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On an Inverse Type Problem for the Heat Equation in Parabolic Regular Graph Domains

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Abstract

In this paper we prove some results concerning inverse/free boundary type problems, below the continuous threshold, for the heat equation in the setting of parabolic regular graph domains.

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1 Introduction

In [D] B. Dahlberg showed that if $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain then the harmonic measure with respect to a fixed point, $d\omega$, and surface measure, $d\sigma$, are mutually absolutely continuous. In fact if $k = d\omega/d\sigma$, then Dahlberg showed that k satisfies a certain reverse type Hölder inequality in L^2 from which it follows that $\log k \in BMO(d\sigma)$, the functions of bounded mean oscillation with respect to surface area on $\partial\Omega$. Later Jerison and Kenig [JK] showed that for a C^1 -domain $\log k \in VMO(d\sigma)$, the functions in $BMO(d\sigma)$ of vanishing mean oscillation. Concerning inverse/free boundary type problems it is natural, in this context, to pose the following problem. *Problem 1. Assume that $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain with Lipschitz constant M and that $\log k \in VMO(d\sigma)$. Is it then true that the outer unit normal to $\partial\Omega$, n , is in $VMO(d\sigma)$?* In fact, Problem 1 has an affirmative answer and to briefly outline the proof of the conclusion that n is in $VMO(d\sigma)$ we note, and this type of arguments is described in detail below, that by an argument by contradiction, combined with a blow-up type argument, Problem 1 can in fact be reduced to the following problem. *Problem 2. Let $\Omega_\infty \subset \mathbf{R}^n$ be an unbounded Lipschitz graph domain with Lipschitz constant M , assume that u_∞ is a non-negative harmonic function in Ω_∞ which vanishes continuously on $\partial\Omega_\infty$. Furthermore, assume that there exists a Radon measure ω_∞ supported on $\partial\Omega_\infty$ such that*

$$\int_{\mathbf{R}^n} u_\infty \Delta \phi dZ = \int_{\partial\Omega_\infty} \phi d\omega_\infty, \quad (1.1)$$

whenever $\phi \in C_0^\infty(\mathbf{R}^n)$. In particular, u_∞ and ω_∞ should be thought of, respectively, as the Green function of Ω_∞ with pole at infinity and as the harmonic measure of Ω_∞ at infinity. Moreover, assume that the limiting function, u_∞ , and the limiting measure, ω_∞ , satisfy,

$$\omega_\infty = \sigma_\infty \text{ on } \partial\Omega_\infty, \quad |\nabla u_\infty(Z)| \leq 1 \text{ whenever } Z \in \Omega_\infty. \quad (1.2)$$

In (1.2) σ_∞ is surface measure on $\partial\Omega_\infty$ and c is a constant, $1 \leq c < \infty$, depending only on n, M . Is it then true that (1.1) and (1.2) implies that

$$\Omega_\infty = \{(z, z_n) : z_n > 0\} \text{ and } u_\infty(z, z_n) = z_n \text{ in an appropriate coordinate system?} \quad (1.3)$$

In fact, to conclude, using the set-up stated in Problem 2, that Ω_∞ must be, as stated in (1.3), a hyperplane one can make use of the result of Caffarelli, see [C], concerning regularity of Lipschitz free boundaries in two-phase free boundary problems for the Laplace equation. Indeed, using [C] one is able to conclude that $\partial\Omega_\infty$ is $C^{1,\sigma}$ -regular, for some $\sigma = \sigma(n, M) \in (0, 1)$, and this is sufficient to conclude that (1.3) holds. Naturally the problems outlined above can also be posed in domains far more general than Lipschitz domains and recently Carlos Kenig and Tatiana Toro have resolved, see [KT,KT1-KT3], this type of problems for the Laplace equation in Reifenberg flat and Ahlfors regular domains. In particular, Kenig and Toro have established characterizations of what they refer to as ‘chord arc domains with vanishing constant’ in terms of the behaviour of the Poisson kernel. We furthermore note that even more recently the author has, in [LN4] and together with John Lewis, established appropriate versions, valid for the p -Laplace equation, $1 < p < \infty$, of the results in [KT,KT1-KT3]. While the results in [KT,KT1-KT3] concern harmonic functions and harmonic measure, i.e., the case $p = 2$, the results proved

in [LN4] are valid for the whole range $1 < p < \infty$. [LN4] extends, to Reifenberg flat and Ahlfors regular domains, the results in [LN1] concerning the regularity and free boundary regularity, below the continuous threshold, for the p -Laplace equation, $1 < p < \infty$, in Lipschitz and C^1 -domains. Furthermore, in [LN2, LN3] the results in [C, C1] concerning general two-phase free boundary problems for the Laplace operator are generalized to the p -Laplace operator, $1 < p < \infty$, and, in particular, in [LN2] the results are also applied to the free boundary-inverse type problems below the continuous threshold in Lipschitz domains studied in [LN1]. In particular, in [LN2] the problem referred to as Problem 2 above is posed and solved in the context of the p -Laplace equation, $1 < p < \infty$.

While the results stated above concern the Laplace equation and the p -Laplace equation, which are, respectively, examples of elliptic and degenerate elliptic equations, it is natural to pose the same type of problems for the parabolic counterparts of these operators. While such a program for equations of p -parabolic type, see [DiB] for the fundamentals of such equations, must be considered as a quite long term research project the heat equation is the first parabolic equation to be considered. In fact, in this context and for the heat equations in time-dependent domains, considerable progress has recently been made concerning these type of problems, see [H], [HL], [LM], [LS], [HLN], [HLN1], [L], [LN], [N1], [N2] and [N3]. The purpose of this paper is twofold. Firstly, we prove that an affirmative answer to Problem 2 above, formulated appropriately for the heat/adjoint heat equation in the parabolic regular graph domains defined below, implies an affirmative answer to the corresponding Problem 1. As such we reduce, in this context, Problem 1 to Problem 2 by performing the appropriate blow-up argument. Then focusing on Problem 2 we can apply the main results in [N3] to give affirmative answers to Problem 2 based on appropriate assumptions on the constants (N, M) . The second purpose of this paper, a purpose which is of small magnitude compared to the first purpose, is to correct some confusing statements made in [N3]. Recall that in [N3] a number of results concerning Problem 2, for the heat/adjoint heat equation in $\text{Lip}(1, 1/2)$ graph domains, are established and hence, apart from the results in [N3], Problem 2 is still not completely understood for the heat/adjoint heat equation. Furthermore, at one point we thought that the recent result in [AW] could be used to further understand Problem 2 in our context, but, and this is discussed below, the result in [AW] does not seem to have any impact, and this is on the contrary to statements made in [AW], on the problems considered in this paper. However, to discuss this further and to properly state the results established in this paper we first need to introduce some notation.

Points in Euclidean $(n + 1)$ -space $\mathbf{R}^{n+1}$ are denoted by $(X, t) = (x_1, \dots, x_n, t)$ or (x, x_n, t) where $x = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ and t represents the time-coordinate. We let $\bar{E}, \partial E$, be the closure and boundary of the set $E \subset \mathbf{R}^{n+1}$. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^n$ and we let $|X| = \langle X, X \rangle^{1/2}$ be the Euclidean norm of X . Given $(X, t), (Y, s) \in \mathbf{R}^{n+1}$ we let $d_p(X, t, Y, s) = |X - Y| + |t - s|^{1/2}$ and we define $d(X, t, E)$ to equal the parabolic distance, defined with respect to $d_p(\cdot, \cdot)$, from $(X, t) \in \mathbf{R}^n$ to E . We let $C_r(X, t) = \{(Y, s) \in \mathbf{R}^{n+1} : |y_i - x_i| < r, |t - s| < r^2\}$ whenever $(X, t) \in \mathbf{R}^{n+1}$, $r > 0$, and we let dx denote Lebesgue n -measure on $\mathbf{R}^n$. We let

$$h(E, F) = \max\{\sup\{d(X, t, E) : X \in F\}, \sup\{d(X, t, F) : (X, t) \in E\}\}, \quad (1.4)$$

denote the parabolic Hausdorff distance between the sets $E, F \subset \mathbf{R}^{n+1}$.

A function $\psi : \mathbf{R}^{n-1} \times \mathbf{R} \rightarrow \mathbf{R}$ is called $\text{Lip}(1,1/2)$ or $\text{Lip}(1,1/2)$ regular, with constant M , if

$$|\psi(x, t) - \psi(y, s)| \leq M(|x - y| + |t - s|^{1/2}) \text{ whenever } (x, t) \in \mathbf{R}^n, (y, s) \in \mathbf{R}^n. \quad (1.5)$$

We call $\Omega \subset \mathbf{R}^{n+1}$ a $\text{Lip}(1,1/2)$ graph domain, with constant M , if

$$\Omega = \Omega_\psi = \{(x, x_n, t) \in \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R} : x_n > \psi(x, t)\} \quad (1.6)$$

for some $\text{Lip}(1,1/2)$ function ψ having $\text{Lip}(1,1/2)$ constant bounded by M . We define the surface measure on $\partial\Omega$, μ , as $d\mu(X, t) = d\sigma(X, t)dt$ where $(X, t) \in \partial\Omega$ and $d\sigma(\cdot, t)$ is the naturally defined surface measure on the boundary of the time-slice $\Omega_t = \Omega \cap (\mathbf{R}^n \times \{t\})$. We also let, for $(X, t) \in \partial\Omega$, $n(X, t)$ denote the outer unit spatial normal to $\partial\Omega_t$ at $X \in \partial\Omega_t$. We let, whenever $(x, t) \in \mathbf{R}^n$, $r > 0$, $Q_r(x, t) = \{(y, s) \in \mathbf{R}^n : |y_i - x_i| < r \text{ for } i \in \{1, \dots, n-1\}, |t - s| < r^2\}$. Given a function $\psi : \mathbf{R}^{n-1} \times \mathbf{R} \rightarrow \mathbf{R}$ which is $\text{Lip}(1,1/2)$ we let, whenever $(z, \tau) \in \mathbf{R}^{n-1} \times \mathbf{R}$, $r > 0$,

$$\kappa_\psi(z, \tau, r) = r^{-(n+3)} \inf_L \int_{Q_r(z, \tau)} |\psi(y, s) - L(y)|^2 dy ds, \quad (1.7)$$

where the infimum is taken with respect to functions $L = L(y)$ which are linear functions of y (only). Furthermore, we introduce a measure ν_ψ , defined on $\mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}_+$, as

$$d\nu_\psi(z, \tau, r) = \kappa_\psi(z, \tau, r) r^{-1} dz d\tau dr, \text{ whenever } (z, \tau, r) \in \mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}_+. \quad (1.8)$$

We say that ν_ψ is a Carleson measure on $Q_r(y, s) \times (0, r)$ if there exists $N < \infty$ such that

$$\nu_\psi(Q_\rho(x, t) \times (0, \rho)) \leq N\rho^{n+1}, \quad (1.9)$$

whenever $(x, t) \in \mathbf{R}^n$ and $Q_\rho(x, t) \subset Q_r(y, s)$. The least such N in (1.9) is called the Carleson norm of ν_ψ on $Q_r(y, s) \times (0, r)$. Furthermore, if (1.9) holds for all $r > 0$, then we write $\|\nu_\psi\|_+$ for the Carleson norm of ν_ψ on $\mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}_+$. Finally, if $\|\nu_\psi\|_+ \leq N < \infty$ for some $N > 0$, then we say that ψ is a parabolic regular function with Carleson norm bounded by N . Using these notions we next pose the following definition.

Definition 1.1 Let $\psi : \mathbf{R}^{n-1} \times \mathbf{R} \rightarrow \mathbf{R}$ be a function which is $\text{Lip}(1,1/2)$ regular, with $\text{Lip}(1,1/2)$ constant bounded by M , and assume that ψ is a parabolic regular function with Carleson norm bounded by N . Given ψ , let $\Omega \subset \mathbf{R}^{n+1}$, $\Omega = \Omega_\psi$, be defined as in (1.6). We then say that Ω is a parabolic regular graph domain with constants (M, N) . Moreover, we say that Ω is a parabolic regular graph domain with vanishing constant if, in addition,

$$\lim_{r \rightarrow 0} \sup_{\substack{(x, t) \in \mathbf{R}^n \\ 0 < \rho \leq r}} [\rho^{-(n+1)} \nu_\psi(Q_\rho(x, t) \times (0, \rho))] = 0. \quad (1.10)$$

Remark 1.1 Given a function $\psi : \mathbf{R}^{n-1} \times \mathbf{R} \rightarrow \mathbf{R}$ which is $\text{Lip}(1,1/2)$ regular we let, whenever $(z, \tau) \in \mathbf{R}^{n-1} \times \mathbf{R}$, $r > 0$,

$$\kappa_{\psi, \infty}(z, \tau, r) = \inf_L \sup_{(y, s) \in Q_r(z, \tau)} \frac{|\psi(y, s) - L(y)|}{r}, \quad (1.11)$$

where the infimum is taken with respect to functions $L = L(y)$ which are linear functions of y (only). Then as in display (2.3) in [HLN] one can prove that

$$(\kappa_{\psi,\infty}(z, \tau, \rho))^{n+3} \leq \|\nu\|_{Q_{800r}(x,t) \times (0,800r)} \quad (1.12)$$

whenever $(x, t) \in \mathbf{R}^n$, $r > 0$ and $(z, \tau) \in Q_{100r}(x, t)$, $0 < \rho \leq 100r$. Here $\|\nu\|_{Q_{800r}(x,t) \times (0,800r)}$ denotes the Carleson norm of ν restricted to the set $Q_{800r}(x, t) \times (0, 800r)$. In particular, this implies that if Ω is a parabolic regular graph domain with vanishing constant, then $\partial\Omega$ gets flatter and flatter, locally and as we shrink the scale, and that locally the $\text{Lip}(1, 1/2)$ constant M can be chosen smaller and smaller in a sense that can be made precise based on (1.12). We omit further details.

