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p -Harmonic Functions in the Heisenberg Group: Boundary Behaviour in Domains Well-approximated by Non-characteristic Hyperplanes

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Abstract

In this paper we study, for given p , $1 < p < \infty$, the boundary behaviour of non-negative p -harmonic functions in the Heisenberg group $\mathbb{H}^n$, i.e., we consider weak solutions to the non-linear and potentially degenerate partial differential equation

$$\sum_{i=1}^{2n} X_i(|Xu|^{p-2} X_i u) = 0$$

where the vector fields $X_1, \dots, X_{2n}$ form a basis for the space of left-invariant vector fields on $\mathbb{H}^n$. In particular, we introduce a set of domains $\Omega \subset \mathbb{H}^n$ which we refer to as domains well-approximated by non-characteristic hyperplanes and in Ω we prove, for $2 \leq p < \infty$, the boundary Harnack inequality as well as the Hölder continuity for ratios of positive p -harmonic functions vanishing on a portion of $\partial\Omega$.

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1 Introduction

In a sequence of recent papers, see [LN], [LN1], [LN2], the author, together with John Lewis, proved a number of results concerning the boundary behaviour of positive p -harmonic functions, $1 < p < \infty$, in bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$. In particular, in these papers the boundary Harnack inequality as well as the Hölder continuity for ratios of positive p -harmonic functions, $1 < p < \infty$, vanishing on a portion of $\partial\Omega$ were established. Furthermore, in [LN3] these results were extended to certain Reifenberg flat and Ahlfors regular NTA-domains. In this paper we show that the techniques and results in [LN3] concerning p -harmonic functions in Reifenberg flat domains can be generalized to p -harmonic functions in the Heisenberg group in certain ‘flat’ domains which are well-approximated by hyperplanes. In particular, we consider weak solutions to the non-linear and potentially degenerate partial differential equation

$$L_p u = \sum_{i=1}^{2n} X_i(|Xu|^{p-2} X_i u) = 0, \quad 1 < p < \infty, \quad (1.1)$$

where the vector fields $X_1, \dots, X_{2n}$ form a basis for the space of left-invariant vector fields on $\mathbb{H}^n$ and $|Xu| = (\sum (X_i u)^2)^{1/2}$. A weak solution to the equation in (1.1), see (1.10) below, will in the following be referred to as a p -harmonic function. Note that when $p = 2$ then (1.1) reduces to the extensively studied sum-of square operator L_2 in the Heisenberg group and in this case solutions are referred to as harmonic functions in $\mathbb{H}^n$. In this paper we consider p -harmonic functions in domains which we refer to as domains well-approximated by non-characteristic or verticle hyperplanes, i.e., hyperplanes which contain a line parallel to the axis in the direction of the variable in the second layer. In particular, these are the approximating hyperplanes which do not contain any characteristic points. While this setting can be generalized to allow for approximation with more general, and characteristic, hyperplanes, see the discussion in Section 9 at the end of the paper, the reason that we in this paper limit ourselves to the case of domains well-approximated by non-characteristic hyperplanes is threefold. Firstly, in this case we can give a complete and novel story concerning the boundary behaviour of p -harmonic functions in $\mathbb{H}^n$. Secondly, in this case some of the technicalities are a bit more simple compared to the case when we allow for approximation with characteristic hyperplanes and this allows us here to focus more on the techniques concerning partial differential equations introduced in the paper. Thirdly, already in this case the domains we introduce supply a new class of (local) NTA-domains which is of independent interest and to our knowledge this is a new way to describe the geometry of domains in $\mathbb{H}^n$. For an account of Reifenberg flat domains in the Euclidean setting, and a refined analysis of harmonic measure and the Poisson kernel associated to the Laplace operator, we refer to [KT], [KT1]-[KT3] and [CKL].

A well established fact is that in the sub-elliptic setting the existence of characteristic points is a new difficulty compared to the elliptic setting. Following [GaP] we note that the subtle role of characteristic points was recognized already in the pioneering works of Fichera [F1], [F2] (who first introduced the notion of characteristic set), Kohn-Nirenberg [KN1] and Bony [Bo]. In the case $p = 2$, in [Je1] Jerison considered the Dirichlet problem for the operator L_2 in the Heisenberg group and constructed an example of a smooth domain for which the Dirichlet problem for L_2 admits a Green function which, in the neighborhood of a characteristic point, is at most Hölder continuous up to the boundary, see also [Je2]. This is in sharp contrast with the classical elliptic theory, in which smooth data on smooth domains produce solutions which are

smooth up to the boundary. We want to stress that while the approach outlined and the results established in this paper are novel so far our results do not contribute to the understanding of the boundary behaviour of p -harmonic functions in the Heisenberg group at characteristic boundary points. In fact, in this context the boundary behaviour at characteristic boundary points still remains a challenging open problem.

To briefly discuss previous developments concerning the boundary behaviour of non-negative p -harmonic functions in the Heisenberg group we are only aware of one paper and that is the recent paper of Garofalo and Cong Phuc, see [GaP], where the study of the boundary behaviour of non-negative p -harmonic functions in the Heisenberg group was initiated. We emphasize that our techniques and results are much different from those established in [GaP]. In particular, in [GaP] the authors establish the boundary Harnack inequality for non-negative p -harmonic functions, $1 < p < \infty$, vanishing on a portion of a (Euclidean) $C^{1,1}$ -domain $\Omega \subset \mathbb{H}^n$. The result is established in neighbourhoods of non-characteristic boundary points. The bulk of the arguments in [GaP] is devoted to proving the non-trivial fact that away from the characteristic boundary points a $C^{1,1}$ -domain $\Omega \subset \mathbb{H}^n$ possesses appropriate uniform families of intrinsic balls which are tangent from the inside to the domain, and whose centers are located along paths which possess a crucial segment property, or quasi-segment property, with respect to the sub-Riemannian distance. Having established this geometrical result for (Euclidean) $C^{1,1}$ -domains $\Omega \subset \mathbb{H}^n$ the authors can proceed using barrier type arguments to prove that any of the p -harmonic functions under consideration behave, at a point near the boundary, as the sub-Riemannian distance to the boundary. Obviously this then gives the boundary Harnack inequality away from characteristic boundary points. We note that the approach in [GaP] is similar to the approach in [AKSZ] where the corresponding result is established, in $C^{1,1}$ -domains, in the case of the Euclidean p -Laplace operator, i.e., in the case when we have $X_i = \partial/\partial x_i$ in (1.1). However, the techniques explored in [AKSZ] and [GaP] do not seem strong enough to prove the Hölder continuity for ratios of non-negative p -harmonic functions vanishing on a portion of $\partial\Omega$. In contrast, the techniques developed by the second author and John Lewis, e.g., see [LN], [LN1], [LN2] and [LN3], concerning the boundary behaviour of p -harmonic functions is much different from the techniques in [AKSZ] and using these techniques the boundary Harnack inequality as well as the Hölder continuity for ratios of positive p -harmonic functions vanishing on a portion of $\partial\Omega$ can be established. The purpose of this paper is to introduce a subset of these technique in the context of the Heisenberg group and in the context of domains well-approximated by non-characteristic hyperplanes.

To properly state the results of this paper we need to introduce some notation and several notions. To start with, recall that the Heisenberg group $\mathbb{H}^n$ is the Lie group which can be identified with $\mathbb{R}^{2n+1}$ equipped with the non-Abelian group multiplication

$$\begin{aligned} g \circ g' &= (x, y, t) \circ (x', y', t') \\ &= (x + x', y + y', t + t' + \frac{1}{2}(\langle x, y' \rangle - \langle x', y \rangle)) \end{aligned} \quad (1.2)$$

where $g = (x, y, t)$, $g' = (x', y', t') \in \mathbb{R}^{2n+1}$. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. We let $e = (0, 0, 0)$ denote the group identity with respect to (1.2) and we note that the inverse of $g = (x, y, t)$, g^{-1} , equals $(-x, -y, -t)$. Let $L_g(g') = g \circ g'$ denote the operator of left-translation on $\mathbb{H}^n$ and let $(L_g)_*$ denote its differential. The Heisenberg algebra admits a decomposition $\mathcal{H}_n = V_1 \oplus V_2$ where $V_1 = \mathbb{R}^{2n} \times \{0\}$, $V_2 = \{0_{\mathbb{R}^{2n}}\} \times \mathbb{R}$, and where $0_{\mathbb{R}^{2n}}$ denotes the identity in $\mathbb{R}^{2n}$. Identifying $\mathcal{H}_n$ with the space of left-invariant vector fields on $\mathbb{H}^n$ we can deduce that a

basis for $\mathcal{H}_n$ is given by the set of vector fields $\{X_1(g), \dots, X_n(g), X_{n+1}(g), \dots, X_{2n}(g), T\}$ where

$$\begin{aligned} X_i(g) &:= (L_g)_*(\partial_{x_i}) = \partial_{x_i} - \frac{y_i}{2}\partial_t, \quad i \in \{1, \dots, n\}, \\ X_{n+i}(g) &:= (L_g)_*(\partial_{y_i}) = \partial_{y_i} + \frac{x_i}{2}\partial_t, \quad i \in \{1, \dots, n\}, \\ T(g) &:= (L_g)_*(\partial_t) = \partial_t, \end{aligned} \tag{1.3}$$

and that the only non-trivial commutation relation is

$$[X_i, X_{n+j}] = T\delta_{ij}, \quad i, j \in \{1, \dots, n\}, \tag{1.4}$$

where δ_{ij} is the Kronecker delta. In particular, $[V_1, V_1] = V_2$ and, since $[V_1, V_2] = \{0\}$, $\mathbb{H}^n$ is a Carnot group of step 2. The subspace V_1 is called the horizontal layer and V_2 is called the vertical layer of the Heisenberg algebra. Elements of V_j , $j = 1, 2$, are assigned the formal degree j . The associated non-isotropic dilations of $\mathbb{H}^n$ are given by

$$\delta_r = (rx, ry, r^2t), \quad r > 0. \tag{1.5}$$

The homogeneous dimension of $\mathbb{H}^n$ with respect to (1.5) is the number $Q = 2n + 2$. Note that, given that Lebesgue measure dg is a left- and right-invariant Haar measure on $\mathbb{H}^n$, it follows that

$$(d \circ \delta_r)(g) = r^Q dg. \tag{1.6}$$

We denote by $d_{cc}(g, g')$ the CC (Carnot-Caratheodory) distance on $\mathbb{H}^n$ associated with the system $\{X_1, \dots, X_{2n}\}$. In the following we often denote a point g in $\mathbb{H}^n$ by (z, t) , i.e., $z = (x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$. Defining

$$N(g) = (|z|^4 + 16t^2)^{1/4}, \quad g \in \mathbb{H}^n, \tag{1.7}$$

it was proved in [Cy] that

$$d(g, g') = N(g^{-1}g') = \left\{ |z' - z|^4 + 16(t' - t + \frac{1}{2}(\langle z, (z')^\perp \rangle))^2 \right\}^{1/4} \tag{1.8}$$

defines a metric on $\mathbb{H}^n$, the so called gauge distance, and that d and d_{cc} are equivalent. Here $(z')^\perp = (x', y')^\perp = (y', -x')$. We let $\bar{E}, \partial E$, be the closure, boundary of the set $E \subset \mathbb{H}^n$ and we define $d(y, E)$ to equal the gauge distance from $y \in \mathbb{H}^n$ to E . We let $|E|$ denote the Lebesgue measure of a measurable set $E \subset \mathbb{H}^n$ and we let $B(g, r) = \{g' \in \mathbb{H}^n : d(g, g') < r\}$ whenever $g \in \mathbb{H}^n$, $r > 0$. Note that $|B(g, r)| = c_n r^Q$ whenever $g \in \mathbb{H}^n$, $r > 0$. Furthermore, we let $d_e(g, g') := (|z - z'|^2 + |t - t'|^2)^{1/2}$ denote the standard Euclidean distance function and we let $d_e(y, E)$ equal the Euclidean distance from $y \in \mathbb{H}^n$ to E . Finally, we let

$$D(E, F) = \max\{\sup\{d(y, E) : y \in F\}, \sup\{d(y, F) : y \in E\}\}, \tag{1.9}$$

be the (gauge) Hausdorff distance between the sets $E, F \subset \mathbb{H}^n$.

Let $O \subset \mathbb{H}^n$ be an open set and let $1 \leq q \leq \infty$. Then by $W^{1,q}(O) = W_H^{1,q}(O)$ we denote the Folland-Stein Sobolev space of equivalence classes of functions f with horizontal distributional

derivatives $X_i f$, for $i \in \{1, \dots, 2n\}$, all of which are integrable to the q -th power on O . Given $f \in W^{1,q}(O)$ we let

$$\|f\|_{1,q} = \|f\|_q + \sum_{i=1}^{2n} \|X_i f\|_q$$

be the norm in $W^{1,q}(O)$ where $\|\cdot\|_q$ denotes the Lebesgue q norm in O . $C_0^\infty(O)$ is the set of infinitely differentiable functions with compact support in O and $W_0^{1,q}(O)$ is the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O) = W_H^{1,q}(O)$. Given $G \subset \mathbb{H}^n$ a bounded domain, i.e., a connected open set, and $1 < p < \infty$, we say that u is p -harmonic in G provided $u \in W^{1,p}(G)$ and

$$\int |Xu|^{p-2} \langle Xu, X\theta \rangle dg = 0 \quad (1.10)$$

whenever $\theta \in W_0^{1,p}(G)$. We will sometimes formally write, using summation convention, that

$$\sum_{i=1}^{2n} X_i(|Xu|^{p-2} X_i u) = 0 \text{ in } G \quad (1.11)$$

and we will refer to this equation as the p -Laplace partial differential equation in the Heisenberg group.

The rest of this paper is organized as follows. In Section 2 we state our main results and briefly outline the ideas of the proofs. Section 3 is of preliminary nature and we here discuss (local) NTA-domains, hyperplanes and half-spaces as well as the geometry of domains well approximated by non-characteristic hyperplanes. In section 4 we discuss the existence and uniqueness of weak solutions to the sub-elliptic p -Laplace equation, we outline important singular solutions to be used in barrier type arguments and we state/establish a number of basic estimates concerning the boundary behaviour of p -harmonic functions in (local) NTA-domains. Section 5 is devoted to the boundary behaviour of non-negative solutions to certain associated linear and degenerate sub-elliptic equations and we here stated recent results from [GN]. In this section we also outline the connection between boundary estimates for p -harmonic functions and the linear and degenerate sub-elliptic equations discussed. In Section 6 we prove our main results in the special case of a half-space defined by a non-characteristic hyperplane. Section 7 is devoted to more refined estimates for p -harmonic functions in domains which are well approximated by non-characteristic hyperplanes. The final proof of our main results is presented in Section 8. Section 9 is devoted to remarks and discussion.

2 Statement of main results

Given $z = (x, y) \in \mathbb{R}^{2n}$, $t \in \mathbb{R}$, and a vector $\nu = (a, b, c) \in \mathbb{R}^{2n+1} \setminus \{0\}$, we let

$$H^\nu(z, t) = \{(z', t') \in \mathbb{H}^n | \langle (z' - z, t' - t), \nu \rangle = \langle a, x' - x \rangle + \langle b, y' - y \rangle + c(t' - t) > 0\} \quad (2.1)$$

and

$$\Pi^\nu(z, t) = \{(z', t') \in \mathbb{H}^n | \langle (z' - z, t' - t), \nu \rangle = \langle a, x' - x \rangle + \langle b, y' - y \rangle + c(t' - t) = 0\}. \quad (2.2)$$

Then $\Pi^\nu(z, t)$ denotes the plane through (z, t) defined by $\nu = (a, b, c)$. Throughout the paper we will assume, without loss of generality, that

$$|\nu|^2 = |a|^2 + |b|^2 + |c|^2 = 1 \quad (2.3)$$

and hence $\nu = (a, b, c)$ is the unit normal to the plane $\Pi^\nu(z, t)$. Furthermore, when we write $\Pi(z, t)$ we mean that $\Pi(z, t) = \Pi^\nu(z, t)$ for some ν as above. We let, for any non-negative integer k , $\mathbb{S}^k$ denote the unit sphere in $\mathbb{R}^k$. In the following we by $\mathcal{S}$ denote a subset of $\mathbb{S}^{2n+1}$. Given $(z, t) \in \mathbb{R}^{2n+1}$, $\mathcal{S} \subset \mathbb{S}^{2n+1}$, we let

$$H^\mathcal{S}(z, t) = \{H^\nu(z, t) : \nu \in \mathcal{S}\}, \quad \Pi^\mathcal{S}(z, t) = \{\Pi^\nu(z, t) : \nu \in \mathcal{S}\} \quad (2.4)$$

be the set of all half-spaces and hyperplanes, defined relative to (z, t) , having unit normal ν in the set $\mathcal{S}$.

Definition 2.1 Consider $\Pi = \Pi^\nu = \Pi^\nu(z, t)$ for some $(z, t) \in \mathbb{R}^{2n+1}$, $\nu \in \mathbb{S}^{2n+1}$. Then Π is said to contain a characteristic point if there exists a point $\hat{g} = (\hat{z}, \hat{t}) \in \Pi$ such that $|X\Pi|(\hat{z}, \hat{t}) = 0$. If no such point $(\hat{z}, \hat{t})$ exists then Π is called a non-characteristic hyperplane.

Note that

$$(|X\Pi|(\hat{z}, \hat{t}))^2 = |a - c\hat{y}/2|^2 + |b + c\hat{x}/2|^2 = |a|^2 + |b|^2 + \frac{|\hat{z}|^2}{4}c^2 - \langle (a, b), (\hat{x}, \hat{y})^\perp \rangle c. \quad (2.5)$$

Hence if $c = 0$ then Π contains no characteristic points. If $c \neq 0$ then the point $\hat{g} = (\hat{z}, \hat{t}) \in \Pi$ with $\hat{z} = (-2b/c, 2a/c)$ is the unique characteristic point located on Π .

Definition 2.2 Let $\Omega \subset \mathbb{H}^n$ be a domain and let $\mathcal{S} \subset \mathbb{S}^{2n+1}$. Let $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Then $\Omega \cap B(g_0, r_0)$ and $\partial\Omega \cap B(g_0, r_0)$ are said to be uniformly (δ, r_0) -approximable by hyperplanes in the set $\mathcal{S}$ in the Heisenberg group, in the sense of the gauge Hausdorff distance, provided there exists, whenever $g \in \partial\Omega \cap B(g_0, r_0)$ and $0 < r < r_0$, a hyperplane $\Pi = \Pi^\nu = \Pi^\nu(g)$, with $\nu \in \mathcal{S}$, such that

$$D(\partial\Omega \cap B(g, r), \Pi \cap B(g, r)) \leq \delta r.$$

Let $E \subset \mathbb{H}^n$, $g \in \mathbb{H}^n$. By $g^{-1} \circ E$ we denote the set $\{g^{-1} \circ \hat{g} : \hat{g} \in E\}$. Using this notation we note that

$$D(\partial\Omega \cap B(g, r), \Pi \cap B(g, r)) = D(\partial(g^{-1} \circ \Omega) \cap B(0, r), (g^{-1} \circ \Pi) \cap B(0, r)). \quad (2.6)$$

Furthermore, Definition 2.2 is only interesting when δ is small and hence we throughout the paper always assume $\delta \in (0, 1/8)$.

Let $\partial\Omega \cap B(g_0, r_0)$ be uniformly (δ, r_0) -approximable by hyperplanes in the set $\mathcal{S}$ in the Heisenberg group, in the sense of the gauge Hausdorff distance, as defined in Definition 2.2. Let $g \in \partial\Omega \cap B(g_0, r_0)$, $0 < r < r_0$, and $\Pi = \Pi^\nu = \Pi^\nu(g)$, with $\nu \in \mathcal{S}$, be as in Definition 2.2. We say that $\partial\Omega$ separates $B(g, r)$, if

$$\{g' \in \Omega \cap B(g, r) : d(g', \partial\Omega) \geq 2\delta r\} \subset \text{one component of } \mathbb{H}^n \setminus \Pi. \quad (2.7)$$

Definition 2.3 Let $\Omega \subset \mathbb{H}^n$ be a domain and let $\mathcal{S} \subset \mathbb{S}^{2n+1}$. Let $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Then $\Omega \cap B(g_0, r_0)$ and $\partial\Omega \cap B(g_0, r_0)$ are said to be (δ, r_0) -flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{S}$, provided $\partial\Omega \cap B(g_0, r_0)$ is uniformly (δ, r_0) -approximable by hyperplanes in the set $\mathcal{S}$ in the Heisenberg group, in the sense of the gauge Hausdorff distance, as defined in Definition 2.2, and provided (2.7) holds whenever $0 < r < r_0$, $g \in \partial\Omega \cap B(g_0, r_0)$.

Definition 2.4 Let $\Omega \subset \mathbb{H}^n$ be a domain and let $\mathcal{S} \subset \mathbb{S}^{2n+1}$. Let $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Assume that $\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{S}$. Let $g \in \partial\Omega \cap B(g_0, r_0/2)$, $0 < r < r_0/2$. We say that $\partial\Omega \cap B(g, r)$ is flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{S}$, and with vanishing constant, if for each $\epsilon > 0$, there exists $\tilde{r} = \tilde{r}(\epsilon) > 0$ with the following property. If $g' \in \partial\Omega \cap B(g, r)$ and $0 < \rho < \tilde{r} < r$, then there exists a plane $\tilde{\Pi} = \tilde{\Pi}^\nu = \Pi^\nu(g', \rho)$, with $\nu \in \mathcal{S}$, such that

$$D(\partial\Omega \cap B(g', \rho), \tilde{\Pi} \cap B(g', \rho)) \leq \epsilon \rho.$$

Definition 2.5 Let

$$\mathcal{V} = \{\nu = (a, b, c) \in \mathbb{S}^{2n+1} : c = 0\}. \quad (2.8)$$

Given $g = (z, t) \in \mathbb{H}^n$, $\Pi^\nu(z, t)$ is the set of all non-characteristic or verticle hyperplanes in the Heisenberg group through the point $g = (z, t)$.

