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On the identification of the unrestricted Thurstonian model for ranking data

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Abstract

The identification issues of the unrestricted Thurstonian model for ranking data is the focus of the current paper. The Thurstonian framework has been proved very influential in modeling ranking data. Within this framework, the objects to be ranked are associated with a latent continuous variable, often interpreted as utility. The unrestricted Thurstonian model has a central role in the related theory development but due to the discrete and comparative nature of ranking data it faces more serious identification problems than the indeterminacy of the latent scale origin and unit. Most researchers resort to the study of the unrestricted model referring to the differences of object utilities but then the inference on object utilities becomes tricky. Maydeu-Olivares & Böckenholt (2005) suggest a strategy to overcome the identification problem of the unrestricted model referring to object utilities but this requires many extra identification constraints, additional to the ones needed for defining the scale origin and unit. In the current paper, we study the suggested identification approach to investigate its general applicability. Our findings indicate that the estimates obtained based on this approach can be seriously biased when the extra constraints deviate from the true values of the parameters. Besides, the effect of the constraints is not uniform on all estimated parameters.

Keywords: unrestricted Thurstonian model; identification.

1 Introduction

Ranking experimental designs find many applications in social and behavioral sciences when the research interest lies in investigating preferences of individuals regarding a set of objects or options (e.g. Hult et al., 1997; Florig et al., 2001; Morgan et al., 2001; Oakes & Slotterback, 2002). Some examples of ranking designs are asking consumers to rank (or order) a set of similar products or services, citizens to rank alternative policies on a certain issue, individuals to rank alternative leisure activities or sports, and voters to rank candidates applying for a certain position. A ranking design can be even used in the case of animals in order to find their preferences in food for example (e.g. Nombekela et al., 1993). Ranking designs aim to answer the questions: which objects are mostly or least preferred among a given set of objects? Is there any dominant preference structure? Are there any relationships among the objects? Objects which are perceived similarly, i.e. have a positive relationship, are expected to have very close or even adjacent ranks systematically. On the contrary, when two objects are systematically apart in rankings, that is an indication of negative relationship.

Ranking data differ from ordinal data despite their common discrete nature. In ranking designs, an integer is assigned to each object to denote its ranking within a set of objects. For ordinal data, the integer denotes the position of an object on a prespecified ordinal scale. In other words, individuals are asked to use a given scale to provide their answers, while answers in ranking data are free of scale. Thus, in ranking designs, different understandings or usages of a scale by respondents that may occur in ordinal designs are completely avoided (Brady, 1989). The price for this advantage is that the provided rankings are only valid for the specific set of objects studied.

For modeling ranking data a variety of models has been suggested; Marden (1995), and Flinger & Verducci (1993) give an extensive overview of these models. In the current study, we concentrate on Thurstonian model class (Thurstone, 1927; Böckenholt, 2006) where the basic idea of modeling and analysis is very similar to factor analysis. Each object is assumed to have a latent continuous utility driving the observed rankings. The higher the utility of an object compared to others, the higher rank it gets. The latent vector of utilities is assumed to follow a multinormal distribution and the inference is based on the estimated mean vector and covariance matrix. In the unrestricted Thurstonian model (e.g. Chan & Bentler, 1998; Maydeu-Olivares, 1999, 2002; Maydeu-Olivares & Böckenholt, 2005; Yao & Böckenholt, 1999) these parameters are allowed to be free of any structure. However, due to the discrete and comparative nature of ranking data, the unrestricted model has more identification problems than the indeterminancy of scale origin and unit (e.g. Chan & Bentler, 1998; Yao & Böckenholt, 1999). Hence, researchers resort to study the unrestricted model referring to the differences of object utilities instead, where the definition of scale unit is enough to identify the model. The disadvantage though, is that inference, especially about object relationships, becomes tricky.

To overcome the identification problems of the unrestricted model written in terms of object utilities, Maydeu-Olivares & Böckenholt (2005) suggest that one could fix additional parameters to those needed for defining the scale origin and unit. Therefore, a natural question that can be raised is whether these extra constraints affect the general suitability of the approach. We expect that the estimates of the free parameters will be biased and inconsistent unless the extra constraints coincide with the true values of the constrained parameters. We also presume that the more the deviation between the true values and the values that the parameters are fixed to, the larger the bias and inconsistency of the estimates will be. To investigate these questions we have conducted a simulation study where the suggested identification approach is employed under different types and levels of misspecification. The effect of sample size and model size have also been considered.

The structure of the current paper is as follows: Section 2 briefly presents the basics about ranking data and Thurstonian ranking modeling followed by an overview of the approach suggested by Maydeu-Olivares & Böckenholt (2005) in Section 3. In Section 4, we discuss the suggested approach and in Section 5 we report the results of our simulation study. Section 6 provides an implication of our results for the factor analytic Thurstonian model and in the last section we summarise the main conclusions.

2 Ranking data & Thurstonian models

Ranking experimental designs

Ranking designs can be distinguished into two main categories, complete and partial (e.g. Marden, 1995). In a complete ranking design, respondents are presented with a set of m objects

$\{O_1, O_2, \ldots, O_m\}$ (also referred to as options, choices, alternatives, items, or stimuli in the literature) listed in a prespecified random order and asked to assign a rank to each object according to their preference or a certain criterion. The set of ranks is $\{1, \ldots, m\}$, where 1 often refers to the best preferred object and m to the least preferred. In partial ranking, respondents are asked to rank only a certain subset of the initial set of objects or provide, in preference order, their k favorite objects ($k < m$) or their k_1 favorite and k_2 least favorite objects ($k_1 + k_2 < m$). In general, any design where not all objects are ranked falls into the category of partial ranking.

Another issue regarding the design is whether ties are allowed (e.g. Marden, 1995). When they are not allowed, respondents are not permitted to show equal preference to two or more objects. That means that each rank should be used only once. Hence, in the case of complete ranking with no ties the response is an m-dimensional vector of permuted integers from 1 to m and the sample space consists of $m!$ ranking vectors. This is the design that we focus on in the current study. Note that ranking and ordering objects is two sides of the same coin. No matter which design is followed, there is one-to-one correspondence between ranking and ordering vectors.

Log-likelihood

Let $\mathbf{r}' = (r_{O_1}, r_{O_2}, \ldots, r_{O_m})$ be the the m-dimensional vector of rankings where r_{O_i} is the rank assigned to object O_i, $i = 1, \ldots, m$. Since for complete ranking with no ties there are $m!$ possible ranking patterns, the log-likelihood for a random sample of size n is:

$$\ln L\left(\boldsymbol{\theta}; \mathbf{r}\right) = \sum_{c=1}^{m!} n_c \ln \pi_c(\boldsymbol{\theta}) \ ,$$

where $\boldsymbol{\theta}$ is a q-dimensional parameter vector (q should be much less than $m!$), n_c is the observed frequency of ranking pattern c, with $\sum_{c=1}^{m!} n_c = n$, and $\pi_c(\boldsymbol{\theta})$ is the probability under the model of ranking pattern c, with $\pi_c(\boldsymbol{\theta}) > 0$ and $\sum_{c=1}^{m!} \pi_c(\boldsymbol{\theta}) = 1$.

The Thurstonian model class

Thurstone (1927) introduced a class of models for fitting ranking data that has been highly influential in the literature (e.g. Böckenholt 1992, 1993, 2006; Chan & Bentler, 1998; Maydeu-Olivares, 1999, 2002; Maydeu-Olivares & Böckenholt, 2005; Yao & Böckenholt, 1999). There are two main assumptions in Thurstonian models: a) the observed ranks assigned to the objects are assumed to be the result of latent continuous variables, often interpreted as object utilities, and b) individual differences in object utility assessments are assumed to follow a multinormal distribution. Based on the first assumption, a higher ranking of an object compared to another reflects that a higher utility is perceived in the first object than in the second. Let

- u_j be the latent utility of object j, $j = 1, \ldots, m$, and
- $\mathbf{u}' = (u_1, u_2, \ldots, u_m)$ be the m-dimensional latent random vector containing all objects utilities,

then, the Thurstonian model for ranking data assumes that:

$$\mathbf{u} \sim N_m(\boldsymbol{\mu}, \Sigma) \ ,$$

where the parameters $\boldsymbol{\mu}$ and Σ may be unrestricted or assumed to follow a specific structure, i.e. they are assumed to be functions of a parameter $\boldsymbol{\kappa}$, $\boldsymbol{\mu}(\boldsymbol{\kappa})$ and $\Sigma(\boldsymbol{\kappa})$. To write the probability of a ranking vector π_c in terms of the parameters $\boldsymbol{\mu}$ and Σ let:

- $O^c_{(h)}$ denote the object having been assigned rank h when the complete ranking vector falls in category c, $h = 1, \ldots, m$, $c = 1, \ldots, m!$,

- $\tilde{u}^c_i = u_{O^c_{(h)}} - u_{O^c_{(h-1)}}$ be the utility difference between objects with adjacent ranks within ranking pattern c, $i = 1, \ldots, m-1$,

- $\tilde{\mathbf{u}}_c$ be the $(m-1)$-dimensional vector containing all the above utility differences, as defined above, implied by ranking pattern c,

- C_c be an $(m-1) \times m$ contrast matrix transforming vector $\mathbf{u}$ into $\tilde{\mathbf{u}}_c$; its exact form clearly depends on the ranking pattern c in question, and

- $D = \text{diag}(C_c \Sigma C_c')^{-1/2}$,

then, the ranking probability π_c can be written as follows:

$$
\begin{aligned}
\pi_c &= Pr(\mathbf{r}_c) = Pr\left(u_{O^c_{(1)}} > u_{O^c_{(2)}} > \ldots > u_{O^c_{(m)}}\right) \\
&= Pr\left(u_{O^c_{(1)}} - u_{O^c_{(2)}} > 0 \ \& \ u_{O^c_{(2)}} - u_{O^c_{(3)}} > 0 \ \& \ \cdots \ \& \ u_{O^c_{(m-1)}} - u_{O^c_{(m)}} > 0\right) \\
&= Pr\left(\tilde{u}^c_1 > 0 \ \& \ \tilde{u}^c_2 > 0 \ \& \ \cdots \ \& \ \tilde{u}^c_{m-1} > 0\right) = Pr(\tilde{\mathbf{u}}_c > \mathbf{0}) = Pr(C_c\mathbf{u} > \mathbf{0}) \\
&= \Phi_{m-1}\left(DC_c\boldsymbol{\mu};\ DC_c\boldsymbol{\Sigma}C_c'D\right), \qquad (1)
\end{aligned}
$$

where $\Phi_{m-1}(DC_c\boldsymbol{\mu}; DC_c\Sigma C_c'D)$ is the $(m-1)$-variate cumulative normal distribution with correlation matrix $DC_c\Sigma C_c'D$ evaluated at $DC_c\boldsymbol{\mu}$.

The identification problems of Thurstonian models

Due the discrete and comparative nature of ranking data, the utility random vector $\mathbf{u}$ is unique only up to a linear transformation. To have a unique solution for $\boldsymbol{\mu}$ and Σ, the origin and the unit of utility scale should be defined. The origin is usually defined by setting the utility mean of an object equal to 0, e.g. $\mu_m = 0$, and the unit by fixing a utility variance equal to 1, e.g. $\sigma_{mm} = 1$. As a consequence, the elements of $\boldsymbol{\mu}$ and Σ should be interpreted in relative terms and the inference should be based on the estimates of standardized parameters, such as correlations ρ_{ij}, variance ratios, e.g. σ_{jj}/σ_{11}, and standardized mean utility differences, e.g. $(\mu_1 - \mu_j)/\sqrt{\sigma_{11} - 2\sigma_{1j} + \sigma_{jj}}$, $j = 2, \ldots, m$. However, in the case of the unrestricted model, where $\boldsymbol{\mu}$ and Σ are free of any structure, the identification problem is more serious than the indeterminacy of scale origin and unit. Note that $\boldsymbol{\mu}$ and Σ are the parameters of an m-dimensional multinormal distribution but the ranking probability π_c is expressed in terms of an $(m-1)$-dimensional cumulative normal distribution, as it can be seen in (1). Therefore, it is only the parameters of this $(m-1)$-dimensional distribution that can be estimated in the case of the unrestricted model (e.g. Chan & Bentler, 1998).

To reformulate model (1) so that the probability π_c is written in terms of $(m-1)$-dimensional parameters, many authors (e.g. Chan & Bentler, 1998; Yao & Böckenholt, 1999) suggested to consider the random vector $\tilde{\mathbf{u}}$ which contains all utility differences with respect to a certain object utility. The reference object is arbitrarily chosen. For example, for $m = 4$ with the first object as a reference object, $\tilde{\mathbf{u}}$ gets the form

$$\tilde{\mathbf{u}} = (u_1 - u_2, u_1 - u_3, u_1 - u_4)$$

and the connection with the initial random vector $\mathbf{u}$ is

$$\tilde{\mathbf{u}} = B\mathbf{u}\,, \tag{2}$$

where B is a contrast matrix of the form

$$B = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

In the general case, with the first object as a reference, B is of dimension $(m-1) \times m$, of form similar as above, and

$$\tilde{\mathbf{u}} \sim N_{m-1}\left(\tilde{\boldsymbol{\mu}}, \tilde{\Sigma}\right) ,$$

where $\tilde{\boldsymbol{\mu}} = B\boldsymbol{\mu}$ and $\tilde{\Sigma} = B\Sigma B'$. Hence, model (1) is modified as follows:

$$\pi_c = \mathbf{\Phi}_{m-1}(\tilde{D}C_c\tilde{\boldsymbol{\mu}};\ \tilde{D}C_c\tilde{\Sigma}C_c'\tilde{D}) , \tag{3}$$

where $\tilde{D} = \text{diag}(C_c\tilde{\Sigma}C_c')^{-1/2}$, C_c is now an $(m-1)\times(m-1)$ contrast matrix transforming vector $\tilde{\mathbf{u}}$ into $\tilde{\mathbf{u}}_c$ in a similar way as before. Since the model now refers to utility differences, only the scale unit is needed to be defined. That can be done by fixing either a difference utility mean, e.g. $\tilde{\mu}_1 = a$, $a \neq 0$, or a difference utility variance, e.g. $\tilde{\sigma}_1 = b$, $b > 0$. As Chan & Bentler (1998) point out, fixing a variance may be preferable because one avoids to decide the sign of constant a. Having defined the scale unit, the unrestricted model in (3) is completely identified.