To continue, we say that $\psi = \psi(x, t) : \mathbf{R}^n \rightarrow \mathbf{R}$ is a time-varying function with parameters b_1 and b_2 if ψ has compact support and satisfies

$$\begin{aligned} (i) \quad & |\psi(x, t) - \psi(y, t)| \leq b_1 |x - y|, x, y \in \mathbf{R}^{n-1}, t \in \mathbf{R}, \\ (ii) \quad & D_{1/2}^t \psi \in BMO(\mathbf{R}^n), \quad \|D_{1/2}^t \psi\|_* \leq b_2 < \infty. \end{aligned} \quad (1.13)$$

Here $D_{1/2}^t \psi(x, t)$ denotes the $1/2$ derivative in t of $\psi(x, \cdot)$, x fixed. This half derivative in time can be defined by way of the Fourier transform or by

$$D_{1/2}^t \psi(x, t) \equiv \hat{c} \int_{\mathbf{R}} \frac{\psi(x, s) - \psi(x, t)}{|s - t|^{3/2}} ds \quad (1.14)$$

for properly chosen $\hat{c}$. $\|\cdot\|_*$ denotes the norm in parabolic $BMO(\mathbf{R}^n)$. For a definition of the space $BMO(\mathbf{R}^n)$ we refer to [HLN1]. In [HLN1] it is proved that if $\Omega = \Omega_\psi$ is a parabolic regular graph domain with constants (M, N) , then there exists a constant c , which is independent of (M, N) , such that

$$\|D_{1/2}^t \psi\|_* \leq c(c(M) + N^2) \text{ with } c(M) \rightarrow 0 \text{ as } M \rightarrow 0. \quad (1.15)$$

Hence, if $\Omega = \Omega_\psi$ is a parabolic regular graph domain, and if ψ has compact support, then ψ is a time-varying function. Furthermore, one can prove that conditions (i), (ii) in (1.13) imply that ψ is $\text{Lip}(1, 1/2)$ with constant $M = M(b_1, b_2)$.

Let $\Omega \subset \mathbf{R}^{n+1}$ be a $\text{Lip}(1, 1/2)$ graph domain and let $(X, t) \in \partial\Omega$, $0 < r < r_0$. Given $(X, t) \in \partial\Omega$ and $r > 0$ we let $\Delta(X, t, r) = C_r(X, t) \cap \partial\Omega$. If $0 < b < 1$ and $(Z, \tau) \in \Delta(X, t, 2r)$ then we let

$$\Gamma(Z, \tau) = \Gamma_b(Z, \tau) = \{(Y, s) \in \Omega : d(Y, s, \partial\Omega) > b(|Z - Y| + |\tau - s|^{1/2})\}. \quad (1.16)$$

Given a measurable function k on $\cup_{(Z, \tau) \in \Delta(X, t, 2r)} \Gamma(Z, \tau)$ we define the non tangential maximal function $N(k) : \Delta(X, t, 2r) \rightarrow \mathbf{R}$ for k as

$$N(k)(Z, \tau) = \sup_{(Y, s) \in \Gamma(Z, \tau) \cap C_{4r}(X, t)} |k|(Y, s) \text{ whenever } (Z, \tau) \in \Delta(X, t, 2r). \quad (1.17)$$

We let $L^q(\Delta(X, t, 2r))$, $1 \leq q \leq \infty$, be the space of functions which are integrable, with respect to the surface measure, μ , to the power q on $\Delta(X, t, 2r)$. Furthermore, given a measurable

function f on $\Delta(X, t, 2r)$ we say that f is of bounded mean oscillation on $\Delta(X, t, r)$, $f \in \text{BMO}(\Delta(X, t, r))$, if there exists A , $0 < A < \infty$, such that

$$\int_{\Delta(Z, \tau, \rho)} |f - f_{\Delta}|^2 d\mu \leq A^2 \mu(\Delta(Z, \tau, \rho)) \quad (1.18)$$

whenever $(Z, \tau) \in \Delta(X, t, r)$ and $0 < \rho \leq r$. Here f_{Δ} denotes the average of f on $\Delta = \Delta(Z, \tau, \rho)$ with respect to the surface measure μ . The least A for which (1.18) holds is denoted by $\|f\|_{\text{BMO}(\Delta(X, t, r))}$. If f is a vector valued function, $f = (f_1, \dots, f_n)$, then $f_{\Delta} = (f_{1,\Delta}, \dots, f_{n,\Delta})$ and the BMO-norm of f is defined as in (1.10) with $|f - f_{\Delta}|^2 = \langle f - f_{\Delta}, f - f_{\Delta} \rangle$. Also, we say that f is of vanishing mean oscillation on $\Delta(X, t, r)$, $f \in \text{VMO}(\Delta(X, t, r))$, provided $f \in \text{BMO}(\Delta(X, t, r))$ and provided for each $\epsilon > 0$ there is a $\delta > 0$ such that (1.18) holds with A replaced by ϵ whenever $0 < \rho < \min(\delta, r)$ and $(Z, \tau) \in \Delta(X, t, r)$. For more on BMO we refer to [S, chapter IV].

Let $\Omega \subset \mathbf{R}^{n+1}$ be a $\text{Lip}(1, 1/2)$ graph domain. Then, it is well known that the bounded continuous Dirichlet problem for the heat equation always has a unique solution in Ω . Given $(\hat{X}, \hat{t}) \in \Omega$ we let $G(\cdot, \cdot, \hat{X}, \hat{t})$ denote Green's function for the heat equation in Ω with pole at $(\hat{X}, \hat{t})$. That is

$$\frac{\partial}{\partial t} G(X, t, \hat{X}, \hat{t}) - \Delta G(X, t, \hat{X}, \hat{t}) = \delta_{(\hat{X}, \hat{t})}(X, t) \text{ in } \Omega \text{ and } G \equiv 0 \text{ on } \partial\Omega. \quad (1.19)$$

Here $\delta_{(\hat{X}, \hat{t})}$ denotes the Dirac delta function at $(\hat{X}, \hat{t})$ and Δ is the Laplacian in X . Furthermore, we note that $G(\hat{X}, \hat{t}, \cdot, \cdot)$ is the Green's function for the adjoint heat equation with pole at $(\hat{X}, \hat{t}) \in \Omega$. That is,

$$-\frac{\partial}{\partial t} G(\hat{X}, \hat{t}, X, t) - \Delta G(\hat{X}, \hat{t}, X, t) = \delta_{(\hat{X}, \hat{t})}(X, t) \text{ in } \Omega \text{ and } G \equiv 0 \text{ on } \partial\Omega. \quad (1.20)$$

We let $\omega(\hat{X}, \hat{t}, \cdot)$ and $\hat{\omega}(\hat{X}, \hat{t}, \cdot)$ be the caloric and adjoint caloric measure, at $(\hat{X}, \hat{t}) \in \Omega$, associated to the heat and adjoint heat equation in Ω . Given $(\hat{X}, \hat{t}) \in \Omega$ we let $G(X, t, \hat{X}, \hat{t}) \equiv 0$ whenever $(X, t) \in (\mathbf{R}^n \times (\hat{t}, \infty)) \setminus \Omega$ and $G(\hat{X}, \hat{t}, X, t) \equiv 0$ whenever $(X, t) \in (\mathbf{R}^n \times (-\infty, \hat{t})) \setminus \Omega$. Then, by the Riesz representation theorem for sub caloric functions and for adjoint sub caloric functions in $\mathbf{R}^{n+1} \setminus \{(\hat{X}, \hat{t})\}$, see [Do], we have that

$$\begin{aligned} \int G(\hat{X}, \hat{t}, X, t) (\Delta\phi - \frac{\partial\phi}{\partial t}) dX dt &= \int \phi d\omega(\hat{X}, \hat{t}, \cdot), \\ \int G(X, t, \hat{X}, \hat{t}) (\Delta\phi + \frac{\partial\phi}{\partial t}) dX dt &= \int \phi d\hat{\omega}(\hat{X}, \hat{t}, \cdot), \end{aligned} \quad (1.21)$$

for all $\phi \in C_0^\infty(\mathbf{R}^{n+1} \setminus \{(\hat{X}, \hat{t})\})$. For (X, t) , $r > 0$, and $A \geq 100$, we define

$$\begin{aligned} \Gamma_A(X, t, r) &= \{(Y, s) : |Y - X|^2 \leq A|s - t|, |s - t| \geq 5r^2\}, \\ \Gamma_A^+(X, t, r) &= \{(Y, s) : |Y - X|^2 \leq A|s - t|, |s - t| \geq 5r^2, s > t\}, \\ \Gamma_A^-(X, t, r) &= \{(Y, s) : |Y - X|^2 \leq A|s - t|, |s - t| \geq 5r^2, s < t\}. \end{aligned} \quad (1.22)$$

Let $(X, t) \in \partial\Omega$ and consider $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A(X, t, 16r)$. Then it is well-known (see [LM], [N]) that $\omega(\hat{X}, \hat{t}, \cdot)$ and $\hat{\omega}(\hat{X}, \hat{t}, \cdot)$ are doubling measures in the sense that there exists a constant

$c = c(n, M, A)$ such that

$$\begin{aligned}\omega(\hat{X}, \hat{t}, \Delta(X, t, 2r)) &\leq c\omega(\hat{X}, \hat{t}, \Delta(X, t, r)) \text{ if } (\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, 16r), \\ \hat{\omega}(\hat{X}, \hat{t}, \Delta(X, t, 2r)) &\leq c\hat{\omega}(\hat{X}, \hat{t}, \Delta(X, t, r)) \text{ if } (\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^-(X, t, 16r).\end{aligned}\quad (1.23)$$

Given $(X, t) \in \partial\Omega$, let $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A(X, t, 16r)$. We will, in the following, use the following convention. If $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, 16r)$, then $\omega^*(\hat{X}, \hat{t}, \cdot) = \omega(\hat{X}, \hat{t}, \cdot)$ and if $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^-(X, t, 16r)$, then $\omega^*(\hat{X}, \hat{t}, \cdot) = \hat{\omega}(\hat{X}, \hat{t}, \cdot)$. To continue our definitions, suppose $\omega^*(\hat{X}, \hat{t}, \cdot)$ is absolutely continuous with respect to the surface measure μ on $\Delta(X, t, 4r)$. We then say that $\omega^*(\hat{X}, \hat{t}, \cdot)$ is an A^∞ -weight with respect to μ on $\Delta(X, t, r)$, $d\omega^*(\hat{X}, \hat{t}, \cdot) \in A^\infty(\Delta(X, t, r), d\mu)$ for short, if for some $\gamma > 0$ there exists $\epsilon = \epsilon(\gamma) > 0$ with the property that if $(Z, \tau) \in \Delta(X, t, r)$, $0 < \rho < r$ and if $E \subset \Delta(Z, \tau, \rho)$, then

$$\frac{\mu(E)}{\mu(\Delta(Z, \tau, \rho))} \geq \gamma \text{ implies that } \frac{\omega^*(\hat{X}, \hat{t}, E)}{\omega^*(\hat{X}, \hat{t}, \Delta(Z, \tau, \rho))} \geq \epsilon. \quad (1.24)$$

Furthermore, if $\omega^*(\hat{X}, \hat{t}, \cdot)$ is absolutely continuous with respect to μ on $\Delta(X, t, 2r)$, then we let $k^*(\hat{X}, \hat{t}, \cdot, \cdot)$ denote the Radon-Nikodym derivative of $\omega^*(\hat{X}, \hat{t}, \cdot)$ with respect to μ . Note that if $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, 16r)$, then we call $k(\hat{X}, \hat{t}, \cdot) = k^*(\hat{X}, \hat{t}, \cdot, \cdot)$ the caloric Poisson kernel at $(\hat{X}, \hat{t})$ and if $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^-(X, t, 16r)$, then we call $\hat{k}(\hat{X}, \hat{t}, \cdot) = k^*(\hat{X}, \hat{t}, \cdot, \cdot)$ the adjoint caloric Poisson kernel at $(\hat{X}, \hat{t})$. Using the notation introduced above we first note the following two theorems.