Remark 2.6 Let $\Omega \subset \mathbb{H}^n$ be a domain, $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Assume that $\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{V}$. In Lemma 3.8 below we prove that there exists $\hat{\delta} = \hat{\delta}(n)$ such that if $\delta < \hat{\delta}$ then Ω is a local NTA-domain at g_0 in the sense of Definition 3.4. In the following $a_r(g)$, $g \in \partial\Omega \cap B(g_0, r_0)$, will denote a non-tangential point of reference as introduced in Definition 3.1.

In this paper we prove the following results.

Theorem 1. Let $\Omega \subset \mathbb{H}^n$ be a domain, $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Assume that $\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{V}$. Given p , $2 \leq p < \infty$, there exists $\delta_1 = \delta_1(p, n)$ such that the following is true. Let $g \in \partial\Omega \cap B(g_0, r_0/2)$, $0 < r < r_0/16$, and $0 < \delta < \delta_1$. Suppose that u, v are positive p -harmonic functions in $\Omega \cap B(g, 4r)$, that u, v are continuous in $\bar{\Omega} \cap B(g, 4r)$, $u = 0 = v$ on $\partial\Omega \cap B(g, 4r)$ and that $v \leq u$. Then there exists $c = c(p, n)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then

$$c^{-1} \frac{u(a_{\tilde{r}}(g)) - v(a_{\tilde{r}}(g))}{v(a_{\tilde{r}}(g))} \leq \frac{u(g') - v(g')}{v(g')} \leq c \frac{u(a_{\tilde{r}}(g)) - v(a_{\tilde{r}}(g))}{v(a_{\tilde{r}}(g))}$$

whenever $g' \in \Omega \cap B(g, r/c)$ and where $a_{\tilde{r}}(g)$ is a point of reference as in Remark 2.6.

Corollary 1.1 Let $\Omega \subset \mathbb{H}^n$, r_0 and g_0 be as in Theorem 1. Given p , $2 \leq p < \infty$, there exists $\delta_1 = \delta_1(p, n)$ such that the following is true. Let $g \in \partial\Omega \cap B(g_0, r_0/2)$, $0 < r < r_0/16$, and $0 < \delta < \delta_1$. Suppose that u, v are positive p -harmonic functions in $\Omega \cap B(g, 4r)$, that u, v are continuous in $\bar{\Omega} \cap B(g, 4r)$ and $u = 0 = v$ on $\partial\Omega \cap B(g, 4r)$. Then there exist $c = c(p, n)$, $1 \leq c < \infty$, and $\alpha = \alpha(p, n)$, $\alpha \in (0, 1)$, such that

$$\left| \log \frac{u(g'_1)}{v(g'_1)} - \log \frac{u(g'_2)}{v(g'_2)} \right| \leq c \left(\frac{d(g'_1, g'_2)}{r} \right)^\alpha$$

whenever $g'_1, g'_2 \in \Omega \cap B(g, r/c)$.

Corollary 1.2 *Let $\Omega \subset \mathbb{H}^n$, r_0 and g_0 be as in Theorem 1. Let $g \in \partial\Omega \cap B(g_0, r_0/2)$, $0 < r < r_0/16$ and suppose that $\partial\Omega \cap B(g, 4r)$ is flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{V}$, and with vanishing constant. Assume for given p , $2 \leq p < \infty$, that u, v are positive p -harmonic functions in $\Omega \cap B(g, 4r)$, that u, v are continuous in $\bar{\Omega} \cap B(g, 4r)$, $u = 0 = v$ on $\partial\Omega \cap B(g, 4r)$ and that $v \leq u$. Then there exist $r^* = r^*(p, n) > 0$, and $c = c(p, n)$, $1 \leq c < \infty$, such that if $g' \in \partial\Omega \cap B(g, r)$ and $0 < r' < r^*$, then*

$$c^{-1} \frac{u(a_{\tilde{r}'}(g')) - v(a_{\tilde{r}'}(g'))}{v(a_{\tilde{r}'}(g'))} \leq \frac{u(g'') - v(g'')}{v(g'')} \leq c \frac{u(a_{\tilde{r}'}(g')) - v(a_{\tilde{r}'}(g'))}{v(a_{\tilde{r}'}(g'))}$$

whenever $g'' \in \Omega \cap B(g', r'/c)$.

Corollary 1.3 *Let $\Omega \subset \mathbb{H}^n$, r_0 and g_0 be as in Theorem 1. Let $g \in \partial\Omega \cap B(g_0, r_0/2)$, $0 < r < r_0/16$ and suppose that $\partial\Omega \cap B(g, 4r)$ is flat in $\mathbb{H}^n$, and with respect to hyperplanes defined by the set $\mathcal{V}$, with vanishing constant. Assume for given p , $2 \leq p < \infty$, that u, v are positive p -harmonic functions in $\Omega \cap B(g, 4r)$, u, v are continuous in $\bar{\Omega} \cap B(g, 4r)$ and $u = 0 = v$ on $\partial\Omega \cap B(g, 4r)$. Then there exist $r^* = r^*(p, n) > 0$, $c = c(p, n)$, $1 \leq c < \infty$, and $\alpha = \alpha(p, n)$, $\alpha \in (0, 1)$, such that if $g' \in \partial\Omega \cap B(g, r)$ and $0 < r' < r^*$, then*

$$\left| \log \frac{u(g'_1)}{v(g'_1)} - \log \frac{u(g'_2)}{v(g'_2)} \right| \leq c \left(\frac{d(g'_1, g'_2)}{r'} \right)^\alpha$$

whenever $g'_1, g'_2 \in \Omega \cap B(g', r'/c)$.

One may argue that to consider domain which are well-approximated by hyperplanes defined by the set $\mathcal{V}$ is restrictive and in many cases too restrictive. However, we note that this set-up may very well turn out relevant when attempting to establish regularity in free boundary problems involving the sub-elliptic p -Laplacian. To elaborate a bit on this we note that in a fine recent paper, see [DGP], Danielli, Garofalo and Petrosyan establish $C^{1,\alpha}$ regularity of the free boundary occurring in the obstacle problem in Carnot groups on step two and their results apply, in particular, in $\mathbb{H}^n$ and to the obstacle problem $L_2 u = \chi_{\{u>0\}}$ where $\chi_{\{u>0\}}$ is the indicator function for the set $\{u > 0\}$. While [DGP] contains many insights, one important insight is that certain global solutions to the relevant obstacle problem are independent of the variables in the second layer. In $\mathbb{H}^n$ this means that the global solutions are independent of the t -variable. Starting from this insight the authors are then able to prove, we here only quote their result in the context of $\mathbb{H}^n$, that if we write $(z, t) = (z_1, \dots, z_{2n}, t) \in \mathbb{H}^n$, then locally the associated free boundary is a graph of the form $z_m = f(z_1, \dots, z_{m-1}, z_{m+1}, \dots, z_{2n}, t)$ for some $m \in \{1, \dots, 2n\}$ and for some Lipschitz function f . Note that f is here Lipschitz in all variables and if the Lipschitz constant is small enough, see (3.38)-(3.42) below, then this graph is well-approximated by hyperplanes defined by the set $\mathcal{V}$ in the sense discussed in the paper. To prove that f is in fact $C^{1,\alpha}$ the authors then, as part of their argument, use a version of Corollary 1.1 valid for linear sub-elliptic equations and essentially established in [CG], see Theorem 7.4 and Corollary 7.5 in [DGP]. Hence we claim that our results could be relevant when attempting to prove regularity of flat free boundaries occurring in free boundary problems involving the sub-elliptic p -Laplacian.

2.1 Outline of the proof of Theorem 1 and Corollary 1.1

In the bulk of the paper we will only prove Theorem 1 and Corollary 1.1. In the following we briefly outline key points in the proof of Theorem 1 and we in particular note that the proofs are built on the following important insights.

Step 0. Let $\nu = (a, b, c) \in \mathbb{R}^{2n+1} \setminus \{0\}$, and consider the linear function $L = L'_{(z,t)}((z', t')) = \langle (z' - z, t' - t), \nu \rangle$. Then, given p , $1 < p < \infty$, we note that L is p -harmonic, L is non-negative in $H^\nu(z, t)$ and L vanishes continuously on the hyperplane $\Pi^\nu(z, t)$. As a starting point for our analysis we establish Corollary 1.1 in the special case when the boundary of Ω coincides with a non-characteristic hyperplane.

Step A. (Uniform non-degeneracy of $|Xu|$ - the ‘fundamental inequality’). Let u be as in the statement of Theorem 1. Then there exist $\delta_1 = \delta_1(p, n)$, $\hat{c}_1 = \hat{c}_1(p, n)$ and $\bar{\lambda} = \bar{\lambda}(p, n)$, such that if $0 < \delta < \delta_1$, then

$$\bar{\lambda}^{-1} \frac{u(g')}{d(g', \partial\Omega)} \leq |Xu(g')| \leq \bar{\lambda} \frac{u(g')}{d(g', \partial\Omega)} \text{ whenever } g' \in \Omega \cap B(g, r/\hat{c}_1). \quad (2.9)$$

If (2.9) holds then we say that $|Xu|$ satisfies the ‘fundamental inequality’ in $\Omega \cap B(g, r/\hat{c}_1)$. Our proof of the ‘fundamental inequality’ uses recent results of Zhong [Z] concerning local Hölder estimates for $X_i u$, $i \in \{1, \dots, 2n\}$. In [Z] such estimates are proved for p in the range $2 \leq p < \infty$. In particular, this is the instance where we have to impose the restriction $2 \leq p < \infty$. Note that there has recently been considerable activity in the study of regularity for quasi-linear equations in the Heisenberg group, see [D], [DM], [MaM], [M], [MZZ] and [Z] and the references in these papers.

Step B. (Extension of $|Xu|^{p-2}$ to an A_2 -weight). Let u be as in Step A. Then there exist $\delta_2 = \delta_2(p, n)$ and $\hat{c}_2 = \hat{c}_2(p, n)$ such that if $0 < \delta < \delta_2$, then $|Xu|^{p-2}$ extends to an $A_2(B(g, r/(\hat{c}_1 \hat{c}_2))$ -weight with constant depending only on p, n . For the definition of A_2 -weights we refer to (5.1) below.

Step C. (Introduction of associated linear equations). In our argument we need to establish estimates for the difference of two p -harmonic functions. In particular, let $\hat{u}, \hat{v}$ be positive p -harmonic functions in an open set $O \subset \mathbb{H}^n$. Suppose also, for some $\hat{\eta} > 1$, that

$$\hat{\eta}^{-1} \frac{u'(g)}{d(g, \partial O)} \leq |Xu'(g)| \leq \hat{\eta} \frac{u'(g)}{d(g, \partial O)}, \text{ whenever } g \in B(g', r), u' \in \{\hat{u}, \hat{v}\}, \quad (2.10)$$

where $\bar{B}(g', 2r) \subset O$. Define

$$e(g) = \hat{u}(g) - \hat{v}(g) \text{ whenever } g \in B(g', r) \quad (2.11)$$

and put

$$u(g, \tau) = \tau \hat{u}(g) + (1 - \tau) \hat{v}(g) \text{ whenever } g \in B(g', r) \text{ and } \tau \in [0, 1]. \quad (2.12)$$

Clearly, $e(g) = u(g, 1) - u(g, 0)$. Using p -harmonicity of $\hat{u}, \hat{v}$ and that

$$|\xi|^{p-2} \xi - |\eta|^{p-2} \eta = \int_0^1 \frac{d\{|t\xi + (1-t)\eta|^{p-2}[t\xi + (1-t)\eta]\}}{dt} dt$$

whenever $\xi, \eta \in \mathbb{R}^{2n} \setminus \{0\}$, it follows that e is a weak solution to

$$\hat{L}e := \sum_{i,j=1}^{2n} X_i(\hat{a}_{ij}(\cdot)X_j e) = 0 \text{ in } B(g', r), \quad (2.13)$$

where, whenever $g \in B(g', r)$,

$$\begin{aligned} \hat{a}_{ij}(g) &= \int_0^1 a_{ij}(g, \tau) d\tau, \\ a_{ij}(g, \tau) &= |Xu(g, \tau)|^{p-4} ((p-2)(X_i u)(g, \tau)(X_j u)(g, \tau) + \delta_{ij} |Xu(g, \tau)|^2). \end{aligned} \quad (2.14)$$

Here $i, j \in \{1, \dots, n\}$ and δ_{ij} is the Kronecker δ . In particular, we observe that $e = \hat{u} - \hat{v}$ is the solution to a symmetric divergence form PDE with ellipticity constant, at $g \in B(g', r)$, estimated by

$$\min\{p-1, 1\}|\xi|^2 \hat{\lambda}(g) \leq \sum_{i,j=1}^{2n} \hat{a}_{ij}(g) \xi_i \xi_j \leq \max\{p-1, 1\}|\xi|^2 \hat{\lambda}(g), \quad (2.15)$$

whenever $\xi \in \mathbb{R}^{2n}$. Here

$$\begin{aligned} \hat{\lambda}(g) &= \int_0^1 |Xu(g, \tau)|^{p-2} d\tau \approx \left(|X\hat{u}(g)| + |X\hat{v}(g)| \right)^{p-2} \\ &\approx \left(\hat{u}(g)/d(g, \partial O) + \hat{v}(g)/d(g, \partial O) \right)^{p-2}, \end{aligned} \quad (2.16)$$

whenever $g \in B(g', r)$. In (2.16) $\approx$ means that the constants of proportionality only depend on p, n and $\hat{\eta}$. Using (2.16) and the Harnack inequality we see that locally L is a linear and uniformly elliptic sub-elliptic pde as studied in generality in [CG]. However, near the boundary the ellipticity may degenerate.

Step D. (Boundary Harnack inequalities for linear degenerate sub-elliptic equations). Let Ω , u and v be as in the statement of Theorem 1. By Step B it will follow that $(|Xu| + |Xv|)^{p-2}$ extends to an $A_2(B(g, r/(\hat{c}_1 \hat{c}_2))$ -weight with constant depending only on p, n . Let $r' = r/(2\hat{c}_1 \hat{c}_2)$, $O = \Omega \cap B(g, r')$ and construct the operator $\hat{L}$ as above using $\hat{u} = u$ and $\hat{v} = v$. Then the operator $\hat{L}$ can be considered as a linear sub-elliptic degenerate operator and using results from [GN] we can conclude that the following is true. Assume that v_1 and v_2 are two positive weak solutions to $\hat{L}v = 0$ in $\Omega \cap B(g, 2r')$ and $v_1 = 0 = v_2$ continuously on $\partial\Omega \cap B(g, 2r')$. Then there exist $c = c(p, n)$, $1 \leq c < \infty$, and $\alpha = \alpha(p, n) \in (0, 1)$ such that

$$\left| \log \frac{v_1(g_1)}{v_2(g_1)} - \log \frac{v_1(g_2)}{v_2(g_2)} \right| \leq c \left(\frac{d(g_1, g_2)}{r'} \right)^\alpha$$

whenever $g_1, g_2 \in \Omega \cap B(g, r'/c)$. In the bulk of the paper we prove that we can then use this, and additional arguments, to conclude the validity of Theorem 1 and Corollary 1.1.

3 Preliminaries

In this section, which is of preliminary nature, we first recall the notion of (local) NTA-domains in the Heisenberg group. We then consider hyperplanes in the set $\mathcal{V}$ and geometry of domains well approximated by hyperplanes in the set $\mathcal{V}$. Recall the gauge distance d introduced in (1.7)-(1.8). Throughout the paper we will mainly measure distances using d and $B(g, r) = \{g' \in \mathbb{H}^n : d(g, g') < r\}$, for $g \in \mathbb{H}^n$, $r > 0$, will be referred to as a gauge ball. However, at instances will also make references to the equivalent distance d_{cc} which we now define. Recall that a piecewise C^1 -smooth curve $\gamma : [0, \ell] \rightarrow \mathbb{H}^n$ is called subunitary in $\mathbb{H}^n$ if at every $t \in [0, \ell]$ at which $\gamma'(t)$ exists one has

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{i=1}^{2n} \langle X_i(\gamma(t)), \xi \rangle^2 \quad (3.1)$$

whenever $\xi \in \mathbb{H}^n$. We note explicitly that the above inequality forces $\gamma'(t)$ to belong to the span of $\{X_1(\gamma(t)), \dots, X_{2n}(\gamma(t))\}$. The subunit length of γ is by definition $l_s(\gamma) = \ell$. Given $g_1, g_2 \in \mathbb{H}^n$, denote by $\mathcal{S}(g_1, g_2)$ the collection of all subunitary $\gamma : [0, \ell] \rightarrow \mathbb{H}^n$ which join g_1 to g_2 . By the accessibility theorem of Chow and Rashevsky, [Ch], [Ra], there exists, for every $g_1, g_2 \in \mathbb{H}^n$, $\gamma \in \mathcal{S}(g_1, g_2)$. As a consequence, if we define

$$d_{cc}(g_1, g_2) = \inf \{l_s(\gamma) \mid \gamma \in \mathcal{S}(g_1, g_2)\}, \quad (3.2)$$

we obtain a distance on $\mathbb{H}^n$, called the Carnot-Carathéodory distance, associated with the system $\{X_1, \dots, X_{2n}\}$. Note that given any compact subset K of $\mathbb{H}^n$, there exists $C \geq 1$ such that

$$C^{-1}|g_1 - g_2| \leq d_{cc}(g_1, g_2) \leq C|g_1 - g_2|^{1/2}, \quad g_1, g_2 \in K. \quad (3.3)$$

Recall that the Heisenberg group $\mathbb{H}^n$ is perhaps the simplest example of a graded nilpotent Lie group of step two, i.e., of a Carnot group of step 2. Finally, given a set $E \subset \mathbb{R}^n$ we let

$$\sup_E v \text{ and } \inf_E v$$

denote the essential supremum and the essential infimum of the function v on E , respectively.

3.1 (Local) NTA-domains

Given a bounded open set $\Omega \subset \mathbb{H}^n$, we recall that a ball $B(g, r)$ is M -non-tangential in Ω (with respect to the metric d) if

$$M^{-1}r < d(B(g, r), \partial\Omega) < Mr.$$

Furthermore, given $g, g' \in \Omega$ a sequence of M -non-tangential balls in Ω , $B(g_1, r_1), \dots, B(g_p, r_p)$, is called a M -Harnack chain of length p , joining g to g' , if $g \in B(g_1, r_1)$, $g' \in B(g_p, r_p)$, and $B(g_i, r_i) \cap B(g_{i+1}, r_{i+1}) \neq \emptyset$ for $i \in \{1, \dots, p-1\}$. We note that in this definition consecutive balls have comparable radii.

Definition 3.1 *Let $\Omega \subset \mathbb{H}^n$ be a bounded domain. We say that Ω is a non-tangentially accessible domain in $\mathbb{H}^n$, a NTA-domain hereafter, if there exist $M \geq 2$, $r_0 > 0$ such that the following holds.*

- (i) (*Interior corkscrew condition*) For any $g_0 \in \partial\Omega$ and $r \leq r_0$ there exists $a_r(g_0) \in \Omega$ such that $r/M < d(a_r(g_0), g_0) \leq r$ and $d(a_r(g_0), \partial\Omega) > r/M$.
- (ii) (*Exterior corkscrew condition*) $\Omega^c = \mathbb{H}^n \setminus \Omega$ satisfies property (i).
- (iii) (*Harnack chain condition*) Whenever $\epsilon > 0$ and $g_1, g_2 \in \Omega$ are such that $d(g_i, \partial\Omega) > \epsilon$, $i \in \{1, 2\}$, and $d(g_1, g_2) < C\epsilon$, for some constant $C \geq 1$, then there exists a M -Harnack chain joining g_1 to g_2 whose length depends on C but not on ϵ .

Remark 3.2 Recall that when d is the standard Euclidean distance the notion of NTA-domain was introduced in [JK] in connection with the study of the boundary behavior of non-negative harmonic functions. The first study of NTA-domains in a sub-Riemannian context was conducted in [CG], where a large effort was devoted to the nontrivial problem of constructing examples, see also [CGN]. In particular, in [CG] it was proved that in every Carnot group of step 2 the gauge balls are NTA-domains. In [CG] a Fatou theory was also developed and, in particular, the doubling condition for harmonic measure, and the comparison theorem for quotients of non-negative solutions of sub-Laplacians were established. Concerning NTA-domains in a sub-Riemannian context we also refer to [MM1].

Remark 3.3 Note that in $\mathbb{H}^n$ every bounded domain whose boundary is (Euclidean) $C^{1,1}$ is an NTA-domain. This result is proved [MM2]. In fact in [MM2] this result is proved for every Carnot group of step 2.

Definition 3.4 Let $\Omega \subset \mathbb{H}^n$ be a domain and let $g_0 \in \partial\Omega$. We say that Ω is a local non-tangentially accessible domain at g_0 , a local NTA-domain at g_0 hereafter, if there exist $M \geq 2, r_0 > 0$ such that the following holds.

- (i) Ω satisfies, for all r , $0 < r < r_0$, and at every point of $\partial\Omega \cap B(g_0, r_0)$, the interior and exterior corkscrew condition with constant M .
- (ii) (*Localized Harnack chain condition*) For every $0 < r \leq r_0$ and $g_1, g_2 \in \Omega \cap B(g_0, r)$, with $d(g_i, \partial\Omega) \geq \epsilon$, $i \in \{1, 2\}$, and $d(g_1, g_2) \leq C\epsilon$, there exists an M -Harnack chain, joining g_1 and g_2 , which is contained in $\Omega \cap B(g_0, Mr)$ and which has a length depending only on C, M and r_0 .

