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SOME PROPERTIES OF TWO-PHASE QUADRATURE DOMAINS

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ABSTRACT. In this paper, we investigate general properties of the two-phase quadrature domains, which recently has been introduced by Emamizadeh-Prajapat-Shahgholian. The concept, which is the generalization of the well-known one-phase case, introduces substantial difficulties with interesting and even richer features than its one-phase counterpart.

For given positive constants $\lambda^\pm$ and two bounded and compactly supported measures $\mu^\pm$, we investigate the uniqueness of the solution of the following free boundary problem

$$\begin{cases} \Delta u = (\lambda^+ \chi_{\Omega^+} - \mu^+) - (\lambda^- \chi_{\Omega^-} - \mu^-), & \text{in } \mathbb{R}^N \quad (N \geq 2), \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega = \Omega^+ \cup \Omega^-$. It is further required that the supports of $\mu^\pm$ should be inside $\Omega^\pm$; this in general may fail and give rise to non-existence of solutions.

Along the lines of various properties that we state and prove here, we also present several conjectures and open problems that we believe should be true.

1. Introduction

The concept of quadrature domains is well known in modern potential theory and concerns generalized form of (sub)mean-value property for (sub)harmonic functions.

The main idea in this paper is to deal with a two-phase version of this concept, introduced in [7]. Our main result concerns uniqueness for two-phase quadrature domains when certain restrictions are made on the sign(s) of the solution function.

This paper is organized as follows. Section 2 contains some background in one-phase case and some fundamental concepts in potential theory. In section 3 we then move to the two-phase case scenario and extract its PDEs formulation and introduce quadrature inequalities and take some examples. In section 4 we note some recently result on existence theory for two phase free boundary problem and finally in the last section we study the uniqueness and prove our main result just by considering some conditions. Also we make some conjectures.

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2. One-phase case

The definition of a quadrature domain is as follows.

Definition 2.1. Let μ be a Radon measure with compact support in $\mathbb{R}^N$. An open connected domain $\Omega \subset \mathbb{R}^N$, ($N \geq 2$) is called *quadrature domain* with respect to μ if

$$(2.1) \quad \int_{\Omega} h dx = \int h d\mu, \quad \forall h \in HL^1(\Omega), \quad \text{supp}(\mu) \subset \Omega,$$

where $HL^1(\Omega)$ is the space of harmonic functions in $L^1(\Omega)$.

We denote by $Q(\mu, HL^1)$ the class of all nonempty domains satisfying (2.1) and we write $\Omega \in Q(\mu, HL^1)$.

A simple example of a quadrature domain (in one-phase case) corresponding to the Dirac measure $\mu = \delta_a$ is the appropriate ball $B(a, r)$, (for instance, let $N = 2$, $a = 0$, $r = \frac{1}{\sqrt{\pi}}$). The mean value theorem for harmonic functions implies

$$\int_{B(a,r)} h dx = h(a) = \int h d\mu.$$

The quadrature identity (2.1) is equivalent to the following identities (see [12]),

$$(2.2) \quad \begin{cases} U^{\chi\Omega} = U^{\mu}, & \text{in } \mathbb{R}^N \setminus \Omega, \\ \nabla U^{\chi\Omega} = \nabla U^{\mu}, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where U^{μ} denotes the Newtonian potential of the measure μ defined by

$$U^{\mu}(x) := (G * \mu)(x) = \int_{\mathbb{R}^N} G(x - y) d\mu(y), \quad x \in \mathbb{R}^N.$$

Here,

$$G(x) = \begin{cases} \frac{1}{N(N-2)\omega_N} |x|^{2-N}, & \text{for } N \geq 3, \\ -\frac{1}{2\pi} \ln |x|, & \text{for } N = 2, \end{cases}$$

denotes the fundamental solution to the Laplace operator and ω_N is the volume of unit sphere in $\mathbb{R}^N$. Thus, $U^{\chi\Omega}$ (from now on U^{Ω} for simplicity) is the Newtonian potential of Ω considered as a body with density one. The second equality in (2.2) is a consequence of the first one except possibly at certain points on $\partial\Omega$. Also we can prove that $-\Delta U^{\mu} = \mu$ in the sense of distributions (see [5], [1]).

It has been explained in [12] and [15] that this problem is equivalent to finding a pair (u, Ω) of the following one-phase free boundary problem:

$$(2.3) \quad \begin{cases} \Delta u = \chi\Omega - \mu, & \text{in } \mathbb{R}^N, \\ u = |\nabla u| = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $u = U^\mu - U^\Omega$ is the so-called modified Schwarz potential (MSP) of the pair (μ, Ω) .

We also can replace the following inequality in (2.1) for the class of subharmonic functions $SL^1(\Omega)$,

$$(2.4) \quad \int h d\mu \leq \int_{\Omega} h dx, \quad \forall h \in SL^1(\Omega),$$

and get a quadrature domain for subharmonic functions. In this case, we call Ω a *subharmonic quadrature domain* with respect to μ and write $\Omega \in Q(\mu, SL^1)$. The authors in [10] and [12] have showed that (2.4) is equivalent to

$$(2.5) \quad \begin{cases} U^\Omega \leq U^\mu, & \text{in } \mathbb{R}^N, \\ U^\Omega = U^\mu, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

which is equivalent to

$$(2.6) \quad \begin{cases} \Delta u = \chi_\Omega - \mu, & \text{in } \mathbb{R}^N, \\ u \geq 0, & \text{in } \mathbb{R}^N, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $u = U^\mu - U^\Omega$. We note that in (2.6) it is not generally true that $u > 0$ in Ω (see [7]). For more details about quadrature domains, [6], [11] and [14] are basic references.