Theorem 1 *Let $\Omega = \Omega_\psi \subset \mathbf{R}^{n+1}$ be a parabolic regular graph domain with constants (M, N) and assume that ψ has compact support. Let $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A(X, t, 16r)$. Then $\omega^*(\hat{X}, \hat{t}, \cdot)$ is absolutely continuous with respect to μ on $\Delta(X, t, 4r)$, $d\omega^*(\hat{X}, \hat{t}, \cdot) \in A^\infty(\Delta(X, t, 2r), d\mu)$ and*

$$\lim_{(Y, s) \in \Gamma(Z, \tau), (Y, s) \rightarrow (Z, \tau)} \nabla_Y G(\hat{X}, \hat{t}, Y, s), \quad \lim_{(Y, s) \in \Gamma(Z, \tau), (Y, s) \rightarrow (Z, \tau)} \nabla_Y G(Y, s, \hat{X}, \hat{t}),$$

exist for μ -almost every $(Z, \tau) \in \Delta(X, t, 4r)$. In particular,

$$k(\hat{X}, \hat{t}, Z, \tau) = \langle \nabla_Y G(\hat{X}, \hat{t}, Z, \tau), N(Z, \tau) \rangle, \quad \hat{k}(\hat{X}, \hat{t}, Z, \tau) = \langle \nabla_Y G(Z, \tau, \hat{X}, \hat{t}), N(Z, \tau) \rangle$$

for μ -almost every $(Z, \tau) \in \Delta(X, t, 4r)$. Furthermore, there exist $q > 1$ and a constant c , $1 \leq c < \infty$, which both only depend on n, M, N and A , such that

$$\begin{aligned}(i) \quad & N(|\nabla G^*|) \in L^q(\Delta(X, t, 2r)), \\ (ii) \quad & \int_{\Delta(X, t, 2r)} (k^*)^q d\mu \leq cr^{(n+1)(1-q)} \left(\int_{\Delta(X, t, 2r)} k^* d\mu \right)^q, \\ (iii) \quad & \log k^* \in BMO(\Delta(X, t, r)), \quad \|\log k^*\|_{BMO(\Delta(X, t, r))} \leq c,\end{aligned}$$

where, either, $G^ = G^*(\cdot, \cdot) = G(\hat{X}, \hat{t}, \cdot, \cdot)$ or $G^* = G^*(\cdot, \cdot) = G(\cdot, \cdot, \hat{X}, \hat{t})$, and $k^* = k^*(\cdot, \cdot) = k^*(\hat{X}, \hat{t}, \cdot, \cdot)$.*

Theorem 2 *Let $\Omega = \Omega_\psi \subset \mathbf{R}^{n+1}$ be a parabolic regular graph domain with constants (M, N) and assume that ψ has compact support. Let $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A(X, t, 16r)$. Assume, in addition, that Ω is a parabolic regular graph domain with vanishing constant. Then*

$$\log k^* \in VMO(\Delta(X, t, r)),$$

where $k^* = k^*(\cdot, \cdot) = k^*(\hat{X}, \hat{t}, \cdot, \cdot)$.

Concerning appropriate references for Theorem 1 and Theorem 2 we first recall, as noted above, that if $\Omega = \Omega_\psi \subset \mathbf{R}^{n+1}$ is a parabolic regular graph domain with constants (M, N) and if ψ has compact support, then ψ is a time-varying function satisfying conditions (i), (ii) in (1.13). Hence, Theorem 1 follows from the results in [LM]. Furthermore, recall that examples of [KW] and [LS] show that caloric and adjoint caloric measure need not be absolutely continuous with respect to the surface measure σ if we only assume that $\Omega = \Omega_\psi$ is a $\text{Lip}(1, 1/2)$ graph domain. Theorem 2 is proved in greater generality in [HLN1] but we note that the proof in [HLN1] uses results from [H], [HL] and [LM].

In this paper we prove the following theorems.

Theorem 3 *Let $\Omega = \Omega_\psi \subset \mathbf{R}^{n+1}$ be a parabolic regular graph domain with constants (M, N) and assume that ψ has compact support. Let $(X, t) \in \partial\Omega$, $(X, t) = (x, \psi(x, t), t)$ for some $(x, t) \in \mathbf{R}^n$, $r > 0$, and let $A \geq 100$, $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A(X, t, 16r)$. Assume that*

$$\log k^* \in VMO(\Delta(X, t, r))$$

where $k^* = k^*(\cdot, \cdot) = k^*(\hat{X}, \hat{t}, \cdot, \cdot)$. Assume that M and N are such that (1.31) and (1.32) stated below imply the conclusion stated in (1.33) below. Then,

$$\lim_{\tilde{r} \rightarrow 0} \sup_{\substack{(z, \tau) \in Q_{\tilde{r}/2}(x, t) \\ 0 < \rho \leq \tilde{r}}} [\rho^{-(n+1)} \nu_\psi(Q_\rho(z, \tau) \times (0, \rho))] = 0.$$

Theorem 4 *Let $\Omega = \Omega_\psi \subset \mathbf{R}^{n+1}$ be a parabolic regular graph domain with constants (M, N) and assume that ψ has compact support. Define $\Omega^1 = \Omega$, $\Omega^2 = \mathbf{R}^{n+1} \setminus \bar{\Omega}$. Let $(X, t) \in \partial\Omega$, $(X, t) = (x, \psi(x, t), t)$ for some $(x, t) \in \mathbf{R}^n$, $r > 0$, and let $A \geq 100$, $(\hat{X}^i, \hat{t}^i) \in \Omega^i \cap \Gamma_A(X, t, 16r)$ for $i \in \{1, 2\}$. Let $(\omega^*)^i = (\omega^*)^i(\hat{X}, \hat{t}, \cdot)$ and $(k^*)^i = (k^*)^i(\cdot, \cdot) = (k^*)^i(\hat{X}, \hat{t}, \cdot, \cdot)$, for $i \in \{1, 2\}$, be the caloric measure and Poisson kernel, at $(\hat{X}^i, \hat{t}^i)$, and with respect to Ω^i . Assume, in addition, that*

$$\log(k^*)^i \in VMO(\Delta(X, t, r)) \text{ for } i \in \{1, 2\}.$$

Then

$$\lim_{\tilde{r} \rightarrow 0} \sup_{\substack{(z, \tau) \in Q_{\tilde{r}/2}(x, t) \\ 0 < \rho \leq \tilde{r}}} [\rho^{-(n+1)} \nu_\psi(Q_\rho(z, \tau) \times (0, \rho))] = 0.$$

To outline the proof of Theorem 3 we note, to start with, that we only prove this theorem in the case $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, 16r)$, $\omega^*(\hat{X}, \hat{t}, \cdot) = \omega(\hat{X}, \hat{t}, \cdot)$, $k^* = k(\hat{X}, \hat{t}, \cdot, \cdot)$. The adjoint

case is proved by analogy. In the following we sometimes use the short notation $G = G(\cdot, \cdot) = G(\hat{X}, \hat{t}, \cdot, \cdot)$, $k = k(\cdot, \cdot) = k(\hat{X}, \hat{t}, \cdot, \cdot)$ for the Green function with pole at $(\hat{X}, \hat{t})$ and for the Poisson kernel at $(\hat{X}, \hat{t})$. We extend G to $\mathbf{R}^{n+1} \setminus \{(\hat{X}, \hat{t})\}$ by putting $G \equiv 0$ on $(\mathbf{R}^{n+1} \setminus \{(\hat{X}, \hat{t})\}) \setminus \Omega$. To prove Theorem 3 we argue by contradiction and in particular we assume that

$$\eta = \lim_{\tilde{r} \rightarrow 0} \sup_{\substack{(z, \tau) \in Q_{\tilde{r}/2}(x, t) \\ 0 < \rho \leq \tilde{r}}} [\rho^{-(n+1)} \nu_\psi(Q_\rho(z, \tau) \times (0, \rho))] \text{ for some } \eta > 0. \quad (1.25)$$

Assuming (1.25) we see that there exists a sequence $\{(X_j, t_j)\}$, $(X_j, t_j) \in \Delta(X, t, r/2)$, $(X_j, t_j) = (x_j, \psi(x_j, t_j), t_j)$, and a sequence of scales, $\{r_j\}$, $r_j \rightarrow 0$, such that

$$\eta = \lim_{j \rightarrow \infty} r_j^{-(n+1)} \nu_\psi(Q_{r_j}(x_j, t_j) \times (0, r_j)). \quad (1.26)$$

Based on $\{(X_j, t_j)\}$, $\{r_j\}$, we let $T_j(Z, \tau) = (X_j + r_j Z, t_j + r_j^2 \tau)$ whenever $(Z, \tau) \in \mathbf{R}^{n+1}$ and we put, for $j \in \{1, 2, \dots\}$,

$$\Omega_j = T_j^{-1}(\Omega) = \{(r_j^{-1}(Z - X_j), r_j^{-2}(\tau - t_j)) : (Z, \tau) \in \Omega\}. \quad (1.27)$$

Furthermore, defining

$$\psi_j(z, \tau) = r_j^{-1}(\psi(r_j z + x_j, r_j^2 \tau + t_j) - \psi(x_j, t_j)) \text{ whenever } (z, \tau) \in \mathbf{R}^n, \quad (1.28)$$

we see that $\Omega_j = \{(z, z_n, \tau) \in \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R} : z_n > \psi_j(z, \tau)\}$, $(0, 0) \in \partial\Omega_j$, $\psi_j(0, 0) = 0$. Moreover, we let $(\hat{X}_j, \hat{t}_j) = T_j^{-1}(\hat{X}, \hat{t})$ for $j \in \{1, 2, \dots\}$ and we let, for $j \in \{1, 2, \dots\}$,

$$u_j(Z, \tau) = \lambda_j G(\hat{X}, \hat{t}, T_j(Z, \tau)) \text{ whenever } (Z, \tau) \in (\mathbf{R}^{n+1} \setminus \{(\hat{X}_j, \hat{t}_j)\}), \quad (1.29)$$

where $\{\lambda_j\}$ is an appropriate sequence of real numbers defined in (3.3) below. Recalling that $\Omega \subset \mathbf{R}^{n+1}$ is a parabolic regular graph domain with constants (M, N) and using that

$$\nu_{\psi_j}(Q_\rho(x, t) \times (0, \rho)) = r_j^{-(n+1)} \nu_\psi(Q_{\rho r_j}(r_j x + x_j, r_j^2 t + t_j) \times (0, \rho r_j)) \leq N \rho^{n+1}, \quad (1.30)$$

whenever $(x, t) \in \mathbf{R}^n$, $\rho > 0$, we then show that subsequences of $\{\Omega_j\}, \{\partial\Omega_j\}$ converge to $\Omega_\infty, \partial\Omega_\infty$, in the parabolic Hausdorff distance sense, where also $\Omega_\infty = \{(z, z_n, \tau) \in \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R} : z_n > \psi_\infty(z, \tau)\}$ for some function ψ_∞ and Ω_∞ is a parabolic regular graph domain with constants (M, N) . Moreover, by our choice of the sequence $\{\lambda_j\}$ it follows that a subsequence of $\{u_j\}$ converges uniformly on compact subsets of $\mathbf{R}^{n+1}$ to u_∞ , a positive adjoint caloric function in Ω_∞ vanishing continuously on $\partial\Omega_\infty$. Defining $d\omega_j = |\nabla u_j| d\mu_j$, where μ_j is surface measure on $\partial\Omega_j$, it will also follow that a subsequence of $\{\omega_j\}$ converges weakly as Radon measures to ω_∞ and that

$$\int_{\mathbf{R}^{n+1}} u_\infty (\Delta \phi - \partial_\tau \phi) dZ d\tau = \int_{\partial\Omega_\infty} \phi d\omega_\infty, \quad (1.31)$$

whenever $\phi \in C_0^\infty(\mathbf{R}^{n+1})$. In particular, u_∞ and ω_∞ should be thought of, respectively, as the Green function of Ω_∞ with pole at $+\infty$ and as the caloric measure of Ω_∞ at $+\infty$. Moreover, we prove that the limiting measure, ω_∞ , and the limiting function, u_∞ , satisfy,

$$\omega_\infty = \mu_\infty \text{ on } \partial\Omega_\infty, \quad |\nabla u_\infty(Z, \tau)| \leq 1 \text{ whenever } (Z, \tau) \in \Omega_\infty. \quad (1.32)$$

In (1.32) μ_∞ is surface measure on $\partial\Omega_\infty$ and c is a constant, $1 \leq c < \infty$, depending only on n, M and N . Hence, to complete the argument by contradiction we have to be able to conclude that (1.31) and (1.32) imply that

$$\Omega_\infty = \{(z, z_n, \tau) : z_n > 0\} \text{ and } u_\infty(z, z_n, \tau) = z_n \text{ in an appropriate coordinate system.} \quad (1.33)$$

Indeed, assuming (1.33) and using (1.26), (1.30) we prove that

$$\begin{aligned} \eta &= \lim_{j \rightarrow \infty} r_j^{-(n+1)} \nu_\psi(Q_{r_j}(x_j, t_j) \times (0, r_j)) \\ &= \lim_{j \rightarrow \infty} \nu_{\psi_j}(Q_1(0, 0) \times (0, 1)) = \nu_{\psi_\infty}(Q_1(0, 0) \times (0, 1)) = 0 \end{aligned} \quad (1.34)$$

where the last equality follows from (1.33). In particular, (1.34) contradicts (1.25) and (1.26) and hence the proof of Theorem 3 is completed once one can conclude that (1.31) and (1.32) imply (1.33). Theorem 4 is proved along the same lines, however in this case we can instead use the imposed two-phase condition to make the conclusion in (1.33).

Based on the discussion above we note that the core of the problem in Theorem 3, which is in fact to understand when (1.31) and (1.32) imply (1.33), remains to be understood further. However, we here recall that in [N3] a number of results in this direction are proved and the question is to what extent the assumption imposed in [N3] can be relaxed. We refer the reader to the discussion below (3.62) for a more through outline of the results in [N3] in this context. Finally, we note that in [AW] regularity of parabolic Bernoulli free boundaries was established and, in particular, a parabolic version of the results in [AC] is established. Naturally, one may ask whether or not the results in [AW] apply in our setting. However, in our opinion the conditions imposed in [AW] are far from transparent and in particular, see [AW, Definition 3.2], to apply the main results of [AW], in our case, we need to be able to verify that there exists a constant $\hat{c}$ such that if $(X, t) \in \partial\Omega_\infty$, and $r > 0$, then

$$\int_{C_r(X, t) \cap \Omega_\infty} |\partial_\tau u_\infty|^2 dZ d\tau \leq \hat{c} r^n. \quad (1.35)$$

This a condition more akin to the analysis presented in [ACS], [ACS1], [CLW], [CLW1] see, in particular, [ACS1] where a number of results concerning caloric functions in $\text{Lip}(1, 1)$ domains are proved. Still, it seems hard to verify (1.35) in our setting.

The rest of the paper is organized as follows. Section 2 is of preliminary nature while we in Section 3 give the details of the proof of Theorem 3 and Theorem 4 and correct some confusing statements made in [N3].