3.2 Hyperplanes in the set $\mathcal{V}$

In the following we let $\nu = (a, b, c) \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}) \setminus \{(0, 0, 0)\}$. We will often write $\nu = (\omega, c)$ where $\omega = (a, b)$. Given ν we let

$$\begin{aligned} H^\nu &= \{(z, t) \in \mathbb{H}^n \mid \langle z, \nu \rangle = \langle x, a \rangle + \langle y, b \rangle + ct > 0\}, \\ \Pi^\nu &= \{(z, t) \in \mathbb{H}^n \mid \langle z, \nu \rangle = \langle x, a \rangle + \langle y, b \rangle + ct = 0\}. \end{aligned} \quad (3.4)$$

Note that $(0, 0, 0) \in \Pi^\nu$. We will always assume (2.3). Consider $g_0 = (z_0, t_0) \in H^\nu$ and let $g = (z, t) \in \Pi^\nu$ be such that

$$d(g_0, \Pi^\nu) = d(g_0, g). \quad (3.5)$$

We want to find $d(g_0, \Pi^\nu)$ and g . To do this we simply consider

$$f(z, t) = d((z_0, t_0), (z, t))^4 = |z - z_0|^4 + 16 \left(t - t_0 + \frac{1}{2} \langle z_0^\perp, z \rangle \right)^2 \quad (3.6)$$

whenever $(z, t) \in \Pi^\nu$. We want, given $g_0 = (z_0, t_0) \in H^\nu$, to minimize the function $f(z, t)$ with respect to the constraint $(z, t) \in \Pi^\nu$. Using Lagrange multipliers we see that the critical points of f , subject to this constraint, are solutions to the following system of equations

$$\begin{aligned} (i) \quad & 4|z - z_0|^2(x - x_0) + 16 \left(t - t_0 + \frac{1}{2} \langle (y_0, -x_0), z \rangle \right) y_0 = \lambda a, \\ (ii) \quad & 4|z - z_0|^2(y - y_0) - 16 \left(t - t_0 + \frac{1}{2} \langle (y_0, -x_0), z \rangle \right) x_0 = \lambda b, \\ (iii) \quad & 32 \left(t - t_0 + \frac{1}{2} \langle (y_0, -x_0), z \rangle \right) = \lambda c, \\ (iv) \quad & \langle a, x \rangle + \langle b, y \rangle + ct = 0. \end{aligned} \quad (3.7)$$

Suppose that $\lambda = 0$. In that case $x = x_0$, $y = y_0$ and $(z_0, t_0) \in \Pi^\nu$ which is not the case. Hence $\lambda \neq 0$. Focusing on hyperplanes in the set $\mathcal{V}$ we let $c = 0$.

The case $c = 0$. In this case we see that

$$\begin{aligned} (i) \quad & 4|z - z_0|^2(x - x_0) + 16 \left(t - t_0 + \frac{1}{2} \langle (y_0, -x_0), z \rangle \right) y_0 = \lambda a, \\ (ii) \quad & 4|z - z_0|^2(y - y_0) - 16 \left(t - t_0 + \frac{1}{2} \langle (y_0, -x_0), z \rangle \right) x_0 = \lambda b, \\ (iv) \quad & \langle a, x \rangle + \langle b, y \rangle = 0, \end{aligned} \quad (3.8)$$

and

$$\lambda = -4|z - z_0|^2 \langle a, x_0 \rangle - 4|z - z_0|^2 \langle b, y_0 \rangle. \quad (3.9)$$

This gives that

$$z = (x, y) = (x_0, y_0) - (\langle z_0, \omega \rangle) \omega, \quad t = t_0 + \frac{1}{2} \langle z_0, \omega \rangle \langle z_0^\perp, \omega \rangle, \quad (3.10)$$

where $\omega = (a, b)$. Furthermore,

$$d(g_0, g) = \langle z_0, \omega \rangle. \quad (3.11)$$

Remark 3.5 Given $g_0 = (z_0, t_0) \in H^\nu$, $\nu = (\omega, c)$, $c = 0$, and $g = (z, t) \in \Pi^\nu$ such that (3.5) holds, we see that $g = (z, t)$ is given by the formulas in (3.10). Furthermore, g belongs to the horizontal plane through g_0 . Recall that the horizontal plane through g_0 is given by

$$\{(z, t) \in \mathbb{H}^n : t = t_0 - \frac{1}{2} \langle z_0^\perp, z \rangle\}. \quad (3.12)$$

From the above argument we also see that

$$d(g_0, g) = d(g_0, \Pi^\nu) = d_e(g_0, \Pi^\nu). \quad (3.13)$$

I.e., the gauge distance from g_0 to Π^ν coincides with the corresponding Euclidean distance.

Remark 3.6 Given $g = (z, t) \in H^\nu$, $\nu \in \mathcal{V}$, consider $\Pi^\nu(g)$ and let $\Pi^{\tilde{\nu}}(0) = g^{-1} \circ \Pi^\nu(g)$. Then $\tilde{\nu} = \nu$, i.e., the set of hyperplanes defined by the set $\mathcal{V}$ is invariant under left translations.

3.3 Domains well approximated by hyperplanes in the set $\mathcal{V}$

In the following we establish results concerning the geometry of the flat domains introduced. In particular, we prove that if these domains are flat enough then they are local NTA-domains. Let $\Omega \subset \mathbb{H}^n$ be a domain, $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Recall that if $\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{V}$, then there exists, whenever $g \in \partial\Omega \cap B(g_0, r_0)$ and $0 < r < r_0$, a hyperplane $\Pi = \Pi^\nu = \Pi^\nu(g)$, with $\nu \in \mathcal{V}$, such that

$$D(\partial\Omega \cap B(g, r), \Pi \cap B(g, r)) \leq \delta r,$$

and such that (2.7) hold. In the following we instead of ν and $\Pi^\nu(g)$ write $\nu_{r,g}$ and $\Pi^{\nu_{r,g}}(g)$ to indicate the scale r and the center g . Note that $\nu_{r,g}$ is a unit vector normal to the plane $\Pi^{\nu_{r,g}}(g)$. We first prove the following lemma.

Lemma 3.7 *Let $\Omega \subset \mathbb{H}^n$ be a domain, $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Assume that $\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{V}$. Then there exists $\hat{\delta} = \hat{\delta}(n)$, $0 < \hat{\delta} < 1/8$, such that if $0 < \delta < \hat{\delta}$, $0 < r < r_0/2$, then, for some constant $c = c(n)$, $1 \leq c < \infty$,*

$$\begin{aligned} (i) \quad & D((g^{-1} \circ (\Pi^{\nu_{r,g}}(g))) \cap B(0, 1), (g^{-1} \circ (\Pi^{\nu_{r/2,g}}(g))) \cap B(0, 1)) \leq c\delta, \\ (ii) \quad & D((g^{-1} \circ (\Pi^{\nu_{r,g}}(g))) \cap B(0, 1), ((g')^{-1} \circ (\Pi^{\nu_{r,g'}}(g')) \cap B(0, 1)) \leq c\delta, \end{aligned} \quad (3.14)$$

whenever $g, g' \in \partial\Omega \cap B(g_0, r_0)$ with $r/2 \leq d(g, g') \leq 2r$. Furthermore,

$$(i) \quad |\nu_{r,g} - \nu_{r/2,g}| \leq c\delta, \quad (ii) \quad |\nu_{r,g} - \nu_{r,g'}| \leq c\delta, \quad (3.15)$$

whenever $g, g' \in \partial\Omega \cap B(g_0, r_0)$ with $r/2 \leq d(g, g') \leq 2r$.

Proof. By (2.6) we have

$$\begin{aligned} (i') \quad & D(\partial(g^{-1} \circ \Omega) \cap B(0, r), (g^{-1} \circ \Pi^{\nu_{r,g}}(g)) \cap B(0, r)) \leq \delta r, \\ (ii') \quad & D(\partial(g^{-1} \circ \Omega) \cap B(0, r/2), (g^{-1} \circ \Pi^{\nu_{r/2,g}}(g)) \cap B(0, r/2)) \leq \delta r/2. \end{aligned} \quad (3.16)$$

Hence, using (3.16) (i') and (ii') we first see that

$$D((g^{-1} \circ (\Pi^{\nu_{r,g}}(g))) \cap B(0, r/2), (g^{-1} \circ (\Pi^{\nu_{r/2,g}}(g))) \cap B(0, r/2)) \leq 2\delta r. \quad (3.17)$$

(3.14) (i) with $B(0, 1)$ replaced by $B(0, 1/2)$ now follows by scaling and elementary geometry.

(3.15) (i) follows immediately from (3.14) (i). Indeed, we assume, as we may, that $g = 0$ and we let $\tilde{g} \in B(0, r) = \{\tilde{g} = (\tilde{z}, \tilde{t}) : (|\tilde{z}|^4 + 16\tilde{t}^2) \leq r^4\}$. Then, using (3.14) (i) we see that

$$|\langle \tilde{z}, \nu_{r,g} \rangle \nu_{r,g} - \langle \tilde{z}, \nu_{r/2,g} \rangle \nu_{r/2,g}| \leq c\delta r. \quad (3.18)$$

If we now consider $\tilde{g} = (\nu_{r,g}r/4, 0)$ then (3.18) implies that

$$(1 - |\langle \nu_{r,g}, \nu_{r/2,g} \rangle|^2) = |\nu_{r,g} - \langle \nu_{r,g}, \nu_{r/2,g} \rangle \nu_{r/2,g}|^2 \leq c\delta. \quad (3.19)$$

Using (3.19) we see that (3.15) (i) holds. To prove (3.14) (ii) we first note, using the definitions and the assumption $r/2 \leq d(g, g') \leq 2r$, $g, g' \in \partial\Omega$, that

$$\begin{aligned} (i'') \quad & D(\partial\Omega \cap B(g', r), \Pi^{\nu_{4r,g}} \cap B(g, 4r)) \leq 4\delta r, \\ (ii'') \quad & D(\partial\Omega \cap B(g', r), \Pi^{\nu_{r,g'}} \cap B(g', r)) \leq \delta r. \end{aligned} \quad (3.20)$$

Hence

$$D(\Pi^{\nu_{4r,g}} \cap B(g, 4r), \Pi^{\nu_{r,g'}} \cap B(g', r)) \leq 5\delta r. \quad (3.21)$$

Since $B(g', r) \subset B(g, 4r)$ we can conclude that

$$D(\Pi^{\nu_{4r,g}} \cap B(g', r), \Pi^{\nu_{r,g'}} \cap B(g', r)) \leq 5\delta r. \quad (3.22)$$

Now, arguing as in (3.18)-(3.19) we see that $|\nu_{4r,g} - \nu_{r,g'}| \leq c\delta$. (3.14) (ii) and (3.15) (ii) follow from this observation, (3.14) (i), (3.15) (i) and scaling. $\square$

We next prove the following crucial lemma.

Lemma 3.8 *Let $\Omega \subset \mathbb{H}^n$ be a domain, $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Assume that $\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{V}$. Then there exists $\hat{\delta} = \hat{\delta}(n)$, $0 < \hat{\delta} < 1/8$, and $M = M(n)$, $1 \leq M < \infty$, such that the following is true. Assume $0 < \delta < \hat{\delta}$ and let $\tilde{g}_0 \in \partial\Omega \cap B(g_0, r_0/2)$, $\tilde{r}_0 = r_0/M$, $0 < r < \tilde{r}_0$. Then Ω is a local NTA-domain at $\tilde{g}_0$ with parameters M and $\tilde{r}_0$.*

Proof. We first focus on (i) of Definition 3.4. Let $\tilde{g}_0 \in \partial\Omega \cap B(g_0, r_0/2)$ and let $\tilde{r}_0 = r_0/M$ where M is to be determined. Let $g \in \partial\Omega \cap B(\tilde{g}_0, \tilde{r}_0)$, $r < \tilde{r}_0$. Using Definition 2.2 and Definition 2.3 for $\mathcal{S} = \mathcal{V}$ we see that there exists a hyperplane $= \Pi^{\nu_{r,g}}$, with $\nu_{r,g} \in \mathcal{V}$, such that

$$D(\partial\Omega \cap B(g, r), \Pi^{\nu_{r,g}} \cap B(g, r)) \leq \delta r$$

and such that (2.7) holds. For $\eta \in (1/4, 1)$ fixed we let

$$A_r(g) = g \circ (\eta r \nu_{r,g}), \quad \hat{A}_r(g) = g \circ (-\eta r \nu_{r,g}). \quad (3.23)$$

Then $d(A_r(g), g) = \eta r = d(\hat{A}_r(g), g)$. Furthermore,

$$\min\{d(A_r(g), \partial B(g, r)), d(\hat{A}_r(g), \partial B(g, r))\} \geq (1 - \eta)r$$

and

$$\min\{d(A_r(g), \partial\Omega), d(\hat{A}_r(g), \partial\Omega)\} \geq r(\eta - \delta).$$

In particular, if we let $\hat{\delta} < \eta/2$ we see that (i) of Definition 3.4 holds with, for example, $M = 2/\eta$. In the following we always let $a_r(g)$ denote the point in the set $\{A_r(g), \hat{A}_r(g)\} = \{g \circ (\eta r \nu_{r,g}), g \circ (-\eta r \nu_{r,g})\}$ which is located in Ω .

We next focus on the localized Harnack chain condition, i.e., (ii) of Definition 3.4 and we start by developing some preliminary results. In particular, let $0 < r \leq \tilde{r}_0$, $\tilde{g} \in \Omega \cap B(\tilde{g}_0, \tilde{r}_0)$ and $\epsilon = d(\tilde{g}, \partial\Omega)$. Let $g \in \partial\Omega$ be such that $\epsilon = d(\tilde{g}, g)$. We first prove that

$$\begin{aligned} (i) \quad & u(\tilde{g}) \approx u(a_\epsilon(g)), \\ (ii) \quad & u(a_\epsilon(g)) \approx u(a_{2\epsilon}(g)), \end{aligned} \quad (3.24)$$

where $\approx$ means that constants of proportionality only depend on p , n and $M = M(\eta)$. Note that we can, using translation invariance, without loss of generality assume that $g = 0$ and we

in the following redefine $\Omega = g^{-1} \circ \Omega$. To prove (3.24) (i) we as above note that there exists $\nu_{8\epsilon,0} \in \mathcal{V}$ such that if $0 < \delta < \hat{\delta}$, for some $\hat{\delta} = \hat{\delta}(\eta)$ small, then

$$H^{\nu_{8\epsilon,0}}(32\delta\epsilon\nu_{8\epsilon,0}) \cap B(0, 8\epsilon) \subset \Omega \cap B(0, 8\epsilon), \quad (3.25)$$

and

$$\tilde{g}, a_\epsilon(0) \in H^{\nu_{8\epsilon,0}}(64\delta\epsilon\nu_{8\epsilon,0}) \cap B(0, 2\epsilon), \quad \min\{d(\tilde{g}, \partial\Omega), d(a_\epsilon(0), \partial\Omega)\} \geq 7\eta\epsilon/8. \quad (3.26)$$

Indeed, (3.25) follows from the separation property in (2.7) and (3.26) is trivially fulfilled for $\tilde{g}$. To verify (3.26) for $a_\epsilon(0)$ recall that $a_\epsilon(0) = \eta\epsilon\nu_{\epsilon,0}$ and that the distance from $a_\epsilon(0)$ to the hyperplane $H^{\nu_{8\epsilon,0}}(0)$ equals $\langle \nu_{\epsilon,0}, \nu_{8\epsilon,0} \rangle \eta\epsilon$. Furthermore, using (3.15) we see that

$$\langle \nu_{\epsilon,0}, \nu_{8\epsilon,0} \rangle \eta\epsilon \geq (1 - c\delta)\eta\epsilon. \quad (3.27)$$

Choosing δ small enough we see that (3.26) holds. In the following we let (z_1, t_1) and (z_2, t_2) denote, respectively, the point $\tilde{g}$ and $a_\epsilon(0)$ and we assume $t_1 \geq t_2$. $d(\tilde{g}, a_\epsilon(0)) \leq 2\epsilon$ implies that

$$\begin{aligned} (\alpha) \quad d((z_2, t_1), (z_2, t_2)) &= 16|t_1 - t_2|^{1/2} \leq 16|t_1 - t_2 + \frac{1}{2}\langle z_1, z_2^\perp \rangle|^{1/2} + 4|\langle z_1, z_2^\perp \rangle|^{1/2} \leq 40\epsilon, \\ (\beta) \quad d((z_1, t_2), (z_2, t_2)) &= (|z_1 - z_2|^4 + 4\langle z_1, z_2^\perp \rangle^2)^{1/4} \leq 40\epsilon. \end{aligned} \quad (3.28)$$

We immediately see that we can connect (z_2, t_2) to (z_2, t_1) using a Harnack chain, with length only depending on n , of balls centered on the line segment $\{(z_2, \lambda t_1 + (1 - \lambda)t_2) : \lambda \in [0, 1]\}$. Hence, using the Harnack inequality we see that $u(z_2, t_2) \approx u(z_2, t_1)$. We next connect the points $(z_1, t_2), (z_2, t_2)$ through the Euclidean line segment $\{(z(\lambda), t_2) = (\lambda z_2 + (1 - \lambda)z_1, t_2) : \lambda \in [0, 1]\}$. Then by construction

$$d((z(\lambda), t_2), \partial\Omega) \geq 7\eta\epsilon/8 \text{ for all } \lambda \in [0, 1]. \quad (3.29)$$

Let $\lambda_k = k\eta^2/10^8$ for all non-negative integers $k \leq k_0$ where k_0 is the smallest positive integer such that $k_0\eta^2/10^8 \geq 1$. We let $\lambda_{k_0} = 1$. Then

$$d((z(\lambda_k), t_2), (z(\lambda_{k-1}), t_2)) \leq 40\epsilon(\lambda_k - \lambda_{k-1})^{1/2} = 40\epsilon\eta/10^4. \quad (3.30)$$

Using (3.28) (β), (3.28) and (3.30) we see that we can connect (z_1, t_2) to (z_2, t_2) using a Harnack chain, with length only depending on n, η , of balls centered on the segment $\{(z(\lambda), t_2) = (\lambda z_2 + (1 - \lambda)z_1, t_2) : \lambda \in [0, 1]\}$. Hence, using the Harnack inequality we see that $u(z_1, t_2) \approx u(z_2, t_2)$. This proves (3.24) (i). To prove (3.24) (ii) one can argue analogously.

After these preliminaries we now focus on (ii) of Definition 3.4. In particular, let $0 < r \leq \tilde{r}_0$ and $\tilde{g}_1, \tilde{g}_2 \in \Omega \cap B(\tilde{g}_0, \tilde{r}_0)$ with $d(\tilde{g}_i, \partial\Omega) \geq \epsilon$, $i \in \{1, 2\}$, and $d(\tilde{g}_1, \tilde{g}_2) \leq 2^l\epsilon$ for some positive integer l . Let $g_1, g_2 \in \partial\Omega$ be such that $\epsilon_i := d(\tilde{g}_i, g_i) = d(\tilde{g}_i, \partial\Omega)$ for $i \in \{1, 2\}$. Using (3.24) (i) we have

$$u(\tilde{g}_i) \approx u(a_{\epsilon_i}(g_i)). \quad (3.31)$$

Following [KT] we divide the proof into the following cases

$$(i) \ \epsilon_1, \epsilon_2 \leq 2^{l-2}\epsilon, \ (ii) \ \epsilon_1 > 2^{l-2}\epsilon, \epsilon_2 \leq 2^{l-2}\epsilon, \ (iii) \ \epsilon_1 > 2^{l-2}\epsilon, \epsilon_2 > 2^{l-2}\epsilon. \quad (3.32)$$

(3.32) **case (i).** Let $j_i, i \in \{1, 2\}$, be such that $2^{l-1}\epsilon \leq 2^{j_i}\epsilon_i < 2^l\epsilon$. Then using (3.31) and (3.24) (ii) we see that

$$\beta^{-j_i}u(a_{2^{j_i}\epsilon_i}(g_i)) \leq u(\tilde{g}_i) \leq \beta^{j_i}u(a_{2^{j_i}\epsilon_i}(g_i)), i \in \{1, 2\}, \quad (3.33)$$

for some $\beta = \beta(p, n, M(\eta))$, $1 \leq \beta < \infty$. Furthermore,

$$\begin{aligned} d(a_{2^{j_1}\epsilon_1}(g_1), a_{2^{j_2}\epsilon_2}(g_2)) &\leq 2^{j_1}\epsilon_1 + 2^{j_2}\epsilon_2 + d(g_1, g_2) \\ &\leq 2^{l+1}\epsilon + d(g_1, \tilde{g}_1) + d(\tilde{g}_1, \tilde{g}_2) + d(\tilde{g}_2, g_2) \leq 2^{l+2}\epsilon. \end{aligned} \quad (3.34)$$

Arguing as in the proof of (3.24) (i), essentially consider a ball centered at the boundary and with radius $8 \cdot 2^l\epsilon$ instead of radius 8ϵ , we can now deduce that

$$u(a_{2^{j_1}\epsilon_1}(g_1)) \approx u(a_{2^{j_2}\epsilon_2}(g_2)). \quad (3.35)$$

This observation and (3.33) completes the proof in this case.

(3.32) **case (ii).** In this case let j_2 be such that $2^{l-1}\epsilon \leq 2^{j_2}\epsilon_i < 2^l\epsilon$. Then using (3.31) and (3.24) (ii) we again see that

$$\beta^{-j_2}u(a_{2^{j_2}\epsilon_2}(g_2)) \leq u(\tilde{g}_2) \leq \beta^{j_2}u(a_{2^{j_2}\epsilon_2}(g_2)), \quad (3.36)$$

for some $\beta = \beta(p, n, M(\eta))$, $1 \leq \beta < \infty$. Furthermore,

$$d(\tilde{g}_1, a_{2^{j_2}\epsilon_2}(g_2)) \leq d(\tilde{g}_1, \tilde{g}_2) + d(\tilde{g}_2, a_{2^{j_2}\epsilon_2}(g_2)) \leq 2^{l+1}\epsilon. \quad (3.37)$$

Again arguing as in the proof of (3.24) (i), see the proof of (3.35), we deduce that $u(\tilde{g}_1) \approx u(a_{2^{j_2}\epsilon_2}(g_2))$. This observation and (3.36) completes the proof in this case.