On the contrary, to identify the model in equation (1), at least m constraints should be imposed on the covariance matrix Σ in addition to these for defining the scale origin and unit. That can be easily seen by observing the relationship connecting the unrestricted covariance of object utilities Σ with the unrestricted covariance of object utility differences $\tilde{\Sigma}$, $\tilde{\Sigma} = B\Sigma B'$. Given $\tilde{\Sigma}$, the matrix Σ cannot be determined uniquely by the equation, which can be seen as a linear system with $m(m-1)/2$ equations and $m(m+1)/2$ unknowns. There are always m more unknowns than equations and therefore, infinite solutions for Σ unless at least m elements of Σ are fixed. On the other hand, the unrestricted object utility mean vector $\boldsymbol{\mu}$ can be derived from the equation $\tilde{\boldsymbol{\mu}} = B\boldsymbol{\mu}$, given $\tilde{\boldsymbol{\mu}}$, by simply employing the constraint defining the scale origin.

Interpretation and limitations of Thurstonian models

Ranking the elements of the mean vector $\boldsymbol{\mu}$ gives the ranking pattern with the highest probability under the model which, in turn, reveals the dominant preference pattern in the population. As mentioned before, the parameters of interest are the standardized ones. The correlations of object utilities reflect the relationships among the objects. High correlations indicate that the corresponding objects are perceived very similarly and are expected to have adjacent or very close ranks regardless of their exact position in the complete ranking. High negative correlations imply that the corresponding objects are seen as contradictory alternatives and are expected to be quite apart in the full ranking vector. Ranking the variances of object utilities or taking their ratios indicate the relative degree of consensus within the population about object utilities. Standardized mean differences show how closely the object utilities are perceived on average.

However, in ranking designs inference is valid only for the specific set of objects studied. Besides, Thurstonian models face certain limitations. The assumption that the object utilities follow multinormal distribution requires that ranking data come from a homogeneous population and forces the modeled multinomial distribution to be unimodal. Thurstone (1927) had already warned that normality is not a safe assumption if a group of respondents is split in some systematic way. If systematic relationships between different types of respondents and different objects are suspected then other types of models like mixture models (e.g. Marden, 1995) and latent class models (e.g. Croon, 1993) are probably more appropriate. Moreover, the modeled multinomial distribution is forced to be to some extent monotone in the sense that the probabilities of ranking patterns are decreasing as rankings depart from the mode ranking in distance. This monotonicity becomes stricter in the case where all covariances of object utilities are assumed to be zero.

3 A suggested approach to identify the unrestricted model with respect to object utilities

Alternative representation of ranking vectors

Maydeu-Olivares & Böckenholt (2005) employ an alternative representation of ranking vectors (see also e.g. Maydeu-Olivares, 1999; 2002) which facilitates the presentation of a Thurstonian ranking model as a Structural Equation Model (SEM) with ordinal observed data. In particular, any m-dimensional complete ranking vector can be written as an $\tilde{m}$-dimensional vector of binary variables, where $\tilde{m} = m(m-1)/2$ is the number of all possible pairs out of m objects. Each binary variable corresponds to a pairwise comparison and takes the value 1 if, for a given complete ranking pattern, the first object of the pair is preferred to the second, and 0 otherwise. Let $y_{c'} = y_{\{i,j\}}$ be the binary variable corresponding to the comparison between object i and j, $i = 1, \ldots, m-1$, $j = i+1, \ldots, m$, j running faster than i, and $c' = 1, \ldots, \tilde{m}$; then

$$y_{c'} = y_{\{i,j\}} = \begin{cases} 1 & \text{if } O_i \text{ is preferred to } O_j \\ 0 & \text{otherwise} \end{cases} .$$

Hence, a ranking vector $\mathbf{r}' = (r_{O_1}, r_{O_2}, \ldots, r_{O_m})$ can be presented in the form $\mathbf{y}' = (y_1, y_2, \ldots, y_{\tilde{m}})$, with both vectors containing exactly the same information. Note that, in the general case of $\tilde{m}$ binary variables, there are $2^{\tilde{m}}$ possible response patterns, but in complete ranking designs only these $m!$ patterns of binary variables corresponding to the $m!$ different ranking vectors are used. To illustrate this alternative representation, let $m = 4$ and let define the 6 binary variables as follows: $y_1 = O_1$ is preferred to O_2, $y_2 = O_1$ is preferred to O_3, $y_3 = O_1$ is preferred to O_4, $y_4 = O_2$ is preferred to O_3, $y_5 = O_2$ is preferred to O_4, and $y_6 = O_3$ is preferred to O_4. Then, the ranking vector $(2, 1, 4, 3)$, for instance, can be written as $(0, 1, 1, 1, 1, 0)$. On contrary, the pattern $(1, 1, 1, 1, 0, 1)$ does not correspond to a complete ranking vector since the answers to variables y_4, y_5, and y_6 do not allow us to order objects O_2, O_3, and O_4. Such a situation can happen in paired-comparison designs and is referred to as intransitivity in the literature (e.g. Maydeu-Olivares, 2002; Maydeu-Olivares & Böckenholt 2005; Tsai & Böckenholt, 2006).

Model formulation

It is assumed that there are $\tilde{m}$ underlying continuous variables partially observed through their corresponding $\tilde{m}$ binary variables and the connection between the two is:

$$y_{c'} = \begin{cases} 1 & \Longleftrightarrow \tilde{u}^*_{c'} > 0 \\ 0 & \Longleftrightarrow \tilde{u}^*_{c'} < 0 \end{cases} , \tag{4}$$

where

$$\tilde{u}^*_{c'} = u_{O_i} - u_{O_j} ,$$

i.e. $\tilde{u}^*_{c'}$ is the utility difference of object O_i from object O_j, both members of pair c', $c' = 1, \dots, \tilde{m}$, $i = 1, \dots, m-1$, and $j = i+1, \dots, m$. The $\tilde{m}$-dimensional vector $\tilde{\mathbf{u}}^*$ of underlying utility differences, in turn, measures the m-dimensional latent vector $\mathbf{u}$ of object utilities in the following way:

$$\tilde{\mathbf{u}}^* = A\mathbf{u}\,, \tag{5}$$

where A is an $\tilde{m} \times m$ contrast matrix forming the utility differences of all possible pairs. Each row of the matrix corresponds to a pair and each column to an object. The elements of row c' which corresponds to the object pair $\{O_i, O_j\}$ take the values:

$$[A]_{c'k} = \begin{cases} 1 & \text{if } k = i \\ -1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}, \; k = 1, \dots, m.$$

To illustrate, for $m = 4$, $\tilde{\mathbf{u}}^* = (u_1 - u_2, u_1 - u_3, u_1 - u_4, u_2 - u_3, u_2 - u_4, u_3 - u_4)$ and A is of value

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Hence, equation (5) is a very special measurement model where the loading matrix is known, equal to A, and there are no measurements errors. It also implies that $\tilde{\mathbf{u}}^* \sim N_{\tilde{m}}\left(\tilde{\boldsymbol{\mu}}^*, \tilde{\boldsymbol{\Sigma}}^*\right)$ with $\tilde{\boldsymbol{\mu}}^* = A\boldsymbol{\mu}$ and $\tilde{\boldsymbol{\Sigma}}^* = A\Sigma A'$. Note that $\tilde{\boldsymbol{\Sigma}}^*$ is a singular covariance matrix, since the rank of matrix A is $m-1$ (Maydeu-Olivares, 1999).

The structural model in the case of the unrestricted Thurstonian model with respect to $\mathbf{u}$ is of the form:

$$\mathbf{u} = \boldsymbol{\mu} + \boldsymbol{\zeta}\,, \text{ where } \boldsymbol{\zeta} \sim N_m(\mathbf{0}, \Sigma)\,. \tag{6}$$

Thus, the SEM consists of both equations (5) and (6) and corresponds to a SEM with only endogenous latent variables. For this specific model, both thresholds and tetrachoric correlations are modeled and they are functions of $\boldsymbol{\mu}$ and Σ as follows:

$$\begin{aligned} \boldsymbol{\tau}_{\tilde{u}^*} &= -[\mathrm{diag}(\Sigma)]^{-1/2} A\boldsymbol{\mu}\,, \text{ and} \\ \mathrm{P}_{\tilde{u}^*} &= -[\mathrm{diag}(A\Sigma A')]^{-1/2} A\Sigma A' - [\mathrm{diag}(A\Sigma A')]^{-1/2}\,, \end{aligned}$$

and the ranking probability π_c can be written in the form:

$$\pi_c = \int_{R_{\tilde{m}}} \cdots \int_{R_1} \phi(\tilde{\mathbf{u}}^*;\ \tilde{\boldsymbol{\mu}}^*, \tilde{\boldsymbol{\Sigma}}^*) d\tilde{\mathbf{u}}^*\,,$$

where $R_i = \begin{cases} (0, \infty) & \text{if } \tilde{u}^*_i > 0 \\ (-\infty, 0) & \text{if } \tilde{u}^*_i < 0 \end{cases}$, $i = 1, \dots, \tilde{m}$ (Maydeu-Olivares & Böckenholt, 2005).

Model identification

As discussed in Section 2, apart from defining the scale origin and unit by fixing, for example, $\mu_1 = 0$ and $\sigma_{11} = 1$, m extra constraints on the unrestricted covariance matrix Σ should be set to get the model identified. Maydeu-Olivares & Böckenholt (2005) suggest that one could fix the covariances of the m^{th} object with all the other objects equal to 0, i.e. $\sigma_{mi} = 0$, $i = 1, \ldots, m-1$, and the variance of the last object σ_{mm} equal to 1.

Model estimation

The big advantage of formulating Thurstonian ranking models as SEMs is that the conventional 3-stage estimating approach of SEM with ordinal data can be applied and the SEM packages can be used for this. In the first stage, the thresholds $\boldsymbol{\tau}_{\tilde{u}^*}$ are estimated by maximizing the likelihood functions of binary rankings. In the second stage, given the estimates of thresholds, the tetrachoric correlations are estimated by maximizing the likelihood functions of trinary rankings and pairs of binary rankings[1]. Finally, in the third stage, given the estimates of stages one and two, the model parameters $\boldsymbol{\mu}$ and Σ are estimated using generalised least squares. Unweighted least squares (ULS) or diagonally weighted least squares (DWLS) are mostly used. The degrees of freedom should be appropriately adjusted as there are redundancies among the thresholds $\boldsymbol{\tau}_{\tilde{u}^*}$ and the tetrachoric correlations coming from the redundancies among the binary and trinary rankings (Brady, 1989; Maydeu-Olivares, 1999). In particular, $df = (\tilde{m}(\tilde{m}+1)/2) - p - s$, where $\tilde{m}(\tilde{m}+1)/2$ is the total number of $\tilde{m}$ thresholds and $\tilde{m}(\tilde{m}-1)/2$ tetrachoric correlations, p is the number of parameters to be estimated, and s is the number of redundancies among the thresholds and tetrachoric correlations which is equal to $m(m-1)(m-2)/6$ (Maydeu-Olivares & Böckenholt, 2005; Maydeu-Olivares, 1999).

4 Discussion of the suggested identification approach

In Thurstonian ranking models, the interest lies in estimating the standardized parameters as accurately and efficiently as possible. In the unrestricted model, we do not want to assume anything about the standardized parameters but rather let the data "speak". Fixing a mean and a variance of an object utility to define the origin and the unit of the utility scale, respectively, makes no assumption about the standardized parameters, since these two constraints result in only rescaling the population distribution and leaving the standardized parameters unchanged. However, any further identification constraints on the elements of covariance matrix Σ imply that some correlations and/or variance ratios are assumed to follow a certain structure or be equal to certain numbers, something which may not hold in the population. Consequently, the model to be estimated may be misspecified and then, the estimates of the free correlations and variance ratios are expected to be biased. Note that any extra constraints in Σ have no impact on the estimates of standardized mean differences $(\mu_1 - \mu_i)/\sqrt{\sigma_{11} - 2\sigma_{1i} + \sigma_{ii}}$, $i = 2, \ldots, m$. In fact, these parameters can be estimated through the unrestricted model with respect to $\tilde{\mathbf{u}}$ as well. They can be written as ratios of a utility difference mean towards the corresponding utility difference variance, i.e. $\tilde{\mu}_j/\sqrt{\tilde{\sigma}_{jj}}$, $j = 1, \ldots, m-1$, given that $\tilde{\mathbf{u}}$ has been defined as in expression (2).

The identification approach suggested by Maydeu-Olivares & Böckenholt (2005) makes, in fact, assumptions about the value of a certain correlation and a certain variance ratio. Let assume that the scale unit is defined by fixing $\sigma_{11} = 1$. The general idea of the suggested approach is to fix additionally the covariances of an object utility with all the other utiltities and one variance, i.e.

[1] Binary rankings refer to the relative ranking within pairs of objects, while trinary rankings refer to the relative ranking within triplets of objects, implied by a given full ranking (see e.g. Brady, 1989). The distributions of binary rankings correspond to the univariate distributions, while these of trinary and pairs of binary rankings to the bivariate distributions.