2 Preliminaries

In this section we let $\Omega \subset \mathbf{R}^{n+1}$ be a $\text{Lip}(1, 1/2)$ graph domain with constant M in the sense of (1.6). Throughout this section and the paper c will denote, unless otherwise stated, a positive constant ≥ 1 , not necessarily the same at each occurrence, depending only on n, M . In general, $c(a_1, \dots, a_m)$ denotes a positive constant ≥ 1 , which may depend only on n, M and $a_1, \dots, a_m$, and which is not necessarily the same at each occurrence. If $A \approx B$ then A/B is bounded from

above and below by constants which, unless otherwise stated, only depend on n, M . Recall that $(X, t), X = (x, x_n, t), t \in \mathbf{R}$ denotes a point in $\mathbf{R}^{n+1}$. If $(X, t) \in \partial\Omega$ we define

$$A_r^+(X, t) = (x', x_n + 100Mr, t + 2r^2), \quad A_r^-(X, t) = (x', x_n + 100Mr, t - 2r^2). \quad (2.1)$$

Moreover, we also let $A_r(X, t) = (x, x_n + 100Mr, t)$.

2.1 Estimates for solutions to the heat and adjoint heat equation

In this section we state a number of estimates for non-negative solutions to the heat and adjoint heat equation in Ω . For proofs of these lemmas we refer the reader to [FS], [FSY], [LM], [N], [HLN1]. In [N] all relevant estimates are stated and proved in the general setting of second order parabolic equations in divergence form.

Lemma 2.1 *Let $\Omega \subset \mathbf{R}^{n+1}$ be a $\text{Lip}(1, 1/2)$ graph domain with constant M . Let $(X, t) \in \partial\Omega$, $0 < r < r_0$. Suppose that u is a non-negative solution to either the heat or the adjoint heat equation in $\Omega \cap C_{2r}(X, t)$ and assume that u vanishes continuously on $\Delta(X, t, 2r)$. Then there exist $\alpha = \alpha(n, M)$, $\alpha \in (0, 1)$, and $c = c(n, M)$, $1 \leq c < \infty$, such that if $(Y, s) \in \Omega \cap C_r(X, t)$ then*

$$u(Y, s) \leq c \left(\frac{d(X, t, Y, s)}{r} \right)^\alpha \sup_{\Omega \cap C_{2r}(X, t)} u$$

Lemma 2.2 *Let $\Omega \subset \mathbf{R}^{n+1}$ be a $\text{Lip}(1, 1/2)$ graph domain with constant M . Let $(X, t) \in \partial\Omega$, $0 < r < r_0$. Suppose that u is a non-negative solution to either the heat or the adjoint heat equation in $\Omega \cap C_{2r}(X, t)$ and assume that u vanishes continuously on $\Delta(X, t, 2r)$. Then there exists $c = c(n, M)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then*

$$\sup_{\Omega \cap C_{\tilde{r}}(X, t)} u(Y, s) \leq cu(A_{\tilde{r}}^+(X, t))$$

when u is a solution to the heat equation while

$$\sup_{\Omega \cap C_{\tilde{r}}(X, t)} u(Y, s) \leq cu(A_{\tilde{r}}^-(X, t))$$

when u is a solution to the adjoint heat equation.

Lemma 2.3 *Let $\Omega \subset \mathbf{R}^{n+1}$ be a $\text{Lip}(1, 1/2)$ graph domain with constant M . Let $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A(X, t, r)$. Then there exists $c = c(n, M, A)$, $1 \leq c < \infty$, such that*

$$\begin{aligned} c^{-1}r^n G(\hat{X}, \hat{t}, A_r^+(X, t)) &\leq \omega(\hat{X}, \hat{t}, \Delta(X, t, r/2)) \leq cr^n G(\hat{X}, \hat{t}, A_r^-(X, t)) \text{ if } \hat{t} \geq t, \\ c^{-1}r^n G(A_r^-(X, t), \hat{X}, \hat{t}) &\leq \hat{\omega}(\hat{X}, \hat{t}, \Delta(X, t, r/2)) \leq cr^n G(A_r^+(X, t), \hat{X}, \hat{t}) \text{ if } \hat{t} \leq t. \end{aligned}$$

Lemma 2.4 *Let $\Omega \subset \mathbf{R}^{n+1}$ be a $\text{Lip}(1, 1/2)$ graph domain with constant M . Let $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A(X, t, r)$. Then there exists $c = c(n, M, A)$, $1 \leq c < \infty$, such that*

$$\begin{aligned} G(\hat{X}, \hat{t}, A_r^-(X, t)) &\leq cG(\hat{X}, \hat{t}, A_r^+(X, t)) \text{ if } \hat{t} \geq t, \\ G(A_r^-(X, t), \hat{X}, \hat{t}) &\leq cG(A_r^+(X, t), \hat{X}, \hat{t}) \text{ if } \hat{t} \leq t. \end{aligned}$$

Lemma 2.5 *Let $\Omega \subset \mathbf{R}^{n+1}$ be a $\text{Lip}(1,1/2)$ graph domain with constant M . Let $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A(X, t, r)$. Then there exists $c = c(n, M, A)$, $1 \leq c < \infty$, such that*

$$\omega^*(\hat{X}, \hat{t}, \Delta(X, t, r)) \leq c\omega^*(\hat{X}, \hat{t}, \Delta(X, t, r/2))$$

where $\omega^* = \omega$ when $\hat{t} \geq t$ while $\omega^* = \hat{\omega}$ when $\hat{t} \leq t$.

Let $(X, t) \in \partial\Omega$, suppose that $0 < r < r_0$, and let $0 < \rho < r/2$. Then a non-negative function u is said to satisfy a strong Harnack inequality in $\Omega \cap C_r(X, t)$ provided that u is a solution to either the heat or adjoint heat equation in $\Omega \cap C_r(X, t)$ and, for some $\lambda \geq 1$,

$$u(X_1, t_1) \leq \lambda u(X_2, t_2) \quad (2.2)$$

whenever $(X_1, t_1), (X_2, t_2) \in C_\rho(Z, \tau)$ and $C_{2\rho}(Z, \tau) \subset \Omega \cap C_r(X, t)$. Note that $\lambda, 1 \leq \lambda < \infty$, is independent of $C_{2\rho}(Z, \tau) \subset \Omega \cap C_r(X, t) \cap \Omega$. Using Lemma 2.3, Lemma 2.4 and Lemma 2.5 it follows that if $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, then $G(\hat{X}, \hat{t}, \cdot, \cdot)$ satisfies a strong Harnack inequality in $C_r(X, t) \cap \Omega$ if $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, r)$, while $G(\cdot, \cdot, \hat{X}, \hat{t})$ satisfies a strong Harnack inequality in $C_r(X, t) \cap \Omega$ if $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^-(X, t, r)$. Moreover, in both these cases we have that $\lambda = \lambda(n, M, A)$. Furthermore, using the notion of u satisfying a strong Harnack inequality the following can be proved.

Lemma 2.6 *Let $\Omega \subset \mathbf{R}^{n+1}$ be a $\text{Lip}(1,1/2)$ graph domain with constant M . Let $(X, t) \in \partial\Omega$, $0 < r < r_0$. Suppose that u, v are non-negative solutions to either the heat or the adjoint heat equation in $\Omega \cap C_{2r}(X, t)$ and assume that u, v vanish continuously on $\Delta(X, t, 2r)$. Moreover, assume that u, v satisfy a strong Harnack inequality in $\Omega \cap C_{2r}(X, t)$ for some constant $\lambda \geq 1$. Then there exist $c = c(n, M, A, \lambda)$, $1 \leq c < \infty$, and $\alpha = \alpha(n, M, A, \lambda)$, $\alpha \in (0, 1)$, such that*

$$\left| \log \left(\frac{u(X_1, t_1)}{v(X_1, t_1)} \right) - \log \left(\frac{u(X_2, t_2)}{v(X_2, t_2)} \right) \right| \leq c \left(\frac{d(X_1, t_1, X_2, t_2)}{r} \right)^\alpha$$

whenever $(X_1, t_1), (X_2, t_2) \in \Omega \cap C_{r/c}(X, t)$.

Lemma 2.7 *Let $\Omega \subset \mathbf{R}^{n+1}$ be a $\text{Lip}(1,1/2)$ graph domain with constant M . Then there exists a pair (u, ω) satisfying the following. u is a non-negative solution to the adjoint heat in Ω , u vanishes continuously on $\partial\Omega$ and there exists a constant $\lambda = \lambda(n, M)$ such that u satisfies a strong Harnack inequality in $\Omega \cap C_r(X, t)$, in the sense of (2.2) and with constant λ , whenever $(X, t) \in \partial\Omega$ and $0 < r < \infty$. ω is a doubling Radon measure with support contained in $\partial\Omega$ and*

$$\int_{\mathbf{R}^{n+1}} u(\Delta\phi - \partial_s\phi) dY ds = \int_{\partial\Omega} \phi d\omega,$$

whenever $\phi \in C_0^\infty(\mathbf{R}^{n+1})$. Moreover, u and ω are unique modulo a constant and u and ω should be referred to, respectively, as a Green function of Ω with pole at $+\infty$ and as the associated caloric measure of Ω at $+\infty$.

Proof. See [N3, Lemma 2.13 and Lemma 2.14]. □

2.2 Implications of the VMO-condition

Lemma 2.8 *Let $\Omega \subset \mathbf{R}^{n+1}$ is a parabolic regular graph domain with constants (M, N) . Let $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, 16r)$. Let $\omega(\hat{X}, \hat{t}, \cdot)$ and $k = k(\cdot, \cdot) = k(\hat{X}, \hat{t}, \cdot, \cdot)$ be the caloric measure and the Poisson kernel, at $(\hat{X}, \hat{t}) \in \Omega$, defined with respect to Ω . Assume that $\log k \in VMO(\Delta(X, t, r))$, let $\hat{r} < r/4$ and let $\eta > 0$ be such that $\|\log k\|_{BMO(\Delta(X, t, 4\hat{r}))} \leq \eta$. Then there exists for $\epsilon \in (0, 1)$ given, $c = c(n, \epsilon, M, A, \eta)$, $1 \leq c < \infty$, such that*

$$c^{-1} \left(\frac{\mu(E)}{\mu(\Delta(X, t, \hat{r}))} \right)^{1+\epsilon} \leq \frac{\omega(\hat{X}, \hat{t}, E)}{\omega(\hat{X}, \hat{t}, \Delta(X, t, \hat{r}))} \leq c \left(\frac{\mu(E)}{\mu(\Delta(X, t, \hat{r}))} \right)^{1-\epsilon},$$

whenever $E \subset \Delta(X, t, \hat{r})$.

Proof. Let $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, 16r)$, and assume that $\|\log k\|_{BMO(\Delta(X, t, 4\hat{r}))} \leq \eta$. To prove Lemma 2.8 we first note that these assumptions imply that the following holds for any $q \in (1, \infty)$. There exists a constant $c = c(p, n, \epsilon, M, q, \eta)$, $1 \leq c < \infty$, such that if $(Z, \tau) \in \Delta(X, t, 2\hat{r})$, $0 < \rho < \hat{r}$, then

$$\begin{aligned} \left(\frac{1}{\mu(\Delta(Z, \tau, \rho))} \int_{\Delta(Z, \tau, \rho)} k^q d\mu \right)^{1/q} &\leq c \frac{1}{\mu(\Delta(Z, \tau, \rho))} \int_{\Delta(Z, \tau, \rho)} k d\mu, \\ \left(\frac{1}{\mu(\Delta(Z, \tau, \rho))} \int_{\Delta(Z, \tau, \rho)} k^{-q} d\mu \right)^{1/q} &\leq c \frac{1}{\mu(\Delta(Z, \tau, \rho))} \int_{\Delta(Z, \tau, \rho)} k^{-1} d\mu. \end{aligned} \quad (2.3)$$

In fact, (2.3) is a consequence of the John-Nirenberg inequality, see [GR], and the fact that $\Delta(X, t, 4\hat{r})$ can be decomposed into a family of (parabolic) dyadic cubes. We let $q = 1/\epsilon$ and we consider $E \subset \Delta(X, t, \hat{r})$. Using (2.3) and Hölder's inequality we see that

$$\begin{aligned} \omega(\hat{X}, \hat{t}, E) &= \int_E k d\mu \leq \left(\int_{\Delta(X, t, \hat{r})} k^q d\mu \right)^{1/q} (\mu(E))^{1-1/q} \\ &\leq c \left(\frac{\mu(E)}{\mu(\Delta(X, t, \hat{r}))} \right)^{1-1/q} \omega(\hat{X}, \hat{t}, \Delta(X, t, \hat{r})). \end{aligned} \quad (2.4)$$

This completes the proof of the right hand side inequality in Lemma 2.8 is proved. The left hand side inequality is proved similarly. We omit further details. $\square$

Lemma 2.9 *Let $\Omega \subset \mathbf{R}^{n+1}$ is a parabolic regular graph domain with constants (M, N) . Let $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, 16r)$. Let $\omega(\hat{X}, \hat{t}, \cdot)$ and $k = k(\cdot, \cdot) = k(\hat{X}, \hat{t}, \cdot, \cdot)$ be the caloric measure and the Poisson kernel, at $(\hat{X}, \hat{t}) \in \Omega$, defined with respect to Ω . Assume that $\log k \in VMO(\Delta(X, t, r))$. Given $\epsilon > 0$ there exist $\tilde{r} = \tilde{r}(\epsilon)$, $0 < \tilde{r} < r$, and $c = c(n, M, N, A)$, $1 \leq c < \infty$, such that the following is true whenever $0 < r' \leq \tilde{r}$. There exists*

a set $G \subset \Delta(X, t, r')$ such that

$$(a) \quad \frac{\mu(G \cap \Delta(X, t, r'))}{\mu(\Delta(X, t, r'))} \geq 1 - \epsilon,$$

$$(b) \quad (1 - \epsilon)b \leq \frac{\omega(\hat{X}, \hat{t}, \Delta(Z, \tau, \rho))}{\mu(\Delta(Z, \tau, \rho))} \leq (1 + \epsilon)b \text{ if } 0 < \rho < r', \ (Z, \tau) \in G \cap \Delta(X, t, r').$$

Here $\log b$ is the average of $\log k$ on $\Delta(X, t, 4r')$.