(3.32) **case (iii).** In this case we can immediately argue as in the proof of (3.24) (i), (3.35), and deduce that $u(\tilde{g}_1) \approx u(\tilde{g}_2)$. this completes the proof in this case. $\square$

We next consider domains well approximated by hyperplanes in the set $\mathcal{V}$ in the case of domains locally given as graphs. In particular, let $\Omega \subset \mathbb{H}^n$ be a domain, $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 = (x_0, t_0) \in \partial\Omega$. We assume that

$$\begin{aligned} \Omega \cap B(g_0, r_0) &= \{(\tilde{z}, \tilde{t}) = (\tilde{z}', \tilde{z}'', \tilde{t}) \in \mathbb{H}^n : \tilde{z}'' > f(\tilde{z}', \tilde{t})\} \cap B(g_0, r_0), \\ \partial\Omega \cap B(g_0, r_0) &= \{(\tilde{z}, \tilde{t}) = (\tilde{z}', \tilde{z}'', \tilde{t}) \in \mathbb{H}^n : \tilde{z}'' = f(\tilde{z}', \tilde{t})\} \cap B(g_0, r_0), \end{aligned} \quad (3.38)$$

where $(\tilde{z}', \tilde{z}'')$ is a splitting of the coordinate $\tilde{z}$ with $\tilde{z}' \in \mathbb{R}^{2n-1}$, $\tilde{z}'' \in \mathbb{R}$. We let $g = (z, t) \in \Omega \cap B(g_0, r_0)$, $0 < r < r_0$. It is natural to try to understand the condition that there exists a hyperplane $\Pi = \Pi^\nu = \Pi^\nu(g)$, with $\nu \in \mathcal{V}$, such that

$$D(\partial\Omega \cap B(g, r), \Pi \cap B(g, r)) \leq \delta r \quad (3.39)$$

in terms of the function f . Let $(\tilde{z}, \tilde{t}) = g \circ (w, s)$, i.e., we translate to the origo. Then

$$\begin{aligned} &\partial(g^{-1} \circ \Omega) \cap B(0, r) \\ &= \{(w, s) = (w', w'', s) \in \mathbb{H}^n : z'' + w'' = f(z' + w', t + s + \frac{1}{2}\langle z, w^\perp \rangle)\} \cap B(0, r). \end{aligned} \quad (3.40)$$

In particular, if f is independent of the coordinate in the second layer then (3.39) simply state a restriction on the Lipschitz constant of the function f . If f is not independent of the coordinate in the second layer then things are more complicated and while this has to be understood further we here simply give an example. Indeed, assume that f is (Euclidean) Lipschitz in all variables with Lipschitz constant M . Then a crude estimate gives that

$$|w''| = |f(z' + w', t + s + \frac{1}{2}\langle z, w^\perp \rangle) - z''| \leq M(2 + \frac{1}{2}|z|)r \quad (3.41)$$

whenever $(w, s) \in B(0, r)$, $r \leq 1$. In particular, if

$$\sup_{g=(z,t) \in \Omega \cap B(g_0, r_0)} M(2 + \frac{1}{2}|z|) \leq \delta \quad (3.42)$$

then $\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{V}$.

4 Basic estimates for p -harmonic functions in the Heisenberg group

Recall that given a domain $G \subset \mathbb{H}^n$, i.e., a connected open set, and $1 < p < \infty$, we say that u is p -harmonic in G provided $u \in W^{1,p}(G)$ and

$$\int |Xu|^{p-2} \langle Xu, X\theta \rangle dg = 0 \quad (4.1)$$

whenever $\theta \in W_0^{1,p}(G)$. Formally, $L_p u = 0$ in G where L_p is the sub-elliptic p -Laplace operator defined in (1.1). If (4.1) holds with $=$ replaced by $\geq$ ($\leq$) for all functions $\theta \in W_0^{1,p}(G)$ such that $\theta \geq 0$ a.e. in G , then u is said to be p -superharmonic (p -subharmonic) in G . $u \in W_{loc}^{1,p}(G)$ if $\theta u \in W^{1,p}(G)$ for any test function θ with support in G . For proofs of the following lemmas we refer to [Da].

Lemma 4.1 *Let $G \subset \mathbb{H}^n$. Given any $\phi \in W^{1,p}(G)$ there exists a unique p -harmonic function $u \in W^{1,p}(G)$ such that $u - \phi \in W_0^{1,p}(G)$.*

Lemma 4.2 *Let $G \subset \mathbb{H}^n$. Let $u, v \in W_{loc}^{1,p}(G)$ be a p -superharmonic function and a p -subharmonic function in G , respectively. If $\inf\{u - v, 0\} \in W_0^{1,p}(G)$, then $u \geq v$ a.e in G .*

Given a domain $G \subset \mathbb{H}^n$ and p , $1 < p < \infty$, a point $g_0 \in \partial G$ is called regular for the Dirichlet problem for the operator L_p if, for every $\phi \in W^{1,p}(G) \cap C(\bar{G})$, the solution u of Lemma 4.1 satisfies

$$\lim_{g \rightarrow g_0, g \in G} u(g) = \phi(g_0).$$

If every $g_0 \in \partial G$ is regular, then we say that G is regular for the Dirichlet problem for L_p . A Wiener type criteria for regularity was proved in [Da] and from this result it follows that a sufficient geometric condition for G to be regular is that its exterior have uniform positive density. This means that there exist $1 \leq c$ and $r_0 > 0$ such that if $g_0 \in \partial \Omega$ and $0 < r < r_0$, then

$$|(\mathbb{H}^n \setminus G) \cap B(g_0, r)| \geq c^{-1} r^Q, \quad Q = 2n + 2. \quad (4.2)$$

In particular, the exterior of any NTA-domain as considered in this paper has uniform positive density and hence the NTA-domains introduced are regular for the Dirichlet problem the sub-elliptic p -Laplace operator L_p .

Let $1 < p < \infty$. A distribution $\Gamma_p(\cdot, g)$ is called a fundamental solution of L_p , with singularity at $g \in \mathbb{H}^n$, if

$$\begin{aligned} (i) \quad & \Gamma_p(\cdot, g) \in W^{1,p}(\mathbb{H}^n \setminus \{g\}), \\ (ii) \quad & |X\Gamma_p(\cdot, g)|^{p-1} \in L^1_{loc}(\mathbb{H}^n), \\ (iii) \quad & \int_{\mathbb{H}^n} |X\Gamma_p(\cdot, g)|^{p-2} \langle X\Gamma_p(\cdot, g), X\phi \rangle = \phi(g) \text{ whenever } \phi \in C_0^\infty(\mathbb{H}^n). \end{aligned} \quad (4.3)$$

Lemma 4.3 *Let $1 < p < \infty$, consider $\mathbb{H}^n$ and let $Q = 2n + 2$. Define*

$$\begin{aligned} \Gamma_p(g, g') &= \Gamma_p(g', g) = \frac{(p-1)}{(Q-p)} \sigma_p^{-\frac{1}{p-1}} d(g, g')^{\frac{p-Q}{p-1}}, \quad p \neq Q, \\ \Gamma_p(g, g') &= \Gamma_p(g', g) = -\sigma_p^{-\frac{1}{p-1}} \log d(g, g'), \quad p = Q, \end{aligned} \quad (4.4)$$

whenever $g \neq g'$ and with

$$\sigma_p = Q\omega_p, \quad \omega_p = \int_{B(e,1)} |Xd(\cdot, e)|^p dg. \quad (4.5)$$

Then $\Gamma_p(g', g)$ is a fundamental solution of (1.11) with singularity at $g \in \mathbb{H}^n$.

Proof. See [CDG]. For the case $p = Q$ see also [HH]. $\square$

4.1 Basic estimates in (Local) NTA-domains

Let $1 < p < \infty$ and let $\Omega \subset \mathbb{H}^n$ be a bounded domain. In this section we collect a number of basic estimates for non-negative p -harmonic functions in Ω assuming that Ω is a (local) NTA-domain with constants M, r_0 . Throughout this section and this paper, unless otherwise stated, c will denote a positive constant ≥ 1 , not necessarily the same at each occurrence, depending only on $p, n, M, .$ In general, $c(a_1, \dots, a_m)$ denotes a positive constant ≥ 1 , which may depend only on p, n, M and $a_1, \dots, a_m$, and which is not necessarily the same at each occurrence. If $A \approx B$ then A/B is bounded from above and below by constants which, unless otherwise stated, only depend on p, n, M .

For the proof of Lemma 4.4 and Lemma 4.5 below we refer to [Da].

Lemma 4.4 *Let $g_0 \in \mathbb{H}^n$. Given $p, 1 < p < \infty$, let u be a positive p -harmonic function in $B(g_0, 2r)$. Then*

$$\begin{aligned} (i) \quad & r^{p-Q} \int_{B(g_0, r/2)} |Xu|^p dx \leq c \left(\sup_{B(g_0, r)} u \right)^p, \\ (ii) \quad & \sup_{B(g_0, r)} u \leq c \inf_{B(g_0, r)} u. \end{aligned}$$

Furthermore, there exists $\alpha = \alpha(p, n) \in (0, 1)$ such that if $g, g' \in B(g_0, r)$, then

$$(iii) \quad |u(g) - u(g')| \leq c(d(g, g')/r)^\alpha \sup_{B(g_0, 2r)} u.$$

Lemma 4.5 *Let $\Omega \subset \mathbb{H}^n$ be a domain, let $g_0 \in \partial\Omega$ and assume that Ω is a local NTA-domain at g_0 with constants M and r_0 . Let $p, 1 < p < \infty$, be given. Let $\tilde{g}_0 \in \partial\Omega \cap B(g_0, r_0)$, $0 < r < r_0/2$, and suppose that u is a non-negative continuous p -harmonic function in $\bar{\Omega} \cap B(\tilde{g}_0, 2r)$ and that $u = 0$ on $\partial\Omega \cap B(\tilde{g}_0, 2r)$. Then*

$$(i) \quad r^{p-Q} \int_{\Omega \cap B(\tilde{g}_0, r/2)} |Xu|^p dx \leq c \left(\sup_{\Omega \cap B(\tilde{g}_0, r)} u \right)^p.$$

Furthermore, there exists $\alpha = \alpha(p, n, M) \in (0, 1)$ such that if $g, g' \in \Omega \cap B(\tilde{g}_0, r)$, then

$$(ii) \quad |u(g) - u(g')| \leq c(d(g, g')/r)^\alpha \sup_{\Omega \cap B(\tilde{g}_0, 2r)} u.$$

Lemma 4.6 *Let $\Omega \subset \mathbb{H}^n$ be a domain, let $g_0 \in \partial\Omega$ and assume that Ω is a local NTA-domain at g_0 with constants M and r_0 . Let $p, 1 < p < \infty$, be given. Let $\tilde{g}_0 \in \partial\Omega \cap B(g_0, r_0)$, $0 < r < r_0/2$, and suppose that u is a non-negative continuous p -harmonic function in $\bar{\Omega} \cap B(\tilde{g}_0, 2r)$ and that $u = 0$ on $\partial\Omega \cap B(\tilde{g}_0, 2r)$. Then there exists $c = c(p, n, M)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then*

$$\sup_{\Omega \cap B(\tilde{g}_0, \tilde{r})} u \leq c u(a_{\tilde{r}}(\tilde{g}_0)).$$

Proof. This follows from the local NTA-character of Ω at g_0 , Lemma 4.4 (ii) and Lemma 4.5 (ii) along the lines of the corresponding proof in [CFMS]. $\square$

Lemma 4.7 *Let $g_0 \in \mathbb{H}^n$. Given $p, 1 < p < \infty$, let u be a non-negative p -harmonic function in $B(g_0, 4r)$. Then there exists $c = c(p, n)$, $1 \leq c < \infty$, such that*

$$c^{-1} \sup_{B(g_0, r)} |Xu| \leq \left(r^{-Q} \int_{B(g_0, 2r)} |Xu|^p dg \right)^{1/p} c \leq c r^{-1} \sup_{B(g_0, 4r)} |u| \quad (4.6)$$

Proof. This is Theorem 1.1 in [Z]. $\square$

Lemma 4.8 *Let $g_0 \in \mathbb{H}^n$. Given $p, 2 \leq p < \infty$, let u be a non-negative p -harmonic function in $B(g_0, 4r)$. Then the horizontal gradient Xu is Hölder continuous. In particular, there exist $c = c(p, n)$, $1 \leq c < \infty$, $\sigma \in (0, 1]$, depending only on p, n , such that if $g, g' \in B(g_0, r/2)$, then*

$$c^{-1} |Xu(g) - Xu(g')| \leq (d(g, g')/r)^\sigma \sup_{B(g_0, r)} |Xu| \leq c r^{-1} (d(g, g')/r)^\sigma \sup_{B(g_0, 4r)} u. \quad (4.7)$$

Proof. This is Theorem 1.2 in [Z]. $\square$

4.2 A technical lemma concerning $|Xu|$

Lemma 4.9 *Let $O \subset \mathbb{H}^n$ be an open set, $2 \leq p < \infty$. Suppose that $\hat{u}, \hat{v}$ are positive p -harmonic functions in O . Let $a \geq 1, g \in O$, and assume that*

$$\frac{1}{a} \frac{\hat{v}(g)}{d(g, \partial O)} \leq |X\hat{v}(g)| \leq a \frac{\hat{v}(g)}{d(g, \partial O)}.$$

Let $\tilde{\epsilon}^{-1} = (ca)^{(1+\sigma)/\sigma}$, where σ is as in Lemma 4.8 and $c = c(p, n)$. Then the following statement is true for $c = c(p, n)$ suitably large. If

$$(1 - \tilde{\epsilon})\hat{L} \leq \frac{\hat{v}}{\hat{u}} \leq (1 + \tilde{\epsilon})\hat{L} \text{ in } B(g, \frac{1}{4}d(g, \partial O))$$

for some $\hat{L}, 0 < \hat{L} < \infty$, then

$$\frac{1}{ca} \frac{\hat{u}(g)}{d(g, \partial O)} \leq |X\hat{u}(g)| \leq ca \frac{\hat{u}(g)}{d(g, \partial O)}.$$

Proof. Let $a \geq 1$ and $g \in O$ be as in the statement of the lemma. Using Lemma 4.8, the fact that the gauge balls are NTA-domains (see Remark 3.2), and the Harnack inequality in Lemma 4.5 (ii), we see that,

$$|X\hat{u}(g_1) - X\hat{u}(g_2)| \leq c\eta^\sigma \sup_{B(g, \eta d(g, \partial O))} |X\hat{u}(\cdot)| \leq c^2\eta^\sigma \hat{u}(g)/d(g, \partial O) \quad (4.8)$$

whenever $g_1, g_2 \in \bar{B}(g, \eta d(g, \partial O))$ and $0 < \eta \leq 1/100$. Here c depends only on p, n . Using (4.8) we see that we only have to prove bounds from below for the gradient of $\hat{u}$ at g . To achieve this we suppose that,

$$|X\hat{u}(g)| \leq \zeta \hat{u}(g)/d(g, \partial O), \quad (4.9)$$

for some small $\zeta > 0$ to be chosen. Using (4.8) with $g_2 = g$, and (4.9), we deduce that

$$|X\hat{u}(g_1)| \leq [\zeta + c^2\eta^\sigma] \hat{u}(g)/d(g, \partial O) \quad (4.10)$$

whenever $g_1 \in B(g, \eta d(g, \partial O))$. Let $\eta = \zeta^{1/\sigma}$ and consider $g' \in B(g, \eta d(g, \partial O))$ subject to the following restriction. Let $d_{cc}(g', g)$ be the Carnot-Carathéodory distance from g' to g , see (3.1)-(3.3). Then we in the following only consider g' such that $d_{cc}(g', g) < \eta d(g, \partial O)$. For such g' we see, by integrating along the path connecting g and g' and using (4.10), that

$$|\hat{u}(g') - \hat{u}(g)| \leq c'\zeta^{1+1/\sigma} \hat{u}(g). \quad (4.11)$$

The constants in (4.10), (4.11) depend only on p, n . Next we note that (4.8) also holds with $\hat{u}$ replaced by $\hat{v}$. Let $\lambda = X\hat{v}(g)/|X\hat{v}(g)|$. Then from (4.8) for $\hat{v}$, and the non-degeneracy assumption on $|X\hat{v}|$ in Lemma 4.9, we find that

$$\langle X\hat{v}(g_1), \lambda \rangle \geq (1 - ca\zeta)|X\hat{v}(g)| \text{ whenever } g_1 \in \bar{B}(g, \zeta^{1/\sigma}d(g, \partial O)), \quad (4.12)$$

for some $c = c(p, n)$. Let $g = (z, t) = (x, y, t)$ and $\hat{\rho} = \zeta^{1/\sigma}d(g, \partial O)$. Based on λ and (4.12) we consider the path

$$\hat{\gamma}(\tau) = (\hat{\gamma}_1(\tau), \dots, \hat{\gamma}_{2n+1}(\tau)) = g \circ (\lambda_1\hat{\rho}\tau, \dots, \lambda_{2n}\hat{\rho}\tau, 0), \quad \tau \in [0, 1]. \quad (4.13)$$

Then $\hat{\gamma}(0) = g$ and $\hat{\gamma}(\tau) \in \bar{B}(g, \zeta^{1/\sigma}d(g, \partial O))$ whenever $\tau \in [0, 1]$. Using summation convention we note that

$$\frac{d}{d\tau} \hat{v}(\hat{\gamma}(\tau)) = \hat{v}_{x_i}(\hat{\gamma}(\tau))\lambda_i\hat{\rho} + \hat{v}_{y_i}(\hat{\gamma}(\tau))\lambda_{n+i}\hat{\rho} + \hat{v}_t(\hat{\gamma}(\tau))(x_i\lambda_{n+i}\hat{\rho} - y_i\lambda_i\hat{\rho}). \quad (4.14)$$

Since,

$$(x_i \lambda_{n+i} \hat{\rho} - y_i \lambda_i \hat{\rho}) = ((x_i + \lambda_i \hat{\rho} \tau) \lambda_{n+i} \hat{\rho} - (y_i + \lambda_{n+i} \hat{\rho} \tau) \lambda_i \hat{\rho}) \quad (4.15)$$

we see that (4.14) can be rewritten as

$$\frac{d}{d\tau} \hat{v}(\hat{\gamma}(\tau)) = \sum_{i=1}^{2n} X_i \hat{v}(\hat{\gamma}(\tau)) \lambda_i \hat{\rho} = \langle X \hat{v}(\hat{\gamma}(\tau)), \lambda \rangle \hat{\rho}. \quad (4.16)$$

Next, using the fundamental theorem of calculus we see that

$$\hat{v}(\hat{\gamma}(\tau)) - \hat{v}(g) = \int_0^\tau \langle X \hat{v}(\hat{\gamma}(s)), \lambda \rangle \hat{\rho} ds. \quad (4.17)$$

We now consider g' such that $d_{cc}(g', g) = \eta d(g, \partial O)/2$. Note that (4.11) holds for this choice of g' . We can also conclude that there exists $\tau_{g'} \in [0, 1]$ such that $\tau_{g'} \approx 1$ and such that $g' = \hat{\gamma}(\tau_{g'}) \in \partial B(g, \zeta^{1/\sigma} d(g, \partial O)/2)$. Then, using (4.12) and (4.17) we can conclude, if $\zeta \leq (2ca)^{-1}$, where c is the constant in (4.12), that

$$c^*(\hat{v}(g') - \hat{v}(g)) \geq a^{-1} \zeta^{1/\sigma} \hat{v}(g) \quad (4.18)$$

where the constant c^* depends only on p, n . From (4.18), (4.11), we see that if $\tilde{\epsilon}$ is as in Lemma 4.9, then

$$\begin{aligned} (1 - \tilde{\epsilon}) \hat{L} &\leq \frac{\hat{u}(g')}{\hat{v}(g')} \leq \left(\frac{1 + c' \zeta^{1+1/\sigma}}{1 + \zeta^{1/\sigma}/(ac^*)} \right) \frac{\hat{u}(g)}{\hat{v}(g)} \\ &\leq (1 + \tilde{\epsilon}) \left(\frac{1 + c' \zeta^{1+1/\sigma}}{1 + \zeta^{1/\sigma}/(ac^*)} \right) \hat{L} < (1 - \tilde{\epsilon}) \hat{L} \end{aligned} \quad (4.19)$$

provided $1/(a\tilde{c})^{1/\sigma} \geq \zeta^{1/\sigma} \geq a\tilde{c}\tilde{\epsilon}$ for some large $\tilde{c} = \tilde{c}(p, n)$. This inequality and (4.18) are satisfied if $\tilde{\epsilon}^{-1} = (\tilde{c}a)^{(1+\sigma)/\sigma}$ and $\zeta^{-1} = \tilde{c}a$. Moreover, if the hypotheses of Lemma 4.9 hold for this $\tilde{\epsilon}$, then in order to avoid the contradiction in (4.19) it must be true that (4.9) is false for this choice of ζ . Hence Lemma 4.9 is true. $\square$

5 Linear degenerate equations in the Heisenberg group

Let $g_0 \in \mathbb{H}^n$, $0 < r$ and let λ be a real valued Lebesgue measurable function defined almost everywhere on $B(g_0, 2r)$. λ is said to belong to the class $A_2(B(g_0, r))$ if there exists a constant γ such that

$$\tilde{r}^{-2Q} \int_{B(\zeta, \tilde{r})} \lambda dg \cdot \int_{B(\zeta, \tilde{r})} \lambda^{-1} dg \leq \gamma \quad (5.1)$$

whenever $\zeta \in B(g_0, r)$ and $0 < \tilde{r} \leq r$. If λ belongs to the class $A_2(B(g_0, r))$ then λ is referred to as an $A_2(B(g_0, r))$ -weight. The smallest γ such that (5.1) holds is referred to as the constant of the weight. In the following we let $\Omega \subset \mathbb{H}^n$ be a NTA-domain, or a local NTA-domain at a

point $g_0 \in \partial\Omega$, with parameters $M, r_0 > 0$. We let $\tilde{g}_0 \in \partial\Omega \cap B(g_0, r_0)$, $0 < r < r_0$, and we consider the operator

$$L = \sum_{i,j=1}^{2n} X_i(\hat{a}_{ij}(g)X_j) \quad (5.2)$$

in $\Omega \cap B(\tilde{g}_0, r)$. We assume that the coefficients $\{\hat{a}_{ij}(g)\}$ are bounded, Lebesgue measurable functions defined almost everywhere on $B(\tilde{g}_0, 2r)$. Moreover, $\hat{a}_{ij} = \hat{a}_{ji}$ for all $i, j \in \{1, \dots, 2n\}$, and

$$c^{-1}\lambda(g)|\xi|^2 \leq \sum_{i,j=1}^{2n} \hat{a}_{ij}(g)\xi_i\xi_j \leq c|\xi|^2\lambda(g), \quad (5.3)$$

for almost every $g \in B(\tilde{g}_0, 2r)$, where $\lambda \in A_2(B(\tilde{g}_0, r))$. By definition L is a degenerate sub-elliptic operator (in divergence form) in $B(\tilde{g}_0, 2r)$ with ellipticity measured by the function λ . If $O \subset B(\tilde{g}_0, 2r)$ is open then we let $S_\lambda^{1,2}(O)$ be the weighted Sobolev space of equivalence classes of functions v with horizontal distributional gradients $X_j v$, $j \in \{1, \dots, 2n\}$, and norm

$$\|v\|_{S_\lambda^{1,2}(O)}^2 := \int_O v^2 \lambda dy + \int_O |Xv|^2 \lambda dy < \infty. \quad (5.4)$$

Let $S_{\lambda,0}^{1,2}(O)$ be the closure of $C_0^\infty(O)$ in the norm defining $S_\lambda^{1,2}(O)$. We say that v is a weak solution to $Lv = 0$ in O provided $v \in S_\lambda^{1,2}(O)$ and

$$\int_O \sum_{i,j} \hat{a}_{ij} X_i v X_j \phi dg = 0 \quad (5.5)$$

whenever $\phi \in S_{\lambda,0}^{1,2}(O)$. $u \in S_\lambda^{1,2}(O)$ is called a subsolution of L if (5.5) holds with $=$ replaced by $\leq$ for all $\phi \in S_{\lambda,0}^{1,2}(O)$ such that $\phi \geq 0$. u is called a supersolution if $-u$ is a subsolution.