Moreover, if we also introduce the class $Q(\mu, AL^1)$ by saying that $\Omega \in Q(\mu, AL^1)$ if and only if $\nabla U^\Omega = \nabla U^\mu$ in Ω^c then

$$Q(\mu, SL^1) \subseteq Q(\mu, HL^1) \subseteq Q(\mu, AL^1).$$

For instance, if $\mu = \delta_0$ then all these classes are equal to $\{B(0, r)\}$, see [10]. The existence and uniqueness theorems in one-phase quadrature domains are established in [14] for class SL^1 .

3. Two-phase Case

In this section our objective is to define two phase quadrature domain and investigate its PDE formulation.

3.1. Definition and basic properties

Let Ω is an open and bounded subset of $\mathbb{R}^N$. We define $\tilde{H}(\Omega)$ by

$$\begin{aligned} \tilde{H}(\Omega) = \{U^\eta : \eta \text{ is a signed Radon measure} \\ \text{with compact support and } \text{supp}(\eta) \subset \Omega^c\}. \end{aligned}$$

Next lemma leads us to have a definition of the two-phase quadrature domain and quadrature identity.

Lemma 3.1. *Let Ω and $\tilde{H}(\Omega)$ be as above.*

- (1) *If $h \in \tilde{H}(\Omega)$ then $h \in L^1_{loc}(\mathbb{R}^N)$.*
- (2) *All functions in $\tilde{H}(\Omega)$ are harmonic in Ω .*
- (3) *For $x \in \Omega^c$ we have $G(x - \cdot) = U^{\delta_x} \in \tilde{H}(\Omega)$.*
- (4) *Suppose that h is harmonic in a bounded open set D such that $\Omega \subset\subset D$. There exists a measure ν with compact support such that $\text{supp}(\nu) \subset D \setminus \bar{\Omega}$ and $h = U^\nu$ in Ω .*

Proof. The items (1), (2) and (3) are immediately verified by the definition of $\tilde{H}(\Omega)$. To prove the last one, suppose that h is a harmonic function on a bounded open set $D \subset \mathbb{R}^N$ such that $\Omega \subset\subset D$. Let $\xi \in C_c^\infty(\mathbb{R}^N)$ such that $\text{supp}(\xi) \subset D$. Moreover, we choose $\xi = 1$ in a neighborhood close enough to Ω . For $x \in \Omega$ one obtains

$$\begin{aligned} h(x) &= (h\xi)(x) = \int \delta_x(h\xi)(y) dy \\ &= \int -\Delta G(x-y)(h\xi)(y) dy \\ &= \int G(x-y)(-\Delta(h\xi))(y) dy. \end{aligned}$$

Now if one sets $d\nu = (-\Delta h\xi)(y)dy$, then $h = U^\nu$ in Ω and $\text{supp}(\nu) \subset D \setminus \bar{\Omega}$. $\square$

Then we have the following definition.

Definition 3.2. Let $\Omega^\pm$ be two open, disjoint and connected subsets of $\mathbb{R}^N$ and $\mu^\pm$ be two positive Radon measures with compact supports. Moreover, suppose that $\lambda^\pm$ are two positive constants. We say that $\Omega = \Omega^+ \cup \Omega^-$ is a *two-phase quadrature domain*, with respect to $\mu^\pm, \lambda^\pm$ and $\tilde{H}(\Omega)$ if $\text{supp}(\mu^\pm) \subseteq \Omega^\pm$, and

$$(3.1) \quad \int_{\Omega^+} \lambda^+ h - \int_{\Omega^-} \lambda^- h = \int h (d\mu^+ - d\mu^-), \quad \forall h \in \tilde{H}(\Omega).$$

We then write $\Omega^\pm \in Q(\mu^\pm, \tilde{H})$ or $\Omega \in Q(\mu, \tilde{H})$ where $\mu = \mu^+ - \mu^-$.

To reach a potential theory interpretation of the two phase quadrature domain let us choose $h(x) = h_y(x) = G(x-y)$ in (3.1), as a harmonic function for $y \in \mathbb{R}^N \setminus \Omega$. Then, we have

$$(3.2) \quad U^f = U^\mu \text{ in } \mathbb{R}^N \setminus \Omega,$$

where $f = \lambda^+ \chi_{\Omega^+} - \lambda^- \chi_{\Omega^-}$, and $\mu = \mu^+ - \mu^-$.

We deal with the following question.

Main question: Whether we can claim that Ω is the unique domain satisfies (3.1)?

It turns out that the uniqueness problems in this case are much more involved than in the one-phase case.

A different formulation (or a different starting point) of our problem would come from the well-known potential theoretic formulation of analyzing gravi-equivalent bodies. Indeed, suppose there are non-empty bounded domains $D = D^+ \cup D^-$ and $\Omega = \Omega^+ \cup \Omega^-$, where

$$D^+ \cap D^- = \Omega^+ \cap \Omega^- = \emptyset,$$

satisfying

$$(3.3) \quad \int_{\Omega^+} \lambda^+ h - \int_{\Omega^-} \lambda^- h = \int_{D^+} \lambda^+ h - \int_{D^-} \lambda^- h, \quad \forall h \in \tilde{H}(\Omega \cup D).$$

Then, we would like to see whether $\Omega^\pm = D^\pm$, or alternatively what kind of properties such domains would possess. The first property that can be derived from the above integral identity is the following lemma where its idea comes from [10].