Proof. This proof is modeled on the corresponding proof in [LN1]. Let $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, 16r)$. In the following we let $\tilde{\epsilon} > 0$ and $r^*(\tilde{\epsilon}) \ll r$ be small positive numbers. For the moment we allow $\tilde{\epsilon}$ and r^* to vary but we shall later fix these numbers to satisfy several conditions depending on ϵ . Using the assumption that $\log k \in VMO(\Delta(X, t, r))$ we see there exists $\hat{r}$, $0 < \hat{r} \leq r^*$, such that $\log k \in BMO(\Delta(X, t, 8\hat{r}))$ with BMO norm less than or equal to $\tilde{\epsilon}^3$. Let $\hat{A}$ denote the average of $f = \log k$ with respect to μ over $\Delta(X, t, 4\hat{r})$. Then, using the definition of BMO , see (1.18), we have

$$\begin{aligned} & \frac{\tilde{\epsilon} \mu(\{(Z, \tau) \in \Delta(X, t, 4\hat{r}) : |f(Z, \tau) - \hat{A}| > \tilde{\epsilon}\})}{\mu(\Delta(X, t, 4\hat{r}))} \\ & \leq (\mu(\Delta(X, t, 4\hat{r})))^{-1} \int_{\Delta(X, t, 4\hat{r})} |f - \hat{A}|^2 d\mu \leq c\tilde{\epsilon}^3. \end{aligned} \quad (2.5)$$

If $b = e^{\hat{A}}$, then from (2.5) we see

$$\begin{aligned} & \text{that there exists a set } E \subset \Delta(X, t, 4\hat{r}) \text{ such that } (1 - c\tilde{\epsilon})b \leq k \leq (1 + c\tilde{\epsilon})b \text{ on } E \\ & \text{and if } F = \Delta(X, t, 4\hat{r}) \setminus E \text{ then } \mu(F) \leq c\tilde{\epsilon}^2 \mu(\Delta(X, t, 4\hat{r})). \end{aligned} \quad (2.6)$$

In (2.6), c is a universal constant. We introduce, for μ integrable functions h defined on $\Delta(X, t, 5\hat{r})$ and for $(Z, \tau) \in \Delta(X, t, 4\hat{r})$, the maximal function

$$M(h)(Z, \tau) = \sup_{0 < s < \hat{r}} \frac{1}{\mu(\Delta(Z, \tau, s))} \int_{\Delta(Z, \tau, \rho)} h d\mu.$$

Let $G = \{(Z, \tau) \in \Delta(X, t, 4\hat{r}) : M(\chi_F)(Z, \tau) \leq \tilde{\epsilon}\}$ where χ_F is the indicator function for the set F introduced in (2.6) and define $K = \Delta(X, t, 4\hat{r}) \setminus G$. Using weak type estimates for the maximal function, see [S], it then follows that

$$\mu(K) \leq c\tilde{\epsilon} \mu(\Delta(X, t, 4\hat{r})). \quad (2.7)$$

Let $(Z, \tau) \in G \cap \Delta(X, t, \hat{r})$, $0 < s \leq \hat{r}$. Using Theorem 1 we see that

$$\omega(\hat{X}, \hat{t}, \Delta(Z, \tau, \rho)) = \int_{\Delta(Z, \tau, \rho)} k d\mu = \int_{E \cap \Delta(Z, \tau, \rho)} k d\mu + \int_{F \cap \Delta(Z, \tau, \rho)} k d\mu = T_1 + T_2. \quad (2.8)$$

From the definitions of the sets E, F, G , we see that

$$(1 - c\tilde{\epsilon}) b\mu(\Delta(Z, \tau, \rho)) \leq T_1 \leq (1 + c\tilde{\epsilon}) b\mu(\Delta(Z, \tau, \rho)), \quad (2.9)$$

for some $c = c(n, M, N, A)$, provided $\tilde{\epsilon}$ is sufficiently small. Let $q > 1$ be as in the statement of Theorem 1. Then, using Hölder's inequality we see that

$$(\mu(\Delta(Z, \tau, \rho)))^{-1} T_2 \leq \left(\frac{1}{\mu(\Delta(Z, \tau, \rho))} \int_{\Delta(Z, \tau, \rho)} k^q d\mu \right)^{1/q} \left(\frac{\mu(F \cap \Delta(Z, \tau, \rho))}{\mu(\Delta(Z, \tau, \rho))} \right)^{1/q}. \quad (2.10)$$

Moreover, using that $(Z, \tau) \in G$ and the reverse Hölder inequality for k in Theorem 1, we see that (2.10) implies that

$$T_2 \leq c\tilde{\epsilon}^{1/q} \omega(\hat{X}, \hat{t}, \Delta(Z, \tau, \rho)). \quad (2.11)$$

Using (2.9) and (2.11) in (2.8), we obtain that

$$(1 - c\tilde{\epsilon}^{1/q})b \leq \frac{\omega(\Delta(Z, \tau, \rho))}{\sigma(\Delta(Z, \tau, \rho))} \leq (1 + c\tilde{\epsilon}^{1/q})b. \quad (2.12)$$

The proof of Lemma 2.9 is now complete. $\square$

3 Proof of Theorem 3 and Theorem 4

Let $\Omega \subset \mathbf{R}^{n+1}$ is a parabolic regular graph domain with constants (M, N) . Let $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, 16r)$. In the following we sometimes use the short notation $G = G(\cdot, \cdot) = G(\hat{X}, \hat{t}, \cdot, \cdot)$, $k = k(\cdot, \cdot) = k(\hat{X}, \hat{t}, \cdot, \cdot)$ for the Green function with pole at $(\hat{X}, \hat{t})$ and for the Poisson kernel at $(\hat{X}, \hat{t})$. Assume that $\log k \in VMO(\Delta(X, t, r))$. We extend G to $\mathbf{R}^{n+1} \setminus \{(\hat{X}, \hat{t})\}$ by putting $G \equiv 0$ on $(\mathbf{R}^{n+1} \setminus \{(\hat{X}, \hat{t})\}) \setminus \Omega$. We now proceed as outlined in (1.25)-(1.27). In particular, we consider a sequence $\{(X_j, t_j)\}$, $(X_j, t_j) \in \Delta(X, t, r/2)$, $(X_j, t_j) = (x_j, \psi(x_j, t_j), t_j)$, and a sequence of scales, $\{r_j\}$, $r_j \rightarrow 0$, based on which we let $T_j(Z, \tau) = (X_j + r_j Z, t_j + r_j^2 \tau)$, whenever $(Z, \tau) \in \mathbf{R}^{n+1}$, and we define, for $j \in \{1, 2, \dots\}$, Ω_j as in (1.27). Moreover, we let $(\hat{X}_j, \hat{t}_j) = T_j^{-1}(\hat{X}, \hat{t})$ for $j \in \{1, 2, \dots\}$. Then, applying Lemma 2.9 with (X, t) , r' replaced by (X_j, t_j) , $2^j r_j$ and with $\epsilon = 2^{-2j^2}$, we see, for j large enough, that there exists a set $\tilde{G}_j \subset \Delta(X_j, t_j, 2^j r_j)$, such that

$$\frac{\mu(\tilde{G}_j \cap \Delta(X_j, t_j, 2^j r_j))}{\mu(\Delta(X_j, t_j, 2^j r_j))} \geq 1 - 2^{-2j^2} \quad (3.1)$$

and

$$(1 - 2^{-2j^2})b_j \leq \frac{\omega(\hat{X}, \hat{t}, \Delta(Z, \tau, \rho))}{\mu(\Delta(Z, \tau, \rho))} \leq (1 + 2^{-2j^2})b_j \quad (3.2)$$

whenever $0 < \rho < 2^j r_j$ and $(Z, \tau) \in \tilde{G}_j \cap \Delta(X, t, 2^j r_j)$. In (3.2) $\log b_j$ denotes the average, with respect to μ , of $\log k(\hat{X}, \hat{t}, \cdot)$ on $\Delta(X_j, t_j, 2^{j+2} r_j)$. We now let, for $j \in \{1, 2, \dots\}$,

$$\begin{aligned} u_j(Z, \tau) &= \lambda_j G(\hat{X}, \hat{t}, T_j(Z, \tau)) \text{ whenever } (Z, \tau) \in (\mathbf{R}^{n+1} \setminus \{(\hat{X}_j, \hat{t}_j)\}), \\ \lambda_j &= (r_j b_j)^{-1}. \end{aligned} \quad (3.3)$$

Using that the heat and adjoint heat equations are invariant under translation and dilation we see that u_j is adjoint caloric in $\Omega_j \cap \{(Z, \tau) : \tau < \hat{t}_j\}$, u_j is continuous in $\mathbf{R}^{n+1} \setminus \{(\hat{X}_j, \hat{t}_j)\}$ and $u_j \equiv 0$ in $\mathbf{R}^{n+1} \setminus \Omega_j$. Furthermore,

$$\int u_j(Z, \tau)(\Delta\phi - \partial_\tau\phi)dZd\tau = \int \phi d\omega_j(\hat{X}_j, \hat{t}_j, \cdot) \quad (3.4)$$

for all $\phi \in C_0^\infty(\mathbf{R}^{n+1} \setminus \{(\hat{X}_j, \hat{t}_j)\})$, where the measure $\omega_j(\hat{X}_j, \hat{t}_j, \cdot)$ is supported on $\partial\Omega_j$ and

$$d\omega_j(\hat{X}_j, \hat{t}_j, \cdot) = |\nabla u_j|d\mu_j(\cdot). \quad (3.5)$$

Here μ_j is the surface measures on $\partial\Omega_j$ and we note that (3.5) follows from Theorem 1. Moreover, the following is true whenever H_j is a Borel subset of $\partial\Omega_j$,

$$\mu_j(H_j) = r_j^{-1-n} \mu(T_j(H_j)), \quad \omega_j(\hat{X}_j, \hat{t}_j, H_j) = \lambda_j r_j^{-n} \omega(X, \hat{t}, T_j(H_j)). \quad (3.6)$$

Let $\tilde{G}_j$ be the set in (3.1) and (3.2) and let $G_j = T_j^{-1}(\tilde{G}_j)$. Then, from (1.27), (3.6), (3.1) and (3.2) we see that $G_j \subset \partial\Omega_j \cap C_{2^j}(0, 0)$ and

$$\begin{aligned} (i) \quad & \frac{\mu_j(G_j \cap \partial\Omega_j \cap C_{2^j}(0, 0))}{\mu_j(\partial\Omega_j \cap C_{2^j}(0, 0))} \geq 1 - 2^{-2j^2}, \\ (ii) \quad & (1 - 2^{-2j^2}) \leq \frac{\omega_j(\hat{X}_j, \hat{t}_j, \partial\Omega_j \cap C_\rho(Z, \tau))}{\mu_j(\partial\Omega_j \cap C_\rho(Z, \tau))} \leq (1 + 2^{-2j^2}) \text{ whenever } 0 < \rho < 2^j \\ & \text{and } (Z, \tau) \in G_j \cap \partial\Omega_j. \end{aligned} \quad (3.7)$$

In the following we will refer to u_j as the Green function for Ω_j , having pole at $(\hat{X}_j, \hat{t}_j)$, and we will refer to $\omega_j(\hat{X}_j, \hat{t}_j, \cdot)$ as the associated caloric measure at $(\hat{X}_j, \hat{t}_j)$ defined with respect to Ω_j . Using this notation we see that if we let $k_j(\hat{X}_j, \hat{t}_j, \cdot)$ be the associated Poisson kernel, at $(\hat{X}_j, \hat{t}_j)$ and defined with respect to Ω_j , then

$$k_j(\hat{X}_j, \hat{t}_j, Y, s) = \lambda_j r_j k(\hat{X}, \hat{t}, T_j(Y, s)) \text{ whenever } (Y, s) \in \partial\Omega_j, \quad (3.8)$$

and we can rewritten (3.4) as

$$\int u_j(Z, \tau)(\Delta\phi - \partial_\tau\phi)dZd\tau = \int \phi k_j(\hat{X}_j, \hat{t}_j, Y, s)d\mu_j(Y, s) \quad (3.9)$$

for all $\phi \in C_0^\infty(\mathbf{R}^{n+1} \setminus \{(\hat{X}_j, \hat{t}_j)\})$. To proceed we let ψ_j be defined as in (1.28), we note that $\psi_j(0, 0) = 0$ and, see (1.30),

$$\Omega_j \subset \mathbf{R}^{n+1} \text{ is a parabolic regular graph domain with constants } (M, N). \quad (3.10)$$

Using the fact that $\psi_j(0, 0) = 0$, (3.10) and a standard compactness argument, we see that exists a subsequence $\{\psi'_j\}$ of $\{\psi_j\}$, with $\psi'_j \rightarrow \psi_\infty$ uniformly on compact subsets of $\mathbf{R}^n$. Furthermore, ψ_∞ is $\text{Lip}(1, 1/2)$ regular with $\text{Lip}(1, 1/2)$ constant bounded by M and

$$\int_{\mathbf{R}^n} \phi \partial_{z_i} \psi'_j dzd\tau = \int_{\mathbf{R}^n} \phi \partial_{z_i} \psi_\infty dzd\tau \text{ for } 1 \leq i \leq n-1 \text{ and } \phi \in C_0^\infty(\mathbf{R}^n). \quad (3.11)$$