$\sigma_{mi} = k_i$, $i = 1, \ldots, m-1$, where k_i's are constants, and $\sigma_{jj} = k_m$, for a specific j, $j = 2, \ldots, m$, and k_m is a constant. These m extra constraints along with $\sigma_{11} = 1$ defining the scale unit lead to the assumptions that

$$\rho_{mj} = \frac{k_j}{\sqrt{k_m}} \text{ and } \frac{\sigma_{jj}}{\sigma_{11}} = k_m \,.$$

As Maydeu-Olivares & Böckenholt (2005) comment, the above set of m extra identification constraints is "not unique and other identification constraints can be specified that yield equivalent model fits". Whichever set of m extra elements of Σ we choose to fix though, a certain structure or value on some correlations and/or variance ratios is implied. If one chooses to fix m covariances instead of $m-1$ and one variance, then a ratio of certain correlations is assumed to be equal to a constant. To illustrate, let $\sigma_{i1} = k_i$, where $i = 2, \ldots, m$, and k_i's are constants, and $\sigma_{st} = a$ for a specific s and t, where $s = t+1, \ldots, m$, $t = 2, \ldots, m-1$, and a is constant. Then, this implies that

$$\frac{\rho_{st}}{\rho_{s1}\rho_{t1}} = \frac{a/\sqrt{\sigma_{ss}\sigma_{tt}}}{\left(k_s/\sqrt{\sigma_{ss}1}\right)\left(k_t/\sqrt{\sigma_{tt}1}\right)} = \frac{a}{k_s k_t} \,.$$

If one chooses to fix instead the rest $m-1$ variances and one covariance, this assumes that all variance ratios and a correlation are equal to certain constants. To illustrate, let $\sigma_{ii} = k_i$, $i = 2, \ldots, m$, k_i's are constants, and $\sigma_{st} = a$ for a specific s and t, $s \neq t = 1, \ldots, m$, and a is constant. The direct implication of these constraints is the assumption that

$$\frac{\sigma_{ii}}{\sigma_{11}} = k_i, \; i = 2, \ldots, m, \text{ and } \rho_{st} = \frac{a}{\sqrt{k_{ss}k_{tt}}} \,.$$

Therefore, any m extra constraints in Σ lead to constraints on some correlations and/or variance ratios. The question that follows naturally is how these constraints affect the estimates of the remaining free correlations and variance ratios. If the constraints coincide with the true values of the parameters, then there is no misspecification and the estimates are expected to be as accurate and efficient as the estimation method permits. But when there are deviations between the fixed and the true values, how large will the bias and MSE of the estimates be? Our simulation study presented in Section 5 tries to give an answer to this question.

It is very important to emphasize that the goodness-of-fit statistics are unable to detect possible misspecifications induced by the extra identification constraints. As mentioned earlier, all possible sets of m extra constraints in Σ "yield equivalent model fits". Hence, we cannot distinguish between constraints where no misspecification occurs from those inducing serious misspecifications. Besides, we noticed that the values of goodness-of-fit statistics are exactly the same for both the unrestricted model with respect to $\mathbf{u}$ and the unrestricted model with respect to $\tilde{\mathbf{u}}$. Both models use the same information from data and estimate the same number of parameters. In fact, if we define the scale unit in the same way for both models, we can pass from the one set of parameters ($\boldsymbol{\mu}$ and Σ) to the other set ($\tilde{\boldsymbol{\mu}}$ and $\tilde{\Sigma}$) by using the relationships $\tilde{\boldsymbol{\mu}} = B\boldsymbol{\mu}$ and $\tilde{\Sigma} = B\Sigma B'$ and applying the m extra constraints adopted for the unrestricted model of $\mathbf{u}$. Hence, if the observed data come indeed from a latent multinormal distribution, then both models will present the same good fit. However, for the unrestricted model with respect to $\mathbf{u}$ there may be serious misspecifications.

5 Simulation study

Set-up of the simulation study

The simulation study focuses on the unrestricted model with respect to $\mathbf{u}$ where the identification constraints suggested by Maydeu-Olivares & Böckenholt (2005) are adopted, i.e. set $\mu_m = 0$ and $\sigma_{11} = 1$ to define the scale origin and unit, respectively, plus $\sigma_{m1} = \ldots = \sigma_{m(m-1)} = 0$ and $\sigma_{mm} = 1$. The objective is to study the impact of the last m extra constraints on the estimates of the standardized parameters (i.e. correlations, variance ratios, and standardized mean differences) in terms of bias and mean squared error (MSE) under different types and levels of misspecification (9 different situations studied). Additionally, we are interested in investigating the effects of model size ($m = 4, 7$) and sample size ($n = 500, 1000$). In total we examine 36 different experimental conditions where 2000 replicates have been carried out for each condition.

The two models considered are referred as Model I and Model II. Model I is a small model with $m = 4$ objects to be ranked, while Model II is larger with $m = 7$. Both models are studied under correct specification. The results under the correctly specified models are used as a benchmark to make comparisons with the results obtained under different types and levels of misspecification. By correct specification we mean that the population model used to generate the data is the same as the model to be estimated, while in the case of misspecification they differ. In our study, in order to generate different types and levels of misspecification, we keep the model to be estimated the same and change the population model. For Model I the estimated model is a four-dimensional normal distribution, and for Model II a seven-dimensional normal distribution with parameters $\boldsymbol{\mu}$ and Σ as specified in Table 1. For both models the implications of the m extra identification constraints are that $\rho_{mi} = 0$, $i = 1, \ldots, m-1$, and $\sigma_{mm}/\sigma_{11} = 1$.

Model I: $\mathbf{u} \sim N_4$ with	Model II: $\mathbf{u} \sim N_7$ with
$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ 0 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 1 & & & \\ \sigma_{21} & \sigma_{22} & & \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \\ 0 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 1 & & & & & & \\ \sigma_{21} & \sigma_{22} & & & & & \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & & & & \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & & & \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} & & \\ \sigma_{61} & \sigma_{62} & \sigma_{63} & \sigma_{64} & \sigma_{65} & \sigma_{66} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Table 1: The models to be estimated

The population models used to generate the data are given in Table 2. Situation 1 corresponds to the case of a correctly specified model, while situations 2-9 to cases where the model is misspecified. Situations 2-4 refer to three different levels of misspecification of covariance σ_{m1}. The true values of σ_{m1} are 0.1, 0.3, and 0.6, respectively but it is fixed to be 0 inr the model to be estimated. In terms of correlations, the true values of ρ_{m1} are 0.1, 0.3, and 0.6, respectively, but it is fixed to be 0. Situations 5-7 refer to three different levels of misspecification of variance σ_{mm}, which, in turn, implies misspecification of the variance ratio σ_{mm}/σ_{11}. The true values of σ_{mm} and of σ_{mm}/σ_{11} are 1.1, 1.5, and 2, respectively, but in the model to be estimated both are constrained to be equal to 1. Finally, situations 8-9 represent the cases where all covariances σ_{mi}, $i = 1, \ldots, m-1$, and the variance σ_{mm} are misspecified. The level of misspecification is low in situation 8 and larger in situation 9 for all the parameters. These misspecifications result in misspecifying all correlations ρ_{mi}, $i = 1, \ldots, m-1$, as well as the variance ratio σ_{mm}/σ_{11}, the true values of which, for each of the situations 8 and 9, are given in Table 3.

To study the effect of sample size, we study all the aforementioned nine situations within each model under a sample of 500 and a sample of 1000 observations.

	Model I: $\mathbf{u} \sim N_4$ with				Model II: $\mathbf{u} \sim N_7$ with						
	$\boldsymbol{\mu} = \begin{pmatrix} -0.3 \\ 0.3 \\ 0.6 \\ 0 \end{pmatrix}$ $\Sigma = \begin{pmatrix} 1 & & & \\ 0.36 & 0.8 & & \\ 0.75 & 0.2 & 1.2 & \\ [\sigma_{41}] & [\sigma_{42}] & [\sigma_{43}] & [\sigma_{44}] \end{pmatrix}$				$\boldsymbol{\mu} = \begin{pmatrix} 0.7 \\ 0.5 \\ 0.3 \\ -0.1 \\ -0.3 \\ -0.6 \\ 0 \end{pmatrix}$ $\Sigma = \begin{pmatrix} 1 & & & & & & \\ 0.4 & 0.8 & & & & & \\ 0.35 & 0.5 & 1.2 & & & & \\ 0.2 & 0.45 & 0.3 & 0.6 & & & \\ 0.15 & 0.3 & 0.2 & 0.3 & 0.9 & & \\ 0.1 & 0.2 & 0.17 & 0.1 & 0.08 & 1.4 & \\ [\sigma_{71}] & [\sigma_{72}] & [\sigma_{73}] & [\sigma_{74}] & [\sigma_{75}] & [\sigma_{76}] & [\sigma_{77}] \end{pmatrix}$						
	σ_{41}	σ_{42}	σ_{43}	σ_{44}	σ_{71}	σ_{72}	σ_{73}	σ_{74}	σ_{75}	σ_{76}	σ_{77}
situation 1	0	0	0	1	0	0	0	0	0	0	1
situation 2	0.1	0	0	1	0.1	0	0	0	0	0	1
situation 3	0.3	0	0	1	0.3	0	0	0	0	0	1
situation 4	0.6	0	0	1	0.6	0	0	0	0	0	1
situation 5	0	0	0	1.1	0	0	0	0	0	0	1.1
situation 6	0	0	0	1.5	0	0	0	0	0	0	1.5
situation 7	0	0	0	2	0	0	0	0	0	0	2
situation 8	0.11	0.15	0.06	1.1	0.05	0.08	0.14	0.13	0.1	0.08	1.1
situation 9	0.85	0.7	1	2	0.57	0.64	0.94	0.72	0.75	0.76	2

Table 2: The population models

	Model I				Model II						
	ρ_{41}	ρ_{42}	ρ_{43}	σ_{44}/σ_{11}	ρ_{71}	ρ_{72}	ρ_{73}	ρ_{74}	ρ_{75}	ρ_{76}	σ_{77}/σ_{11}
situation 8	0.105	0.160	0.052	1.1	0.048	0.085	0.122	0.160	0.101	0.064	1.1
situation 9	0.601	0.553	0.645	2	0.403	0.506	0.607	0.657	0.559	0.454	2

Table 3: The true values of the misspecified standardized parameters in situations 8 and 9

Data generation and analysis

The following steps have been taken in order to generate data for each experimental condition:

1. An m-dimensional random vector $\mathbf{u}$ is generated from $N_m(\boldsymbol{\mu}, \Sigma)$ with $\boldsymbol{\mu}$ and Σ as defined for each situation within each model in Table 2.

2. The random vector $\mathbf{u}$ is transformed to the $\tilde{m}$-dimensional vector $\tilde{\mathbf{u}}^*$ using equation (5).

3. The $\tilde{m}$-dimensional vector $\tilde{\mathbf{u}}^*$ is transformed to the $\tilde{m}$-dimensional vector of binary variables $\mathbf{y}$ using expression (4).

4. Steps 1-3 are repeated n times to generate the required sample size within each experimental condition.

Then, the analysis proceeds as follows:

1. For each of the 2000 generated data sets, the corresponding model given in Table 1 is estimated. For this we used Mplus (version 5.21), and the same input syntax file as Maydeu-Olivares & Böckenholt (2005) used in their study, provided on the webpage http://supp.apa.org/psycarticles/supplemental/met_10_3_285/met_10_3_285_supp.html.

2. All 2000 replications are screened for non-admissible solutions, which in our case this is non-positive estimated variances and correlations whose absolute value is larger than 1. The replications with improper solutions are considered as invalid observations and are eliminated from the rest of the analysis. Note that the estimates of the free standardized parameters (i.e. standardized mean differences, variance ratios, and correlations) are calculated based on the estimates of $\boldsymbol{\mu}$ and Σ.

3. After having calculated the true values of the standardized parameters for each situation within each model, the bias and MSE of estimated standardized parameters are computed following the formulas provided below.

Performance criteria

Taking into account only the valid replications within each experimental condition, the bias and MSE of the estimated standardized parameters are calculated as follows:

$$Bias = \frac{1}{R}\sum_{i=1}^{R}\left(\hat{\theta}_i - \theta\right) \ , \text{ and}$$

$$MSE = \frac{1}{R}\sum_{i=1}^{R}\left(\hat{\theta}_i - \theta\right)^2 \ ,$$

where R is the number of valid replicates, $\hat{\theta}_i$ the estimate of a standardized parameter at the i^{th} valid replication, and θ its corresponding true value.

Results

There are only three experimental conditions where the problem of non-admissible solutions occurs and all three concern Model II and situations 8-9, i.e. the situations where all m extra constraints deviate from the true values of the parameters. The specific cases and the corresponding

	Valid replications
Model II - situation 8 - $n = 500$	99.70%
Model II - situation 9 - $n = 1000$	42.05 %
Model II - situation 9 - $n = 500$	36.50 %

Table 4: Experimental conditions with valid replications less than 100%

percentage of valid replications are given in Table 4. As it can be seen, the percentage is remarkably low when all m parameters are seriously misspecified (situation 9). Moreover, once the level of misspecification is relatively high, the sample size has a small effect in ameliorating the problem of non-admissibility. As it can be seen, for Model II - situation 9, doubling the sample size caused an increase of the percentage from 36.5% to only 42.05%.

The bias and MSE for all standardized parameters and all experimental conditions are reported in Tables 7-14 given in the Appendix. In these tables, for each parameter it is reported: the true value, the mean of the estimates, the estimated bias, and the MSE. Figures 1-9 depict the bias and MSE of correlations and variance ratios for all experimental conditions. For getting a better picture of the impact of different misspecifications, the average bias and MSE over the parameters of same type within the same experimental condition has been calculated. Table 5 reports the results. Based on all these tables and figures the major conclusions have as follows:

- With no surprise, the bias and MSE for all parameters, for both models, within situation 1 (correctly specified model) are very close to zero and decreasing with the sample size.

- As expected, the bias and MSE of the standardized mean differences are not affected by any kind and level of misspecification. They are very close to zero for all experimental conditions.

- As expected, the bias and MSE of variance ratios and correlations are negatively affected by the different types and levels of misspecification. In particular, they increase (the bias in absolute value) as the degree of misspecification increases. However, the increase of bias (in absolute value) and MSE is not uniform for all parameters. An interesting pattern can be seen in Figure 5, which depicts the effect of misspecification of σ_{71} on correlations in Model II; the bias and MSE of ρ_{i1}, $i = 2, \ldots, 6$ are on average smaller than the bias and MSE of the rest of the correlations.

- When only one parameter is misspecified, the bias and MSE of variance ratios seem to be much more affected when the misspecified parameter is a covariance rather than a variance, while these two types of misspecification seem to cause similar impact on the bias and MSE of correlations. Moreover, regarding variance ratios, when the variance σ_{mm} is misspecified (situations 5, 6, and 7), for model I and for both sample sizes, the average bias and average MSE do not seem to be affected by the degree of misspecification. On the contrary, they are very close to the level of average bias and average MSE observed in situation 1. For Model II, this observation applies only for the average MSE.

- When only the covariance σ_{m1} is misspecified, both variance ratios and correlations are systematically overestimated.

- When only the variance σ_{mm} is misspecified, the correlations are overestimated systematically, while the variance ratios are overestimated when their true values are smaller than 1 and underestimated when their true values are larger than 1.

