillation.

Figure 1: Numerical approximation of Down and out digital barrier option with one discontinuous dividend

We can see clearly from the figures above, there exist oscillations near the discontinuity of the Digital barrier option. Later on we will introduce the result of an oscillation-free method.

Furthermore, for the explicit method with upwind differencing, there is a necessary condition for stability of the solution on the first derivative, and in our case the Courant-Friedrichs-Lewy condition (CFL-condition) is:

$$| \frac{u_1(S) \cdot \Delta t}{\Delta S} | \leq 1$$  \hspace{1cm} (26)

where $u_1(S)$ is the coefficient of the first derivative.

And also for the second derivative, there is another necessary condition for stability of an explicit time implementation:

$$| \frac{u_2(S) \cdot \Delta t}{\Delta S^2} | \leq 1$$  \hspace{1cm} (27)

where $u_2(S)$ is the coefficient of the second derivative.

For the implicit method and the splitting method(also implicit), only condition (26) is required.
5 Accuracy of numerical methods

5.1 Local truncation error

Let us look at the color equation (10) again. If $u(S)$ is a function of $S$, then the local truncation error using flux limiters can be found in [5] as

$$-\frac{1}{2}[\Delta S - u(S)\Delta t]u'(S)V_S(S, t) + \mathcal{O}(\Delta S^2)$$  \hspace{1cm} (28)

And the local truncation error for upwind method is [5]

$$\frac{1}{2}\Delta S u(S)V_{SS}(S, t) + \mathcal{O}(\Delta S^2)$$  \hspace{1cm} (29)

We can see that the upwind method is of first-order accuracy. The flux limiter method is also of first order, when $u(S)$ is not constant e.g. in the case of Black-Scholes PDE, and when $u(S)$ is constant, which means $u'(S) \equiv 0$, the flux limiter method is second order accurate e.g. in the case of transferred equation.

Then let us do some error experiments with a European option using the implicit method with Black-Scholes equation and the transferred one.

5.2 Error comparison

Now we will test the accuracy of the numerical methods described above, mainly with a European call option and Digital barrier option using both Black-Scholes PDE and the transferred PDE. Furthermore, here we will calculate the Euclidean norm error and the maximum norm error, supposing that $e_j$ is the error of $j$th space point at the initial time $t = 0$, and $e$ is the vector consisting of all space points. The definitions of the terms are:

**Euclidean norm error**

$$\sqrt{\frac{\sum_{j=1}^{M+1} e_j^2}{M+1}}$$

**Maximum norm error**

$$\max_j |e_j|$$

25
5.2.1 Black-Scholes PDE

First, let us test these methods with the Black-Scholes PDE.

**European call option** The analytical solution of the Black-Scholes PDE for the European call option is:

\[
C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)}
\]

\[
d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}
\]

\[
d_2 = d_1 - \sigma \sqrt{T-t}
\]

Here \(N(\cdot)\) is the cumulative distribution of the standard normal distribution, more details can be obtained from [3]. Using this exact solution, we can compare the errors of the approximated solutions using different methods.
Considering the max norm error, we can observe from (a) and (b) that explicit method has the smallest max norm error with both upwind differencing and minmod limiters. But explicit method is not as stable as the other two implicit methods as we discussed in Section 4.7 even it seems to be a bit better than the others with a not so small space step. Moreover, we can see that when $\sigma$ is some value between 0.1 and 0.2 the max value of max norm error is obtained in both (a) and (b), but the max value is between 0.25 and 0.3 in (b), obviously bigger than the max value in (a) which is between 0.1 and 0.15. The situation with Euclidean norm error is similar to this, the max error in (d) is also larger than (c). Hence, we can say that the error with minmod limiters is larger than...
the one with upwind differencing. The local truncation error in (28) and (29) are both of first order accuracy. We can see that the figure is very flat which means $V_{SS}(S,t) = 0$ everywhere but $V_S(S,t) \neq 0$ and the error of upwind differencing can be smaller than the one with flux limiters. Furthermore, the error of implicit method and splitting method are almost the same.

Now let us reduce the time step to a sufficiently small value. Take $N = 1000$ and $\Delta t = T/N$ so the time discretization error should be very small.