We note that L is a special case of the more general linear sub-elliptic degenerate operators considered in [GN]. In particular, for the following results we refer to Lemma 2.3, Lemma 2.1 and Lemma 4.1 in [GN].

Lemma 5.1 (*Weak maximum principle*) *Let $\Omega \subset \mathbb{H}^n$ be a bounded domain. Let u and v be a supersolution and a subsolution to L in Ω , respectively. Assume that $\inf\{u - v, 0\} \in S_{\lambda,0}^{1,2}(\Omega)$. Then $u \geq v$ a.e. in Ω .*

Lemma 5.2 *Let $g_0 \in \mathbb{H}^n$, $0 < r$ and let λ be an $A_2(B(g_0, 4r))$ -weight with constant γ . Suppose that v is a positive weak solution to $Lv = 0$ in $B(g_0, r)$. Then there exists a constant $c = c(n, X, \gamma)$, $1 \leq c < \infty$ such that if $\tilde{g}_0 \in \mathbb{H}^n$, $0 < \tilde{r}$, $B(\tilde{g}_0, 2\tilde{r}) \subset B(g_0, r)$, then*

$$\begin{aligned} (i) \quad & c^{-1}\tilde{r}^2 \int_{B(\tilde{g}_0, \tilde{r}/2)} |Xv|^2 \lambda dx \leq c \left(\int_{B(\tilde{g}_0, \tilde{r})} \lambda dx \right) \left(\sup_{B(\tilde{g}_0, \tilde{r})} v \right)^2 \leq c \int_{B(\tilde{g}_0, 2\tilde{r})} |v|^2 \lambda dx, \\ (ii) \quad & \sup_{B(\tilde{g}_0, \tilde{r})} v \leq c \inf_{B(\tilde{g}_0, \tilde{r})} v. \end{aligned}$$

Furthermore, there exists $\alpha = \alpha(n, X, \gamma) \in (0, 1)$ such that if $g, g' \in B(\tilde{g}_0, \tilde{r})$ then

$$(iii) \quad |v(g) - v(g')| \leq c \left(\frac{d(g, g')}{\tilde{r}} \right)^\alpha \sup_{B(\tilde{g}_0, 2\tilde{r})} v.$$

Lemma 5.3 *Let $\Omega \subset \mathbb{H}^n$ be a domain, let $g_0 \in \partial\Omega$ and assume that Ω is a local NTA-domain at g_0 with constants M and r_0 . Let $0 < r \leq r_0/2$, $\tilde{g}_0 \in \partial\Omega \cap B(g_0, r_0/2)$, and assume that λ is a $A_2(B(\tilde{g}_0, 4r))$ -weight with constant γ . Suppose that v is a non-negative weak solution to $Lv = 0$ in $\Omega \cap B(\tilde{g}_0, 2r)$ and that $v = 0$ continuously on $\partial\Omega \cap B(\tilde{g}_0, 2r)$. Then there exists $c = c(n, M, X, \gamma)$, $1 \leq c < \infty$, such that the following holds with $\tilde{r} = r/c$.*

$$(i) \quad r^2 \int_{\Omega \cap B(\tilde{g}_0, r/2)} |Xv|^2 \lambda dx \leq c \int_{\Omega \cap B(\tilde{g}_0, r)} |v|^2 \lambda dx,$$

$$(ii) \quad \sup_{\Omega \cap B(\tilde{g}_0, \tilde{r})} v \leq cv(a_{\tilde{r}}(\tilde{g}_0)).$$

Furthermore, there exists $\alpha = \alpha(n, M, X, \gamma) \in (0, 1)$ such that if $g, g' \in \Omega \cap B(\tilde{g}_0, \tilde{r})$, then

$$(iii) \quad |v(g) - v(g')| \leq c \left(\frac{d(g, g')}{r} \right)^\alpha \sup_{\Omega \cap B(\tilde{g}_0, 2\tilde{r})} v.$$

Lemma 5.4 *Let $\Omega \subset \mathbb{H}^n$ be a domain, let $g_0 \in \partial\Omega$ and assume that Ω is a local NTA-domain at g_0 with constants M and r_0 . Let $0 < r \leq r_0/2$, $\tilde{g}_0 \in \partial\Omega \cap B(g_0, r_0/2)$, and assume that λ is a $A_2(B(\tilde{g}_0, r))$ -weight with constant γ . Suppose that v_1 and v_2 are two non-negative weak solutions to $Lv = 0$ in $\Omega \cap B(\tilde{g}_0, 2r)$ and $v_1 = 0 = v_2$ continuously on $\partial\Omega \cap B(\tilde{g}_0, 2r)$. Then there exist $c = c(n, M, X, \gamma)$, $1 \leq c < \infty$, and $\alpha = \alpha(n, M, X, \gamma) \in (0, 1)$ such that*

$$\left| \log \frac{v_1(g)}{v_2(g)} - \log \frac{v_1(g')}{v_2(g')} \right| \leq c \left(\frac{d(g, g')}{r} \right)^\alpha$$

whenever $g, g' \in \Omega \cap B(\tilde{g}_0, r/c)$.

Proof. This is essentially Theorem 5.2 in [GN] formulated in the setting of local NTA-domains. $\square$

5.1 Connection with p -harmonic functions

Let in the following $\Omega \subset \mathbb{H}^n$ be a local NTA-domain at $g_0 \in \partial\Omega$ with parameters $M \geq 2, r_0 > 0$. Let $\tilde{g}_0 \in \partial\Omega \cap B(g_0, r_0)$, $0 < r < r_0/2$, and suppose that $\hat{u}, \hat{v}$ are positive p -harmonic functions in $\Omega \cap B(\tilde{g}_0, 4r)$, continuous in $B(\tilde{g}_0, 4r)$ with $\hat{u} = \hat{v} = 0$ on $B(\tilde{g}_0, 4r) \setminus \Omega$.

Assumption 1. There exists c_0 , $1 \leq c_0 < \infty$, such if $\hat{r} = r/c_0$, then for some $\mu \geq 1$,

$$\mu^{-1} \frac{h(g)}{d(g, \partial\Omega)} \leq |Xh(g)| \leq \mu \frac{h(g)}{d(g, \partial\Omega)} \text{ whenever } g \in \Omega \cap B(\tilde{g}_0, 4\hat{r}), \ h \in \{\hat{u}, \hat{v}\}. \quad (5.6)$$

Repeating the outline in (2.11)-(2.16) we see that if we define

$$e(g) = \hat{u}(g) - \hat{v}(g) \text{ whenever } g \in \Omega \cap B(\tilde{g}_0, 4\hat{r}) \quad (5.7)$$

and if we put

$$u(g, \tau) = \tau \hat{u}(g) + (1 - \tau) \hat{v}(g) \text{ whenever } g \in \Omega \cap B(\tilde{g}_0, 4\hat{r}) \text{ and } \tau \in [0, 1], \quad (5.8)$$

then e is a weak solution to

$$Le := \sum_{i,j=1}^{2n} X_i(\hat{a}_{ij}(\cdot)X_j e) = 0 \text{ in } \Omega \cap B(\tilde{g}_0, 4\hat{r}), \quad (5.9)$$

where, whenever $g \in \Omega \cap B(\tilde{g}_0, 4\hat{r})$, $\hat{a}_{ij}(g)$ is defined as in (2.14). Again the ellipticity constant of L , at $g \in \Omega \cap B(\tilde{g}_0, 4\hat{r})$, can be estimated by

$$\min\{p-1, 1\}|\xi|^2 \hat{\lambda}(g) \leq \sum_{i,j=1}^{2n} \hat{a}_{ij}(g) \xi_i \xi_j \leq \max\{p-1, 1\}|\xi|^2 \hat{\lambda}(g), \quad (5.10)$$

whenever $\xi \in \mathbb{R}^{2n}$ and where

$$\hat{\lambda}(g) = \int_0^1 |Xu(g, \tau)|^{p-2} d\tau \approx \left(|X\hat{u}(g)| + |X\hat{v}(g)| \right)^{p-2}. \quad (5.11)$$

In (5.11) $\approx$ means that the constants of proportionality only depend on p, n .

Assumption 2. There exists $\tilde{c}_0 \geq 1$ such that if $r^* = \hat{r}/\tilde{c}_0$, then $(|X\hat{u}| + |X\hat{v}|)^{p-2}$ extends to an A_2 -weight in $B(\tilde{g}_0, 4r^*)$ with A_2 -constant γ .

Lemma 5.5 *Let $\Omega \subset \mathbb{H}^n$, g_0 , M , r_0 , $\tilde{g}_0$, r , p , $\hat{u}$, $\hat{v}$ be as above and assume Assumption 1 and Assumption 2. Assume in addition that $\hat{v} \leq \hat{u}$ and let r^* be as in Assumption 2. Then there exists $c \geq 1$, $c = c(p, n, M, c_0, \mu, \tilde{c}_0, \gamma)$ such that if $\tilde{r} = r^*/c$, then*

$$c^{-1} \frac{\hat{u}(a_{\tilde{r}}(\tilde{g}_0)) - \hat{v}(a_{\tilde{r}}(\tilde{g}_0))}{\hat{v}(a_{\tilde{r}}(\tilde{g}_0))} \leq \frac{\hat{u}(g) - \hat{v}(g)}{\hat{v}(g)} \leq c \frac{\hat{u}(a_{\tilde{r}}(\tilde{g}_0)) - \hat{v}(a_{\tilde{r}}(\tilde{g}_0))}{\hat{v}(a_{\tilde{r}}(\tilde{g}_0))}$$

whenever $g \in \Omega \cap B(\tilde{g}_0, \tilde{r})$.

Proof. We first prove the lefthand inequality in Lemma 5.5. To do so we show the existence of Λ , $1 \leq \Lambda < \infty$, and $\hat{c} \geq 1$, such that if $r' = r^*/\hat{c}$ and if

$$e(g) = \Lambda \left(\frac{\hat{u}(g) - \hat{v}(g)}{\hat{u}(a_{r^*}(\tilde{g}_0)) - \hat{v}(a_{r^*}(\tilde{g}_0))} \right) - \frac{\hat{v}(g)}{\hat{v}(a_{r^*}(\tilde{g}_0))} \quad (5.12)$$

for $g \in \Omega \cap B(\tilde{g}_0, r^*)$, then

$$e(g) \geq 0 \text{ whenever } g \in \Omega \cap B(\tilde{g}_0, 2r'). \quad (5.13)$$

To do this, we initially allow $\Lambda, \hat{c} \geq 1$ to vary in (5.12). $\Lambda, \hat{c}$, are then fixed near the end of the argument. Put

$$\begin{aligned} u'(g) &= \frac{\Lambda \hat{u}(g)}{\hat{u}(a_{r^*}(\tilde{g}_0)) - \hat{v}(a_{r^*}(\tilde{g}_0))}, \\ v'(g) &= \frac{\Lambda \hat{v}(g)}{\hat{u}(a_{r^*}(\tilde{g}_0)) - \hat{v}(a_{r^*}(\tilde{g}_0))} + \frac{\hat{v}(g)}{\hat{v}(a_{r^*}(\tilde{g}_0))}. \end{aligned}$$

Observe from (5.12) that $e = u' - v'$. Let $\tilde{L}$ be defined as in (5.9) using u', v' , instead of $\hat{u}, \hat{v}$, and let e_1, e_2 be the solutions to $Le_i = 0, i = 1, 2$, in $\Omega \cap B(\tilde{g}_0, r^*)$, with continuous boundary values

$$e_1(g) = \frac{\hat{u}(g) - \hat{v}(g)}{\hat{u}(a_{r^*}(\tilde{g}_0)) - \hat{v}(a_{r^*}(\tilde{g}_0))}, \quad e_2(g) = \frac{\hat{v}(g)}{\hat{v}(a_{r^*}(\tilde{g}_0))}, \quad (5.14)$$

whenever $g \in \partial(\Omega \cap B(\tilde{g}_0, r^*))$. Using Assumption 2 we see that Lemma 5.4 can be applied and we get, for some $c_+ \geq 1$ and $r_+ = r^*/c_+$, that

$$c_+^{-1} \frac{e_1(a_{r_+}(\tilde{g}_0))}{e_2(a_{r_+}(\tilde{g}_0))} \leq \frac{e_1(g)}{e_2(g)} \leq c_+ \frac{e_1(a_{r_+}(\tilde{g}_0))}{e_2(a_{r_+}(\tilde{g}_0))} \quad (5.15)$$

whenever $g \in \Omega \cap B(\tilde{g}_0, 2r_+)$. We now put

$$\hat{c} = c_+, \quad r' = r_+, \quad \text{and} \quad \Lambda = \hat{c} \frac{e_2(a_{r'}(\tilde{g}_0))}{e_1(a_{r'}(\tilde{g}_0))},$$

and observe from (5.15) that

$$\Lambda e_1(g) - e_2(g) \geq 0 \text{ whenever } g \in \Omega \cap B(\tilde{g}_0, 2r'). \quad (5.16)$$

Let $\hat{e} = \Lambda e_1 - e_2$ and note from linearity of $\tilde{L}$ that $\hat{e}, e$, both satisfy the same linear locally uniformly elliptic sub-elliptic pde in $\Omega \cap B(\tilde{g}_0, r^*)$ and also that these functions have the same continuous boundary values on $\partial(\Omega \cap B(\tilde{g}_0, r^*))$. Hence, using the maximum principle for the operator $\tilde{L}$, see Lemma 5.1, it follows that $e = \hat{e}$ and then by (5.16) that $e(y) \geq 0$ in $\Omega \cap B(\tilde{g}_0, 2r')$. To complete the proof of the left-hand inequality in Lemma 5.5 with $\tilde{r} = 2r'$ we prove that

$$\Lambda \leq c(p, n, M, c_0, \mu, \tilde{c}_0, \gamma). \quad (5.17)$$

In fact let L denote the operator corresponding to $\hat{u} - \hat{v}$ and defined as in (5.9). Then from the Harnack inequality in Lemma 5.2 (ii) for L , applied to $\hat{u} - \hat{v}$, and the definition of r' , we deduce the existence of $\zeta \in \Omega \cap \partial B(\tilde{g}_0, r^*)$ with $d(\zeta, \partial\Omega) \geq r/c$ and such that $e_1 \geq c^{-1}$ on $\partial B(\tilde{g}_0, r^*) \cap B(\zeta, d(\zeta, \partial\Omega)/4)$. Note that we here have used the fact that the gauge balls are NTA-domains (see Remark 3.2). Using this we get, essentially just using Lemma 5.3 (iii) and the Harnack inequality in Lemma 5.2 applied to the function e_1 , that $e_1(a_{r'}(\tilde{g}_0)) \geq \bar{c}^{-1}$. Also from Lemma 4.6 and the Harnack inequality applied to $\hat{v}$ we get $e_2(a_{r'}(\tilde{g}_0)) \leq \bar{c}$ for some $\bar{c} = \bar{c}(p, n, M, c_0, \mu, \tilde{c}_0, \gamma)$. Thus (5.17) is true and the proof of the left hand inequality in Lemma 5.5 is complete. To prove the right hand inequality in Lemma 5.5, one can proceed similarly and in this case one needs to prove, for e_1, e_2 as above, that $e_1(a_{r'}(\tilde{g}_0)) \leq \bar{c}$ and $e_2(a_{r'}(\tilde{g}_0)) \geq \bar{c}$. The proof of the second inequality follows, as above, essentially from Lemma 5.3 (iii) and the Harnack inequality in Lemma 5.2 applied to the function e_2 . The first inequality follows from Lemma 5.3 (iii), (ii) for L , applied to $\hat{u} - \hat{v}$, and the Harnack inequality. This completes the proof of Lemma 5.5. $\square$

Lemma 5.6 *Let $\Omega \subset \mathbb{H}^n$, $g_0, M, r_0, \tilde{g}_0, r, p, \hat{u}, \hat{v}$ be as above and assume Assumption 1 and Assumption 2. Let r^* be as in Assumption 2. There exist $c \geq 1$, $c = c(p, n, M, c_0, \mu, \tilde{c}_0, \gamma)$, and $\alpha = \alpha(p, n, M, c_0, \mu, \tilde{c}_0, \gamma)$, $\alpha \in (0, 1)$, such that if $\tilde{r} = r^*/c$, then*

$$\left| \log \frac{\hat{u}(g'_1)}{\hat{v}(g'_1)} - \log \frac{\hat{u}(g'_2)}{\hat{v}(g'_2)} \right| \leq c \left(\frac{d(g'_1, g'_2)}{r} \right)^\alpha$$

whenever $g'_1, g'_2 \in \Omega \cap B(\tilde{g}_0, \tilde{r})$.