Model I ($m = 4$)	Variance ratios		Correlations	
$n = 500$	Average Bias	Average MSE	Average Bias	Average MSE
situation 1	0.009	0.033	0.001	0.007
situation 2	0.224	0.093	0.025	0.007
situation 3	0.875	0.857	0.082	0.017
situation 4	3.087	10.004	0.195	0.049
situation 5	0.007	0.032	0.030	0.008
situation 6	0.001	0.028	0.116	0.020
situation 7	0.001	0.026	0.192	0.045
situation 8	0.021	0.050	-0.040	0.012
situation 9	0.014	0.182	-0.297	0.116
$n = 1000$				
situation 1	0.005	0.017	0.001	0.004
situation 2	0.232	0.075	0.028	0.005
situation 3	0.870	0.805	0.081	0.015
situation 4	3.027	9.428	0.194	0.047
situation 5	0.008	0.016	0.029	0.004
situation 6	0.003	0.015	0.116	0.018
situation 7	0.001	0.016	0.194	0.045
situation 8	0.015	0.029	-0.037	0.006
situation 9	-0.002	0.152	-0.297	0.109

Model II ($m = 7$)	Variance ratios		Correlations	
$n = 500$	Average Bias	Average MSE	Average Bias	Average MSE
situation 1	0.008	0.039	-0.004	0.010
situation 2	0.240	0.110	0.043	0.010
situation 3	0.879	0.883	0.113	0.024
situation 4	3.062	9.953	0.196	0.054
situation 5	0.019	0.038	0.033	0.011
situation 6	0.016	0.035	0.146	0.028
situation 7	0.018	0.036	0.242	0.065
situation 8	-0.099	0.052	-0.084	0.025
situation 9	-0.295	0.149	-0.419	0.263
$n = 1000$				
situation 1	0.002	0.020	-0.002	0.005
situation 2	0.228	0.078	0.044	0.007
situation 3	0.862	0.805	0.112	0.021
situation 4	3.013	9.449	0.196	0.052
situation 5	0.004	0.019	0.033	0.006
situation 6	0.007	0.019	0.145	0.025
situation 7	0.014	0.022	0.242	0.063
situation 8	-0.107	0.032	-0.081	0.015
situation 9	-0.337	0.162	-0.458	0.298

Table 5: Average, over parameters of same type, bias and MSE for each experimental condition

- When all parameters σ_{mi}, $i = 1, \ldots, m$, and σ_{mm} are misspecified, then almost all variance ratios and correlations are underestimated.
- Interestingly, once the model is misspecified, the increase of sample size seems to play marginal role in decreasing MSE and bias. This can be seen in Figures 1-9 where the shapes of the lines and the level of values of bias and MSE are similar in graphs referring to the same experimental condition except the sample size.

To get a better idea about the effect of sample size we calculated the relative absolute difference of average bias (RAD_{Bias}) and the relative difference of average MSE (RD_{MSE}) for the two sample sizes within the same misspecification situation and model,

$$RAD_{Bias} = \frac{|Average\, Bias_{n=1000}| - |Average\, Bias_{n=500}|}{|Average\, Bias_{n=500}|}, \text{ and}$$

$$RD_{MSE} = \frac{Average\, MSE_{n=1000} - Average\, MSE_{n=500}}{Average\, MSE_{n=500}}.$$

A minus sign is interpreted as a decrease in absolute average bias or average MSE after increasing the sample size. Table 6 presents the results. As it can be seen, the RAD_{Bias} for the correlations are very close to zero for almost all situations of misspecification in both models. This implies that doubling the sample size played almost no role in decreasing the average bias of the correlations in absolute terms. Using the relative differences referring to situation 1 as a benchmark, we observe that, after doubling the sample size, the decrease in average bias and MSE is smaller for most of the situations of misspecification compared to the decrease observed in situation 1. Finally, there are few cases where the relative difference is positive implying that despite the double sample size the average bias and the average MSE did not get any smaller at all.

6 Implication of the simulation results for the factor analytic Thurstonian model

The main conclusions of Section 5 are valid for the factor analytic Thurstonian model written with respect to **u** since similar identification constraints as those for the unrestricted model of **u** are needed. The factor analytic Thurstonian model belongs to the general category of restricted Thurstonian models since the covariance matrix Σ is assumed to follow the structure imposed by factor analysis, namely $\Sigma = \Lambda\Phi\Lambda' + \Psi$, where Ψ is a diagonal matrix. The motivation for this model, and in general for restricted Thurstonian models, is to overcome the indeterminacy of the unrestricted covariance matrix Σ. However, due to the discrete and comparative nature of ranking data, the factor analytic Thurstonian model has more identification issues than a conventional exploratory factor model with ordinal data (e.g. Brady, 1989; Maydeu-Olivares & Böckenholt, 2005). Apart from the rotational indeterminacy of the loading matrix Λ and the scale unit of factors, one needs to define the origin of each common latent dimension as well as the scale origin and unit of the unique factors specific to each particular object. To define the origin of each common latent dimension the most straightforward way is to set one loading for each dimension equal to 0. The last constraints along with these needed to solve the rotational indeterminacy of

Model I ($m = 4$)	Variance ratios		Correlations	
	RAD_{Bias}	RD_{MSE}	RAD_{Bias}	RD_{MSE}
situation 1	-0.387	-0.486	0.028	-0.499
situation 2	0.036	-0.195	0.137	-0.344
situation 3	-0.006	-0.061	-0.019	-0.136
situation 4	-0.019	-0.057	-0.003	-0.039
situation 5	0.137	-0.515	-0.022	-0.461
situation 6	3.301	-0.448	0.003	-0.120
situation 7	0.349	-0.404	0.011	-0.018
situation 8	-0.291	-0.414	-0.090	-0.455
situation 9	-0.887	-0.164	0.002	-0.061
Model II ($m = 7$)				
situation 1	-0.743	-0.501	-0.598	-0.517
situation 2	-0.051	-0.287	0.019	-0.346
situation 3	-0.019	-0.088	-0.008	-0.110
situation 4	-0.016	-0.051	0.000	-0.032
situation 5	-0.817	-0.516	0.005	-0.459
situation 6	-0.586	-0.469	-0.010	-0.128
situation 7	-0.213	-0.391	0.003	-0.033
situation 8	0.081	-0.380	-0.036	-0.373
situation 9	0.141	0.086	0.092	0.132

Table 6: Relative absolute difference of average bias (ARD_{Bias}) and relative difference of average MSE (RD_{MSE}) between the two sample sizes within the same misspecification situation and model

Λ, may result in a Λ with the following form:

$$\Lambda = \begin{pmatrix} 0 & \dots & 0 \\ \lambda_{21} & \ddots & \vdots \\ & \ddots & 0 \\ \vdots & & \lambda_{uv} \\ & & \vdots \\ \lambda_{w1} & \dots & \lambda_{wv} \end{pmatrix},$$

i.e. the upper triangular part of Λ consists of 0. Such a matrix Λ along with a diagonal Ψ result in a covariance matrix Σ which has a row with all covariances equal to 0. In other words, it is assumed that the covariances of an object with all the other objects are equal to 0, same assumption as that made by the extra identification constraints for the unrestricted Thurstonian model of $\mathbf{u}$. To avoid such an assumption, one may consider the factor analytic Thurstonian model written with respect to $\tilde{\mathbf{u}}$ instead. Since that model refers to utility differences, there is no need to define the origin of each latent dimension, and the structure of $\tilde{\Sigma}$ is such that all its elements are assumed to follow a factor structure.

7 Summary and Conclusions

Thurstonian framework offers a class of models for ranking data with an easy interpretation. The mean vector $\boldsymbol{\mu}$ reveals the dominant preference structure, the standardized mean differences $(\mu_i - \mu_j)/\sqrt{\sigma_{ii} - 2\sigma_{ij} + \sigma_{jj}}$ show how similarly object utilities are perceived on average, the variance ratios σ_{ii}/σ_{jj} describe the relative variability of perceived utility for each object, and the correlations ρ_{ij} give information about the relationships among the objects. Any Thurstonian model can be written with respect to either object utilities or object utility differences. The parameter sets of the two models are connected with a certain deterministic relationship. Our aim is to estimate the aforementioned standardized parameters referring to the object utility model because then, the interpretation and inference is straightforward. However, that is not a simple task for the unrestricted Thurstonian model referring to object utilities due to serious identification problems. Although, in the unrestricted model, we are willing to let the data "speak", the identification issues force us to make assumptions about the values of some standardized parameters, additional to these constraints defining the scale origin and unit. Hence, one has to decide either to deal only with the unrestricted model referring to object utility differences which is well identified after defining the scale unit or to make the extra assumptions needed for the unrestricted model with respect to object utilities. The disadvantage of the first approach is that the inference about object utilities becomes tricky and complicated. Maydeu-Olivares & Böckenholt (2005) suggest a set of identification constraints in order to get the unrestricted model with respect to object utilities identified. In the current paper, we have studied this approach in order to investigate its general applicability. Our findings show that the approach leads to reliable estimates of standardized parameters, and subsequently inference, as long as the extra identification constraints coincide with or at least are very close to the true values of the constrained parameters. However, when this condition does not hold, the estimates of almost all correlations and variance ratios parameters are seriously biased. The level of bias and MSE increases with the misspecification level and not in a uniform way for all parameters. The increase of sample size seems to have very marginal effect in decreasing the bias and MSE. Besides, the goodness-of-fit statistics are unable to detect the misspecifications induced by the identification constraints. That means that the approach should be used with great caution. Before adopting any extra constraints one should resort to theoretical reasons or previous empirical results to verify that the extra constraints are reasonable and justified for the population in question. A research on statistical procedures testing whether the extra identifications are plausible with the observed data could enhance the general suitability of the suggested approach.

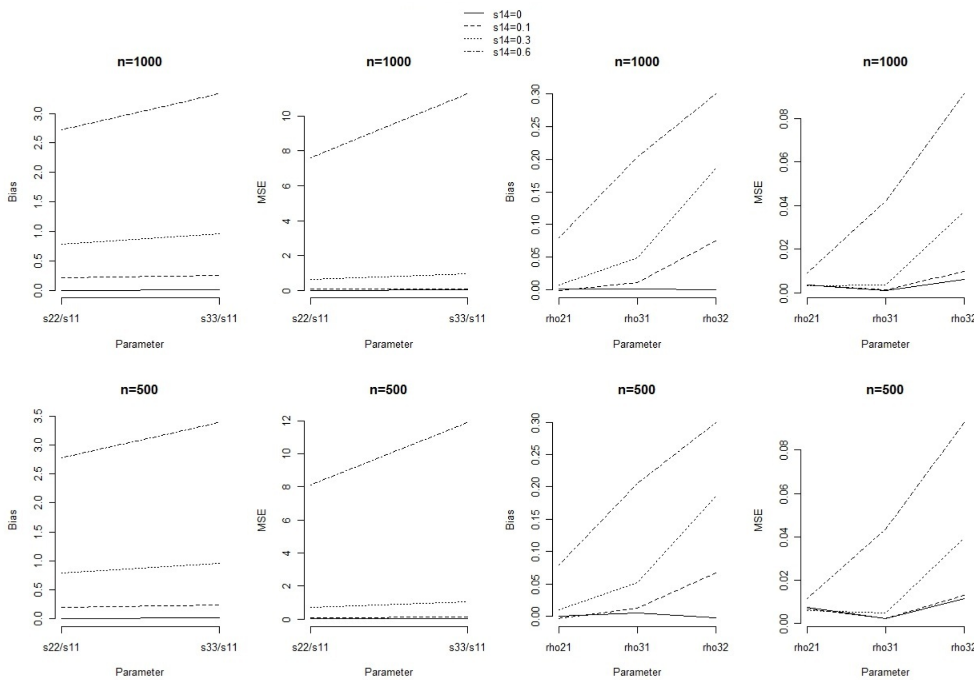

Figure 1: The effect of σ_{14} misspecification on bias and MSE of variance ratios and correlations, $m = 4$

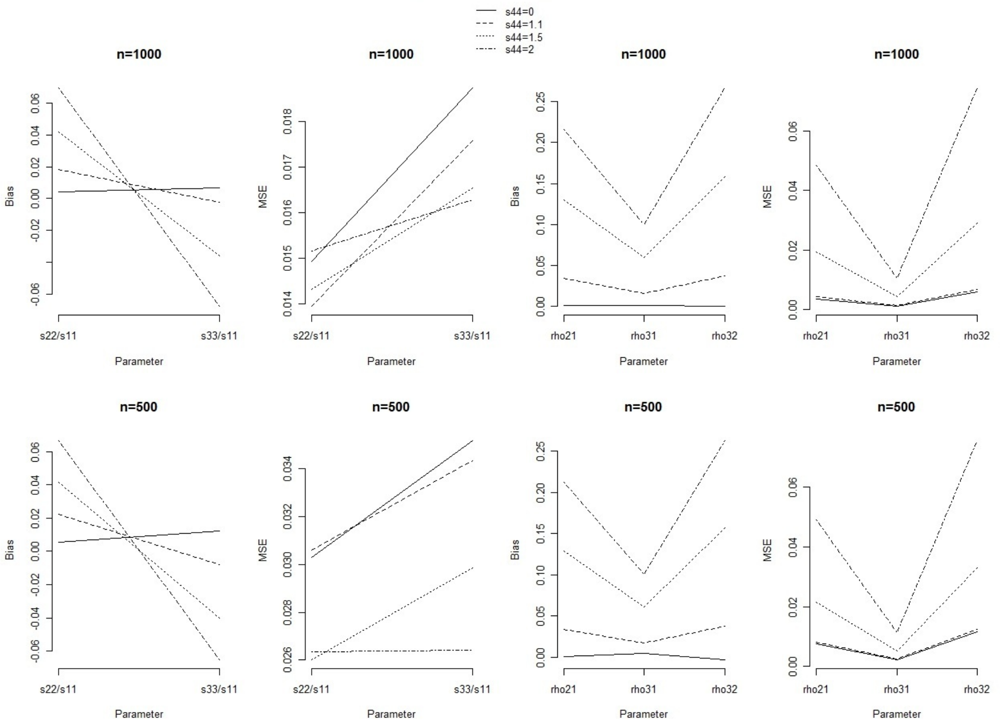

Figure 2: The effect of σ_{44} misspecification on bias and MSE of variance ratios and correlations, $m = 4$