Let $\Omega'_j = \{(z, z_n, \tau) \in \mathbf{R}^{n+1} : z_n > \psi'_j(z, \tau)\}$, $\Omega_\infty = \{(z, z_n, \tau) \in \mathbf{R}^{n+1} : z_n > \psi_\infty(z, \tau)\}$, and let μ'_j and μ_∞ denote, respectively, the surface measure on $\partial\Omega'_j$ and $\partial\Omega_\infty$. Similarly, let $n'_j(\cdot, \tau)$ and $n_\infty(\cdot, \tau)$, for $\tau \in \mathbf{R}$, denote, respectively, the outer unit normal to boundary of the time slice $\Omega'_j \cap (\mathbf{R}^n \times \{\tau\})$ and $\Omega_\infty \cap (\mathbf{R}^n \times \{\tau\})$. From (3.11) we find that

$$\begin{aligned} (i) \quad & h(\partial\Omega'_j \cap C_R(0, 0), \partial\Omega_\infty \cap C_R(0, 0)) \rightarrow 0 \text{ as } j \rightarrow \infty \text{ for every } R > 0, \\ (ii) \quad & \int \langle n'_j, \Phi \rangle d\mu'_j = \int \langle n_\infty, \Phi \rangle d\mu_\infty \text{ whenever } \Phi = (\Phi_1, \dots, \Phi_n) \\ & \text{with } \Phi_i \in C_0^\infty(\mathbf{R}^n) \text{ for } 1 \leq i \leq n. \end{aligned} \quad (3.12)$$

In (3.12) (ii) we have used the fact that if $(Z, \tau) = (z, \psi'_j(z, \tau), \tau) \in \partial\Omega'_j \cap C_{2j}(0, 0)$, then

$$n'_j(Z, \tau) d\mu'_j(Z, \tau) = (\nabla_z \psi'_j(z, \tau), -1).$$

Furthermore,

$$\Omega_\infty \text{ is a parabolic regular graph domain with constants } (M, N). \quad (3.13)$$

In fact, using (1.30) we have that

$$\nu_{\psi_j}(Q_\rho(x, t) \times (0, \rho)) = \int_{Q_\rho(x, t)} \int_0^\rho \kappa_{\psi_j}(z, \tau, r) r^{-1} dz d\tau dr \leq N \rho^{n+1}, \quad (3.14)$$

whenever $(x, t) \in \mathbf{R}^n$, $\rho > 0$, and hence, by the Lebesgue theorem on dominated convergence it is enough to prove that

$$\limsup_{j \rightarrow \infty} \kappa_{\psi_j}(z, \tau, r) = \kappa_{\psi_\infty}(z, \tau, r) \text{ for almost all } (z, \tau, r) \in \mathbf{R}^n \times \mathbf{R}_+. \quad (3.15)$$

However, using the definition in (1.7), and that $\psi'_j \rightarrow \psi_\infty$ uniformly on compact subsets of $\mathbf{R}^n$, we immediately see that (3.15) holds. We next decompose the details of the blow-up argument into the following lemmas.

Lemma 3.1 *Let $\Omega \subset \mathbf{R}^{n+1}$ be a parabolic regular graph domain with constants (M, N) as stated in Theorem 3 and let (X, t) , r , $\{(X_j, t_j)\}$, $\{r_j\}$, $\{\Omega_j\}$, $\{u_j\}$, $\{\omega_j\}$, $\{\Omega'_j\}$, Ω_∞ be as above. Then there exists a subsequence $(u'_j, \omega'_j) \rightarrow (u_\infty, \omega_\infty)$ such that $u'_j \rightarrow u_\infty$ uniformly on compact subsets, u_∞ is a non-negative adjoint caloric function in Ω_∞ with $u_\infty = 0$ continuously on $\partial\Omega_\infty$. Moreover, $u_\infty \equiv 0$ on $\mathbf{R}^{n+1} \setminus \Omega_\infty$. Furthermore, $\omega'_j \rightarrow \omega_\infty$ weakly and*

$$\int_{\mathbf{R}^{n+1}} u_\infty (\Delta \phi - \partial_\tau \phi) dZ d\tau = \int_{\partial\Omega_\infty} \phi d\omega_\infty, \quad (3.16)$$

whenever $\phi \in C_0^\infty(\mathbf{R}^{n+1})$.

Lemma 3.2 *Let $\Omega \subset \mathbf{R}^{n+1}$ be a parabolic regular graph domain with constants (M, N) as stated in Theorem 3. Let (X, t) , r , $\{(X_j, t_j)\}$, $\{r_j\}$, $\{\Omega_j\}$, $\{u_j\}$, $\{\omega_j\}$, $\{\Omega'_j\}$, Ω_∞ , $\{u'_j\}$, $\{\omega'_j\}$, u_∞ , ω_∞ be as in Lemma 3.1. Then $d\omega_\infty = |\nabla u_\infty| d\mu_\infty$ μ_∞ -almost everywhere on $\partial\Omega_\infty$ and*

$$|\nabla u_\infty| \geq 1 \text{ holds } \mu_\infty\text{-almost everywhere on } \partial\Omega_\infty. \quad (3.17)$$

Lemma 3.3 Assume that the assumptions in Lemma 3.2 are fulfilled. Then

$$|\nabla u_\infty(Z, \tau)| \leq 1 \text{ for all } (Z, \tau) \in \Omega_\infty. \quad (3.18)$$

Lemma 3.4 Assume that the assumptions in Lemma 3.3 are fulfilled. Then

$$\mu'_j \rightarrow \mu_\infty \text{ weakly as Radon measures.} \quad (3.19)$$

Proof of Theorem 3. Lemma 3.1-Lemma 3.4 prove that (1.31), (1.32) hold and hence the proof of Theorem 3 follows, as outlined in the introduction, by the assumption stated in Theorem 3 that (1.31), (1.32) implies (1.33). $\square$

Proof of Theorem 4. To prove Theorem 4 we simply repeat the blow-up argument outlined above with Ω replaced by Ω^1 and Ω^2 . Then in the limit we get domains $\Omega_\infty^1 = \Omega_\infty$, $\Omega_\infty^2 = \mathbf{R}^{n+1} \setminus \overline{\Omega_\infty}$, where Ω_∞ is a parabolic regular graph domain with constants (M, N) , and functions u_∞^i , $i \in \{1, 2\}$, such that u_∞^i is a positive adjoint caloric function in Ω_∞^i vanishing continuously on $\partial\Omega_\infty^i = \partial\Omega_\infty$. Moreover, there exist Radon measures ω_∞^i , $i \in \{1, 2\}$, such that

$$\int_{\mathbf{R}^{n+1}} u_\infty^i(\Delta\phi - \partial_\tau\phi) dZ d\tau = \int_{\partial\Omega_\infty} \phi d\omega_\infty^i, \quad (3.20)$$

whenever $\phi \in C_0^\infty(\mathbf{R}^{n+1})$. Furthermore, the limiting measures, ω_∞^i , and the limiting functions, u_∞^i , satisfy,

$$\omega_\infty^i = \mu_\infty \text{ on } \partial\Omega_\infty, \quad |\nabla u_\infty^i(Z, \tau)| \leq 1 \text{ whenever } (Z, \tau) \in \Omega_\infty^i. \quad (3.21)$$

We now let, whenever $(Z, \tau) \in \mathbf{R}^{n+1}$, $v_\infty(Z, \tau) = u_\infty^1(Z, \tau) - u_\infty^2(Z, \tau)$, where we have defined $u_\infty^1(Z, \tau) \equiv 0$ on Ω_∞^2 , $u_\infty^2(Z, \tau) \equiv 0$ on Ω_∞^1 . Using (3.20) and (3.21) we then see that

$$\int_{\mathbf{R}^{n+1}} v_\infty(\Delta\phi - \partial_\tau\phi) dZ d\tau = 0 \quad (3.22)$$

whenever $\phi \in C_0^\infty(\mathbf{R}^{n+1})$. In particular, v_∞ is continuous in $\mathbf{R}^{n+1}$ and (3.22) states that v_∞ is weakly adjoint caloric function in $\mathbf{R}^{n+1}$. Hence, v_∞ is adjoint caloric in $\mathbf{R}^{n+1}$ and we also note that $v_\infty(0, 0) = 0$. By standard estimates for the adjoint heat equation we therefore see that

$$|\nabla_Z^k \partial_\tau^l v_\infty(Z, \tau)| \leq \frac{C_{kl}}{R^{k+2l}} \max_{C_R(0,0)} |v_\infty| \quad (3.23)$$

for all $k, l \in \mathbf{Z}_+$ whenever $(Z, \tau) \in C_{R/2}(0, 0)$. Furthermore, using Lemma 2.7, Lemma 2.2, Lemma 2.4, Lemma 2.3 and (3.21) we also see that

$$\max_{C_R(0,0)} |v_\infty| \leq c \max\{u_\infty^1(A_R^1(0, 0)), u_\infty^2(A_R^2(0, 0))\} \leq c^2 R, \quad (3.24)$$

where $A_R^i(0, 0) \in \Omega_\infty^i$, $i \in \{1, 2\}$, is a point of reference defined with respect to Ω_∞^i and $c = c(n, M)$. Combining (3.23) and (3.24) we can conclude that

$$|\nabla_Z^k \partial_\tau^l v_\infty(Z, \tau)| \leq \frac{c}{R^{k+2l-1}} \quad (3.25)$$

for all $k, l \in \mathbf{Z}_+$ and whenever $(Z, \tau) \in C_{R/2}(0, 0)$. Letting $R \rightarrow \infty$ in (3.25), we see that $|\nabla_Z^k \partial_\tau^l v_\infty(Z, \tau)| = 0$, for all $(Z, \tau) \in \mathbf{R}^{n+1}$, and whenever (k, l) are such that $k + 2l - 1 > 0$. We can therefore conclude that v_∞ is in fact a linear function of the space variables only and hence (1.33) must hold. This completes the proof of Theorem 4. $\square$

3.1 Proof of Lemma 3.1-Lemma 3.4

Proof of Lemma 3.1. Let $\{u'_j\}, \{\omega'_j\}$ be subsequences of $\{u_j\}, \{\omega_j\}$, corresponding to $\{\Omega'_j\}$. Then from Lemma 2.1 - Lemma 2.5 applied to u'_j and (3.7) (ii) we deduce that u'_j is bounded, on compact subsets of $\mathbf{R}^n$ with norms of all functions bounded above by constants which are independent of j . Moreover, if $C_{2\rho}(Z, \tau) \subset \Omega_\infty$, then for large j we see from (3.12) (i) that u'_j is adjoint caloric and bounded in $C_\rho(Z, \tau)$. Furthermore, the bound can be chosen independent of j . Thus we assume, as we may, that $\{u'_j\}$ converges uniformly and weakly on compact subsets of $\mathbf{R}^{n+1}$ to u_∞ , that $u_\infty \geq 0$, and that u_∞ is adjoint caloric in Ω_∞ and continuous on $\mathbf{R}^{n+1}$, with $u_\infty \equiv 0$ on $\mathbf{R}^{n+1} \setminus \Omega_\infty$. Furthermore, if ω_∞ denotes the measure associated with u_∞ in the sense of Lemma 2.7, then

$$\begin{aligned} \int_{\mathbf{R}^{n+1}} \phi d\omega_\infty &= \int_{\mathbf{R}^{n+1}} u_\infty (\Delta\phi - \partial_\tau\phi) dZ d\tau \\ &= \lim_{j \rightarrow \infty} \int_{\mathbf{R}^{n+1}} u'_j (\Delta\phi - \partial_\tau\phi) dZ d\tau = \lim_{j \rightarrow \infty} \int_{\mathbf{R}^{n+1}} \phi d\omega'_j \end{aligned} \quad (3.26)$$

for all $\phi \in C_0^\infty(\mathbf{R}^{n+1})$. Thus $\{\omega'_j\}$ converges weakly to ω_∞ and the proof of Lemma 3.1 is complete. $\square$

Proof of Lemma 3.2. Using Theorem 1 we see that it is enough to prove that

$$\mu_\infty \leq \omega_\infty. \quad (3.27)$$

To prove (3.27) we first note, using the definition of ω'_j in (3.5), (3.7) (ii), and differentiation theory, that

$$1 - 2^{-2j^2} \leq |\nabla u'_j| \leq 1 + 2^{-2j^2} \quad (3.28)$$

μ'_j -almost everywhere on $\partial\Omega'_j \cap G'_j \cap C_{2j}(0, 0)$, where $\{G'_j\}$ is the subsequence of $\{G_j\}$ corresponding to $\{\Omega'_j\}$. Let $\phi \in C_0^\infty(\mathbf{R}^{n+1})$, $\phi \geq 0$. Then, using (3.26), (3.28) and (3.7) (i) we find that

$$\begin{aligned} \int \phi d\omega_\infty &= \lim_{j \rightarrow \infty} \int_{\partial\Omega'_j} \phi |\nabla u'_j| d\mu'_j \geq \liminf_{j \rightarrow \infty} \int_{G'_j \cap \partial\Omega'_j} \phi |\nabla u'_j| d\mu'_j \\ &\geq \liminf_{j \rightarrow \infty} (1 - 2^{-2j}) \int_{G'_j \cap \partial\Omega'_j} \phi d\mu'_j = \liminf_{j \rightarrow \infty} \int_{G'_j \cap \partial\Omega'_j} \phi d\mu'_j. \end{aligned} \quad (3.29)$$