Lemma 3.3. *Suppose that $\Omega = \Omega^+ \cup \Omega^-$ and $D = D^+ \cup D^-$. If (3.3) holds, then for a measure ν ,*

$$\Omega, D \in Q(\nu, \tilde{H}),$$

with $\text{supp}(\nu) \subseteq \overline{\Omega \cap D}$.

Observe that here, the support of the measure can not be expected stay in the set $\overline{\Omega^\pm \cap D^\pm}$, as will be seen from the argument in the proof.

Proof. Define

$$U^\Omega = G * (\lambda^+ \chi_{\Omega^+} - \lambda^- \chi_{\Omega^-}),$$

and U^D correspondingly. Hence, we can define a new function U on $\mathbb{R}^N$ by

$$(3.4) \quad U = \begin{cases} U^\Omega, & \text{in } \Omega^c, \\ U^D, & \text{in } D^c, \\ \text{"arbitrary"}, & \text{in } \Omega \cap D, \end{cases}$$

where "arbitrary" means a suitable function and as smooth as possible. The definition of U on $\Omega \cap D$ can be chosen such that $-\Delta U \in L^\infty(\mathbb{R}^N)$. Let $\nu = -\Delta U$, we have $U = U^\nu$ (because U behaves like a potential at infinity) and it follows that

$$\begin{aligned} U^\Omega &= U^\nu, & \text{in } \Omega^c, \\ U^D &= U^\nu, & \text{in } D^c. \end{aligned}$$

Then by (3.3), $U^\Omega = U^D$ in $\mathbb{R}^N \setminus (\Omega \cup D)$. This proves the lemma with respect to (3.2). $\square$

Remark 1. Observe that $\text{supp}(\nu) \subseteq \overline{\Omega \cap D}$.

Corollary 3.4. *For $\Omega = \Omega^+ \cup \Omega^-$ and $D = D^+ \cup D^-$ admitting the quadrature identity (3.3) we have the intersection $\overline{\Omega} \cap \overline{D}$ is non-void.*

Proof. Suppose that $\overline{\Omega} \cap \overline{D} = \emptyset$ and consider the function U defined by (3.4) in Lemma 3.3. Hence we find that U is harmonic in $\mathbb{R}^N$, i.e.,

$$(3.5) \quad \Delta U = 0, \quad \text{in } \mathbb{R}^N.$$

On the other hand for an arbitrary Radon measure μ one can describe the behavior of potential U^μ as follows (see [13])

$$(3.6) \quad |U^\mu(x)| = O(|x|^{2-N}) \rightarrow 0 \text{ as } |x| \rightarrow \infty \text{ if } N \geq 3,$$

and

$$(3.7) \quad U^\mu(x) = -\frac{1}{2\pi} \ln |x| \int d\mu + O(|x|^{-1}) \text{ as } |x| \rightarrow \infty \text{ if } N = 2.$$

Now with respect to theses properties, we deduce that in the case $N \geq 3$, U is bounded and has logarithmic growth for $N = 2$. By considering (3.5), Liouville's theorem states that $U = c$ where c is a constant. To get a contradiction suppose that B_R is a ball such that $\Omega \subset B_R$. we know that U^Ω is a super solution in B_R and $U = U^\Omega = c$ in $B_R \setminus \Omega$. The strong minimum principle gives us

$$U^\Omega = c, \quad \text{in } B_R,$$

and consequently $\Delta U^\Omega = 0$ in B_R which is a contradiction to $\Delta U^\Omega = -1$ in Ω . $\square$

Remark 2. It remains an open question whether in the above corollary we can conclude that both intersections $\Omega^\pm \cap D^\pm$ are non-void.

Remark 3. (Weighted volume equality) If one takes $h = 1$ in (3.3), then

$$(3.8) \quad \lambda^+ |\Omega^+| - \lambda^- |\Omega^-| = \lambda^+ |D^+| - \lambda^- |D^-|,$$

where $|\Omega|$ denotes the volume of Ω . We shall use this simple property in the proof of some results later.

3.2. PDE formulation

For $\Omega = \Omega^+ \cup \Omega^- \in Q(\mu^\pm, \tilde{H})$, we can define $u = U^\mu - U^{\lambda^+ \chi_{\Omega^+} - \lambda^- \chi_{\Omega^-}}$. Then by (3.1) with $u = 0$ in $\mathbb{R}^N \setminus \Omega$, we have the following free boundary problem

$$(3.9) \quad \begin{cases} \Delta u = (\lambda^+ \chi_{\Omega^+} - \mu^+) - (\lambda^- \chi_{\Omega^-} - \mu^-), & \text{in } \mathbb{R}^N, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega \text{ and } \text{supp}(\mu^\pm) \subset \Omega^\pm, \end{cases}$$

which is a two-phase version of (2.3).

The next theorem verifies the connection between the potential theory formula and the PDE formulation.

Theorem 3.5. *The quadrature identity (3.1) and the potential theory interpretation (3.2) and PDE formulation (3.9) are equivalent.*

Proof. (3.1) $\Rightarrow$ (3.2) $\Rightarrow$ (3.9): This is clear.