Figure 3: Error comparison of European call option with B-S equation when $N = 1000, M = 50$

Let us first take a look at the max norm error. When $\sigma$ is between 0 and 0.1 the maximum value of max norm error is obtained in (a) which is above 0.1 but with the
same $\sigma$ in (b) the max norm error is much smaller which is less than 0.03. The max value of max norm error is less than 0.06 when $\sigma = 0.8$ in (b) still smaller than the one in (a). Then turn to the Euclidean norm error, the max value in (d) is above 0.02, a bit bigger than the one in (c) which is under 0.018. Hence, we can say that the max norm error with minmod limiters is smaller than the one with upwinding, but the situation is different with the Euclidean norm error. And also, we can see that all the time stepping methods have almost the same error since all of them are on top of each other, indicating that the time discretization error is small.

**Digital barrier option**  We do not have the analytical solution for the digital barrier option, so here we use the option with sufficiently small step in both space and time as the exact solution, e.g $M = 2000, N = 2000$. 
Comparing (a) and (b) we can see that the max norm errors of the implicit method and the one with minmod limiters are almost the same. From (a) the plots of the three time discretization methods are on top of each other. Hence these methods have almost the same max norm error. And the peak of max norm error appears between 9 and 10 when $\sigma$ is around 0.2. The situation of Euclidean norm error is quite similar. All of the methods have almost the same shape in (c) and the implicit method is a bit better than the other two in (d).

We compare the errors using different methods with a sufficiently small time step e.g $N = 1000, M = 50$. 

Figure 4: Error comparison of Digital barrier option with B-S equation when $N = 100, M = 20$
Figure 5: Error comparison of Digital barrier option with B-S equation when N=1000, M=50

Compared to Figure 4, both the max norm error and the Euclidean error are smaller now because of the smaller step in time. Also compare (a) and (b), (c) and (d) we can see that this is quite similar to Figure 4. The max norm errors and the Euclidean norm errors of these three methods are almost the same in (a) and (c). With this smaller time step, it is more obvious that both the max norm error and the Euclidean error of the implicit method are smaller than the errors of the other two methods.
5.3 Transferred partial equation

Then we will compare these methods for the transferred partial equation to see how they work.

5.3.1 European call option

First look at the European call option, the analytical solution is already obtained in Section 5.2.1.

![Graphs showing error comparison for European call option with transferred equation.](image)

(a) Max norm error with upwind differencing  (b) Max norm error with Minmod limiters

(c) Euclidean norm error with upwind differencing  (d) Euclidean norm error with Minmod limiters

Figure 6: Error comparison of European call option with transferred equation when N=100, M=20

In general, we can observe in figure 6 that in all of the four sub-figures, these three time methods have the same max norm error and the Euclidean norm error since they
are on top of each other. From (a) and (b), the max norm error increases as $\sigma$ increases from 0 to 0.8. But in (a) the maximum value of the max norm error is just above 1, almost half of the maximum value in (b), which is between 1.8 and 2.0. Furthermore, the Euclidean norm error is smaller than the max norm error, in (c) the maximum value is between 0.35 and 0.4, also smaller than the one in (d), which is around 0.45. Hence, we can say that the max norm error with upwinding differencing is smaller than the one with minmod limiters, the result is the same for the Euclidean norm error.

We compare the errors using different methods with a sufficiently small time step e.g $N = 1000, M = 100$ in figure 7.

![Graphs showing error comparison for different methods with upwind differencing and Minmod limiters.](image)

(a) Max norm error with upwind differencing  (b) Max norm error with Minmod limiters

(c) Euclidean norm error with upwind differencing  (d) Euclidean norm error with Minmod limiters

Figure 7: Error comparison of European call option with transferred equation when $N=1000, M=100$
With a sufficiently small time step, all the time stepping methods still have the same error in all sub-figures. But the error does not increase as $\sigma$ increases, in (a) the minimum value of the max norm error is achieved around $\sigma = 0.15$ and the minimum value is a bit less than 0.05. From (b), the minimum value is achieved around $\sigma = 0.4$ and the minimum value is very close to 0, smaller than the one in (a). Moreover, when $\sigma$ values between 0.2 and 0.5, the max norm error with minmod limiters is smaller than the one with upwind differencing. However, the maximum value in (b) is around 0.65, much larger than the one in (a), which is less than 0.3 when $\sigma = 0.8$.

### 5.3.2 Digital barrier option

Then we turn to the digital barrier option with the transferred PDE. The exact solution is defined as in Section 5.2.1.