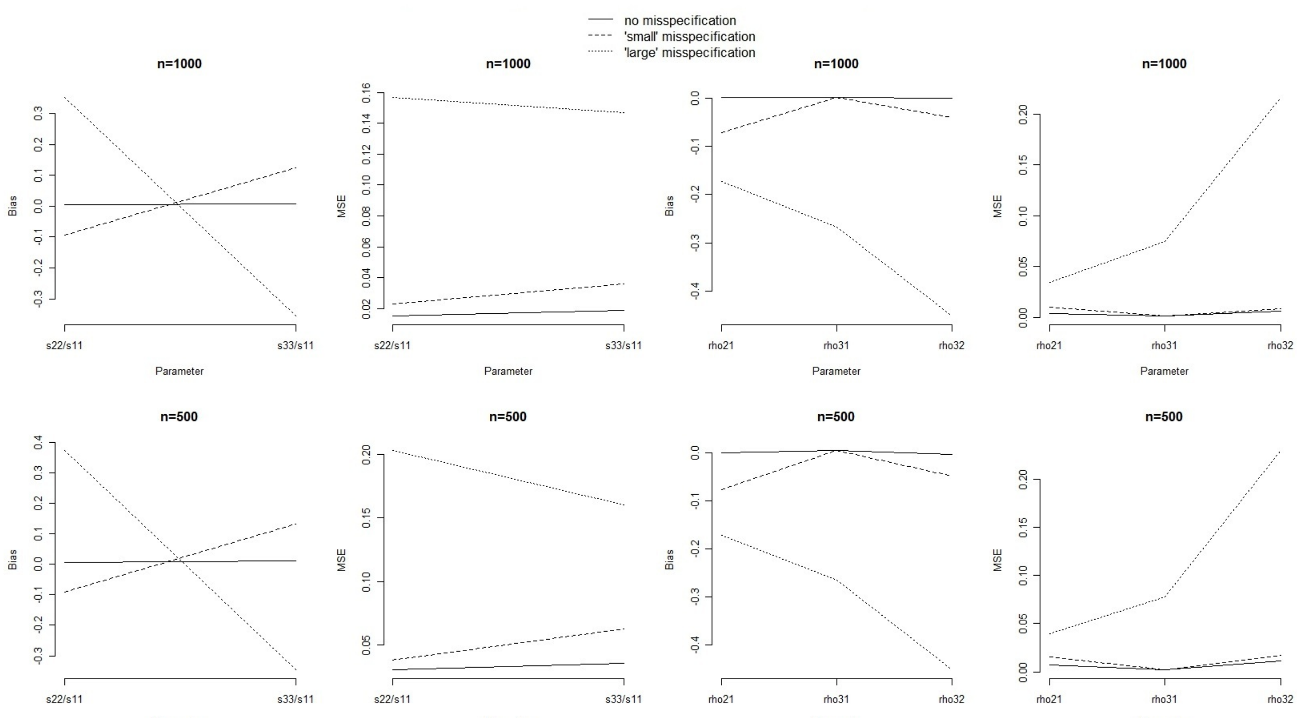

Figure 3: The effect of $\sigma_{14}, \sigma_{24}, \sigma_{34}, \sigma_{44}$ misspecifications on bias and MSE of variance ratios and correlations, $m = 4$

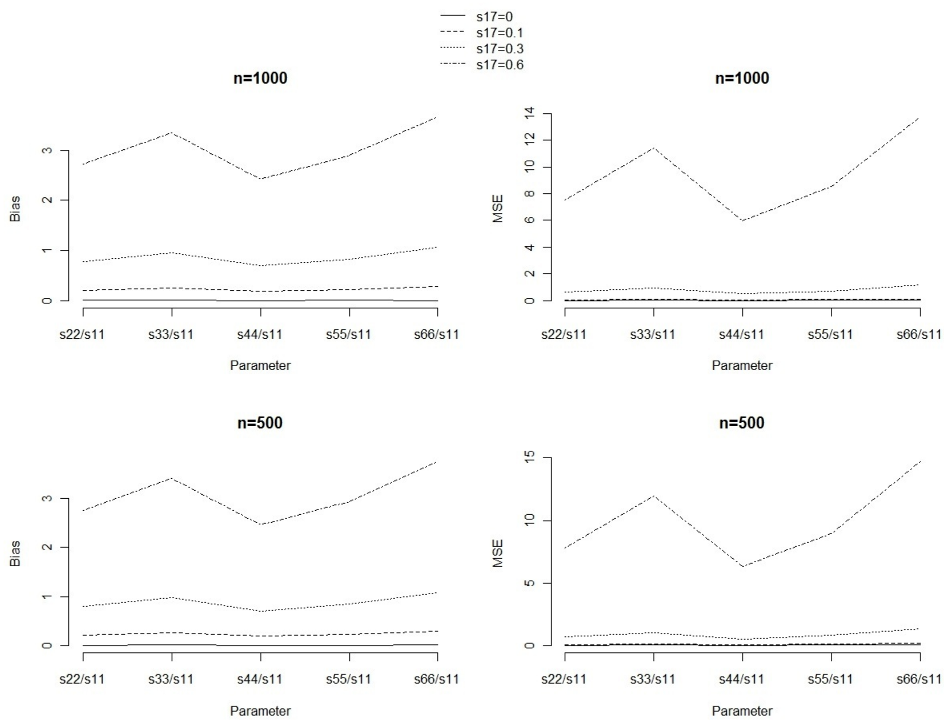

Figure 4: The effect of σ_{17} misspecification on bias and MSE of variance ratios, $m = 7$

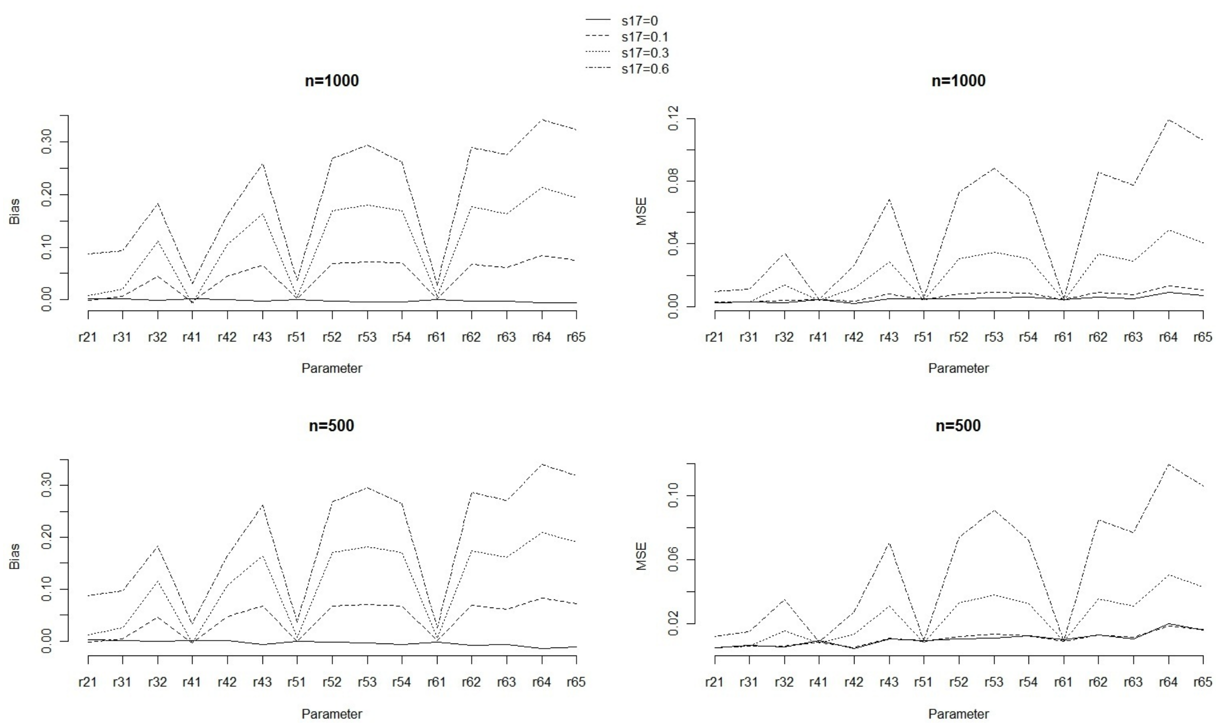

Figure 5: The effect of σ_{17} misspecification on bias and MSE of correlations, $m = 7$

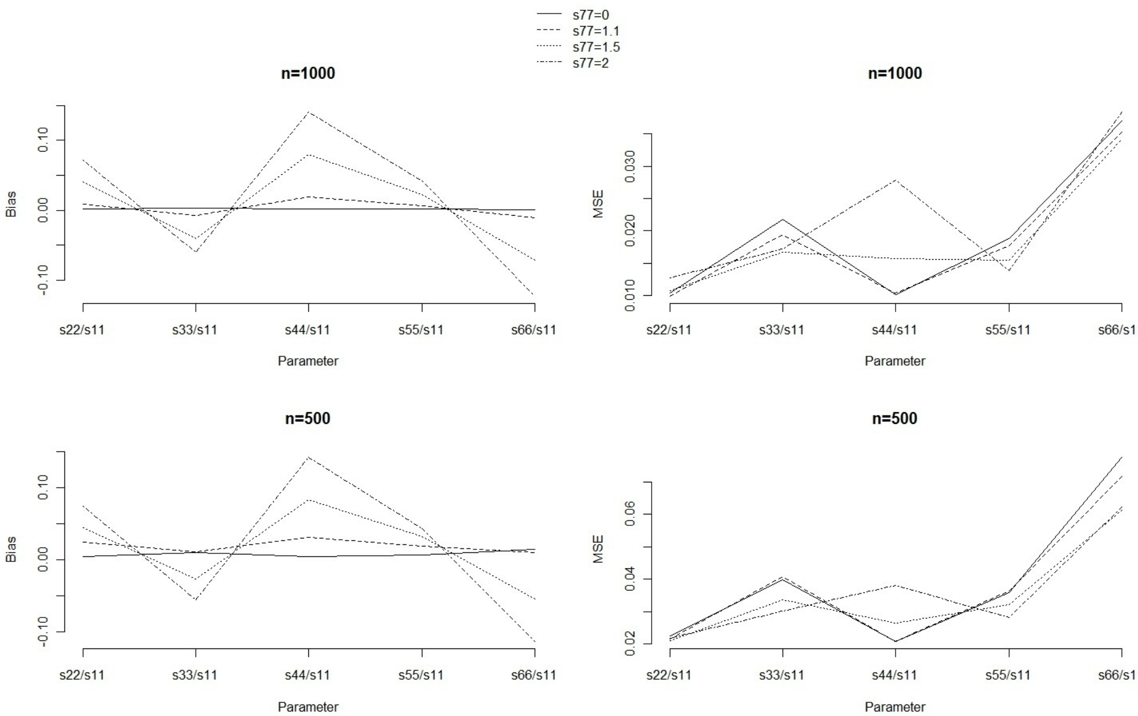

Figure 6: The effect of σ_{77} misspecification on bias and MSE of variance ratios, $m = 7$

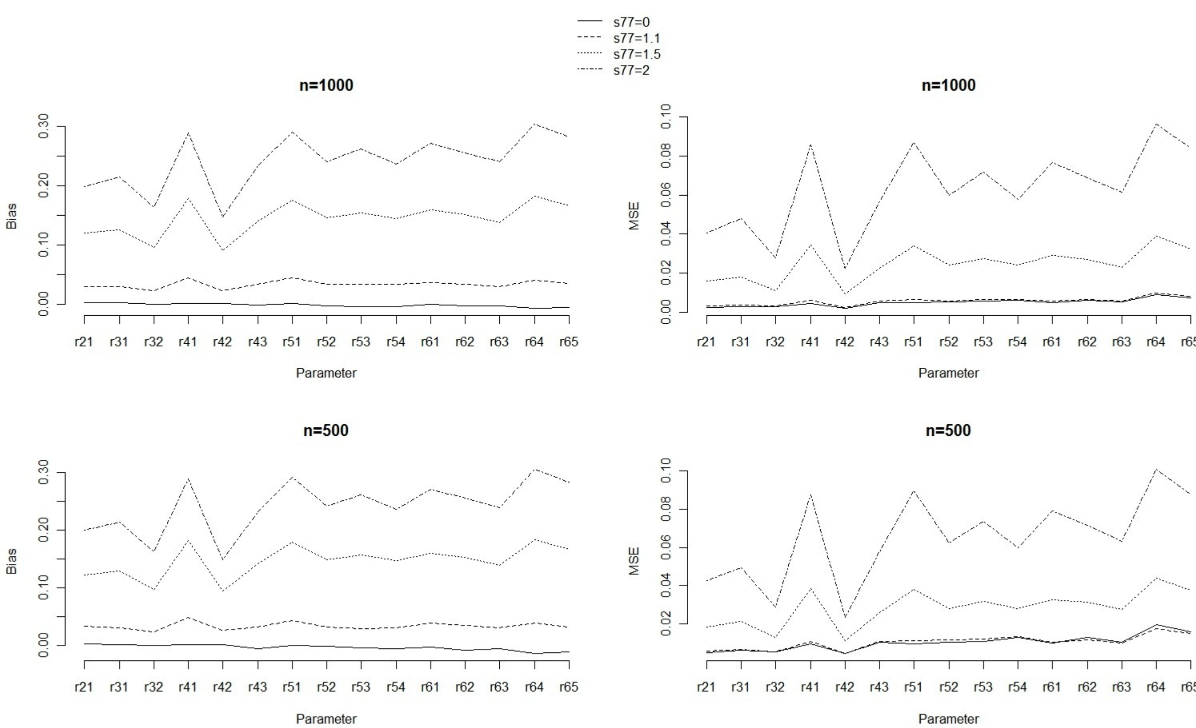

Figure 7: The effect of σ_{77} misspecification on bias and MSE of correlations, $m = 7$