(3.9) $\Rightarrow$ (3.1): Suppose that (3.9) is given. For all $h = U^\eta \in \tilde{H}(\Omega)$ and $\nu = (\lambda^+ \chi_{\Omega^+} - \mu^+) - (\lambda^- \chi_{\Omega^-} - \mu^-)$, Fubini's theorem gives

$$(3.10) \quad \int U^\eta d\nu = \int U^\nu d\eta = \int_{\Omega} U^\nu d\eta + \int_{\Omega^c} U^\nu d\eta.$$

We prove that U^ν vanishes in Ω^c and consequently the second term of (3.10) is zero.

Suppose that $y \in \Omega^c$. Let $R > 0$ and B_R be a ball such that $y \in B_R, \Omega \subset B_R$. Then by assumption on ν

$$\begin{aligned} U^\nu(y) &= \int_{B_R} G(x-y) d\nu(x) = \int_{B_R} G(x-y) \Delta u dx \\ &= \int_{B_R} \left(G(x-y) \Delta u - \Delta G(x-y) u \right) dx + \int_{B_R} \Delta G(x-y) u dx \\ &= \int_{\partial B_R} \left(\frac{\partial G}{\partial n} u - \frac{\partial u}{\partial n} G \right) ds + \int_{B_R} \delta_y(x) u(x) dx \\ &= u(y) \\ &= 0. \end{aligned}$$

On the other hand $\text{supp}(\eta) \subset \Omega^c$ then the first term of (3.10) is also zero. Therefore we have

$$\int U^\eta d\nu = 0, \quad \text{for all } U^\eta \in \tilde{H}(\Omega),$$

which is (3.1). □

3.3. Quadrature inequalities

The corresponding quadrature inequality (2.4) is more subtle in two-phase case. To derive such an inequality suppose that η is a signed Radon measure with compact support. We define:

$$\begin{aligned} S^+(B) &= \{U^\eta : \eta|_B \leq 0\}, \\ S^-(B) &= \{U^\eta : \eta|_B \geq 0\}. \end{aligned}$$

In other words all functions in $S^+(B)$ and $S^-(B)$ are subharmonic and superharmonic on B respectively.

Suppose that $\Omega^\pm \subset \{u^\pm \geq 0\}$. Let

$$\tilde{S}(\Omega) := S^+(\Omega^+) \cap S^-(\Omega^-) = \{U^\eta : \eta|_{\Omega^+} \leq 0, \eta|_{\Omega^-} \geq 0\},$$

and consequently for all $h = U^\eta \in \tilde{S}(\Omega)$ one gets

$$(3.11) \quad \int_{\Omega} u \Delta h = \int_{\Omega^+} u^+(-\eta) + \int_{\Omega^-} u^-(-\eta) \geq 0,$$

where $u = u^+ - u^-$.

Now again suppose that $\nu = \Delta u = (\lambda^+ \chi_{\Omega^+} - \mu^+) - (\lambda^- \chi_{\Omega^-} - \mu^-)$ and $h = U^\eta \in \tilde{S}(\Omega)$. We claim that

$$(3.12) \quad \int U^\eta d\nu = \int U^\nu d\eta \geq 0.$$

To prove this let B_R be a ball contains Ω . We apply Green's formula and get

$$\begin{aligned} \int_{B_R} U^\eta d\nu &= \int_{B_R} h \Delta u \\ &= \int_{B_R} \left(h \Delta u - u \Delta h \right) + \int_{B_R} u \Delta h \\ &= \int_{\partial B_R} \left(h \frac{\partial u}{\partial n} - u \frac{\partial h}{\partial n} \right) + \int_{B_R} u \Delta h \quad (u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B_R) \\ &= \int_{B_R} u \Delta h = \int_{\Omega} u \Delta h \geq 0. \quad (\text{by (3.11)}) \end{aligned}$$

Set now $h_y(x) = -|x - y|^{2-N}$ for $y \in \Omega^+$, $N > 2$ and $h_y(x) = -\ln|x - y|$ for $y \in \Omega^+$, $N = 2$. Then, it is clear that $h_y(x)$ is subharmonic in Ω^+ and it is harmonic in Ω^- , and consequently (3.12) reads

$$(3.13) \quad U^f(y) \leq U^\mu(y) \quad \text{in } \Omega^+.$$

Similarly, if we choose $-h_y(x)$, $y \in \Omega^-$ as a test function in the inequality (3.12), then

$$(3.14) \quad U^f(y) \geq U^\mu(y) \quad \text{in } \Omega^-.$$

Now we are able to make a reasonable definition.

Definition 3.6. Suppose that $\Omega, \mu^\pm, \lambda^\pm$ are the same as in the definition 3.2 and let $f = \lambda^+ \chi_{\Omega^+} - \lambda^- \chi_{\Omega^-}$, and $\mu = \mu^+ - \mu^-$, such that

$$\begin{cases} U^f \leq U^\mu, & \text{in } \mathbb{R}^N \setminus \Omega^-, \\ U^f \geq U^\mu, & \text{in } \mathbb{R}^N \setminus \Omega^+, \end{cases}$$

then we say that Ω is a *two-phase quadrature domain* for the class $\tilde{S}(\Omega)$ and we write $\Omega \in Q(\mu, \tilde{S})$. It is immediately verified that $Q(\mu, \tilde{S}) \subset Q(\mu, \tilde{H})$.