Let $\nu \in \mathbf{S}^{n-1}$ where $\mathbf{S}^{n-1}$ denotes the unit sphere in $\mathbf{R}^n$. Then, using the divergence theorem on each time-slice $\Omega'_j \cap (\mathbf{R}^n \times \{\tau\})$, we see

$$\begin{aligned} \int_{\partial\Omega'_j} \phi d\mu'_j &= \int_{-\infty}^{\infty} \left(\int_{\partial(\Omega'_j \cap (\mathbf{R}^n \times \{\tau\}))} \phi d\sigma'_j(Z, \tau) \right) d\tau \\ &\geq \int_{-\infty}^{\infty} \left(\int_{\partial(\Omega'_j \cap (\mathbf{R}^n \times \{\tau\}))} \phi \langle n'_j(Z, \tau), \nu \rangle d\sigma'_j(Z, \tau) \right) d\tau, \end{aligned}$$

and hence that

$$\int_{\partial\Omega'_j} \phi d\mu'_j \geq \int_{-\infty}^{\infty} \left(\int_{\Omega'_j \cap (\mathbf{R}^n \times \{\tau\})} \operatorname{div}(\phi\nu) dZ \right) d\tau = \int_{\Omega'_j} \operatorname{div}(\phi\nu) dZ d\tau. \quad (3.30)$$

Moreover,

$$\begin{aligned} \liminf_{j \rightarrow \infty} \left(\int_{\Omega'_j} \operatorname{div}(\phi\nu) dZ d\tau \right) &= \int_{\Omega_\infty} \operatorname{div}(\phi\nu) dZ d\tau = \int_{\partial\Omega_\infty} \phi \langle n_\infty, \nu \rangle d\mu_\infty \\ &= \int_{\partial\Omega_\infty} \phi d\mu_\infty - \frac{1}{2} \int_{\partial\Omega_\infty} \phi |n_\infty - \nu|^2 d\mu_\infty \geq \int_{\partial\Omega_\infty} \phi d\mu_\infty. \end{aligned} \quad (3.31)$$

Furthermore,

$$\int_{G'_j \cap \partial\Omega'_j} \phi d\mu'_j = \int_{\partial\Omega'_j} \phi d\mu'_j + A'_j, \quad (3.32)$$

where, for fixed ϕ and for j large enough, we see, using (3.7) (i) and the fact that Ω'_j is a $\operatorname{Lip}(1,1/2)$ graph domain, with $\operatorname{Lip}(1,1/2)$ constant independent of j , that

$$|A'_j| \leq \|\phi\|_\infty 2^{-2j^2} \mu'_j(\partial\Omega'_j \cap C_{2j}(0,0)) \leq c \|\phi\|_\infty 2^{-2j^2} 2^{(n+1)j}. \quad (3.33)$$

Combining (3.29)-(3.33) we therefore see that

$$\begin{aligned} \int \phi d\omega_\infty &\geq \liminf_{j \rightarrow \infty} \int_{G'_j \cap \partial\Omega'_j} \phi d\mu'_j = \liminf_{j \rightarrow \infty} \int_{\partial\Omega'_j} \phi d\mu'_j \\ &\geq \liminf_{j \rightarrow \infty} \left(\int_{\Omega'_j} \operatorname{div}(\phi\nu) dx \right) \geq \int \phi d\mu_\infty, \end{aligned} \quad (3.34)$$

whenever $\phi \in C_0^\infty(\mathbf{R}^{n+1})$, $\phi \geq 0$. Hence (3.27) is true and the proof of Lemma 3.2 complete. $\square$

Proof of Lemma 3.3. In the following we let $\hat{\rho}_j = 2^{j/4}$, $A \geq 100$, and we let $A'_j \in \Omega'_j \cap \Gamma_A^-(0,0,16\hat{\rho}_j)$ be a point such that $\lambda^{-1}\hat{\rho}_j \leq d_p((0,0), A'_j) \leq \lambda\hat{\rho}_j$, $d(A'_j, \partial\Omega'_j) \geq \lambda^{-1}\hat{\rho}_j$, for some $\lambda = \lambda(n, M, A)$, $1 \leq \lambda < \infty$. The existence of A'_j and such a $\lambda = \lambda(n, M, A)$ follows from the fact that the $\operatorname{Lip}(1,1/2)$ constant of Ω'_j is bounded by M . Let $\rho_j = 4\lambda\hat{\rho}_j$. Then we can assume, without loss of generality, that u'_j is a positive adjoint caloric function in $\Omega'_j \cap C_{8\rho_j}(0,0)$, u'_j is continuous in $\bar{\Omega}'_j \cap \bar{C}_{8\rho_j}(0,0)$ and $u'_j = 0$ on $\partial\Omega'_j \cap C_{8\rho_j}(0,0)$. Next, let $\hat{\Omega}'_j = \Omega'_j \cap C_{2\rho_j}(0,0)$ and note that $A'_j \in \hat{\Omega}'_j$. To prove Lemma 3.3 we let $(Z, \tau) \in \Omega_\infty$ and we consider, as we may, j large enough to ensure that

$$d((Z, \tau), (0,0)) \leq \hat{\rho}_j/4 \text{ and } d(Z, \tau, \partial\hat{\Omega}'_j) \geq \frac{1}{2}d(Z, \tau, \partial\Omega_\infty). \quad (3.35)$$

The last assumption is permissible as we see from (3.12) (i). Next we let $\phi \in C_0^\infty(C_{3\hat{\rho}_j/2}(0,0))$, be such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $C_{\hat{\rho}_j}(0,0)$ and $\|\nabla_Z^k \phi\|_\infty \leq c_k 2^{-kj/4}$ for $k = 1, 2, \dots$ and $\|\partial_\tau \phi\|_\infty \leq c 2^{-k/2}$. We let $\xi \in \partial B(0,1)$ and, for fixed j , we set $v = \langle \nabla u'_j, \xi \rangle \phi$. Furthermore, let $\hat{\omega}'_j(Y, s, \cdot)$ denote the adjoint caloric measure at $(Y, s) \in \hat{\Omega}'_j$ defined with respect to $\hat{\Omega}'_j$. Then, using Theorem 1 we first that $\hat{\omega}'_j(A'_j, \cdot)$ and μ'_j are mutually absolutely continuous on $\Gamma'_j := \partial\Omega'_j \cap C_{2\hat{\rho}_j}(0,0) = \partial\hat{\Omega}'_j \cap C_{2\hat{\rho}_j}(0,0)$. Let $(X, t) \in \Gamma'_j$, $t > \tau$. Then using Lemma 2.3 and Lemma 2.4 we first see, for some $c = c(n, M)$, that

$$\frac{d\hat{\omega}'_j(Z, \tau, X, t)}{d\hat{\omega}_j(A'_j, X, t)} = \lim_{\varepsilon \rightarrow 0} \frac{\hat{\omega}'_j(Z, \tau, \partial\Omega'_j \cap C_\varepsilon(X, t))}{\hat{\omega}_j(A'_j, \partial\Omega'_j \cap C_\varepsilon(X, t))} \leq c \lim_{\varepsilon \rightarrow 0} \frac{\hat{g}'_j(A_\varepsilon(X, t), Z, \tau)}{\hat{g}_j(A_\varepsilon(X, t), A'_j)} \leq c \quad (3.36)$$

and then, using Lemma 2.2 and the Harnack inequality for adjoint caloric functions, we conclude, for some $\hat{c} = c(n, M, A)$,

$$\lim_{\varepsilon \rightarrow 0} \frac{\hat{g}'_j(A_\varepsilon(X, t), Z, \tau)}{\hat{g}_j(A_\varepsilon(X, t), A'_j)} \leq \hat{c}. \quad (3.37)$$

Hence, we see, in particular, that $\hat{\omega}'_j(Z, \tau, \cdot)$ is absolutely continuous, on Γ'_j , with respect to $\hat{\omega}'_j(A'_j, \cdot)$ and then, absolutely continuous, on Γ'_j , with respect to μ'_j . Using this we can conclude that v also has non-tangential limits $\hat{\omega}'_j(Z, \tau, \cdot)$ -almost everywhere on Γ'_j . Therefore, we see that

$$\begin{aligned} v(Z, \tau) &= \int_{\hat{\Omega}'_j} \hat{g}'_j(Z, t, Y, s) (\Delta_Y v(Y, s) + \partial_s v(Y, s)) dY ds \\ &+ \int_{\Gamma'_j} v(X, t) d\hat{\omega}'_j(Z, \tau, X, t) = T_1(Z, \tau) + T_2(Z, \tau). \end{aligned} \quad (3.38)$$

To estimate $T_2(Z, \tau)$ we first note that

$$\begin{aligned} |T_2(Z, \tau)| &\leq \int_{\Gamma'_j \cap G'_j} |v(X, t)| d\hat{\omega}'_j(Z, \tau, X, t) + \int_{\Gamma'_j \setminus G'_j} |v(X, t)| d\hat{\omega}'_j(Z, \tau, X, t) \\ &\leq (1 + 2^{-2j^2}) + \int_{\Gamma'_j \setminus G'_j} |v(X, t)| d\hat{\omega}'_j(Z, \tau, X, t) \end{aligned} \quad (3.39)$$

as we see using (3.28) and the fact that $\hat{\omega}'_j(Z, \tau, \cdot)$ is a probability measure. To proceed we define $\hat{k}'_j(A'_j, X, t) = d\hat{\omega}'_j(A'_j, X, t)/d\mu'_j(X, t)$ for μ'_j -almost every $(X, t) \in \Gamma'_j$. Then, using Theorem 1 we know that $\hat{k}'_j(A'_j, \cdot)$ satisfies a scale invariant L^q -reverse Hölder inequality on Γ'_j , for some $q \in (1, \infty)$. In the following we let $\tilde{q}$ denote the index dual to q . Using (3.37) and the Cauchy-Schwarz inequality we see that

$$\begin{aligned} \int_{\Gamma'_j \setminus G'_j} |v(X, t)| d\hat{\omega}'_j(Z, \tau, X, t) &\leq c \int_{\Gamma'_j \setminus G'_j} |v(X, t)| d\hat{\omega}'_j(A'_j, X, t) \\ &= c \int_{\Gamma'_j \setminus G'_j} |\nabla u'_j(X, t)| \hat{k}'_j(A'_j, X, t) d\mu'_j(X, t), \end{aligned} \quad (3.40)$$

and then, by the Cauchy-Schwarz inequality,

$$\begin{aligned} & \int_{\Gamma'_j \setminus G'_j} |v(X, t)| d\hat{\omega}'_j(Z, \tau, X, t) \\ & \leq c \left(\int_{\Gamma'_j \setminus G'_j} |\nabla u'_j(X, t)|^{\tilde{q}} d\mu'_j \right)^{1/\tilde{q}} \left(\int_{\Gamma'_j \setminus G'_j} |\hat{k}'_j(A'_j, X, t)|^q d\mu'_j \right)^{1/q}. \end{aligned} \quad (3.41)$$

We next note that $\log |\nabla u'_j| \in \text{VMO}(\partial\Omega'_j \cap C_{4\hat{\rho}_j}(0, 0))$ and hence $|\nabla u'_j|$ is in $L^{\hat{q}}(\Gamma'_j, d\mu'_j)$ for any $\hat{q} \in (1, \infty)$. Furthermore, $|\nabla u'_j|$ also satisfies a $L^{\hat{q}}$ -reverse Hölder type inequality, with respect to μ'_j , on Γ'_j for any $\hat{q} \in (1, \infty)$. Let $\hat{q} = 2\tilde{q}$ where $\tilde{q}$ is as above. Then,

$$\left(\int_{\Gamma'_j \setminus G'_j} |\nabla u'_j(X, t)|^{\tilde{q}} d\mu'_j \right)^{1/\tilde{q}} \leq c(\mu'_j(\Gamma'_j \setminus G'_j))^{2/\hat{q}} \left(\frac{\omega'_j(\hat{X}_j, \hat{t}_j, \Gamma'_j)}{\mu'_j(\Gamma'_j)} \right)^{1/(p-1)}. \quad (3.42)$$

Furthermore, using Theorem 1 we have

$$\left(\int_{\Gamma'_j \setminus G'_j} |\hat{k}'_j(A'_j, X, t)|^q d\mu'_j \right)^{1/q} \leq c(\mu'_j(\Gamma'_j))^{1/q} \left(\frac{\hat{\omega}'_j(A'_j, \Gamma'_j)}{\mu'_j(\Gamma'_j)} \right) \leq c(\mu'_j(\Gamma'_j))^{1/q-1} \quad (3.43)$$

where we again have used that $\hat{\omega}'_j(A'_j, \cdot)$ is a probability measure. Combining (3.41), (3.42), (3.43) and using (3.7) we see that

$$\int_{\Gamma'_j \setminus G'_j} |v(X, t)| d\hat{\omega}'_j(Z, \tau, X, t) \leq c 2^{-4j^2/\hat{q}} 2^{2j(n-1)/\hat{q}} 2^{j(n-1)(1/q-1)}. \quad (3.44)$$

In particular, (3.39) and (3.44) proves that $T_2(Z, \tau) \leq 1$ as $j \rightarrow \infty$.