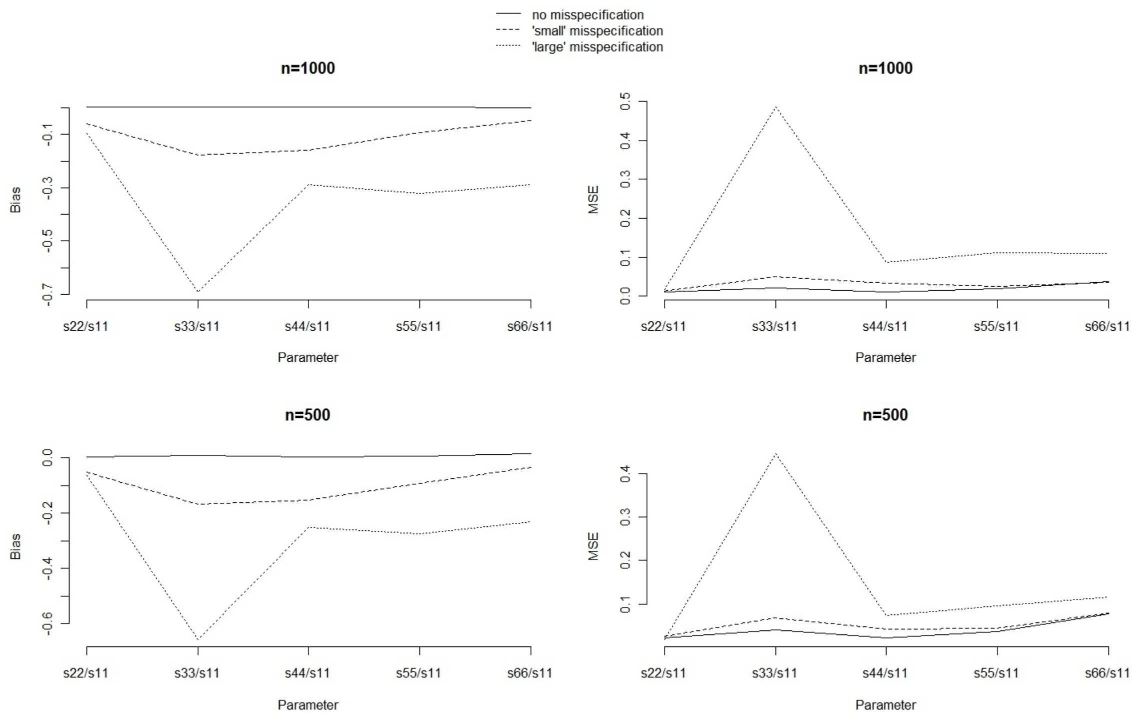

Figure 8: The effect of $\sigma_{17}, \sigma_{27}, \ldots, \sigma_{77}$ misspecifications on bias and MSE of variance ratios, $m = 7$

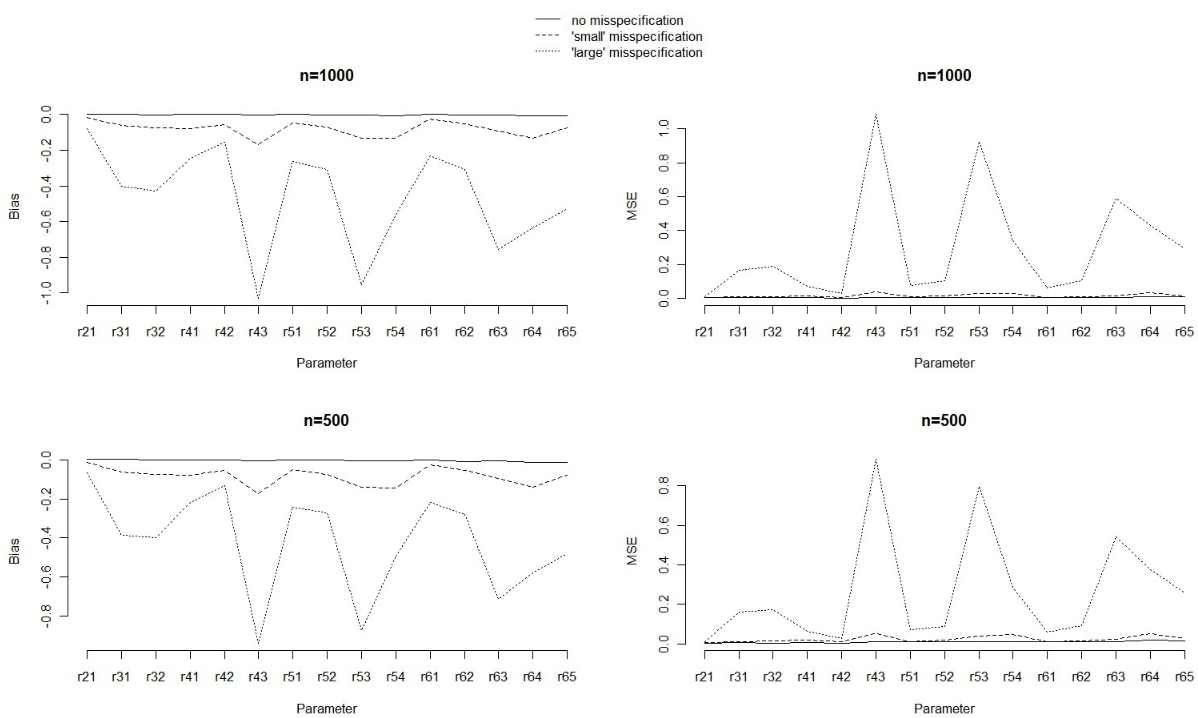

Figure 9: The effect of $\sigma_{17}, \sigma_{27}, \ldots, \sigma_{77}$ misspecifications on bias and MSE of correlations, $m = 7$

Appendix

4 objects $n = 1000$	Situation 1: Correctly specified model			
	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.804	0.004	0.015
σ_{33}/σ_{11}	1.2	1.207	0.007	0.019
ρ_{21}	0.402	0.403	0.001	0.004
ρ_{31}	0.685	0.686	0.001	0.001
ρ_{32}	0.204	0.204	0.000	0.006
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	-0.577	-0.579	-0.002	0.002
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	-1.076	-1.076	-0.001	0.002
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	-0.212	-0.212	0.001	0.001

	Situation 2 ($\sigma_{14} = 0.1$)				Situation 3 ($\sigma_{14} = 0.3$)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	1.011	0.211	0.063	0.8	1.586	0.786	0.656
σ_{33}/σ_{11}	1.2	1.454	0.254	0.087	1.2	2.154	0.954	0.954
ρ_{21}	0.402	0.402	0.000	0.003	0.402	0.410	0.007	0.003
ρ_{31}	0.685	0.696	0.011	0.001	0.685	0.734	0.049	0.003
ρ_{32}	0.204	0.279	0.074	0.010	0.204	0.390	0.186	0.037
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	-0.577	-0.577	0.001	0.002	-0.577	-0.577	0.000	0.002
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	-1.076	-1.078	-0.003	0.003	-1.076	-1.081	-0.005	0.002
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	-0.224	-0.223	0.001	0.002	-0.254	-0.253	0.001	0.001

	Situation 4 ($\sigma_{14} = 0.6$)				Situation 5 ($\sigma_{44} = 1.1$)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	3.526	2.726	7.603	0.8	0.818	0.018	0.014
σ_{33}/σ_{11}	1.2	4.527	3.327	11.253	1.2	1.198	-0.002	0.018
ρ_{21}	0.402	0.482	0.080	0.009	0.402	0.437	0.034	0.004
ρ_{31}	0.685	0.887	0.203	0.042	0.685	0.700	0.016	0.001
ρ_{32}	0.204	0.503	0.299	0.091	0.204	0.241	0.037	0.007
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	-0.577	-0.576	0.001	0.002	-0.577	-0.578	-0.001	0.002
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	-1.076	-1.078	-0.003	0.002	-1.076	-1.077	-0.002	0.002
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	-0.335	-0.335	0.001	0.002	-0.207	-0.207	0.000	0.001

	Situation 6 ($\sigma_{44} = 1.5$)				Situation 7 ($\sigma_{44} = 2$)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.842	0.042	0.014	0.8	0.870	0.070	0.015
σ_{33}/σ_{11}	1.2	1.164	-0.036	0.017	1.2	1.132	-0.068	0.016
ρ_{21}	0.402	0.533	0.130	0.019	0.402	0.618	0.216	0.048
ρ_{31}	0.685	0.744	0.060	0.004	0.685	0.784	0.100	0.011
ρ_{32}	0.204	0.362	0.158	0.029	0.204	0.471	0.267	0.075
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	-0.577	-0.577	0.000	0.002	-0.577	-0.578	-0.001	0.002
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	-1.076	-1.076	0.000	0.002	-1.076	-1.077	-0.001	0.002
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	-0.190	-0.189	0.001	0.001	-0.173	-0.172	0.001	0.001

	Situation 8 (small missp/tion in all σ_{4i})				Situation 9 (larger missp/tion in all σ_{4i})			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.707	-0.093	0.023	0.8	1.153	0.353	0.157
σ_{33}/σ_{11}	1.2	1.323	0.123	0.036	1.2	0.844	-0.356	0.147
ρ_{21}	0.402	0.331	-0.072	0.010	0.402	0.230	-0.173	0.035
ρ_{31}	0.685	0.687	0.002	0.001	0.685	0.417	-0.267	0.075
ρ_{32}	0.204	0.163	-0.041	0.008	0.204	-0.248	-0.452	0.217
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	-0.577	-0.578	-0.001	0.002	-0.577	-0.578	0.000	0.002
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	-1.076	-1.077	-0.002	0.002	-1.076	-1.075	0.000	0.002
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	-0.219	-0.218	0.001	0.001	-0.263	-0.262	0.002	0.002

Table 7: Bias & MSE of the estimates of standardized parameters, $n = 1000$, 4 objects

4 objects $n=500$	Situation 1: Correctly specified model			
	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.805	0.005	0.030
σ_{33}/σ_{11}	1.2	1.212	0.012	0.035
ρ_{21}	0.402	0.403	0.000	0.007
ρ_{31}	0.685	0.689	0.005	0.002
ρ_{32}	0.204	0.201	-0.003	0.012
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	-0.577	-0.577	0.000	0.003
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	-1.076	-1.080	-0.004	0.005
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	-0.212	-0.210	0.002	0.003

	Situation 2 ($\sigma_{14}=0.1$)				Situation 3 ($\sigma_{14}=0.3$)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	1.003	0.203	0.078	0.8	1.588	0.788	0.695
σ_{33}/σ_{11}	1.2	1.446	0.246	0.109	1.2	2.163	0.963	1.019
ρ_{21}	0.402	0.399	-0.004	0.007	0.402	0.412	0.010	0.006
ρ_{31}	0.685	0.697	0.012	0.002	0.685	0.736	0.052	0.005
ρ_{32}	0.204	0.271	0.067	0.013	0.204	0.390	0.186	0.040
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	-0.577	-0.576	0.001	0.003	-0.577	-0.578	-0.001	0.003
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	-1.076	-1.079	-0.003	0.005	-1.076	-1.081	-0.005	0.005
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	-0.224	-0.220	0.004	0.003	-0.254	-0.253	0.000	0.003

	Situation 4 ($\sigma_{14}=0.6$)				Situation 5 ($\sigma_{44}=1.1$)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	3.585	2.785	8.120	0.8	0.822	0.022	0.031
σ_{33}/σ_{11}	1.2	4.588	3.388	11.887	1.2	1.192	-0.008	0.034
ρ_{21}	0.402	0.481	0.079	0.011	0.402	0.436	0.034	0.008
ρ_{31}	0.685	0.890	0.205	0.044	0.685	0.702	0.018	0.002
ρ_{32}	0.204	0.504	0.300	0.093	0.204	0.242	0.038	0.013
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	-0.577	-0.575	0.002	0.003	-0.577	-0.578	-0.001	0.004
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	-1.076	-1.080	-0.004	0.005	-1.076	-1.080	-0.004	0.005
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	-0.335	-0.336	-0.001	0.003	-0.207	-0.205	0.002	0.003

	Situation 6 ($\sigma_{44}=1.5$)				Situation 7 ($\sigma_{44}=2$)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.842	0.042	0.026	0.8	0.867	0.067	0.026
σ_{33}/σ_{11}	1.2	1.160	-0.040	0.030	1.2	1.135	-0.065	0.026
ρ_{21}	0.402	0.531	0.129	0.022	0.402	0.615	0.213	0.049
ρ_{31}	0.685	0.746	0.061	0.005	0.685	0.785	0.101	0.011
ρ_{32}	0.204	0.362	0.158	0.033	0.204	0.467	0.263	0.076
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	-0.577	-0.577	0.001	0.003	-0.577	-0.576	0.001	0.003
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	-1.076	-1.078	-0.002	0.005	-1.076	-1.080	-0.004	0.005
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	-0.190	-0.188	0.002	0.003	-0.173	-0.172	0.002	0.003

	Situation 8 (small missp/tion in all σ_{4i})				Situation 9 (larger missp/tion in all σ_{4i})			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.710	-0.090	0.038	0.8	1.174	0.374	0.203
σ_{33}/σ_{11}	1.2	1.333	0.133	0.063	1.2	0.855	-0.345	0.160
ρ_{21}	0.402	0.326	-0.077	0.016	0.402	0.230	-0.173	0.039
ρ_{31}	0.685	0.689	0.004	0.003	0.685	0.419	-0.265	0.078
ρ_{32}	0.204	0.156	-0.048	0.017	0.204	-0.248	-0.452	0.230
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	-0.577	-0.576	0.001	0.003	-0.577	-0.578	-0.001	0.003
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	-1.076	-1.081	-0.005	0.005	-1.076	-1.079	-0.004	0.005
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	-0.219	-0.217	0.001	0.003	-0.263	-0.262	0.002	0.003

Table 8: Bias & MSE of the estimates of standardized parameters, $n=500$, 4 objects