Proof. Let $\tilde{u}, \tilde{v}$ be the p -harmonic functions in $\Omega \cap B(\tilde{g}_0, 2r^*)$ with

$$\tilde{u} = \sup\{\hat{u}, \hat{v}\} \text{ and } \tilde{v} = \inf\{\hat{u}, \hat{v}\} \text{ on } \partial[\Omega \cap B(\tilde{g}_0, 2r^*)].$$

From the maximum principle for p -harmonic functions we have $\tilde{u} \geq \tilde{v}$ and hence we can apply Lemma 5.5 to conclude that

$$c^{-1} \frac{\tilde{u}(a_{\tilde{r}}(\tilde{g}_0))}{\tilde{v}(a_{\tilde{r}}(\tilde{g}_0))} \leq \frac{\tilde{u}(g)}{\tilde{v}(g)} \leq c \frac{\tilde{u}(a_{\tilde{r}}(\tilde{g}_0))}{\tilde{v}(a_{\tilde{r}}(\tilde{g}_0))}$$

whenever $g \in \Omega \cap B(\tilde{g}_0, \tilde{r})$. Moreover, using the definition of $\tilde{u}, \tilde{v}$, and the inequalities in the last display we can conclude that

$$\frac{\hat{u}(g_1)}{\hat{v}(g_1)} \leq c \frac{\hat{u}(g_2)}{\hat{v}(g_2)} \text{ whenever } g_1, g_2 \in \Omega \cap B(\tilde{g}_0, \tilde{r}). \quad (5.18)$$

Next if $\hat{g}_0 \in \partial\Omega \cap B(\tilde{g}_0, \tilde{r}/8)$, then we let

$$M(\rho) = \sup_{B(\hat{g}_0, \rho)} \frac{\hat{u}}{\hat{v}} \text{ and } m(\rho) = \inf_{B(\hat{g}_0, \rho)} \frac{\hat{u}}{\hat{v}}$$

when $0 < \rho < \tilde{r}$. If ρ is fixed we can apply Lemma 5.5 with $\hat{u} = \hat{u}, \hat{v} = m(\rho)\hat{v}$, and $2r$ replaced by ρ to conclude the existence of c^*, c_* , such that if $\tilde{\rho} = \rho/c^*$, then

$$M(\tilde{\rho}) - m(\rho) \leq c_*(m(\tilde{\rho}) - m(\rho)). \quad (5.19)$$

Likewise, we can apply Lemma 5.5 with $\hat{u} = M(\rho)\hat{v}$ and $\hat{v} = \hat{u}$ to conclude

$$(M(\rho)\hat{v} - \hat{u})/\hat{u} \approx \text{constant on } \Omega \cap B(\tilde{g}_0, \tilde{\rho}).$$

Using this inequality together with (5.18) it follows that

$$(M(\rho)\hat{v} - \hat{u})/\hat{v} \approx \text{constant on } \Omega \cap B(\tilde{g}_0, \tilde{\rho}).$$

Thus if c_* is large enough, then

$$M(\rho) - m(\tilde{\rho}) \leq c_*(M(\rho) - M(\tilde{\rho})). \quad (5.20)$$

If $\text{osc}(t) := M(t) - m(t)$, then we can add (5.19) and (5.20) and deduce that

$$\text{osc}(\tilde{\rho}) \leq \frac{c_* - 1}{c_* + 1} \text{osc}(\rho). \quad (5.21)$$

We can now use (5.21), since c^* is independent of ρ , in an iterative argument. Doing this we can conclude that

$$\text{osc}(s) \leq c(s/t)^\theta \text{osc}(t) \text{ whenever } 0 < s < t \leq r/2 \quad (5.22)$$

for some $\theta > 0, c \geq 1$. (5.22), (5.18), along with arbitrariness of $\hat{g}_0 \in \partial\Omega \cap B(\tilde{g}_0, \tilde{r}/8)$ and the interior Hölder continuity-Harnack inequalities in Lemma 5.2 applied to $\hat{u}, \hat{v}$, are easily seen to imply Lemma 5.6. $\square$

6 Proof of the main results in the case of a hyperplane in the set $\mathcal{V}$

The purpose is to prove Theorem 1 and, in particular, Corollary 1.1 in the case of a hyperplane in the set $\mathcal{V}$. To do this we first introduce certain cubes defined relative to a given hyperplane. The notation introduced will be used extensively in the forthcoming sections. To start with we note that if $\nu \in \mathcal{V}$, then $\nu = (\omega, 0)$ for some $\omega \in \mathbb{S}^n$ and hence we can identify ν with ω . In the following we identify $g \in \mathbb{H}^n$ with $(z, t) = (z_1, \dots, z_{2n}, t) \in \mathbb{R}^{2n} \times \mathbb{R}$ and we let e_1 be the unit vector in $\mathbb{R}^{2n}$ which points in the z_1 -coordinate direction. Given $r_1, r_2 > 0$ we introduce the sets

$$\begin{aligned} Q_{r_1, r_2} &= \{(z, t) : |z_1| < r_1, |z_i| < r_2, i \in \{2, \dots, 2n\}, |t| < r_2^2\}, \\ Q_{r_1, r_2}^+ &= Q_{r_1, r_2} \cap \{(z, t) : 0 < z_1 < r_1\}, \\ Q_{r_1, r_2}^- &= Q_{r_1, r_2} \cap \{(z, t) : -r_1 < z_1 < 0\}. \end{aligned} \quad (6.1)$$

Let $\omega \in \mathbb{S}^{2n}$ and let $R^\omega : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be a mapping which maps e_1 onto ω . We define

$$\begin{aligned} Q_{r_1, r_2}^\omega &= \{(R^\omega z, t) : (z, t) \in Q_{r_1, r_2}\}, \\ Q_{r_1, r_2}^{\omega, +} &= \{(R^\omega z, t) : (z, t) \in Q_{r_1, r_2}^+\}, \\ Q_{r_1, r_2}^{\omega, -} &= \{(R^\omega z, t) : (z, t) \in Q_{r_1, r_2}^-\}. \end{aligned} \quad (6.2)$$

Given $g_0 \in \mathbb{H}^n$ we let $A(g_0) = \{g_0 \circ g : g \in A\}$ for $A \in \{Q_{r_1, r_2}^\omega, Q_{r_1, r_2}^{\omega, +}, Q_{r_1, r_2}^{\omega, -}\}$. Furthermore, if $r_1 = r_2 = r$ then we write $Q_r^\omega(g_0)$, $Q_r^{\omega, +}(g_0)$, $Q_r^{\omega, -}(g_0)$, respectively, for $Q_{r, r}^\omega(g_0)$, $Q_{r, r}^{\omega, +}(g_0)$, $Q_{r, r}^{\omega, -}(g_0)$. We first note the following two lemmas.

Lemma 6.1 *Let $\nu = (\omega, 0) \in \mathcal{V}$, $r_1, r_2 > 0$, $g_0 \in \mathbb{H}^n$ and p , $1 < p < \infty$. Then the domains $Q_{r_1, r_2}^\omega(g_0)$, $Q_{r_1, r_2}^{\omega, +}(g_0)$ and $Q_{r_1, r_2}^{\omega, -}(g_0)$ are regular for the Dirichlet problem for the sub-elliptic p -Laplace operator L_p .*

Proof. Note that the exterior of any halfspace has uniform positive density in the sense of (4.2). The conclusion of Lemma 6.1 follows immediately from this observation and the Wiener criterium proved in [Da]. $\square$

Lemma 6.2 *Let $\nu = (\omega, 0) \in \mathcal{V}$, $r_1, r_2 > 0$ and consider $Q_{r_1, r_2}^{\omega, +}$. There exists $M = M(n) \geq 2$ such that if $Mr_0 \leq \min\{r_1, r_2\}/8$ then $Q_{r_1, r_2}^{\omega, +}$ is a local NTA-domain at 0 with parameters M , r_0 .*

Proof. This is an immediate consequence of Lemma 3.8 and its proof. $\square$

Given $\nu = (\omega, 0) \in \mathcal{V}$ we let $v^\omega(z, t) = \langle z, \omega \rangle$ and we introduce coordinates with respect to ω . In particular, given $g = (z, t)$ we write $z = (z', z'')$ where $z'' = \langle z, \omega \rangle$ and $z' = z - z''$. Furthermore, given $g_0 \in \mathbb{H}^n$ we let $a_r^\omega(g_0) = g_0 \circ \delta_r(a_1^\omega)$, $r > 0$, where $a_1^\omega = (z', z'', t)$, $z' = 0$, $z'' = 1/(4M)$ with M as in Lemma 6.2 and $t = -\frac{1}{2}\langle z, \omega \rangle \langle z^\perp, \omega \rangle$. By this construction we see, using (3.10) and (3.13), that

$$d(a_r^\omega(g_0), g_0) = d(a_r^\omega(g_0), \Pi^\nu(g_0)) = r/(4M). \quad (6.3)$$

Lemma 6.3 *Let $\nu = (\omega, 0) \in \mathcal{V}$, $g_0 \in \mathbb{H}^n$, $r > 0$, p , $1 < p < \infty$, be given. Assume that u , v are non-negative p -harmonic functions in $Q_r^{\omega,+}(g_0)$, continuous on the closure of $Q_r^{\omega,+}(g_0)$, $u = 0 = v$ on $\partial Q_r^{\omega,+}(g_0) \cap \Pi^\nu(g_0)$. Then there exist $c_1 = c_1(p, n)$, $1 \leq c_1 < \infty$, such that*

$$c_1^{-1} \frac{u(a_r^\omega(g_0))}{v(a_r^\omega(g_0))} \leq \frac{u(g)}{v(g)} \leq c_1 \frac{u(a_r^\omega(g_0))}{v(a_r^\omega(g_0))}$$

whenever $g \in Q_{r/(4M)}^{\omega,+}(g_0)$.

Proof. By invariance with respect to left translations, see Remark 3.6, and with respect to dilations we see that we can assume, without loss of generality, that $r = 1$ and $g_0 = 0$. In particular, we assume that u , v are non-negative p -harmonic functions in $Q_1^{\omega,+}$, continuous on the closure of $Q_1^{\omega,+}$, and with $u = 0 = v$ on $\partial Q_1^{\omega,+} \cap \{(z, t) \in \mathbb{H}^n : \langle z, \omega \rangle = 0\}$. We note that it is enough to prove the lemma with $v = v^\omega$ and that we can assume, without loss of generality, that $u(a_1^\omega) = 1$. Hence our goal is to prove that there exist $c_1 = c_1(p, n)$, $1 \leq c_1 < \infty$, such that

$$c_1^{-1} \leq \frac{u(g)}{v^\omega(g)} \leq c_1 \quad (6.4)$$

whenever $g \in Q_{1/(4M)}^{\omega,+}$. The proof is based on barrier type estimates. For $z = (z', z'', t) \in Q_{1/(4M)}^{\omega,+}$ with $z'' \leq 1/(100M)$ we let

$$\hat{z} = (z', 1/(8M)) \text{ and } \hat{t} = t - \frac{1}{2} \langle \hat{z}, \omega \rangle \langle \hat{z}^\perp, \omega \rangle. \quad (6.5)$$

Then, using the Harnack inequality for p -harmonic functions, and the fact that $Q_1^{\omega,+}$ is a local NTA-domain at 0 with constant M , we see that

$$u(\hat{z}, \hat{t}) \approx u(0, 1/(8M), 0). \quad (6.6)$$

Using this observation we let $\hat{u}$ be p -harmonic in $D = B((\hat{z}, \hat{t}), 1/(8M)) \setminus \bar{B}((\hat{z}, \hat{t}), 1/(100M))$ with continuous boundary values $\hat{u} = u(0, 1/(8M), 0)$ on $\partial B((\hat{z}, \hat{t}), 1/(100M))$ while $\hat{u} = 0$ on $\partial B((\hat{z}, \hat{t}), 1/(8M))$. Note that the any gauge ball is regular for the Dirichlet problem for the operator L_p . Note that $\partial B((\hat{z}, \hat{t}), 1/(8M)) \cap \Pi^\nu(0) = (z', 0, t)$ as we see from (3.10), (3.11) and Remark 3.5. Let

$$\tilde{u}(g) = a_p \Gamma_p(g, (\hat{z}, \hat{t})) + b_p \quad (6.7)$$

whenever $g \in D$ where Γ_p is the fundamental solution defined in Lemma 4.3 and where we choose a_p and b_p so that $\tilde{u}(g) = u(0, 1/(8M), 0)$ if $g \in \partial B((\hat{z}, \hat{t}), 1/(100M))$, $\tilde{u}(g) = 0$ if $g \in \partial B((\hat{z}, \hat{t}), 1/(8M))$. By the comparison principle we see that $\hat{u} = \tilde{u}$ on D . We claim that there exists $c = c(n, p)$ such that

$$c\hat{u}(g) = c\tilde{u}(g) \geq u(0, 1/(8M), 0)z'' \quad (6.8)$$

whenever $g = (z', z'', t) \in D$ with $z'' \leq 1/(100M)$. To prove (6.9) assume $p \neq Q$. Now we see, by simply applying the mean value theorem to the function $\rho^{\frac{p-Q}{p-1}}$, that

$$\frac{\tilde{u}(g)}{u(0, 1/(8M), 0)} \geq cd(g, (z', 0, t)) = cd_e((z', z''), (z', 0)) = cz'' \quad (6.9)$$

whenever $g = (z', z'', t) \in D$ with $z'' \leq 1/(100M)$. The case $p = Q$ can be handled similarly. Next, using the maximum principle and (6.6) we see that $\hat{u} \leq cu$ in D for some constant $c = c(p, n)$. Using these facts we can conclude that

$$c^{-1}u(0, 1/(8M), 0) \leq \frac{u(g)}{z''} \text{ whenever } g = (z', z'', t) \in Q_{1/(4M)}^{\omega, +}, \quad z'' \leq 1/(100M), \quad (6.10)$$

for some constant $c = c(p, n)$. Then, using the Harnack inequality for p -harmonic functions we see that (6.10) in fact holds whenever $g = (z', z'', t) \in Q_{1/(4M)}^{\omega, +}$. Using that $u(a_1^\omega) = 1$ we can also conclude that $u(0, 1/(8M), 0) \approx 1$ by the Harnack inequality. This proves the left hand side inequality in (6.4). We next prove the inequality in the opposite direction. Using the notation introduced above we consider $z = (z', z'', t) \in Q_{1/(4M)}^{\omega, +}$ with $z'' \leq 1/(100M)$ and in this case we let

$$\hat{z} = (z', -1/(8M)) \text{ and } \hat{t} = t - \frac{1}{2}\langle \hat{z}, \omega \rangle \langle \hat{z}^\perp, \omega \rangle. \quad (6.11)$$

Furthermore, we let $\hat{u}$ be p -harmonic in $D = B((\hat{z}, \hat{t}), 1/(4M)) \setminus \bar{B}((\hat{z}, \hat{t}), 1/(8M))$ with continuous boundary values $\hat{u} = u(0, 1/(8M), 0)$ on $\partial B((\hat{z}, \hat{t}), 1/(4M))$ while $\hat{u} = 0$ on $\partial B((\hat{z}, \hat{t}), 1/(8M))$. Using Lemma 4.6 and the Harnack inequality we then see that

$$u(g) \leq c\hat{u}(g) \text{ whenever } g = (z', z'', t) \in D \cap Q_{1/(4M)}^{\omega, +}, \quad z'' \leq 1/(100M), \quad (6.12)$$

As above we then deduce that $u(g) \leq u(0, 1/(8M), 0)z''$ whenever $g = (z', z'', t) \in D \cap Q_{1/(4M)}^{\omega, +}$, $z'' \leq 1/(100M)$. An application of the Harnack inequality then completes the proof of (6.4). $\square$

Lemma 6.4 *Let $\nu = (\omega, 0) \in \mathcal{V}$, $g_0 \in \mathbb{H}^n$, $r > 0$, $p, 1 < p < \infty$, be given. Assume that $\hat{u}, \hat{v}$ are non-negative p -harmonic functions in $Q_r^{\omega, +}(g_0)$, continuous on the closure of $Q_r^{\omega, +}(g_0)$, $\hat{u} = 0 = \hat{v}$ on $\partial Q_r^{\omega, +}(g_0) \cap \Pi^\nu(g_0)$ and that $\hat{v} \leq \hat{u}$. Furthermore, assume that there exist $\bar{\lambda} = \bar{\lambda}(p, n)$, and $\hat{c} = \hat{c}(p, n)$, such that*

$$\bar{\lambda}^{-1} \frac{\tilde{u}(g)}{d(g, \Pi^\nu(g_0))} \leq |X\tilde{u}(g)| \leq \bar{\lambda} \frac{\tilde{u}(g)}{d(g, \Pi^\nu(g_0))} \text{ for } \tilde{u} \in \{\hat{u}, \hat{v}\} \quad (6.13)$$

whenever $g \in Q_{r/\hat{c}}^{\omega, +}(g_0)$. Then there exist $c \geq 1$, $\hat{c}_2 \geq 1$ and $\alpha \in (0, 1)$, all depending only on p, n and $\bar{\lambda}$, such that if $\tilde{r} = r/(\hat{c}\hat{c})$, then

$$\left| \log \frac{\hat{u}(g'_1)}{\hat{v}(g'_1)} - \log \frac{\hat{u}(g'_2)}{\hat{v}(g'_2)} \right| \leq \hat{c}_2 \left(\frac{d(g'_1, g'_2)}{r} \right)^\alpha$$

whenever $g'_1, g'_2 \in Q_{\tilde{r}}^{\omega, +}(g_0)$.

Proof. We can again assume that $g_0 = 0$ and that $r = 1$. To prove this we see that we can essentially directly use Lemma 5.6 once Assumption 2 is verified. In the following we can assume, without loss of generality, that $\hat{u}(a_1^\omega(g_0)) = 1$. Assume first that $p \geq 2$. Then, using (6.13) and Lemma 6.3 we immediately see that $(|X\hat{u}| + |X\hat{v}|)^{p-2} \approx 1$ in $Q_{1/\hat{c}}^{\omega, +}(0)$. In particular, $(|X\hat{u}| + |X\hat{v}|)^{p-2}$ easily extends to an A_2 -weight on $Q_{1/\hat{c}}^{\omega, +}(0)$ and we can control the A_2 -constant γ . Hence Lemma 6.4 follows from Lemma 5.6. Note that in this case we also have the conclusion of Lemma 5.6 by simply quoting the corresponding result in [CG] concerning uniformly elliptic sub-elliptic pdes in divergence form. The case $1 < p < 2$ is treated similarly. $\square$

Lemma 6.5 *Let $\nu = (\omega, 0) \in \mathcal{V}$, $g_0 \in \mathbb{H}^n$, $r > 0$, p , $2 \leq p < \infty$, be given. Assume that u , v are non-negative p -harmonic functions in $Q_r^{\omega,+}(g_0)$, continuous on the closure of $Q_r^{\omega,+}(g_0)$, $u = 0 = v$ on $\partial Q_r^{\omega,+}(g_0) \cap \Pi^\nu(g_0)$. Then there exist $c_2 = c_2(p, n)$, $1 \leq c_2 < \infty$, and $\alpha = \alpha(p, n)$, $\alpha \in (0, 1)$, such that*

$$\left| \log \frac{u(g'_1)}{v(g'_1)} - \log \frac{u(g'_2)}{v(g'_2)} \right| \leq c_2 \left(\frac{d(g'_1, g'_2)}{r} \right)^\alpha$$

whenever $g'_1, g'_2 \in Q_{r/c_2}^{\omega,+}(g_0)$.

Proof. As in the proof of Lemma 6.3 we can assume that $r = 1$, $g_0 = 0$, $v = v^\omega$, and that $u(a_1^\omega) = 1$. Using Lemma 6.3,

$$c_1^{-1} \leq \frac{u(g)}{v^\omega(g)} \leq c_1 \quad (6.14)$$

whenever $g \in Q_{1/(4M)}^{\omega,+}$. Hence, if we let $\hat{u} = 2c_1 u$ then

$$v^\omega(g) \leq \frac{1}{2} \hat{u}(g) \leq c_1^2 v^\omega(g) \quad (6.15)$$

whenever $g \in Q_{1/(4M)}^{\omega,+}$. We now define $u(\cdot, \tau)$, $0 \leq \tau \leq 1$, to be p -harmonic in $Q_{1/(4M)}^{\omega,+}$ and such that, for $0 \leq \tau \leq 1$,

$$u(g, \tau) = \tau \hat{u}(g) + (1 - \tau) v^\omega(g), \text{ for } g \in \partial Q_{1/(4M)}^{\omega,+}. \quad (6.16)$$

Recall that $Q_{1/(4M)}^{\omega,+}$ is regular for the Dirichlet problem, see Lemma 6.1. Using (6.15) we observe that

$$c^{-1} u(\cdot, \tau_1) \leq \frac{u(\cdot, \tau_2) - u(\cdot, \tau_1)}{\tau_2 - \tau_1} = \hat{u} - v^\omega \leq c u(\cdot, \tau_1) \quad (6.17)$$

on $\partial(Q_{1/(4M)}^{\omega,+})$ whenever $\tau_1, \tau_2 \in [0, 1]$. From the boundary maximum principle for p -harmonic functions this inequality also holds in $Q_{1/(4M)}^{\omega,+}$. Let $\epsilon_0 = \tilde{\epsilon}$ where $\tilde{\epsilon}$ is as in Lemma 4.9. We then note that there exists ϵ'_0 , $0 < \epsilon'_0 \leq \epsilon_0$, with the same dependence as ϵ_0 , such that if $|\tau_2 - \tau_1| \leq \epsilon'_0$, then

$$1 - \epsilon_0/2 \leq \frac{u(\cdot, \tau_2)}{u(\cdot, \tau_1)} \leq 1 + \epsilon_0/2 \text{ in } Q_{1/(4M)}^{\omega,+}. \quad (6.18)$$

Let $\xi_1 = 0 < \xi_2 < \dots < \xi_m = 1$ and consider $[0, 1]$ as divided into $\{[\xi_k, \xi_{k+1}]\}$, $1 \leq k \leq m - 1$. We assume that all of these intervals have a length of $\epsilon'_0/2$ with the possible exception of the interval containing $\xi_m = 1$ which is of length $\leq \epsilon'_0/2$. We let $r_1 = 1/(16M)$ and we note that it follows, from our choice of ϵ'_0 and the fact that $u(\cdot, \xi_1) = v^\omega$, that we can apply Lemma 4.9 to conclude that also $u(\cdot, \xi_2)$ verifies (6.13). Hence, we can use Lemma 6.4 to conclude that

$$\begin{aligned} \left| \log \left(\frac{u(g'_1, \xi_2)}{v^\omega(g'_1)} \right) - \log \left(\frac{u(g'_2, \xi_2)}{v^\omega(g'_2)} \right) \right| &= \left| \log \left(\frac{u(g'_1, \xi_2)}{u(g'_1, \xi_1)} \right) - \log \left(\frac{u(g'_2, \xi_2)}{u(g'_2, \xi_1)} \right) \right| \\ &\leq \hat{c}_2 \left(\frac{d(g'_1, g'_2)}{r_1} \right)^\alpha \end{aligned} \quad (6.19)$$

whenever $g'_1, g'_2 \in Q_{r_2}^{\omega,+}$ and where we have introduced $r_2 = r_1/\hat{c}$. We can now continue by induction as in the proof of Theorem 2 in [LN1] to complete the proof. We here supply the details

for the convenience of the reader. Indeed, assume that we have shown, for some $2 \leq k \leq m$, that

$$\left| \log \left(\frac{u(g'_1, \xi_k)}{v^\omega(g'_1)} \right) - \log \left(\frac{u(g'_2, \xi_k)}{v^\omega(g'_2)} \right) \right| \leq (k-1) \hat{c}_2 \left(\frac{d(g'_1, g'_2)}{r_{k-1}} \right)^\alpha \quad (6.20)$$

whenever $g'_1, g'_2 \in Q_{r_k}^{\omega,+}$. Here $r_k \leq r_{k-1} \leq \hat{c} r_k$ for some $\hat{c} = \hat{c}(p, n) > 1$. If $k = m$ we quit but assuming $k < m$, we choose, using the induction hypothesis in (6.20), $\hat{\sigma}$ and $r'_k \leq r_k$ so that

$$\left| \frac{u(g'_1, \xi_k)}{v^\omega(g'_1)} - \frac{u(g'_2, \xi_k)}{v^\omega(g'_2)} \right| \leq \hat{\sigma} \frac{u(g'_1, \xi_k)}{u(g'_1)}$$

whenever $g'_1, g'_2 \in Q_{r'_k}^{\omega,+}$. We fix a $g'_1 \in Q_{r'_k}^{\omega,+}$ and choose $\hat{\sigma} > 0$ small enough to ensure that if $\tau \in [\xi_k, \xi_{k+1}]$, $g'_2 \in Q_{r'_k}^{\omega,+}$, then

$$(1 - \epsilon_0) \frac{u(g'_1, \xi_k)}{v^\omega(g'_1)} \leq \frac{u(g'_2, \tau)}{v^\omega(g'_2)} \leq (1 + \epsilon_0) \frac{u(g'_1, \xi_k)}{v^\omega(g'_1)}. \quad (6.21)$$