![Graphs showing error comparison for digital barrier option](image)

(a) Max norm error with upwind differencing (b) Max norm error with Minmod limiters

![Graphs showing error comparison for digital barrier option](image)

(c) Euclidean norm error with upwind differencing (d) Euclidean norm error with Minmod limiters

Figure 8: Error comparison of Digital barrier option with transferred equation when $N=100$, $M=20$
In general, the implicit method and the splitting method have the same error with upwind differencing and minmod limiters. When $\sigma$ values between 0 and 0.4, the explicit method has the smallest max norm error and the Euclidean norm error, while $\sigma$ values between 0.4 and 0.8 the explicit method has the largest max norm error and Euclidean norm error. Moreover, there is not much difference between the methods with upwind differencing and the ones with minmod limiters by comparing (a) and (b), also (c) and (d).

Next let us choose a sufficiently small time step to see how these methods work e.g $N=1000, M=100$ in figure 9.

(a) Max norm error with upwind differencing  
(b) Max norm error with Minmod limiters

(c) Euclidean norm error with upwind differencing  
(d) Euclidean norm error with Minmod limiters

Figure 9: Error comparison of Digital barrier option with transferred equation when $N=1000, M=100$
Here we can observe clearly the explicit method has the largest max norm error and the Euclidean norm error among these three methods when $\sigma$ values from 0 to 0.8. The implicit method and the splitting method still have the same error in all sub-figures. Comparing (a) and (b) according to max norm error, and also (c) and (d) according to the Euclidean norm error, the figures are almost the same. Hence we can say that in this case the methods with upwind differencing and the ones with minmod limiters have almost the same max norm error and the Euclidean norm error.

### 5.4 Order of accuracy

Since we have introduced the local truncation of the upwind differencing method and the one using flux limiters in Section 5.1, now we will do some experiments to verify the order of accuracy. And here we will use the implicit method since the explicit method is not so stable as the implicit method and the splitting method according to Section 4.7. Moreover, the implicit method and the splitting method have almost the same max norm error and Euclidean norm error.

#### 5.4.1 Black-Scholes equation

First let us compute the Euclidean norm error with the Black-Scholes equation using different methods.

<table>
<thead>
<tr>
<th>Time step N is fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Upwind Differencing</strong></td>
</tr>
<tr>
<td>N</td>
</tr>
<tr>
<td>5000</td>
</tr>
<tr>
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</table>
Fix the number of the time steps to a sufficiently large number e.g. $N = 5000$ then the time step $\Delta t = T/N$ is sufficiently small and double the number of the space steps. From Table 1, we can see that when the number of time steps $N$ is fixed the Euclidean norm error decreases to half as the number of space steps $M$ doubles both with upwind differencing and flux limiter. So both of them are first-order accurate in space.

Then fix the number of the space steps to a sufficiently large number e.g. $M = 5000$ then the time step $\Delta t = T/N$ is sufficiently small and double the number of the space steps. According to Table 2, the Euclidean norm error using the upwind differencing decreases a bit then stays almost the same as the number of the time steps doubles. Almost the same thing happens with the Euclidean norm error using a flux limiter. Compared to the Euclidean error in Table 1 we can see that the Euclidean error is much smaller with a large $M$ than the one with a sufficiently large $N$. The main error here is caused by the space discretization, the Euclidean norm error will not change a lot when $N$ doubles.

### 5.4.2 The transferred partial equation

Then we will compute the Euclidean norm error with the transferred partial equation. First let us fix the number of the time steps to a sufficiently large number e.g. $N = 5000$ then the time step $\Delta t = T/N$ is sufficiently small and double the number of the space
steps. From table 3 the upwind method is still first-order accuracy and second-order accuracy is achieved with the flux limiter method with $u(S) \equiv 0$ in the transferred equation.