Furthermore, by $\Omega \in Q(\mu^\pm, \tilde{A})$ we mean $\nabla U^f = \nabla U^\mu$ in $\Omega^c \setminus (\partial\Omega^+ \cap \partial\Omega^-)$ and consequently one has

$$Q(\mu, \tilde{S}) \subseteq Q(\mu, \tilde{H}) \subseteq Q(\mu, \tilde{A}).$$

Remark 4. It is clear that (3.11) is still valid, if one chooses $h \in S^+(\Omega^+) \cap \tilde{H}(\Omega^-)$ and it reads

$$U^f \leq U^\mu, \quad \text{in } \mathbb{R}^N \setminus \Omega^-.$$

Similarly, if $h \in S^-(\Omega^-) \cap \tilde{H}(\Omega^+)$, then

$$U^f \geq U^\mu, \quad \text{in } \mathbb{R}^N \setminus \Omega^+.$$

Proposition 3.7. *Consider two non-negative bounded Radon measures $\mu^\pm$ with compact supports and two positive constants $\lambda^\pm$. Moreover, suppose that U^μ is the Newtonian potential of $\mu = \mu^+ - \mu^-$ and $f = \lambda^+ \chi_{\Omega^+} - \lambda^- \chi_{\Omega^-}$. Then, the following statements are equivalent:*

- (1) $\int_{\Omega^+} \lambda^+ h - \int_{\Omega^-} \lambda^- h \geq \int h d\mu^+ - \int h d\mu^-, \quad \forall h \in \tilde{S}(\Omega).$
 - (2) $\Omega \in Q(\mu, \tilde{S}(\Omega)).$
 - (3) If $u = U^\mu - U^f$, then
- $$(3.15) \quad \begin{cases} \Delta u = (\lambda^+ \chi_{\Omega^+} - \mu^+) - (\lambda^- \chi_{\Omega^-} - \mu^-), & \text{in } \mathbb{R}^N, \\ \Omega^\pm \subset \{\pm u \geq 0\}. \end{cases}$$

Proof. (1) $\Rightarrow$ (2): It is clear by considering the equations (3.13) and (3.14).

(2) $\Rightarrow$ (3): It is an immediate consequence of Definition 3.6 and the fact that $-\Delta U^f = f$, $-\Delta U^\mu = \mu$.

(3) $\Rightarrow$ (1): By considering (3.12) one obtains

$$\int_{\Omega} (\lambda^+ \chi_{\Omega^+} - \lambda^- \chi_{\Omega^-} - (\mu^+ - \mu^-)) h = \int_{\Omega} h \Delta u \geq 0,$$

which is equivalent to (1). $\square$

Remark 5. Obviously, by taking $\tilde{\Omega}^\pm = \{u^\pm > 0\}$, the free boundary problem (3.15) can be written as

$$(3.16) \quad \begin{cases} \Delta u = (\lambda^+ \chi_{\{u>0\}} - \mu^+) - (\lambda^- \chi_{\{u<0\}} - \mu^-), & \text{in } \mathbb{R}^N, \\ \tilde{\Omega}^\pm = \{u^\pm \geq 0\}, \end{cases}$$

provided $\text{supp}(\mu^\pm) \subset \tilde{\Omega}^\pm$. This free boundary problem have been studied in [7].

3.4. Some examples

In the special case of (3.9) with $\mu^\pm = 0$, $\lambda^\pm = 1$ one can show that the function $u = \frac{(x_1^+)^2}{2} - \frac{(x_1^-)^2}{2}$, where $x_1^\pm := \max(\pm x_1, 0)$, is a solution of

$$\Delta u = \chi_{\{u>0\}} - \chi_{\{u<0\}}, \quad \text{in } \mathbb{R}^N.$$

Now suppose that $\mu^- = 0$, $\mu^+ \neq 0$ and $\Omega^\pm = \{x : \pm u(x) > 0\}$ then consequently the PDE formulations (3.9) turns

$$(3.17) \quad \begin{cases} \Delta u = \lambda^+ \chi_{\{u>0\}} - \mu^+ - \lambda^- \chi_{\{u<0\}}, & \text{in } \mathbb{R}^N, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

If $\partial\Omega^- \neq \emptyset$ then in Ω^-

$$\begin{cases} \Delta u = -\lambda^- \leq 0 & \text{in } \mathbb{R}^N \setminus \Omega^+, \\ u < 0, & \text{in } \Omega^-, \\ u = 0, & \text{on } \partial\Omega^-, \end{cases}$$

which is an obvious contradiction according to the minimum principle. Therefore (3.17) has no solution.

Example 3.8. ($N = 1$) Suppose that $\mu^\pm = a^\pm \delta_{x^\pm}$, $x^+ > 0$, $x^- = -x^+$, $a^\pm > 0$. Hence for $r_1, r_2 > 0$ one has $\Omega^+ = (x^+ - r_1, x^+ + r_1)$ and $\Omega^- = (x^- - r_2, x^- + r_2)$ and they meet each other at $x^+ - r_1 = x^- + r_2$. In other words, $2x^+ = r_1 + r_2$. Regarding (3.9) and the continuity conditions one gets $r_1 = \frac{2a^-x^+}{a^-+a^+}$, $r_2 = \frac{2a^+x^+}{a^-+a^+}$ and

$$u(x) = \begin{cases} \frac{(x-x^+)^2}{2} - a^+(x-x^+)H(x-x^+) + a^-r_2 - \frac{1}{2}r_1^2, & \text{in } \Omega^+, \\ -\frac{(x-x^-)^2}{2} + a^-(x-x^-)H(x-x^-) + \frac{1}{2}r_2^2, & \text{in } \Omega^-, \end{cases}$$

where $H(x)$ is the Heaviside function.