To estimate the magnitude of $\hat{\sigma}$ we observe that if $\tau \in [\xi_k, \xi_{k+1}]$ then

$$\frac{u(g'_2, \tau)}{v^\omega(g'_2)} = \frac{u(g'_2, \xi_{k+1})}{u(g'_2, \xi_k)} \cdot \frac{u(g'_2, \xi_k)}{v^\omega(g'_2)} \leq (1 + \epsilon_0/2)(1 + \hat{\sigma}) \frac{u(g'_1, \xi_k)}{v^\omega(g'_1)}.$$

Thus if $\hat{\sigma} = \epsilon_0/4$ and if ϵ_0 is small enough then the right hand inequality in (6.21) is valid. A similar argument gives the left hand inequality in (6.21) when $\hat{\sigma} = \epsilon_0/4$ and ϵ_0 is small enough. Also since α is independent of k , $k \leq 2/\epsilon'_0$, and $\epsilon'_0 = \epsilon'_0(p, n)$, we deduce from (6.20) that one can take $r'_k = r_k/\tilde{c}$ for $\tilde{c} = \tilde{c}(p, n)$ large enough. From (6.21) we find that we can now apply Lemma 4.9 to conclude the validity of (6.13) for the pair $\{u(\cdot, \xi_k), u(\cdot, \xi_{k+1})\}$ and then that Lemma 6.4 applies to the pair $\{u(\cdot, \xi_k), u(\cdot, \xi_{k+1})\}$. In particular, as

$$\begin{aligned} \left| \log \left(\frac{u(g'_1, \xi_{k+1})}{v^\omega(g'_1)} \right) - \log \left(\frac{u(g'_2, \xi_{k+1})}{v^\omega(g'_2)} \right) \right| &\leq \left| \log \left(\frac{u(g'_1, \xi_{k+1})}{u(g'_1, \xi_k)} \right) - \log \left(\frac{u(g'_2, \xi_{k+1})}{u(g'_2, \xi_k)} \right) \right| \\ &\quad + \left| \log \left(\frac{u(g'_1, \xi_k)}{v^\omega(g'_1)} \right) - \log \left(\frac{u(g'_2, \xi_k)}{v^\omega(g'_2)} \right) \right| \end{aligned}$$

we can, defining $r_{k+1} = r'_k/\tilde{c}_2 = r_k/(\tilde{c}_2\tilde{c})$, make use of the induction hypothesis to conclude that if $g'_1, g'_2 \in Q_{r_{k+1}}^{\omega,+}$, then

$$\begin{aligned} \left| \log \left(\frac{u(g'_1, \xi_{k+1})}{v^\omega(g'_1)} \right) - \log \left(\frac{u(g'_2, \xi_{k+1})}{v^\omega(g'_2)} \right) \right| &\leq \hat{c}_2 \left(\frac{d(g'_1, g'_2)}{r_k} \right)^\alpha + (k-1) \hat{c}_2 \left(\frac{d(g'_1, g'_2)}{r_{k-1}} \right)^\alpha \\ &\leq k \hat{c}_2 \left(\frac{d(g'_1, g'_2)}{r_k} \right)^\alpha. \end{aligned} \quad (6.22)$$

From (6.22) and induction we conclude that (6.20) is valid for all $k \in \{2, \dots, m\}$. In particular, applying (6.20) with $k = m$ we see that Lemma 6.5 is valid. $\square$

Remark 6.6 *Note that in the case of the Euclidean p -Laplace operator, and hence for non-negative p -harmonic functions, $1 < p < \infty$, the proof of Lemma 6.5 is quite straight forward. Indeed, in this case one first notes that it is, after a possible rotation, enough to let $v = x_{2n}$ and one then establishes a version of Lemma 6.3. To get from Lemma 6.3 to actually proving that u/x_{2n} is Hölder up to the boundary $x_{2n} = 0$ one can then use Schwartz reflection and Hölder estimates for the gradient of p -harmonic functions. In $\mathbb{H}^n$ it is not clear to us that there is such an elementary approach and hence we have given a proof of Lemma 6.5 which may, down the road, seem a bit too involved.*

7 Estimates in domains well approximated by hyperplanes in the set $\mathcal{V}$

7.1 Non-degeneracy of $|Xu|$

Lemma 7.1 *Let $\nu = (\omega, 0) \in \mathcal{V}$, $g_0 \in \mathbb{H}^n$, $r > 0$, p , $2 \leq p < \infty$, be given. Assume that u is a non-negative p -harmonic functions in $Q_r^{\omega,+}(g_0)$, continuous on the closure of $Q_r^{\omega,+}(g_0)$, $u = 0$ on $\partial Q_r^{\omega,+}(g_0) \cap \Pi^\nu(g_0)$. Then there exist $\hat{c} = \hat{c}(p, n)$ and $\bar{\lambda} = \bar{\lambda}(p, n)$, such that*

$$\bar{\lambda}^{-1} \frac{u(g)}{d(g, \Pi^\nu(g_0))} \leq |Xu(g)| \leq \bar{\lambda} \frac{u(g)}{d(g, \Pi^\nu(g_0))} \text{ whenever } g \in Q_{r/\hat{c}}^{\omega,+}(g_0).$$

Proof. Note that this is in fact proved as part of the proof of Lemma 6.5. However, we here give a direct proof based on the conclusion of Lemma 6.5. We can assume, without loss of generality, that $r = 1$ and $g_0 = 0$. Furthermore, note that if $g = (z, t) \in H^\nu(0)$ then $d(g, \Pi^\nu(0)) = \langle z, \omega \rangle$. Let $v^\omega(z, t) = \langle z, \omega \rangle$. Applying Lemma 6.5 to the functions u, v^ω we see that

$$\left| \log \left(\frac{v^\omega(g'_1)}{u(g'_1)} \right) - \log \left(\frac{v^\omega(g'_2)}{u(g'_2)} \right) \right| \leq c_2 d(g'_1, g'_2)^\alpha \quad (7.1)$$

whenever $g'_1, g'_2 \in Q_{1/c_2}^{\omega,+}$. Exponentiation of this inequality yields the equivalent inequality

$$\left| \frac{v^\omega(g'_1)}{u(g'_1)} - \frac{v^\omega(g'_2)}{u(g'_2)} \right| \leq c' \frac{v^\omega(g'_2)}{u(g'_2)} d(g'_1, g'_2)^\alpha \quad (7.2)$$

whenever $g'_1, g'_2 \in Q_{1/c_2}^{\omega,+}(0)$. Let $O = Q_{1/c_2}^{\omega,+}$ and note, since $|Xv^\omega|(z, t) \equiv |\omega| = 1$, that if $g'_2 \in Q_{1/(2c_2)}^{\omega,+}$ then, obviously,

$$\frac{1}{\tilde{a}} \frac{v^\omega(g'_2)}{d(g'_2, \partial O)} \leq |Xv^\omega(g'_2)| \leq \tilde{a} \frac{v^\omega(g'_2)}{d(g'_2, \partial O)} \quad (7.3)$$

for some $\tilde{a} = \tilde{a}(n)$. Let ρ be defined through the relation $c'\rho^\alpha = \frac{1}{2} \min\{\tilde{\epsilon}, d(g'_2, \partial O)\}$ where $\tilde{\epsilon}$ is as in Lemma 4.9. Using (7.2) we then see that

$$(1 - \tilde{\epsilon}/2) \frac{v^\omega(g'_2)}{u(g'_2)} \leq \frac{v^\omega(g'_1)}{u(g'_1)} \leq (1 + \tilde{\epsilon}/2) \frac{v^\omega(g'_2)}{u(g'_2)} \quad (7.4)$$

whenever $g'_1 \in B(g'_2, \rho)$. From (7.3), (7.4), and Lemma 4.9 we conclude that Lemma 7.1 is true. $\square$

Lemma 7.2 *Let $\Omega \subset \mathbb{H}^n$ be a domain, $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Assume that $\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{V}$. Let p , $2 \leq p < \infty$, $g \in \partial\Omega \cap B(g_0, r_0/2)$, $\tilde{r}_0 = r_0/(8M)$, $0 < r < \tilde{r}_0$. Suppose that u is a positive p -harmonic function in $\Omega \cap B(g, 4r)$, that u is continuous in $\bar{\Omega} \cap B(g, 4r)$ and $u = 0$ on $\partial\Omega \cap B(g, 4r)$. Then there exist $\hat{\delta} = \hat{\delta}(p, n)$, $\hat{c} = \hat{c}(p, n)$ and $\bar{\lambda} = \bar{\lambda}(p, n)$, such that if $0 < \delta \leq \hat{\delta}$, then*

$$\bar{\lambda}^{-1} \frac{u(g')}{d(g', \partial\Omega)} \leq |Xu(g')| \leq \bar{\lambda} \frac{u(g')}{d(g', \partial\Omega)} \text{ whenever } g' \in \Omega \cap B(g, r/\hat{c}).$$

Proof. Recall that Ω is a local NTA-domain at g . Suppose that u is a positive p -harmonic functions in $\Omega \cap B(g, 4r)$, that u is continuous in $\bar{\Omega} \cap B(g, 4r)$ and $u = 0$ on $\partial\Omega \cap B(g, 4r)$. Let $u \equiv 0$ in $B(g, 4r) \setminus \Omega$. Then u is continuous in $B(g, 4r)$. Let $c_1 = \hat{c}$ be as in Lemma 7.1 and choose $c' \geq 100c_1$ so that if $\hat{g} \in \Omega \cap B(g, r/c')$, $s = 4c_1 d(\hat{g}, \partial\Omega)$, and $g' \in \partial\Omega$ with $d(\hat{g}, g') = d(\hat{g}, \partial\Omega)$, then

$$\sup_{B(g', 4s)} u \leq cu(\hat{g}) \quad (7.5)$$

for some $c = c(p, n)$. That this can be achieved follows from Lemma 4.6 and the Harnack inequality. Using Definition 2.2 with g, r replaced by $g', 4s$, we see that there exists $\nu = (\omega, 0) \in \mathcal{V}$ and a verticle hyperplane $\Pi^\nu = \Pi^\nu(g')$ such that

$$D(\partial\Omega \cap B(g', 4s), \Pi^\nu \cap B(g', 4s)) \leq 4\delta s. \quad (7.6)$$

For the moment we allow $\hat{\delta}$ in Lemma 7.2 to vary but shall later fix it as a number satisfying several conditions. Using (2.7) we deduce that

$$\{\tilde{g} \in \Omega \cap B(g', 4s) : d(\tilde{g}, \partial\Omega) \geq 8\delta s\} \subset \text{one component of } \mathbb{H}^n \setminus \Pi^\nu. \quad (7.7)$$

To proceed we let $\tilde{s}$ be the largest $\tilde{s}$ such that $Q_{4\tilde{s}}^\omega(g') \subset B(g', 4s)$. Then, (7.6), (7.7) imply that

$$D(\partial\Omega \cap Q_{4\tilde{s}}^\omega(g'), \Pi^\nu \cap Q_{4\tilde{s}}^\omega(g')) \leq 4c(n)\delta\tilde{s}, \quad (7.8)$$

and

$$\{\tilde{g} \in \Omega \cap Q_{4\tilde{s}}^\omega(g') : d(\tilde{g}, \partial\Omega) \geq 8c(n)\delta\tilde{s}\} \subset \text{one component of } \mathbb{H}^n \setminus \Pi^\nu. \quad (7.9)$$

We next note that we may, without loss of generality, assume that $g' = 0$ and hence that $\Pi^\nu = \{(z, t) \in \mathbb{H}^n : \langle z, \omega \rangle = 0\}$. In particular, we can, without loss of generality, assume that

$$\{\tilde{g} \in \Omega \cap Q_{4\tilde{s}}^\omega : d(\tilde{g}, \partial\Omega) \geq 8c(n)\delta\tilde{s}\} \subset \{(z, t) \in \mathbb{H}^n : \langle z, \omega \rangle > 0\}. \quad (7.10)$$

We next introduce an auxiliary function in the domain $Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{20c(n)\delta\tilde{s}, 4\tilde{s}}^{\omega,+}}$. In particular, let v be a p -harmonic function in $Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{20c(n)\delta\tilde{s}, 4\tilde{s}}^{\omega,+}}$ with continuous boundary values on $\partial(Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{20c(n)\delta\tilde{s}, 4\tilde{s}}^{\omega,+}})$ and such that $v \leq u$ on $\partial(Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{20c(n)\delta\tilde{s}, 4\tilde{s}}^{\omega,+}})$. Moreover, we choose v so that

$$\begin{aligned} v(\tilde{g}) &= u(\tilde{g}) \text{ whenever } \tilde{g} \in \partial(Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{20c(n)\delta\tilde{s}, 4\tilde{s}}^{\omega,+}}) \setminus \overline{Q_{40c(n)\delta\tilde{s}, 4\tilde{s}}^{\omega,+}}, \\ v(\tilde{g}) &= 0 \text{ whenever } \tilde{g} \in \partial(Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{20c(n)\delta\tilde{s}, 4\tilde{s}}^{\omega,+}}) \cap \overline{Q_{30c(n)\delta\tilde{s}, 4\tilde{s}}^{\omega,+}}. \end{aligned}$$

Using (7.5), Harnack's inequality and Lemma 4.5, Lemma 4.6, we see that $u \leq v + c\delta^{\tilde{\sigma}}u(\hat{g})$ on $\partial(Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{20c(n)\delta\tilde{s}, 4\tilde{s}}^{\omega,+}})$. In particular, again using the maximum principle for p -harmonic functions we conclude that

$$u \leq v + c\delta^{\tilde{\sigma}}u(\hat{g}) \text{ in } Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{20c(n)\delta\tilde{s}, 4\tilde{s}}^{\omega,+}}. \quad (7.11)$$

Thus, using (7.11) and (7.5) we see that

$$1 \leq \frac{u(g)}{v(g)} \leq (1 - c\delta^{\tilde{\sigma}})^{-1} \text{ whenever } g \in Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{20c(n)\delta\tilde{s}, 4\tilde{s}}^{\omega,+}}. \quad (7.12)$$

Using Lemma 7.1 and the construction we now note that we also have

$$\hat{\lambda}^{-1} \frac{v(\hat{g})}{d(\hat{g}, \partial\Omega)} \leq |Xv(\hat{g})| \leq \hat{\lambda} \frac{v(\hat{g})}{d(\hat{g}, \partial\Omega)} \quad (7.13)$$

for some $\hat{\lambda} = \hat{\lambda}(p, n)$. In particular, from (7.12), (7.13) we see that for $0 < \delta < \hat{\delta}$, and $\hat{\delta} = \hat{\delta}(p, n)$ suitably small, the hypotheses of Lemma 4.9 are satisfied with $O = Q_{4\hat{s}}^{\omega,+} \setminus \overline{Q_{20c(n)\delta\hat{s},4\hat{s}}^{\omega,+}}$ and $\tilde{a} = \hat{\lambda}$. We now fix $\hat{\delta}$ and from Lemma 4.9 we conclude, for some $\bar{\lambda} = \bar{\lambda}(p, n)$, that

$$\bar{\lambda}_1^{-1} \frac{u(\hat{g})}{d(\hat{g}, \partial\Omega)} \leq |Xu(\hat{g})| \leq \bar{\lambda}_1 \frac{u(\hat{g})}{d(\hat{g}, \partial\Omega)}$$

Since $\hat{g} \in \Omega \cap B(g, r/c')$ is arbitrary, the proof of Lemma 7.2 is complete. $\square$

Lemma 7.3 *Let $\Omega \subset \mathbb{H}^n$ be a domain, $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Assume that $\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{V}$. Let p , $1 < p < \infty$, $g \in \partial\Omega \cap B(g_0, r_0/2)$, $\tilde{r}_0 = r_0/(8M)$, $0 < r < \tilde{r}_0$. Suppose that u is a positive p -harmonic function in $\Omega \cap B(g, 4r)$, that u is continuous in $\bar{\Omega} \cap B(g, 4r)$ and $u = 0$ on $\partial\Omega \cap B(g, 4r)$. Then there exist, for $\epsilon > 0$ given, $\hat{\delta} = \hat{\delta}(p, n, \epsilon) > 0$ and $c = c(p, n, \epsilon)$, $1 \leq c < \infty$, such that*

$$c^{-1} \left(\frac{\hat{r}}{r} \right)^{1+\epsilon} \leq \frac{u(a_{\hat{r}}(g))}{u(a_r(g))} \leq c \left(\frac{\hat{r}}{r} \right)^{1-\epsilon}$$

whenever $0 < \delta \leq \hat{\delta}$ and $0 < \hat{r} < r/4$.

Proof. Again we extend u to $B(g, 4r) \setminus \Omega$ by putting $u \equiv 0$ in this set and note that u is now continuous in $B(g, 4r)$. Moreover, by familiar arguments we can assume that $r = 4$, $g = 0$, and that

$$D(\partial\Omega \cap B(0, 4), \Pi^\nu(0) \cap B(0, 4)) \leq 4\hat{\delta},$$

for some $\nu = (\omega, 0) \in \mathcal{V}$. We again let $\tilde{s}$ be the largest $\tilde{s}$ such that $Q_{4\tilde{s}}^\omega(0) \subset B(0, 4)$ and we note that

$$D(\partial\Omega \cap Q_{4\tilde{s}}^\omega(0), \Pi^\omega(0) \cap Q_{4\tilde{s}}^\omega(0)) \leq 4c(n)\hat{\delta}.$$

Furthermore, if $\bar{\delta} = 4c(n)\hat{\delta}$ is small enough, then we may assume, as in (7.9), that

$$Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{2\bar{\delta}\tilde{s},4\tilde{s}}^{\omega,+}} \subset \Omega, \quad Q_{4\tilde{s}}^{\omega,-} \setminus \overline{Q_{2\bar{\delta}\tilde{s},4\tilde{s}}^{\omega,-}} \subset \mathbb{H}^n \setminus \Omega. \quad (7.14)$$

Finally we assume, as we may, that $u(a_1(0)) = 1$. Thus, to prove Lemma 7.3 it suffices to show that

$$c^{-1}\hat{r}^{1+\epsilon} \leq u(a_{\hat{r}}(0)) \leq c\hat{r}^{1-\epsilon} \text{ whenever } 0 < \hat{r} < 1. \quad (7.15)$$

To prove (7.15) we introduce two auxiliary functions u^+ and u^- . In particular, we define u^+ to be p -harmonic in $Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{2\bar{\delta}\tilde{s},4\tilde{s}}^{\omega,+}}$ with boundary values on $\partial(Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{2\bar{\delta}\tilde{s},4\tilde{s}}^{\omega,+}})$ defined as follows,

$$\begin{aligned} u^+(\tilde{g}) &= u(\tilde{g}) & \text{if } \tilde{g} \in \partial Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{4\bar{\delta}\tilde{s},4\tilde{s}}^{\omega,+}}, \\ u^+(\tilde{g}) &= \frac{(\tilde{g}_n - 2\hat{\delta})}{2\hat{\delta}} u(\tilde{g}) & \text{if } \tilde{g} \in \partial Q_{4\tilde{s}}^{\omega,+} \cap (\overline{Q_{4\bar{\delta}\tilde{s},4\tilde{s}}^{\omega,+}} \setminus \overline{Q_{2\bar{\delta}\tilde{s},4\tilde{s}}^{\omega,+}}), \\ u^+(\tilde{g}) &= 0 & \text{if } \tilde{g} \in \partial Q_{4\tilde{s}}^{\omega,+} \setminus \overline{Q_{4\bar{\delta}\tilde{s},4\tilde{s}}^{\omega,+}}. \end{aligned}$$

Similarly, u^- is the p -harmonic function in $Q_{4\bar{s}}^{\omega,+} \setminus \overline{Q_{2\bar{s},4\bar{s}}^{\omega,-}}$ with $u^- = u$ on $\partial(Q_{4\bar{s}}^{\omega,+} \setminus \overline{Q_{2\bar{s},4\bar{s}}^{\omega,-}})$. Observe from the maximum principle for p -harmonic functions and (7.14) that $u^+ \leq u \leq u^-$. We also define

$$\begin{aligned} v^+(g) &= v^+(z, t) := \langle z, \omega \rangle - 2\bar{\delta}\tilde{s}, \\ v^-(g) &= v^-(z, t) := \langle z, \omega \rangle + 2\bar{\delta}\tilde{s}, \end{aligned} \quad (7.16)$$

whenever $g \in \mathbb{H}^n$. Hence v^+ and v^- are p -harmonic functions which grow linearly in the direction defined by ω . We first focus on the right hand inequality in (7.15). Using the normalization of u stated above (7.15), Lemma 4.5, the maximum principle for p -harmonic functions, and Lemma 6.2 applied to the functions u^- , v^- in $Q_{4\bar{s}}^{\omega,+} \setminus \overline{Q_{2\bar{s},4\bar{s}}^{\omega,-}}$, we see for $\bar{\delta}$ small enough that

$$u(\tilde{g}) \leq u^-(\tilde{g}) \leq cv^-(\tilde{g}) = c(\langle \tilde{z}, \omega \rangle + 2\bar{\delta}\tilde{s}) \quad (7.17)$$

whenever $\tilde{g} = (\tilde{z}, \tilde{t}) \in \Omega \cap B(0, 1/(4c_1))$. Thus, $u(a_{\bar{\delta}}(0)) \leq \hat{c}\bar{\delta}$. Suppose by way of induction that we have shown,

$$u(a_{\bar{\delta}^k}(0)) \leq (\hat{c}\bar{\delta})^k \quad (7.18)$$

where $\hat{c}$ depends only on p, n . Using the flatness assumption we see there exists a plane $\Pi^{\hat{\nu}} = \Pi^{\hat{\nu}}(0)$, for some $\hat{\nu} = (\hat{\omega}, 0) \in \mathcal{V}$, such that

$$D(\partial\Omega \cap B(0, 4\bar{\delta}^k), \Pi^{\hat{\nu}} \cap B(0, 4\bar{\delta}^k)) \leq 4\hat{\delta}\bar{\delta}^k.$$