To estimate $T_1(Z, \tau)$ we let $\zeta'_j = \langle \nabla u'_j, \xi \rangle$ and we note that $(\Delta \zeta'_j + \partial_s \zeta'_j) = 0$ on the support of ϕ . Using this we see that

$$\begin{aligned} T_1(Z, \tau) &= \int_{\hat{\Omega}'_j} \hat{g}'_j(Z, t, Y, s) \zeta'_j(Y, s) (\Delta_Y + \partial_s) \phi(Y, s) dY ds \\ &= \int_{\hat{\Omega}'_j} \hat{g}'_j(Z, t, Y, s) \langle \nabla \zeta'_j(Y, s), \nabla \phi(Y, s) \rangle dY ds = T_{11}(Z, \tau) + T_{12}(Z, \tau). \end{aligned} \quad (3.45)$$

We let $\hat{F}'_j = \hat{\Omega}'_j \cap (C_{2\hat{\rho}_j}(0, 0) \setminus C_{\hat{\rho}_j}(0, 0))$ and recall, see (3.35), that $(Z, \tau) \in \hat{\Omega}'_j \cap C_{\hat{\rho}_j/4}(0, 0)$. Using this, and elementary Schauder estimates, we see that

$$\begin{aligned} |T_{11}(Z, \tau)| &\leq c \hat{\rho}_j^{-2} \int_{\hat{F}'_j} |\hat{g}'_j(Z, t, Y, s)| |u'_j(Y, s)| (d(Y, s, \partial \hat{\Omega}'_j))^{-1} dY ds, \\ |T_{12}(Z, \tau)| &\leq c \hat{\rho}_j^{-1} \int_{\hat{F}'_j} |\hat{g}'_j(Z, t, Y, s)| |u'_j(Y, s)| (d(Y, s, \partial \hat{\Omega}'_j))^{-2} dY ds. \end{aligned} \quad (3.46)$$

Hence

$$|T_{11}(Z, \tau)| + |T_{12}(Z, \tau)| \leq c\hat{\rho}_j^{-1} \int_{\hat{F}'_j} |\hat{g}'_j(Z, t, Y, s)| |u'_j(Y, s)| (d(Y, s, \partial\hat{\Omega}'_j))^{-2} dY ds. \quad (3.47)$$

Let $\hat{E}'_j = \hat{F}'_j \cap \{Y, s : d(Y, s, \partial\hat{\Omega}'_j) \geq \hat{\rho}_j/\hat{\gamma}\}$, where $\hat{\gamma} \gg 1$ is a fixed constant, and let

$$\begin{aligned} A_1(Z, \tau) &= c\hat{\rho}_j^{-1} \int_{\hat{E}'_j} |\hat{g}'_j(Z, t, Y, s)| |u'_j(Y, s)| (d(Y, s, \partial\hat{\Omega}'_j))^{-2} dY ds, \\ A_2(Z, \tau) &= c\hat{\rho}_j^{-1} \int_{\hat{F}'_j \setminus \hat{E}'_j} |\hat{g}'_j(Z, t, Y, s)| |u'_j(Y, s)| (d(Y, s, \partial\hat{\Omega}'_j))^{-2} dY ds. \end{aligned} \quad (3.48)$$

Using Lemma 2.1-Lemma 2.3 we see that

$$\hat{g}'_j(Z, t, Y, s) \leq c\hat{g}'_j(Z, t, A'_j) \leq c \left(\frac{d(Z, \tau, \partial\hat{\Omega}'_j)}{\hat{\rho}_j} \right)^\alpha \hat{\rho}_j^{-n} \text{ whenever } (Y, s) \in \hat{E}'_j. \quad (3.49)$$

Furthermore, using Lemma 2.2 and (3.49) we can conclude that

$$A_1(Z, \tau) \leq c2^{-j/4} u'_j(A'_j) \left(\frac{d(Z, \tau, \partial\hat{\Omega}'_j)}{\hat{\rho}_j} \right)^\alpha. \quad (3.50)$$

We next consider $(Y, s) \in \hat{F}'_j \setminus \hat{E}'_j$. Arguing as in the deduction in (3.49) we see that

$$\begin{aligned} \hat{g}'_j(Z, t, Y, s) &\leq c \left(\frac{d(Y, s, \partial\hat{\Omega}'_j)}{\hat{\rho}_j} \right)^\alpha g'_j(Z, \tau, A'_j) \\ &\leq c \left(\frac{d(Z, \tau, \partial\hat{\Omega}'_j)}{\hat{\rho}_j} \right)^\alpha \left(\frac{d(Y, s, \partial\hat{\Omega}'_j)}{\hat{\rho}_j} \right)^\alpha \hat{\rho}_j^{-n}. \end{aligned} \quad (3.51)$$

Hence,

$$\begin{aligned} A_2(Z, \tau) &\leq c\hat{\rho}_j^{-n-1-\alpha} u'_j(A'_j) \left(\frac{d(Z, \tau, \partial\hat{\Omega}'_j)}{\hat{\rho}_j} \right)^\alpha \int_{\hat{F}'_j \setminus \hat{E}'_j} (d(Y, s, \partial\hat{\Omega}'_j))^{-2+\alpha} dY ds \\ &\leq c2^{-j/4} u'_j(A'_j) \left(\frac{d(Z, \tau, \partial\hat{\Omega}'_j)}{\hat{\rho}_j} \right)^\alpha. \end{aligned} \quad (3.52)$$

In particular, using the above deductions we can conclude that

$$|T_1(Z, \tau)| \leq |A_1(Z, \tau)| + |A_2(Z, \tau)| \leq c2^{-j/4} u'_j(A'_j) \left(\frac{d(Z, \tau, \partial\hat{\Omega}'_j)}{\hat{\rho}_j} \right)^\alpha. \quad (3.53)$$

To estimate $u'_j(A'_j)$ we note, using Lemma 2.4 and the Harnack inequality, that

$$u'_j(A'_j) \leq c\hat{\rho}_j^{-n} \omega'_j(\partial\hat{\Omega}'_j \cap C_{\hat{\rho}_j}(0, 0)). \quad (3.54)$$

Making estimates in (3.54) using (3.7) we see that

$$\hat{\rho}_j^{-1} u'_j(A'_j) \leq c \hat{\rho}_j^{-n+1} \omega'_j(\partial\Omega'_j \cap C_{\hat{\rho}_j}(0,0)) \leq \hat{c} \quad (3.55)$$

for some constant $\hat{c}$ which is independent of j . Hence, for (Z, τ) fixed we can conclude that

$$|T_1(Z, \tau)| \leq c \left(\frac{d(Z, \tau, \partial\hat{\Omega}'_j)}{\hat{\rho}_j} \right)^\alpha \rightarrow 0 \text{ as } j \rightarrow \infty. \quad (3.56)$$

This completes the proof of Lemma 3.3. $\square$

Proof of Lemma 3.4. Using Theorem 1, Lemma 3.2 and Lemma 3.3 we can conclude that $d\mu_\infty = d\omega_\infty$. Hence, for $\phi \in C_0^\infty(\mathbf{R}^{n+1})$ we see that

$$\begin{aligned} \int \phi d\mu_\infty &= \lim_{j \rightarrow \infty} \int_{\partial\Omega'_j} \phi |\nabla u'_j| d\mu'_j \geq \limsup_{j \rightarrow \infty} \int_{\partial G'_j \cap \partial\Omega'_j} \phi |\nabla u'_j| d\mu'_j \\ &\geq \limsup_{j \rightarrow \infty} (1 - 2^{-2j^2}) \int_{\partial G'_j \cap \partial\Omega'_j} \phi d\mu'_j = \limsup_{j \rightarrow \infty} \int_{\partial\Omega'_j} \phi d\mu'_j \end{aligned} \quad (3.57)$$

where the last equality follows from (3.32) and (3.33). Moreover, using (3.30) and (3.31) we have that

$$\int_{\partial\Omega_\infty} \phi d\mu_\infty \leq \liminf_{j \rightarrow \infty} \int_{\partial\Omega'_j} \phi d\mu'_j \text{ whenever } \phi \in C_0^\infty(\mathbf{R}^{n+1}), \phi \geq 0. \quad (3.58)$$

Combining (3.57) and (3.58) we see that

$$\mu'_j \rightarrow \mu_\infty \text{ weakly as } j \rightarrow \infty. \quad (3.59)$$

This completes the proof of Lemma 3.4. $\square$

3.2 Remarks

In this section we correct some confusing statements made in [N3]. To start with we note that one of the main results proved [N3] can be formulated as follows.

Theorem 3.5 ([N3, Theorem 1.4]) *Let $\Omega \subset \mathbf{R}^{n+1}$ a $Lip(1,1/2)$ graph domain, with constant M , and assume that u is a positive adjoint caloric function in Ω vanishing continuously on $\partial\Omega$. Moreover, assume that there exists a constant c such that*

$$c^{-1} d(Y, s, \partial\Omega) \leq u(Y, s) \leq c d(Y, s, \partial\Omega) \text{ whenever } (Y, s) \in \Omega. \quad (3.60)$$

Then Ω is a parabolic regular graph domain with constants (M, N) for some $N = N(n, c, M)$.

The proof of Theorem 3.5 presented in [N3] is flawless. We note, see Theorem 1, that Theorem 3.5 implies that if we know (3.60), for some u as in Theorem 3.5, then the caloric and adjoint caloric measures are mutually absolutely continuous with respect to the surface

measure on $\partial\Omega$, μ . Furthermore, assume now, in addition, that there exists a Radon measure ω , with support on $\partial\Omega$, such that

$$\int_{\mathbf{R}^{n+1}} u(\Delta\phi - \partial_\tau\phi) dZ d\tau = \int_{\partial\Omega} \phi d\omega \text{ whenever } \phi \in C_0^\infty(\mathbf{R}^{n+1}), \quad (3.61)$$

and that ω satisfies,

$$\omega = \mu \text{ on } \partial\Omega. \quad (3.62)$$

In [N3, Theorem 1.5] we then proved that if $n = 1$, then Ω is in fact a hyperplane containing a line parallel to the time-axis. In [N3, Theorem 1.6] we came to the same conclusion if $n \geq 2$, but in this case we had to assume, in addition, that Ω is a parabolic regular graph domain with constants (M, N) and we had to impose smallness conditions on the constants (M, N) . We note that the statements of Theorem 1.5 and Theorem 1.6 in [N3] may seem slightly confusion as we first assume (3.60), and then (3.61), (3.62), and as (3.62) implies (3.60) as we see from the lemmas in Section 2. The reason for this formulation is, as mentioned above, that we first use Theorem 3.5 and Theorem 1 to conclude that ω and μ are absolutely continuous. However, as ω exists independent of μ we today think that the following is the appropriate statement of Theorem 1.5 in [N3]. The statement of Theorem 1.6 in [N3] should be modified accordingly.

Theorem 3.6 ([N3, Theorem 1.5]) *Let $\Omega = \{(x, t) \in \mathbf{R}^2 : x > \psi(t)\} \subset \mathbf{R}^2$ be a $\text{Lip}(1/2)$ graph domain and assume that $\psi(0) = 0$. Assume that u is a positive adjoint caloric function in Ω vanishing continuously on $\partial\Omega$ and assume that there exists a Radon measure ω such that (3.61) holds. Moreover, assume that ω satisfies (3.62). Then $\psi \equiv 0$ and $u = x$.*

We also note that the proofs of all of the main results in [N3], i.e., [N3, Theorem 1.3], [N3, Theorem 1.4], [N3, Theorem 1.5] and [N3, Theorem 1.6] are flawless. As such we see, for example, that Theorem 3.6 can be used to conclude the validity of (1.33) in the case $n = 1$. While we now have discussed the main results established in [N3] that paper contains much more analysis and in particular a version of the blow-up argument presented in this paper was also outlined in [N3]. While the blow-up argument in [N3] differs from the argument presented in this paper the idea is the same: in the blow-up limit we should achieve the situation stated in (1.31) and (1.32). However, in [N3] there are a few formulations, which are not necessarily incorrect, but which confuse the reader. Moreover, there is also one error in [N3] which adds to the confusion. The confusion alluded to stems from the fact that while we in the main theorems in [N3] only assume that Ω is $\text{Lip}(1, 1/2)$ regular we in the blow-up argument in fact are assuming that Ω is a parabolic regular graph domain with constants (M, N) for some $N = N(n, c, M)$. However, the latter assumption is sometimes formulated sort of implicitly like ‘assume that the Poisson kernel exists’ etcetera. To be systematic we first claim that what is stated on [N3, p.1233-1254] is correct. Then, on p.1255 we would like to add the statement ‘Assume that $\Omega \subset \mathbf{R}^{n+1}$ is a parabolic regular graph domain with constants (M, N) ’ at the beginning of Lemma 2.15. We need this, as in (3.38), to ensure that we can take appropriate non-tangential limits of the spatial gradient of the Green function at the boundary of Ω . Strictly speaking, the statement of Lemma 2.15 in [N3] does not ensure that. Note that in [N3] we use δ_0 instead on M . With this modification of [N3, Lemma 2.15] we claim that what is stated

on [N3, p.1255-1263] is correct up to the statement of Lemma 3.3. However, there is an error in the proof of Lemma 3.3. In fact, as stated in the proof of [N3, Lemma 3.3] we can not at that stage conclude that $\mu'_j \rightarrow \mu_\infty$ weakly as Radon measures. Apart from that error the rest of the analysis in the proof is correct. Still the error made has consequences and the result is that some estimates are carried out in the wrong order and in particular this error allowed us to claim, in [N3], that we in [N3] do not really need the refined estimate on the gradient of u_∞ proved in Lemma 3.3 and in [N3, Lemma 3.9]. However, this is not true and the correct order to prove (1.31) and (1.32) is in the order given by Lemma 3.1-Lemma 3.4 above. Fortunately, the error has limited effects on the proceeding analysis in [N3] though and the proceeding analysis, starting from the proof of Lemma 3.3 in [N3] and to the end of the paper, is flawless and remains in tact. The conclusion is that due to the careless statement in the proof of Lemma 3.3 in [N3], the proof of the blow-up argument presented in [N3] is not complete. However, in this paper a different, shorter and complete argument is given.

4 References

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