Example 3.9. Suppose that m denotes the Lebesgue measure and $\mu^\pm$ are two uniformly distributed surface measure on $|x| = 2, 4$ respectively such that $d\mu^\pm = \rho^\pm dm$ for some $\rho^\pm > 0$ to be decided below. Let

$$u_i^\pm = \pm \frac{1}{2N} |x|^2 + a_i^\pm |x|^{2-N} + b_i^\pm \quad \text{for } i = 1, 2.$$

Now one can choose $a_i^\pm, b_i^\pm$ and $\rho^\pm$ in a such way so that the function u defined as

$$u = \begin{cases} u_1^+, & \text{in } 1 < |x| < 2, \\ u_2^+, & \text{in } 2 \leq |x| < 3, \\ u_1^-, & \text{in } 3 \leq |x| < 4, \\ u_2^-, & \text{in } 4 \leq |x| < 5, \end{cases}$$

is continuous in $1 \leq |x| \leq 5$ and satisfies the two phase free boundary equation (3.9). Therefore $\Omega = \Omega^+ \cup \Omega^- = \{1 < |x| < 3\} \cup \{3 < |x| < 5\}$ is a two phase quadrature domain with respect to $\mu^\pm$. Here, the densities are given by the difference of the normal derivatives of the left- and right-hand sides limits. For existence of these quadrature domains see [14].

4. Discussion on existence theory

In general an existence result of two-phase quadrature domains, is not so easy to obtain. It seems that one needs rather strong assumptions on the densities $\lambda^\pm$ as well as the measures $\mu^\pm$ to ensure the existence of a solution. For example, in the simpler one phase case the crucial assumption is that the measure should be non-negative and sufficiently concentrated, (see [10]).

In other words to ensure the existence of a solution for (3.16), one has to make a balance between measures. However making such balance conditions are a challenging problem and is under research. As far as we know, the Sakai's concentration condition together with estimates of the one phase solutions of $\mu^\pm$ is a sufficient condition (see [7]). For more existence result see the recent article [8].

In the two-phase case, it is far from obvious that such an assumption would be sufficient to guarantee the existence of a solution. Indeed, if one of the measures $\mu^\pm$ is so large that it would "eat up" the other one, i.e, large concentration of one of the two measures, force the support of the other to shrink. This can already be seen in one dimension. For instance, see Example 1.1 in [7].

Our objective in this section is to present some known existence result for the problem

$$(4.1) \quad \Delta u = (\lambda^+ \chi_{\Omega^+} - \mu^+) - (\lambda^- \chi_{\Omega^-} - \mu^-),$$

with the crucial sign properties $\Omega^\pm = \{\pm u > 0\}$. One of the few paper discussing existence results in a simpler case is [7]. The authors of [7] apply the minimization technique to show the existence of solution of

$$(4.2) \quad \Delta u = (\lambda^+ - \mu^+) \chi_{\{u>0\}} - (\lambda^- - \mu^-) \chi_{\{u<0\}}, \quad \text{in } \mathbb{R}^N,$$

which implies a weaker form of the two-phase problem (4.1) with the sign assumptions. Remarkably, it is not so easy to find appropriate conditions to obtain (4.1) by considering (4.2). In other words, it is a challenging problem to find conditions such that

$$\mu^\pm = \mu^\pm \chi_{\{\pm u > 0\}},$$

i.e., $\text{supp}(\mu^\pm) \subset \text{supp}(\pm u)$. In the one phase case, the authors of [12] have established some conditions to guarantee $\text{supp}(\mu) \subset \text{supp}(u)$, but for the two-phase case the problem is almost completely open.

One can easily show the Euler-Lagrange equation for the functional

$$(4.3) \quad J_\Omega(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - g(x)u^+ + h(x)u^- \right) dx,$$

coincides in the following two-phase free boundary problem:

$$(4.4) \quad -\Delta u = g(x) \chi_{\{u>0\}} - h(x) \chi_{\{u<0\}}.$$

The existence of a minimizer for (4.3) in appropriate functional space depends on the existence of the minimizers for the two functionals in one phase case

$$J_+(u) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - g(x)u^+ \right) dx, \quad J_-(u) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + h(x)u^- \right) dx,$$

on the sets $W^{\pm} = \{u \in W_0^{1,2}(\Omega), \pm u \geq 0\}$ respectively.

Theorem 4.1. (*Proposition 2.1 in [7]*) *Assume that Ω is a bounded domain. The functional J_{Ω} has a minimizer u in the space $W_0^{1,2}$ and it satisfies the following inequality*

$$U^- \leq u \leq U^+,$$

where $U^{\pm}$ are the minimizers of $J_{\pm}$.

Using Theorem (4.1) with $g = \mu^+ - \lambda^+$, $h = \lambda^- - \mu^-$, we get the existence of solution for (4.2), see [7].

5. Uniqueness results

In this section, we try to prove the uniqueness for (3.9) in some specific cases. To be more clear and from the potential theory point of view, we can rewrite the main question as follows. First we define a solid domain which is important in the uniqueness problem.

Definition 5.1. By a *solid* domain we mean a domain $\mathcal{M}$ such that it is bounded, $\mathcal{M} = (\overline{\mathcal{M}})^{\circ}$ and the complement of $\overline{\mathcal{M}}$, i.e. $(\overline{\mathcal{M}})^c$ is connected.