Table 3: Time step $N$ is fixed

<table>
<thead>
<tr>
<th>$N$</th>
<th>$M$</th>
<th>Euclidean norm error</th>
<th>ratio</th>
<th>$N$</th>
<th>$M$</th>
<th>Euclidean norm error</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000</td>
<td>100</td>
<td>0.011627712751436</td>
<td>1</td>
<td>100</td>
<td>0.003939184892028</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>0.006102323671270</td>
<td>0.5248</td>
<td>200</td>
<td>0.000687764164051</td>
<td>0.1746</td>
<td></td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>0.003161965845195</td>
<td>0.2719</td>
<td>400</td>
<td>0.000165587529048</td>
<td>0.0420</td>
<td></td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>0.001586444099637</td>
<td>0.1364</td>
<td>800</td>
<td>0.000045496613468</td>
<td>0.0115</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Space step $M$ is fixed

<table>
<thead>
<tr>
<th>$M$</th>
<th>$N$</th>
<th>Euclidean norm error</th>
<th>ratio</th>
<th>$M$</th>
<th>$N$</th>
<th>Euclidean norm error</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000</td>
<td>100</td>
<td>1e-3*0.103370479433037</td>
<td>1</td>
<td>100</td>
<td>1e-4*0.97169583692552</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>1e-3*0.087740485245335</td>
<td>0.0091</td>
<td>200</td>
<td>1e-4*0.446193263128922</td>
<td>0.4336</td>
<td></td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>1e-3*0.087832707245047</td>
<td>0.0091</td>
<td>400</td>
<td>1e-4*0.462172971586348</td>
<td>0.4491</td>
<td></td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>1e-3*0.089482630556937</td>
<td>0.0092</td>
<td>800</td>
<td>1e-4*0.555477129236248</td>
<td>0.5398</td>
<td></td>
</tr>
</tbody>
</table>

Here we can see that with the upwind differencing, the Euclidean error still decreases to half as the number of the space steps $M$ doubles, while with the flux limiter, the Euclidean error is reduced to a quarter as $M$ doubles. This is what we expected in Section 5.1.

Then fix the space step to a sufficiently large number and double time step. From table 4 again the error remain almost the same because the space discretization dominates the
error.
In Table 4, the Euclidean norm error decreases a bit and then stays at a certain value as the number of the time steps $N$ doubles similar to the behaviour in Table 2. The main error here is also caused by the space discretization.

6 Numerical results

From the error comparison of these methods in Section 5.2, when dealing with the European call option the methods with upwind differencing mostly have smaller error than the ones with flux limiters for the chosen space step there. With sufficiently many grid points, second order accuracy is achieved using the oscillation-free method with flux limiters based on the transferred PDE. Also according to Section 4.7 the explicit method is not as stable as the implicit methods. Here we will use the implicit method with upwind differencing to approximate some kinds of options with one discontinuous dividend to see how this oscillation-free method works with these different kinds of options. Again we will deal with both the Black-Scholes equation and the transferred partial equation.

6.1 Black-Scholes equation

First let us look at the Black-Scholes equation in Section 2.1.
6.1.1 Down and out digital barrier option

(a) Price of option at the dividend time and the initial time

(b) Price of option in the time interval

Figure 10: Numerical approximation of Down and out digital barrier option with one discontinuous dividend when $\sigma = 0.25$

In Section 4.7, we can see that oscillations will appear near the discontinuity when central differencing is used in the first derivative approximation. The figure can be found in Figure 1. Here we can see from (a), there is no oscillation near the discontinuity at the dividend time and the initial time. Furthermore, we can see how the value of the option changes with $S$ in the time interval $[0, T]$ in (b), there is no oscillation in the whole time interval.

Then we reduce the volatility to a smaller value 0.025, which may cause oscillations and check this oscillation-free method again.
(a) Price of the option at the dividend time and the initial time

Figure 11: Numerical approximation of Down and out digital barrier option and $\sigma = 0.025$

There is still no oscillation in the whole time interval in Figure 11. Hence we can say that this oscillation free method works well with the down and out digital barrier option.

6.1.2 Down and out barrier call option

This kind of option is combined of European call option and Down and out barrier option. More details can be found in Section 2.3. The solutions are found in Figure 12.

(a) Price of option at the dividend time and the initial time

(b) Price of option in the time interval

Figure 12: Down and out barrier call option with B-S equation
6.1.3 Up and out digital barrier option

The price of up and out digital barrier option changes with stock price \( S \).

(a) Price of option at the dividend time and the initial time

(b) Price of option in the time interval

Figure 13: Up and out digital barrier option with B-S equation

The results are simulated for this type of option in Figure 13.

6.1.4 Up and out barrier call option

This kind of option is combined of European call option and Up and out barrier option. More details can be found in Section 2.3.