7 objects $n = 1000$	Situation 1: Correctly specified model			
	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.802	0.002	0.010
σ_{33}/σ_{11}	1.2	1.203	0.003	0.022
σ_{44}/σ_{11}	0.6	0.602	0.002	0.010
σ_{55}/σ_{11}	0.9	0.902	0.002	0.019
σ_{66}/σ_{11}	1.4	1.401	0.001	0.037
ρ_{21}	0.447	0.449	0.002	0.003
ρ_{31}	0.320	0.322	0.002	0.003
ρ_{32}	0.510	0.510	-0.001	0.003
ρ_{41}	0.258	0.260	0.002	0.005
ρ_{42}	0.650	0.650	0.001	0.002
ρ_{43}	0.354	0.352	-0.002	0.005
ρ_{51}	0.158	0.159	0.001	0.005
ρ_{52}	0.354	0.351	-0.002	0.005
ρ_{53}	0.192	0.188	-0.004	0.006
ρ_{54}	0.408	0.403	-0.005	0.006
ρ_{61}	0.085	0.085	0.000	0.005
ρ_{62}	0.189	0.187	-0.002	0.006
ρ_{63}	0.131	0.128	-0.003	0.005
ρ_{64}	0.109	0.103	-0.006	0.009
ρ_{65}	0.071	0.065	-0.006	0.007
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	0.2	0.200	0.000	0.001
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	0.327	0.327	0.000	0.001
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	0.730	0.730	0.000	0.002
$\frac{\mu_1-\mu_5}{\sqrt{\sigma_{11}-2\sigma_{15}+\sigma_{55}}}$	0.791	0.792	0.001	0.002
$\frac{\mu_1-\mu_6}{\sqrt{\sigma_{11}-2\sigma_{16}+\sigma_{66}}}$	0.876	0.876	0.000	0.002
$\frac{\mu_1-\mu_7}{\sqrt{\sigma_{11}-2\sigma_{17}+\sigma_{77}}}$	0.495	0.494	-0.001	0.001

Table 9: Bias & MSE of the estimates of standardized parameters for the baseline model, $n = 1000$, 7 objects

7 objects $n = 1000$	Situation 2 ($\sigma_{17} = 0.1$)				Situation 3 ($\sigma_{17} = 0.3$)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	1.002	0.202	0.055	0.8	1.581	0.781	0.634
σ_{33}/σ_{11}	1.2	1.452	0.252	0.090	1.2	2.152	0.952	0.955
σ_{44}/σ_{11}	0.6	0.782	0.182	0.047	0.6	1.294	0.694	0.507
σ_{55}/σ_{11}	0.9	1.121	0.221	0.072	0.9	1.726	0.826	0.722
σ_{66}/σ_{11}	1.4	1.682	0.282	0.127	1.4	2.459	1.059	1.208
ρ_{21}	0.447	0.446	-0.001	0.003	0.447	0.455	0.008	0.003
ρ_{31}	0.320	0.326	0.006	0.003	0.320	0.340	0.021	0.003
ρ_{32}	0.510	0.555	0.045	0.004	0.510	0.622	0.112	0.014
ρ_{41}	0.258	0.254	-0.005	0.005	0.258	0.252	-0.006	0.004
ρ_{42}	0.650	0.694	0.044	0.003	0.650	0.755	0.106	0.012
ρ_{43}	0.354	0.420	0.067	0.008	0.354	0.517	0.163	0.029
ρ_{51}	0.158	0.160	0.002	0.005	0.158	0.163	0.005	0.004
ρ_{52}	0.354	0.422	0.069	0.008	0.354	0.523	0.169	0.031
ρ_{53}	0.192	0.264	0.071	0.009	0.192	0.372	0.180	0.035
ρ_{54}	0.408	0.479	0.071	0.009	0.408	0.578	0.169	0.031
ρ_{61}	0.085	0.087	0.002	0.005	0.085	0.091	0.007	0.004
ρ_{62}	0.189	0.257	0.068	0.009	0.189	0.365	0.176	0.034
ρ_{63}	0.131	0.192	0.061	0.008	0.131	0.294	0.163	0.029
ρ_{64}	0.109	0.193	0.084	0.013	0.109	0.323	0.214	0.049
ρ_{65}	0.071	0.146	0.075	0.011	0.071	0.266	0.194	0.041
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	0.2	0.201	0.001	0.001	0.2	0.201	0.001	0.001
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	0.327	0.326	-0.001	0.001	0.327	0.327	0.000	0.001
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	0.730	0.732	0.001	0.001	0.730	0.730	0.000	0.002
$\frac{\mu_1-\mu_5}{\sqrt{\sigma_{11}-2\sigma_{15}+\sigma_{55}}}$	0.791	0.791	0.000	0.002	0.791	0.790	-0.001	0.002
$\frac{\mu_1-\mu_6}{\sqrt{\sigma_{11}-2\sigma_{16}+\sigma_{66}}}$	0.876	0.876	-0.001	0.002	0.876	0.875	-0.001	0.002
$\frac{\mu_1-\mu_7}{\sqrt{\sigma_{11}-2\sigma_{17}+\sigma_{77}}}$	0.522	0.521	-0.001	0.001	0.592	0.592	0.000	0.002
	Situation 4 ($\sigma_{17} = 0.6$)				Situation 5 ($\sigma_{77} = 1.1$)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	3.529	2.729	7.554	0.8	0.808	0.008	0.010
σ_{33}/σ_{11}	1.2	4.551	3.351	11.421	1.2	1.193	-0.007	0.019
σ_{44}/σ_{11}	0.6	3.025	2.425	5.982	0.6	0.620	0.020	0.010
σ_{55}/σ_{11}	0.9	3.793	2.893	8.530	0.9	0.907	0.007	0.018
σ_{66}/σ_{11}	1.4	5.067	3.667	13.757	1.4	1.390	-0.010	0.035
ρ_{21}	0.447	0.534	0.087	0.010	0.447	0.477	0.029	0.003
ρ_{31}	0.320	0.413	0.093	0.011	0.320	0.349	0.029	0.004
ρ_{32}	0.510	0.693	0.183	0.034	0.510	0.533	0.022	0.003
ρ_{41}	0.258	0.289	0.031	0.005	0.258	0.303	0.045	0.006
ρ_{42}	0.650	0.812	0.162	0.027	0.650	0.673	0.023	0.002
ρ_{43}	0.354	0.613	0.260	0.068	0.354	0.387	0.033	0.006
ρ_{51}	0.158	0.194	0.036	0.005	0.158	0.202	0.044	0.006
ρ_{52}	0.354	0.622	0.268	0.073	0.354	0.387	0.033	0.006
ρ_{53}	0.192	0.487	0.295	0.088	0.192	0.226	0.034	0.007
ρ_{54}	0.408	0.671	0.263	0.070	0.408	0.441	0.033	0.006
ρ_{61}	0.085	0.112	0.027	0.005	0.085	0.121	0.037	0.006
ρ_{62}	0.189	0.479	0.290	0.085	0.189	0.222	0.033	0.007
ρ_{63}	0.131	0.406	0.275	0.077	0.131	0.161	0.030	0.006
ρ_{64}	0.109	0.452	0.343	0.119	0.109	0.149	0.040	0.010
ρ_{65}	0.071	0.394	0.323	0.106	0.071	0.106	0.034	0.008
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	0.2	0.200	0.000	0.001	0.2	0.201	0.001	0.001
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	0.327	0.326	0.000	0.001	0.327	0.325	-0.001	0.001
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	0.730	0.731	0.001	0.002	0.730	0.730	0.000	0.002
$\frac{\mu_1-\mu_5}{\sqrt{\sigma_{11}-2\sigma_{15}+\sigma_{55}}}$	0.791	0.790	0.000	0.002	0.791	0.792	0.001	0.002
$\frac{\mu_1-\mu_6}{\sqrt{\sigma_{11}-2\sigma_{16}+\sigma_{66}}}$	0.876	0.877	0.000	0.002	0.876	0.876	0.000	0.002
$\frac{\mu_1-\mu_7}{\sqrt{\sigma_{11}-2\sigma_{17}+\sigma_{77}}}$	0.783	0.785	0.002	0.002	0.483	0.481	-0.002	0.001

Table 10: Bias & MSE of the estimates of standardized parameters, $n = 1000$, 7 objects

7 objects	Situation 6 ($\sigma_{77} = 1.5$)				Situation 7 ($\sigma_{77} = 2$)			
$n = 1000$	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.841	0.041	0.011	0.8	0.872	0.072	0.013
σ_{33}/σ_{11}	1.2	1.160	-0.040	0.017	1.2	1.141	-0.059	0.017
σ_{44}/σ_{11}	0.6	0.680	0.080	0.016	0.6	0.741	0.141	0.028
σ_{55}/σ_{11}	0.9	0.923	0.023	0.015	0.9	0.942	0.042	0.014
σ_{66}/σ_{11}	1.4	1.329	-0.071	0.034	1.4	1.278	-0.122	0.038
ρ_{21}	0.447	0.568	0.121	0.016	0.447	0.646	0.199	0.041
ρ_{31}	0.320	0.446	0.126	0.018	0.320	0.534	0.215	0.048
ρ_{32}	0.510	0.607	0.096	0.011	0.510	0.674	0.163	0.028
ρ_{41}	0.258	0.437	0.178	0.035	0.258	0.548	0.290	0.086
ρ_{42}	0.650	0.741	0.091	0.009	0.650	0.797	0.148	0.022
ρ_{43}	0.354	0.494	0.140	0.023	0.354	0.587	0.234	0.057
ρ_{51}	0.158	0.334	0.176	0.034	0.158	0.449	0.291	0.087
ρ_{52}	0.354	0.499	0.146	0.024	0.354	0.595	0.241	0.060
ρ_{53}	0.192	0.346	0.154	0.027	0.192	0.455	0.262	0.072
ρ_{54}	0.408	0.553	0.145	0.024	0.408	0.645	0.237	0.058
ρ_{61}	0.085	0.244	0.159	0.029	0.085	0.356	0.271	0.077
ρ_{62}	0.189	0.340	0.151	0.027	0.189	0.445	0.256	0.069
ρ_{63}	0.131	0.269	0.137	0.023	0.131	0.372	0.241	0.061
ρ_{64}	0.109	0.292	0.183	0.039	0.109	0.414	0.304	0.096
ρ_{65}	0.071	0.237	0.166	0.033	0.071	0.354	0.283	0.084
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	0.2	0.200	0.000	0.001	0.2	0.201	0.001	0.001
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	0.327	0.327	0.000	0.001	0.327	0.327	0.001	0.001
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	0.730	0.730	0.000	0.001	0.730	0.732	0.002	0.002
$\frac{\mu_1-\mu_5}{\sqrt{\sigma_{11}-2\sigma_{15}+\sigma_{55}}}$	0.791	0.792	0.001	0.002	0.791	0.790	-0.001	0.002
$\frac{\mu_1-\mu_6}{\sqrt{\sigma_{11}-2\sigma_{16}+\sigma_{66}}}$	0.876	0.876	0.000	0.002	0.876	0.877	0.001	0.002
$\frac{\mu_1-\mu_7}{\sqrt{\sigma_{11}-2\sigma_{17}+\sigma_{77}}}$	0.443	0.443	0.000	0.001	0.404	0.404	0.000	0.001
	Situation 8 (small missp/tion in all σ_{7i})				Situation 9 (larger missp/tion in all σ_{7i}) (841 valid replications out of 2000)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.741	-0.059	0.014	0.8	0.705	-0.095	0.017
σ_{33}/σ_{11}	1.2	1.023	-0.177	0.050	1.2	0.507	-0.693	0.486
σ_{44}/σ_{11}	0.6	0.440	-0.160	0.035	0.6	0.312	-0.288	0.087
σ_{55}/σ_{11}	0.9	0.806	-0.094	0.025	0.9	0.579	-0.321	0.112
σ_{66}/σ_{11}	1.4	1.353	-0.047	0.036	1.4	1.112	-0.288	0.110
ρ_{21}	0.447	0.430	-0.017	0.003	0.447	0.371	-0.077	0.009
ρ_{31}	0.320	0.258	-0.061	0.007	0.320	-0.080	-0.400	0.167
ρ_{32}	0.510	0.435	-0.075	0.009	0.510	0.083	-0.427	0.191
ρ_{41}	0.258	0.178	-0.080	0.013	0.258	0.013	-0.245	0.069
ρ_{42}	0.650	0.594	-0.056	0.007	0.650	0.494	-0.156	0.029
ρ_{43}	0.354	0.184	-0.169	0.040	0.354	-0.674	-1.027	1.086
ρ_{51}	0.158	0.112	-0.046	0.008	0.158	-0.106	-0.264	0.077
ρ_{52}	0.354	0.282	-0.071	0.012	0.354	0.046	-0.307	0.104
ρ_{53}	0.192	0.060	-0.133	0.026	0.192	-0.759	-0.952	0.924
ρ_{54}	0.408	0.276	-0.132	0.029	0.408	-0.151	-0.560	0.339
ρ_{61}	0.085	0.062	-0.023	0.005	0.085	-0.147	-0.232	0.060
ρ_{62}	0.189	0.138	-0.051	0.009	0.189	-0.119	-0.308	0.103
ρ_{63}	0.131	0.039	-0.092	0.015	0.131	-0.625	-0.756	0.588
ρ_{64}	0.109	-0.023	-0.133	0.032	0.109	-0.526	-0.635	0.429
ρ_{65}	0.071	-0.004	-0.076	0.015	0.071	-0.454	-0.526	0.294
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	0.2	0.202	0.002	0.001	0.2	0.199	-0.001	0.001
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	0.327	0.328	0.001	0.001	0.327	0.328	0.002	0.001
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	0.730	0.731	0.001	0.002	0.730	0.737	0.006	0.002
$\frac{\mu_1-\mu_5}{\sqrt{\sigma_{11}-2\sigma_{15}+\sigma_{55}}}$	0.791	0.790	0.000	0.002	0.791	0.797	0.006	0.002
$\frac{\mu_1-\mu_6}{\sqrt{\sigma_{11}-2\sigma_{16}+\sigma_{66}}}$	0.876	0.877	0.000	0.002	0.876	0.882	0.005	0.002
$\frac{\mu_1-\mu_7}{\sqrt{\sigma_{11}-2\sigma_{17}+\sigma_{77}}}$	0.495	0.495	0.000	0.001	0.513	0.517	0.004	0.001

Table 11: Bias & MSE of the estimates of standardized parameters, $n = 1000$, 7 objects