Question: Suppose that μ is a positive measure with compact support. Can $Q(\mu, \tilde{H})$ contain two distinct domains $\Omega = \Omega^+ \cup \Omega^-$, $D = D^+ \cup D^-$ for solid domains $\Omega^{\pm}$ and $D^{\pm}$?

If one does not consider "solid" assumption on the domains, uniqueness can fail. For instance, in [10] and [14] one can find examples which indicate a non-uniqueness for the one-phase case without such assumptions.

It should be remarked that uniqueness in one-phase case is already a challenging problem and there are studies on it such as [16] and [17]. The following theorem provides uniqueness under the special sign assumptions.

For simplicity we assume that every domain $\mathcal{M}$ in this section satisfies $\mathcal{M} = (\overline{\mathcal{M}})^{\circ}$.

Theorem 5.2. *Let u, v be two solutions of (3.9) and suppose that*

$$\Omega^{\pm} := \{\pm u > 0\}, \quad D^{\pm} := \{\pm v > 0\}.$$

Then, $\Omega^{\pm} = D^{\pm}$ and $u \equiv v$.

Proof. Set $w := u - v$ in $\Omega^+ \cup D^-$. Then, in $\Omega^+ \cup D^-$ we have

$$(4.1) \quad \begin{aligned} \Delta w &= \Delta u - \Delta v = (\lambda^+ \chi_{\Omega^+} - \lambda^- \chi_{\Omega^-}) - (\lambda^+ \chi_{D^+} - \lambda^- \chi_{D^-}) \\ &= \lambda^+ (\chi_{\{\Omega^+ \setminus D^+\}} - \chi_{\{D^+ \setminus \Omega^+\}}) + \lambda^- (\chi_{\{D^- \setminus \Omega^-\}} - \chi_{\{\Omega^- \setminus D^-\}}) \\ &= \lambda^+ \chi_{\{\Omega^+ \setminus D^+\}} + \lambda^- \chi_{\{D^- \setminus \Omega^-\}} \geq 0. \end{aligned}$$

For the boundary of the union one has

$$(4.2) \quad \partial(\Omega^+ \cup D^-) = (\partial\Omega^+ \setminus D^-) \cup (\partial D^- \setminus \Omega^+) := L_1 \cup L_2.$$

Now, we have

$$(4.3) \quad w = u - v = -v \leq 0, \quad \text{on } L_1,$$

since $v \geq 0$ outside D^- . Similarly

$$(4.4) \quad w = u - v = u \leq 0, \quad \text{on } L_2,$$

since $u \leq 0$ outside Ω^+ . Totally we get

$$(4.5) \quad w = u - v \leq 0, \quad \text{on } \partial(\Omega^+ \cup D^-).$$

Then, by the maximum principle

$$u \leq v, \quad \text{in } \Omega^+ \cup D^-.$$

Suppose that $|\cdot|$ denotes the volume of a set. In Ω^+ , we have $0 < u \leq v$ which gives $\Omega^+ \subset D^+$ and $|\Omega^+| < |D^+|$, unless $D^+ = \Omega^+$. In D^- , we have $u \leq v < 0$ which gives $D^- \subset \Omega^-$ and $|D^-| < |\Omega^-|$, unless $D^- = \Omega^-$.

Then, we get

$$\lambda^+ |\Omega^+| - \lambda^- |\Omega^-| < \lambda^+ |D^+| - \lambda^- |D^-|.$$

The latter inequality contradicts the weighted volume equality (3.8), otherwise $D^\pm = \Omega^\pm$. This proves the theorem. $\square$

Remark 6. If $\lambda^\pm = 1$ in Theorem 5.2, then one will get Theorem 4.7(b) of [8].

Now, we present a generalization of the previous theorem.

Theorem 5.3. *Suppose that u, v are two solutions of (3.9) with the corresponding regions $\Omega^\pm, D^\pm$ respectively. Let*

$$(4.6) \quad \Omega^- \subseteq \{u < 0\} \quad \text{and} \quad D^+ \subseteq \{v > 0\}.$$

Furthermore, suppose that $(\Omega^- \cup D^+)^c$ is connected. Then $\Omega^\pm = D^\pm$ with $u \equiv v$.

Proof. Set $w := u - v$ then in $\Omega^+ \cup D^-$, we have (4.1). Here Ω^+ and D^- do not necessarily have the sign property, but still we can conclude that $v \geq 0$ outside D^- and that $u \leq 0$ outside Ω^+ . This shows that the equations (4.2)-(4.5) in Theorem 5.2 are still valid. Then, again by using the maximum principle we obtain

$$(4.7) \quad w \leq 0, \quad \text{in } \Omega^+ \cup D^-.$$

By assumption (4.6) and (4.7) one concludes that

$$(4.8) \quad w \leq 0, \quad \text{in } \mathbb{R}^N.$$

Let $L = B_R \setminus \overline{(D^+ \cup \Omega^-)}$, where $\Omega \cup D \subset\subset B_R$. By the assumption on $\Omega^- \cup D^+$, we deduce that L is a non empty connected domain. Moreover, we have

$$\begin{cases} \Delta w \geq 0, & \text{in } L, \\ w \leq 0, & \text{on } \partial L, \end{cases} \quad (\text{by (4.8)}).$$