The value of up and out barrier call option \( V \) changes with \( S \) and \( t \).

(a) Price of option at the dividend time and the initial time

(b) Price of option in the time interval

Figure 14: Up and out barrier call option with B-S equation
Still there are no oscillations for this option in Figure 14.

6.1.5 European call option

As we discussed in Section 2.3, the holder of the European call option has the right to buy the underlying assets and this option can only be exercised at the expiry time \( T \) which is defined in advance. Here we will approximate the European call option with one discontinuous dividend in Figure 15.

![Graph showing the price of European option changes with stock price S before and after the dividend](image1)

(a) Price of option at the dividend time and the initial time

![Graph showing the value of European option V changes with S and t](image2)

(b) Price of option in the time interval

Figure 15: European call option with B-S equation

The solutions exhibit no oscillations.

6.1.6 American put option

As discussed in Section 2.3, the holder of American put option has the right to sell the underlying assets before \( T \), and this kind of option can be exercised at any time before the expiry time, which is called early exercise. This means there will be a free boundary of the stock price for the American option. Here we will price the American put option with one discontinuous dividend along with the free boundary. To deal with the free boundary of the American put option, we use the iterative PSOR algorithm from [4]. The results below are exactly the same as described in [6].
The price of American option changes with stock price $S$

Before the dividend

Final condition before the dividend

After the dividend

Final condition after the dividend

(a) Price of option at the dividend time and the initial time

(b) Free boundary of American put option in the time interval

(c) Price of option in the time interval

Figure 16: Down and out barrier call option with B-S equation

Based on the approximation results above, we can say that this oscillation-free method works well with Black-Scholes equation to avoid oscillations near the discontinuities.

6.2 Transferred equation

Then we will use the transferred PDE to approximate some kinds of options with this oscillation-free method.
The price of transferred option changes with the transferred stock price $X$

Before the dividend
Final condition before the dividend
After the dividend
Final condition after the dividend

(a) Price of option at the dividend time and the initial time

Figure 17: European call option with the transferred equation

The value of transferred option $C$ changes with $X$ and $t$

(b) Price of option in the time interval

(a) Price of option at the dividend time and the initial time

Figure 18: Down and out digital barrier option with the transferred equation
The price of transferred option C changes with the transferred stock price $X$.

Before the dividend

Final condition before the dividend

After the dividend

Final condition after the dividend

(a) Price of option at the dividend time and the initial time

(b) Price of option in the time interval

Figure 19: Down and out barrier call option with the transferred equation

The value of transferred option C changes with $X$ and $t$.

(a) Price of option at the dividend time and the initial time

(b) Price of option in the time interval

Figure 20: Up and out digital barrier option with the transferred equation
The price of transferred option C changes with the transferred stock price X.

Before the dividend
Final condition before the dividend
After the dividend
Final condition after the dividend

(a) Price of option at the dividend time and the initial time

(b) Price of option in the time interval

Figure 21: Up and out barrier call option with the transferred equation

From the results above, the oscillation free method can also work well with the transferred PDE with different kinds of options.

7 Conclusion

In this thesis, we have implemented methods to solve different kinds of options, such as the European option and the American option, with discontinuous dividends in the underlying asset. Then we moved to the barrier digital option, which has a discontinuity in the final condition and this may cause oscillations near the discontinuity with standard finite difference methods. The oscillations are caused by the first derivative of the stock price, when central difference approximation is used. Instead, we computed the price of the digital barrier options with the Black-Scholes equation with oscillation free methods. During this procedure, we have implemented an explicit method, an implicit method and also a splitting method for the time approximation. As the space approximation, we use upwind differencing instead of central difference approximation, and also a method with flux limiters to improve the accuracy. These methods will remove the oscillations near the discontinuity. Furthermore, when the dividend is paid, the stock price will reduce and then the new price may not lie exactly on the mesh, which is used in the finite difference
method. We also had to approximate the option price of it using the grid points close to it. Finally, we analyze the accuracy of the methods mentioned above based on the Black-Scholes PDE and also the transferred equation.

From the results, we can see that these oscillation-free methods can solve the options with discontinuities nicely. Also from the error analysis, we can see that the methods with flux limiters can achieve second-order accuracy based on the transferred PDE. Furthermore, with these methods, we do not need more grid points near the discontinuities, which may cause the efficiency problem.

References