7 objects $n = 500$	Situation 1: Correctly specified model			
	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.804	0.004	0.022
σ_{33}/σ_{11}	1.2	1.210	0.010	0.040
σ_{44}/σ_{11}	0.6	0.604	0.004	0.021
σ_{55}/σ_{11}	0.9	0.907	0.007	0.036
σ_{66}/σ_{11}	1.4	1.414	0.014	0.078
ρ_{21}	0.447	0.450	0.003	0.005
ρ_{31}	0.320	0.321	0.001	0.006
ρ_{32}	0.510	0.509	-0.001	0.005
ρ_{41}	0.258	0.259	0.000	0.009
ρ_{42}	0.650	0.650	0.001	0.005
ρ_{43}	0.354	0.347	-0.007	0.011
ρ_{51}	0.158	0.157	-0.001	0.010
ρ_{52}	0.354	0.351	-0.002	0.010
ρ_{53}	0.192	0.188	-0.004	0.011
ρ_{54}	0.408	0.402	-0.007	0.013
ρ_{61}	0.085	0.082	-0.003	0.010
ρ_{62}	0.189	0.181	-0.008	0.013
ρ_{63}	0.131	0.125	-0.006	0.010
ρ_{64}	0.109	0.095	-0.014	0.020
ρ_{65}	0.071	0.060	-0.011	0.016
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	0.200	0.202	0.002	0.003
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	0.327	0.324	-0.003	0.003
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	0.730	0.730	-0.001	0.003
$\frac{\mu_1-\mu_5}{\sqrt{\sigma_{11}-2\sigma_{15}+\sigma_{55}}}$	0.791	0.790	0.000	0.003
$\frac{\mu_1-\mu_6}{\sqrt{\sigma_{11}-2\sigma_{16}+\sigma_{66}}}$	0.876	0.876	-0.001	0.004
$\frac{\mu_1-\mu_7}{\sqrt{\sigma_{11}-2\sigma_{17}+\sigma_{77}}}$	0.495	0.493	-0.002	0.003

Table 12: Bias & MSE of the estimates of standardized parameters for the baseline model, $n = 500$, 7 objects

7 objects $n = 500$	Situation 2 ($\sigma_{17} = 0.1$)				Situation 3 ($\sigma_{17} = 0.3$)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	1.010	0.210	0.072	0.8	1.592	0.792	0.682
σ_{33}/σ_{11}	1.2	1.465	0.265	0.125	1.2	2.179	0.979	1.058
σ_{44}/σ_{11}	0.6	0.794	0.194	0.065	0.6	1.303	0.703	0.546
σ_{55}/σ_{11}	0.9	1.131	0.231	0.101	0.9	1.747	0.847	0.803
σ_{66}/σ_{11}	1.4	1.700	0.300	0.185	1.4	2.475	1.075	1.327
ρ_{21}	0.447	0.445	-0.002	0.005	0.447	0.459	0.011	0.005
ρ_{31}	0.320	0.324	0.004	0.006	0.320	0.345	0.025	0.006
ρ_{32}	0.510	0.556	0.046	0.006	0.510	0.626	0.115	0.016
ρ_{41}	0.258	0.254	-0.004	0.009	0.258	0.255	-0.003	0.008
ρ_{42}	0.650	0.697	0.048	0.005	0.650	0.758	0.108	0.013
ρ_{43}	0.354	0.420	0.067	0.011	0.354	0.518	0.164	0.031
ρ_{51}	0.158	0.157	-0.001	0.009	0.158	0.166	0.007	0.008
ρ_{52}	0.354	0.421	0.068	0.012	0.354	0.524	0.171	0.033
ρ_{53}	0.192	0.263	0.071	0.013	0.192	0.375	0.182	0.038
ρ_{54}	0.408	0.476	0.068	0.013	0.408	0.578	0.170	0.033
ρ_{61}	0.085	0.085	0.000	0.009	0.085	0.090	0.006	0.009
ρ_{62}	0.189	0.257	0.068	0.013	0.189	0.363	0.174	0.036
ρ_{63}	0.131	0.192	0.061	0.011	0.131	0.292	0.161	0.031
ρ_{64}	0.109	0.191	0.082	0.018	0.109	0.319	0.210	0.050
ρ_{65}	0.071	0.144	0.072	0.016	0.071	0.263	0.191	0.043
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	0.2	0.200	0.000	0.003	0.2	0.202	0.002	0.003
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	0.327	0.325	-0.001	0.003	0.327	0.327	0.001	0.003
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	0.730	0.729	-0.002	0.003	0.730	0.732	0.001	0.003
$\frac{\mu_1-\mu_5}{\sqrt{\sigma_{11}-2\sigma_{15}+\sigma_{55}}}$	0.791	0.788	-0.002	0.003	0.791	0.790	0.000	0.003
$\frac{\mu_1-\mu_6}{\sqrt{\sigma_{11}-2\sigma_{16}+\sigma_{66}}}$	0.876	0.875	-0.001	0.003	0.876	0.877	0.000	0.003
$\frac{\mu_1-\mu_7}{\sqrt{\sigma_{11}-2\sigma_{17}+\sigma_{77}}}$	0.522	0.520	-0.001	0.003	0.592	0.590	-0.001	0.003
	Situation 4 ($\sigma_{17} = 0.6$)				Situation 5 ($\sigma_{77} = 1.1$)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	3.557	2.757	7.812	0.8	0.824	0.024	0.021
σ_{33}/σ_{11}	1.2	4.602	3.402	11.988	1.2	1.211	0.011	0.041
σ_{44}/σ_{11}	0.6	3.067	2.467	6.301	0.6	0.631	0.031	0.021
σ_{55}/σ_{11}	0.9	3.837	2.937	8.969	0.9	0.919	0.019	0.036
σ_{66}/σ_{11}	1.4	5.145	3.745	14.695	1.4	1.411	0.011	0.072
ρ_{21}	0.447	0.535	0.088	0.012	0.447	0.480	0.033	0.006
ρ_{31}	0.320	0.416	0.096	0.015	0.320	0.349	0.030	0.007
ρ_{32}	0.510	0.694	0.184	0.035	0.510	0.533	0.023	0.005
ρ_{41}	0.258	0.292	0.034	0.009	0.258	0.306	0.048	0.011
ρ_{42}	0.650	0.814	0.164	0.028	0.650	0.675	0.026	0.005
ρ_{43}	0.354	0.615	0.261	0.070	0.354	0.385	0.032	0.011
ρ_{51}	0.158	0.194	0.036	0.009	0.158	0.201	0.043	0.011
ρ_{52}	0.354	0.622	0.269	0.074	0.354	0.385	0.031	0.012
ρ_{53}	0.192	0.489	0.296	0.091	0.192	0.222	0.029	0.012
ρ_{54}	0.408	0.673	0.265	0.072	0.408	0.438	0.030	0.013
ρ_{61}	0.085	0.110	0.025	0.009	0.085	0.123	0.038	0.010
ρ_{62}	0.189	0.475	0.286	0.085	0.189	0.223	0.034	0.012
ρ_{63}	0.131	0.402	0.271	0.077	0.131	0.161	0.030	0.010
ρ_{64}	0.109	0.449	0.340	0.119	0.109	0.148	0.039	0.017
ρ_{65}	0.071	0.390	0.319	0.106	0.071	0.103	0.032	0.015
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	0.2	0.199	-0.001	0.003	0.2	0.202	0.002	0.003
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	0.327	0.327	0.000	0.003	0.327	0.325	-0.002	0.003
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	0.730	0.728	-0.002	0.003	0.730	0.733	0.003	0.003
$\frac{\mu_1-\mu_5}{\sqrt{\sigma_{11}-2\sigma_{15}+\sigma_{55}}}$	0.791	0.788	-0.003	0.004	0.791	0.792	0.002	0.003
$\frac{\mu_1-\mu_6}{\sqrt{\sigma_{11}-2\sigma_{16}+\sigma_{66}}}$	0.876	0.875	-0.001	0.003	0.876	0.878	0.002	0.004
$\frac{\mu_1-\mu_7}{\sqrt{\sigma_{11}-2\sigma_{17}+\sigma_{77}}}$	0.783	0.784	0.002	0.004	0.483	0.483	0.000	0.003

Table 13: Bias & MSE of the estimates of standardized parameters, $n = 500$, 7 objects

7 objects $n = 500$	Situation 6 ($\sigma_{77} = 1.5$)				Situation 7 ($\sigma_{77} = 2$)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.845	0.045	0.021	0.8	0.875	0.075	0.022
σ_{33}/σ_{11}	1.2	1.173	-0.027	0.034	1.2	1.144	-0.056	0.030
σ_{44}/σ_{11}	0.6	0.684	0.084	0.026	0.6	0.742	0.142	0.038
σ_{55}/σ_{11}	0.9	0.933	0.033	0.032	0.9	0.944	0.044	0.028
σ_{66}/σ_{11}	1.4	1.345	-0.055	0.061	1.4	1.287	-0.113	0.062
ρ_{21}	0.447	0.569	0.122	0.018	0.447	0.647	0.200	0.042
ρ_{31}	0.320	0.448	0.129	0.021	0.320	0.533	0.214	0.049
ρ_{32}	0.510	0.608	0.097	0.013	0.510	0.673	0.163	0.029
ρ_{41}	0.258	0.439	0.181	0.039	0.258	0.547	0.289	0.087
ρ_{42}	0.650	0.744	0.095	0.011	0.650	0.798	0.148	0.023
ρ_{43}	0.354	0.495	0.141	0.026	0.354	0.585	0.232	0.058
ρ_{51}	0.158	0.336	0.178	0.038	0.158	0.449	0.291	0.090
ρ_{52}	0.354	0.501	0.148	0.028	0.354	0.594	0.241	0.062
ρ_{53}	0.192	0.348	0.156	0.032	0.192	0.453	0.260	0.073
ρ_{54}	0.408	0.555	0.147	0.028	0.408	0.645	0.236	0.060
ρ_{61}	0.085	0.244	0.160	0.032	0.085	0.355	0.270	0.079
ρ_{62}	0.189	0.342	0.153	0.031	0.189	0.444	0.255	0.071
ρ_{63}	0.131	0.270	0.139	0.027	0.131	0.370	0.238	0.063
ρ_{64}	0.109	0.291	0.182	0.044	0.109	0.414	0.305	0.101
ρ_{65}	0.071	0.237	0.166	0.038	0.071	0.354	0.283	0.088
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	0.2	0.203	0.003	0.003	0.2	0.201	0.001	0.003
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	0.327	0.328	0.001	0.003	0.327	0.327	0.000	0.003
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	0.730	0.732	0.001	0.003	0.730	0.729	-0.001	0.003
$\frac{\mu_1-\mu_5}{\sqrt{\sigma_{11}-2\sigma_{15}+\sigma_{55}}}$	0.791	0.793	0.002	0.003	0.791	0.789	-0.001	0.003
$\frac{\mu_1-\mu_6}{\sqrt{\sigma_{11}-2\sigma_{16}+\sigma_{66}}}$	0.876	0.876	0.000	0.003	0.876	0.875	-0.001	0.003
$\frac{\mu_1-\mu_7}{\sqrt{\sigma_{11}-2\sigma_{17}+\sigma_{77}}}$	0.443	0.444	0.001	0.003	0.404	0.403	-0.002	0.003
	Situation 8 (small missp/tion in all σ_{7i}) (1994 valid replications out of 2000)				Situation 9 (larger missp/tion in all σ_{7i}) (730 valid replications out of 2000)			
	True	Mean	Bias	MSE	True	Mean	Bias	MSE
σ_{22}/σ_{11}	0.8	0.750	-0.050	0.025	0.8	0.738	-0.062	0.019
σ_{33}/σ_{11}	1.2	1.034	-0.166	0.068	1.2	0.544	-0.656	0.445
σ_{44}/σ_{11}	0.6	0.446	-0.154	0.042	0.6	0.349	-0.251	0.073
σ_{55}/σ_{11}	0.9	0.807	-0.093	0.044	0.9	0.626	-0.274	0.095
σ_{66}/σ_{11}	1.4	1.366	-0.034	0.079	1.4	1.168	-0.232	0.115
ρ_{21}	0.447	0.433	-0.014	0.006	0.447	0.381	-0.066	0.010
ρ_{31}	0.320	0.258	-0.062	0.011	0.320	-0.067	-0.387	0.163
ρ_{32}	0.510	0.436	-0.075	0.014	0.510	0.112	-0.398	0.174
ρ_{41}	0.258	0.178	-0.080	0.020	0.258	0.039	-0.219	0.065
ρ_{42}	0.650	0.597	-0.053	0.011	0.650	0.517	-0.133	0.027
ρ_{43}	0.354	0.179	-0.175	0.055	0.354	-0.585	-0.939	0.935
ρ_{51}	0.158	0.108	-0.050	0.013	0.158	-0.083	-0.241	0.071
ρ_{52}	0.354	0.279	-0.074	0.020	0.354	0.083	-0.270	0.089
ρ_{53}	0.192	0.054	-0.139	0.038	0.192	-0.682	-0.875	0.799
ρ_{54}	0.408	0.266	-0.143	0.050	0.408	-0.084	-0.492	0.286
ρ_{61}	0.085	0.059	-0.025	0.010	0.085	-0.133	-0.217	0.059
ρ_{62}	0.189	0.135	-0.054	0.017	0.189	-0.089	-0.278	0.094
ρ_{63}	0.131	0.036	-0.095	0.022	0.131	-0.583	-0.714	0.541
ρ_{64}	0.109	-0.033	-0.142	0.054	0.109	-0.470	-0.579	0.377
ρ_{65}	0.071	-0.009	-0.080	0.026	0.071	-0.409	-0.481	0.260
$\frac{\mu_1-\mu_2}{\sqrt{\sigma_{11}-2\sigma_{12}+\sigma_{22}}}$	0.2	0.203	0.003	0.003	0.2	0.201	0.001	0.003
$\frac{\mu_1-\mu_3}{\sqrt{\sigma_{11}-2\sigma_{13}+\sigma_{33}}}$	0.327	0.327	0.000	0.003	0.327	0.329	0.002	0.003
$\frac{\mu_1-\mu_4}{\sqrt{\sigma_{11}-2\sigma_{14}+\sigma_{44}}}$	0.730	0.732	0.001	0.003	0.730	0.739	0.009	0.003
$\frac{\mu_1-\mu_5}{\sqrt{\sigma_{11}-2\sigma_{15}+\sigma_{55}}}$	0.791	0.790	0.000	0.003	0.791	0.797	0.006	0.003
$\frac{\mu_1-\mu_6}{\sqrt{\sigma_{11}-2\sigma_{16}+\sigma_{66}}}$	0.876	0.876	0.000	0.003	0.876	0.882	0.006	0.004
$\frac{\mu_1-\mu_7}{\sqrt{\sigma_{11}-2\sigma_{17}+\sigma_{77}}}$	0.49497	0.49447	-0.00050	0.00283	0.51326	0.51754	0.00428	0.00250

Table 14: Bias & MSE of the estimates of standardized parameters, $n = 500$, 7 objects

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