The strong maximum principle for w in L states that either $w < 0$ in L , or $w = 0$ in L . But $w = 0$ in $L \setminus (D \cup \Omega)$, hence $w = 0$ in L and consequently

$$(4.9) \quad u \equiv v, \quad \text{in } L.$$

On the other hand one has

$$(4.10) \quad \Delta w = \Delta u - \Delta v = \lambda^+ \chi_{\Omega^+} + \lambda^- \chi_{D^-}, \quad \text{in } L.$$

Now in virtue of (4.9) and (4.10) we get a contradiction unless $u \equiv v \equiv 0$ in L . It means that $(\Omega^+ \cup D^-) \cap L = \emptyset$ which is equivalent to

$$(4.11) \quad \Omega^+ \cup D^- \subset D^+ \cup \Omega^-.$$

We also know that $\Omega^+ \cap \Omega^- = D^+ \cap D^- = \emptyset$ and consequently (4.11) reads

$$\Omega^+ \subset D^+ \quad \text{and} \quad D^- \subset \Omega^-,$$

which is impossible due to the weighted volume equality (3.8). $\square$

The next proposition states that with no sign assumption, there are always stationary points $\{\nabla u = 0\}$ in $\Omega^\pm$ provided $\partial\Omega^\pm$ are locally $C^{1,\alpha}$ away from the so called branch points, (see [16]). This is a sufficient condition to use Hopf's lemma (see [18]).

Proposition 5.4. (*Special Points*) Suppose that u, v are two different solutions of (3.9) and $\Omega^\pm, D^\pm$ are the corresponding regions respectively. Moreover, suppose that

$$(4.12) \quad \left. \frac{\partial u}{\partial n} \right|_{\partial\Omega^+} \leq 0 \quad \text{and} \quad \left. \frac{\partial v}{\partial n} \right|_{\partial D^-} \geq 0,$$

where n is the outward normal vector on $\partial\Omega^+(\partial D^-)$ pointing into $D^+ \setminus \Omega^+(\Omega^- \setminus D^-)$. Then at least one of the following holds.

- (1) v attains its minimum in $D^+ \setminus \Omega$.
- (2) u attains its maximum in $\Omega^- \setminus D$.

Remark 7. Assumptions (4.12) are made to avoid extremely unreasonable techniques to achieve our results. Indeed this assumption is fulfilled in one phase case if $\{u > 0\}$ in the interior neighborhood of $\partial\Omega^+$ and similarly $\{v > 0\}$ in the interior neighborhood of ∂D^+ .

Proof. Consider $K = \overline{[(D^+ \cup \Omega^-) \setminus (\Omega^+ \cup D^-)]}$, $\Omega = \Omega^+ \cup \Omega^-$ and $D = D^+ \cup D^-$. Set $L = B_R \setminus K$ with $\Omega \cup D \subset\subset B_R$, so $w := u - v$ is a subsolution in L and consequently, w attains its maximum on ∂L , say at x_0 . Set n be the outward normal vector on ∂L . According to the Hopf's lemma one obtains

$$(4.13) \quad \partial_n w(x_0) > 0,$$

and therefore

$$(4.14) \quad \partial_n u(x_0) > \partial_n v(x_0).$$

We can deduce that $x_0 \notin \partial B_R$ since

$$\partial_n w = \partial_n u = \partial_n v = 0, \quad \text{on } \partial B_R,$$

which contradicts (4.13). Therefore $x_0 \in \partial K$ and obviously either $x_0 \in \partial\Omega$ or $x_0 \in \partial D$.

- Case (1): Let $x_0 \in \partial\Omega$, then either $x_0 \in D^+$ or $x_0 \in D^-$ but we know that $D^- \cap K = \emptyset$, hence $x_0 \in D^+$. In virtue of (4.12) and (4.14), we deduce that $\partial_n v(x_0) < 0$. Summarizing, we have

$$\begin{cases} x_0 \in D^+ \cap \partial\Omega, \\ \partial_n v(x_0) < 0. \end{cases}$$

It means that v decreases in $D^+ \setminus \Omega$, (n is the normal vector pointing into D^+). Therefore we have $y_0 \in D^+ \setminus \Omega$ such that

$$v(y_0) = \min_{D^+ \setminus \Omega} v \quad \text{and} \quad \nabla v(y_0) = 0.$$

- Case (2): Let $x_0 \in \partial D$, then by a similar discussion as Case (1), u will achieve its maximum value in $\Omega^- \setminus D$, (observe not minimum value).

□

The conclusion is that even in the case of non-uniqueness we have special points in $\Omega^\pm$ and $D^\pm$.

6. Conjectures

Conjecture 6.1. Theorem 5.3, should still be valid if either $\Omega^- := \{u < 0\}$ or $D^+ := \{v > 0\}$.

Conjecture 6.2. The uniqueness for the solution of (3.9) can be obtained by considering only $\Omega^\pm = \{\pm u > 0\}$ without sign properties for $D^\pm$.

Conjecture 6.3. It would be an interesting problem to generalize Theorem (5.3) for the p -Laplacian operator, i.e, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ for $1 < p < \infty$. According to the comparison principle for p -Laplacian (see [4]), it is

straightforward to prove the uniqueness theorem for this operator with all sign properties. We guess that one is able to prove our main result (Theorem 5.3) for the p -Laplacian operator. For more information on p -Laplacian properties and its relation with free boundary problems see [2], [3] and [4] for instance.

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