Repeating the above argument, with Π^{ν} replaced by $\Pi^{\hat{\nu}}$ and with 4 replaced by $4\bar{\delta}^k$, we find that

$$u(a_{\bar{\delta}^{k+1}}(0)) \leq \hat{c}\bar{\delta} u(a_{\bar{\delta}^k}(0)) \leq (\hat{c}\bar{\delta})^{k+1}.$$

Thus by induction the above inequality is true for all positive integers k . Let $\bar{\delta}^{-\epsilon} = \hat{c}$ where $\hat{c}$ is the constant in the above display. Given $0 < \hat{r} < 1$, let k be the smallest integer such that $\bar{\delta}^k \leq \hat{r}$. Then from (7.18), our choice of $\bar{\delta}$, and elementary estimates we see that $u(a_{\hat{r}}(0)) \leq c\hat{r}^{1-\epsilon}$, for some $c = c(p, n, \epsilon)$. Thus the proof of Lemma 7.3 is complete in one direction. To finish the proof we focus on the left hand inequality in (7.15). Indeed, applying Lemma 6.3 to the functions u^+ , v^+ in $Q_{4\bar{s}}^{\omega,+} \setminus \overline{Q_{2\bar{s},4\bar{s}}^{\omega,-}}$ and using the Harnack inequality we see, provided $\bar{\delta}$ is small enough, that

$$\frac{u^+(a_{32\bar{s}\bar{\delta}}(0))}{v^+(a_{32\bar{s}\bar{\delta}}(0))} \approx \frac{u^+(a_{\bar{s}/(16M)}(0))}{v^+(a_{\bar{s}/(16M)}(0))} \approx \frac{u^+(a_{\bar{s}/(16M)}(0))}{\eta - \bar{\delta}}, \quad (7.19)$$

where M is the constant in Lemma 6.2 and $\eta = \eta(n, p) \in (0, 1)$. Using that $u^+(a_{\bar{s}/(16M)}(0)) \geq c^{-1}$ as well as the maximum principle we see that (7.19) implies

$$u^+(a_{32\bar{s}\bar{\delta}}(0)) \geq c^{-1}v^+(a_{32\bar{s}\bar{\delta}}(0)) \geq \bar{c}^{-1}\bar{\delta}.$$

for some $\bar{c} = c(p, n)$, $1 \leq \bar{c} < \infty$. Using this inequality and arguing as in the proof of the right hand inequality in (7.15) we see that

$$u(a_{(32\bar{s}\bar{\delta})^k}(0)) \geq (\bar{c}^{-1}\bar{\delta})^k. \quad (7.20)$$

Let $\bar{\delta}$ be so small that $\bar{c}^{-1}\bar{\delta} \geq (32\bar{\delta})^{1+\epsilon}$ and assume that $\hat{r} \in [(32\bar{s}\bar{\delta})^{k+1}, (32\bar{s}\bar{\delta})^k]$. With $\bar{\delta}(p, n, \epsilon)$ now fixed, it follows from Harnack's inequality for p -harmonic functions that

$$u(a_{\hat{r}}(0)) \geq c^{-1}u(a_{(32\bar{s}\bar{\delta})^k}(0)) \geq c^{-1}(32\bar{\delta})^{k(1+\epsilon)} \geq c^{-1}\hat{r}^{(1+\epsilon)}. \quad (7.21)$$

for some $c = c(p, n, \epsilon)$. (7.21) completes the proof of (7.15) and of Lemma 7.3. $\square$

7.2 Local extension of $|Xu|^{p-2}$ to an A_2 -weight

Lemma 7.4 *Let $\Omega \subset \mathbb{H}^n$ be a domain, $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Assume that $\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$, with respect to hyperplanes defined by the set $\mathcal{V}$. Let p , $2 \leq p < \infty$, $g \in \partial\Omega \cap B(g_0, r_0/2)$, $\tilde{r}_0 = r_0/(8M)$, $0 < r < \tilde{r}_0$. Suppose that u is a positive p -harmonic functions in $\Omega \cap B(g, 2r)$, that u is continuous in $\bar{\Omega} \cap B(g, 2r)$ and $u = 0$ on $\partial\Omega \cap B(g, 2r)$. There exist $\delta' = \delta'(p, n)$, and $c = c(p, n) \geq 1$ such that if $0 < \delta < \delta'$, and $\hat{r} = r/c$, then $|Xu|^{p-2}$ extends to an $A_2(B(w, \hat{r}))$ -weight with constant depending only on p, n .*

Proof. Let in the following $\hat{r} \ll r$ be a radius to be determined. Following [FS] and [Je3] we see that there exists a constant $\beta \geq 1$, and a set of balls $B_j = B(g_j, r_j)$, $j = 1, 2, \dots$, such that

$$\begin{aligned} (i) \quad & (\mathbb{H}^n \setminus \bar{\Omega}) \cap B(g, 8\hat{r}) \subset \cup_j B_j, \\ (ii) \quad & (\mathbb{H}^n \setminus \bar{\Omega}) \cap B(g, 8\hat{r}) \cap B_j \neq \emptyset \text{ for all } j, \\ (iii) \quad & B(g_i, r_i/4) \cap B(g_j, r_j/4) = \emptyset \text{ for all } i \neq j, \\ (iv) \quad & r_j = 10^{-3}d(B_j, \partial\Omega), \\ (v) \quad & \sum_j \chi_{4B_j}(x) \leq \beta \chi_{(\mathbb{H}^n \setminus \bar{\Omega})}(x) \text{ if } x \in (\mathbb{H}^n \setminus \bar{\Omega}) \cap B(g, 8\hat{r}). \end{aligned} \quad (7.22)$$

In (7.22) χ_E is the indicator function for the set E . Furthermore, we let $\{\eta_j\}$ be a partition of unit associated to $\{B_j\} = \{B(g_j, r_j)\}$. Recall that if δ' is small enough, then Ω is a local NTA-domain at g with constant $M = M(n)$, see Lemma 3.8. Thus there exists a constant $\tilde{c} = \tilde{c}(n)$, $1 \leq \tilde{c} < \infty$, such that if $B_j \cap B(g, r) \neq \emptyset$, then there is a $g'_j \in \Omega \cap B(g, \tilde{c}r)$ for which $d(g'_j, \partial\Omega) \sim d(g'_j, g_j) \sim d(g_j, \partial\Omega)$. Here $A \sim B$ means that A/B is bounded from above and below by constants which only depend on n . Let $\hat{r} = r/\max\{\tilde{c}, c\}$ where $c = c(p, n)$, $1 \leq c < \infty$, is large enough to ensure that if $B_j \cap B(g, 8\hat{r}) \neq \emptyset$, then

$$c_*^{-1} \frac{u(g')}{d(g', \partial\Omega)} \leq |Xu(g')| \leq c_* \frac{u(g')}{d(g', \partial\Omega)} \text{ whenever } g' \in B(g'_j, d(g'_j, \partial\Omega)/2), \quad (7.23)$$

and where $c_* = \bar{\lambda}$ is the constant in Lemma 7.2. In particular, using Lemma 7.2 we see that such a $c = c(p, n)$ exists provided δ' is small enough. Next we define $\lambda(g') = |Xu(g')|^{p-2}$ whenever $g' \in \Omega \cap B(g, 4\hat{r})$ and we let Γ be the set of all j such that if $j \in \Gamma$ then $B_j \cap B(g, 4\hat{r}) \neq \emptyset$. Moreover, if $j \in \Gamma$ then we first choose $g'_j \in \Omega \cap B(g, \tilde{c}r)$ as above and we then define $\lambda(g')$, for $g' \in B(g, 4\hat{r}) \setminus \bar{\Omega}$, through the relation

$$\lambda(g') = \sum_j \eta_j(g') \lambda(g'_j), \quad g' \in B(g, 4\hat{r}) \setminus \bar{\Omega}.$$

This defines λ almost everywhere on $B(g, 4\hat{r})$ with respect to the relevant Lebesgue measure, since for δ small enough, $\partial\Omega \cap B(g, r)$ has zero such measure. Using (7.23) and the Harnack inequality for p -harmonic functions we see that

$$\lambda(g') = \lambda(g'_j) \approx \lambda(z) \text{ whenever } g' \in B_j \text{ and } z \in B(g'_j, d(g'_j, \partial\Omega)/2). \quad (7.24)$$

To complete the proof of Lemma 7.4 we prove that the inequality in (5.1) holds for this choice of λ and for all balls $B(\tilde{g}, \tilde{r})$ such that $\tilde{g} \in B(g, \hat{r})$, $\tilde{r} \leq \hat{r}$, with $\gamma = \gamma(p, n)$. To do this we consider

several cases. If $\tilde{r} < d(\tilde{g}, \partial\Omega)/2$, then (5.1) is true as we see using (7.23), (7.24) and the Harnack inequality. On the other hand, if $\tilde{r} \geq d(\tilde{g}, \partial\Omega)/2$ then we choose $\hat{g} \in \partial\Omega$ with $d(\tilde{g}, \partial\Omega) = d(\tilde{g}, \hat{g})$ and we note that, in this case, $B(\tilde{g}, \tilde{r}) \subset B(\hat{g}, 3\tilde{r}) \subset B(g, 8\tilde{r})$. From Hölder's inequality, Lemma 4.5, Lemma 4.6, and Harnack's inequality we see that

$$\begin{aligned} \int_{B(\tilde{g}, \tilde{r})} \lambda dg' &\leq \int_{B(\hat{g}, 3\tilde{r})} \lambda dg' \leq c \int_{\Omega \cap B(\hat{g}, c^* \tilde{r})} |Xu|^{p-2} dg' \\ &\leq c \left(\int_{\Omega \cap B(\hat{g}, c^* \tilde{r})} |Xu|^p dg' \right)^{(1-2/p)} \tilde{r}^{2Q/p} \leq cu(a_{\tilde{r}}(\hat{g}))^{p-2} \tilde{r}^{Q+2-p}. \end{aligned} \quad (7.25)$$

Let $\eta = \min\{1, |p-2|^{-1}\}/20$. To estimate the part of (5.1) involving λ^{-1} we observe from Lemma 7.3 and Harnack's inequality, that if $g' \in \Omega \cap B(\hat{g}, c^* \tilde{r})$, and δ' is small enough, then

$$cu(g') \geq u(a_{\tilde{r}}(\hat{g})) \left(\frac{d(g', \partial\Omega)}{\tilde{r}} \right)^{1+\eta}. \quad (7.26)$$

Therefore, using (7.23) and (7.26) in a deduction similar to (7.25) we see that

$$\int_{B(\tilde{g}, \tilde{r})} \lambda^{-1} dg' \leq c \tilde{r}^{(1+\eta)(p-2)} u(a_{\tilde{r}}(\hat{g}))^{2-p} \int_{\Omega \cap B(\hat{g}, c^* \tilde{r})} d(g', \partial\Omega)^{-\eta(p-2)} dg'. \quad (7.27)$$

To complete the estimate in (7.27) we need to estimate the integral involving the distance function. To do this we define

$$I(z, s) = \int_{\Omega \cap B(z, s)} d(g', \partial\Omega)^{-\eta(p-2)} dg'$$

whenever $z \in \partial\Omega \cap B(g, r)$, $0 < s < r$. Let

$$E_k = \Omega \cap B(z, s) \cap \{g' : d(g', \partial\Omega) \leq \delta^k s\} \text{ for } k = 1, 2, \dots$$

Then since $\partial\Omega$ is δ -flat we deduce that

$$\int_{E_k} dg' \leq c_+^k \delta^k s^Q \quad (7.28)$$

where $c_+ = c_+(p, n)$. Indeed, from δ -flatness it is easily seen that this statement holds for E_1 . Moreover, E_1 can be covered by at most c/δ^{Q-1} balls of radius $100\delta s$ with centers in $\partial\Omega \cap B(z, s)$. We can then repeat the argument in each ball to get that (7.28) holds for E_2 . Continuing in this way we get (7.28) for all positive integers k . Using (7.28) and writing $I(z, s)$ as a sum over $E_k \setminus E_{k+1}$, $k = 1, 2, \dots$ we get

$$I(z, s) \leq cs^{Q-\eta(p-2)} + \delta^{\eta(p-2)-Q} \sum_{k=1}^{\infty} (c_+^k \delta^k s)^{Q-\eta(p-2)} < \tilde{c}s^{Q-\eta(p-2)},$$

where $\tilde{c} = \tilde{c}(p, n)$, provided δ' is small enough. Using this estimate with $s = \tilde{r}$, we can continue our calculation in (7.27) and conclude that

$$\int_{B(\tilde{g}, \tilde{r})} \lambda^{-1} dg' \leq c \tilde{r}^{Q+p-2} u(a_{\tilde{r}}(\hat{g}))^{2-p}. \quad (7.29)$$

To complete the proof of Lemma 7.4, in the case $p > 2$, we simply combine (7.25) and (7.29). Note that the case $p = 2$ is trivial. $\square$

8 Proof of Theorem 1 and Corollary 1.1

To prove Theorem 1 and Corollary 1.1 we see, based on Lemma 5.5 and Lemma 5.6, that we only have to verify Assumption 1 and Assumption 2 used in these lemmas for u and v . Indeed, let $\Omega \subset \mathbb{H}^n$, r_0 and g_0 be as in Theorem 1. Given p , $2 \leq p < \infty$, $g \in \partial\Omega \cap B(g_0, r_0/2)$, $0 < r < r_0/16$, suppose that u, v are positive p -harmonic functions in $\Omega \cap B(g, 4r)$, that u, v are continuous in $\bar{\Omega} \cap B(g, 4r)$, $u = 0 = v$ on $\partial\Omega \cap B(g, 4r)$ and that $v \leq u$ in $\Omega \cap B(g, 4r)$. Using Lemma 7.2 we see that there exist $\hat{\delta} = \hat{\delta}(p, n)$, $\hat{c} = \hat{c}(p, n)$ and $\bar{\lambda} = \bar{\lambda}(p, n)$, such that if $0 < \delta \leq \hat{\delta}$, then

$$\bar{\lambda}^{-1} \frac{h(g')}{d(g', \partial\Omega)} \leq |Xh(g')| \leq \bar{\lambda} \frac{h(g')}{d(g', \partial\Omega)} \text{ whenever } g' \in \Omega \cap B(g, r/\hat{c}), h \in \{u, v\}.$$

In particular, Assumption 1 holds with $\tilde{g}_0$, $\hat{r} = r/c_0$, $\hat{u}$, $\hat{v}$, replaced by g , $\hat{r} = r/\hat{c}$, u , v provided we choose δ small enough. Furthermore, by the same argument as used in the proof of Lemma 7.4 we see that there exist $\delta' = \delta'(p, n)$, and $c = c(p, n) \geq 1$ such that if $0 < \delta < \delta'$, and $r^* = \hat{r}/c$, then $(|Xu| + |Xv|)^{p-2}$ extends to an $A_2(B(w, 4r^*))$ -weight with constant depending only on p, n . In particular, Assumption 2 holds with this choice of r^* and with $\tilde{g}_0$, $\hat{u}$, $\hat{v}$, replaced by g , u , v . In particular, Theorem 1 and Corollary 1.1 now follows from Lemma 5.5 and Lemma 5.6. $\square$

9 Remarks and discussion

Let $\Omega \subset \mathbb{H}^n$ be a domain. Let $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. In this paper we have developed a theory concerning the boundary behaviour of non-negative solution to the sub-elliptic p -Laplace equation under the assumption that $\Omega \cap B(g_0, r_0)$, $\partial\Omega \cap B(g_0, r_0)$ are (δ, r_0) -flat in $\mathbb{H}^n$ with respect to hyperplanes defined by the set $\mathcal{S} = \mathcal{V}$. $\mathcal{V}$ is the set of all non-characteristic or verticle hyperplanes in the Heisenberg group. Naturally one can attempt to consider domains which are flat in the sense of approximation with different sets $\mathcal{S}$. To discuss this further let us return to the optimization problem in (3.7). Below (3.7) we pursued the case $c = 0$ which is equivalent with approximation by non-characteristic hyperplanes. We here consider the case $c \neq 0$, i.e., the case when the hyperplanes contain characteristic points. Recall that we consider hyperplanes defined by the direction $\nu = (\omega, c)$. In the case $c \neq 0$ we see that

$$|z - z_0|^2(z - z_0) = -4 \left(\frac{1}{2} \langle z_0^\perp, z \rangle - \frac{1}{c} \langle \omega, z \rangle - t_0 \right) (z_0^\perp - \frac{2}{c} \omega). \quad (9.1)$$

Hence

$$z - z_0 = \gamma \bar{\nu}, \quad \bar{\nu} = (z_0^\perp - \frac{2}{c} \omega) \text{ for some } \gamma. \quad (9.2)$$

Using this we see that

$$\gamma^3 |\bar{\nu}|^2 + 2\gamma |\bar{\nu}|^2 + 2 \langle z_0, \bar{\nu} \rangle - 4t_0 = 0. \quad (9.3)$$

Assume that $|\bar{\nu}| = 0$. Then $z = z_0$. Assume $|\bar{\nu}| \neq 0$ and let

$$A = 4|\bar{\nu}|^{-2} \left(\frac{1}{2} \langle z_0^\perp, \bar{\nu} \rangle - \frac{1}{c} \langle \omega, \bar{\nu} \rangle \right), \quad B = -4|\bar{\nu}|^{-2} t_0. \quad (9.4)$$

Then

$$\gamma^3 + A\gamma + B = 0. \quad (9.5)$$

Note that

$$A = 2|\bar{\nu}|^{-2} \left(\langle z_0^\perp, z_0^\perp \rangle - \frac{4}{c} \langle z_0^\perp, \omega \rangle + \frac{4}{c^2} \langle \omega, \omega \rangle \right) = 2|\bar{\nu}|^{-2} |z_0^\perp - \frac{2}{c} \omega|^2 = 2. \quad (9.6)$$

Furthermore, $-4A^3 - 27B^2 < 0$. This implies that the equation has one real root. We note that the solutions are given as

$$\gamma = u - \frac{2}{3u}, \quad u = \left(-\frac{B}{2} + \sqrt{\frac{B^2}{4} + \frac{8}{27}} \right)^{1/3} = \left(\frac{2t_0}{|\bar{\nu}|^2} + \sqrt{\left(\left(\frac{2t_0}{|\bar{\nu}|^2} \right)^2 + \frac{8}{27} \right)} \right)^{1/3}. \quad (9.7)$$

Hence,

$$\begin{aligned} z &= z_0 + \gamma \bar{\nu}, \quad \bar{\nu} = (z_0^\perp - \frac{2}{c} \omega), \\ t &= -\frac{1}{c} \left(\langle z_0, \omega \rangle + \gamma \langle z_0^\perp, \omega \rangle - \frac{2\gamma}{c} |\omega|^2 \right). \end{aligned} \quad (9.8)$$

Consider now the special case when $\omega' = (0, 0)$ and $c = 1$. Then (9.8) reduces to $t = 0$ and $z = z_0 + \gamma z_0^\perp$. In this case we see that

$$\begin{aligned} d(g_0, g) &= \left(\gamma^4 |z_0|^4 + 16 \left(-t_0 + \frac{\gamma}{2} |z_0|^2 \right)^2 \right)^{1/4}, \\ \gamma &= u - \frac{2}{3u}, \quad u = \left(\frac{2t_0}{|z_0|^2} + \sqrt{\left(\left(\frac{2t_0}{|z_0|^2} \right)^2 + \frac{8}{27} \right)} \right)^{1/3}. \end{aligned} \quad (9.9)$$

Let $\eta = 2t_0/|z_0|^2$. Then

$$d(g_0, g) = |z_0| \gamma (1 + \gamma^2)^{1/4}, \quad \gamma = u - \frac{2}{3u}, \quad u = \left(\eta + \sqrt{\left(\eta^2 + \frac{8}{27} \right)} \right)^{1/3}. \quad (9.10)$$

A simple analysis shows that $\gamma = \gamma(\eta)$ is non-decreasing as a function of η , that $\gamma(0) = 0$, and that $\gamma \rightarrow \infty$ as $\eta \rightarrow \infty$. Furthermore,

$$\frac{|z_0| d(g_0, g)}{2t_0} = \frac{\gamma}{\eta} (1 + \gamma^2)^{1/4} := h(\eta). \quad (9.11)$$

By studying the function $h(\eta)$ one can prove that there exists η_0 such that $h(\eta)$ is non-decreasing on $(0, \eta_0]$. In particular, $h(\eta)$ exists as $\eta \rightarrow 0$ and we denote the limit by $h(0)$. Hence,

$$h(0) \leq \frac{|z_0| d(g_0, g)}{2t_0} \leq h(\eta_0) \text{ whenever } 0 \leq 2t_0/|z_0|^2 \leq \eta_0. \quad (9.12)$$

In particular,

$$\frac{2h(0)}{|z_0|} t_0 \leq d((z_0, t_0), \Pi^{(0,0,1)}) \leq \frac{2h(\eta_0)}{|z_0|} t_0 \text{ whenever } 0 \leq t_0 \leq \eta_0 |z_0|^2 / 2. \quad (9.13)$$

Definition 9.1 Given $\eta_1, \eta_2 \in (0, \infty)$, $\eta_1 < \eta_2$, $g = (z, t) \in \mathbb{H}^n$, we say that $\nu = (a, b, c) \in \mathbb{S}^{2n+1}$ is a (η_1, η_2) -admissible characteristic direction at g if $c \neq 0$ and

$$\eta_1 \leq |z + c^{-1}(a, b)^\perp| \leq \eta_2. \quad (9.14)$$

If $\nu = (a, b, c)$ is a (η_1, η_2) -admissible characteristic direction at g then $\Pi^\nu(g)$ is called a (η_1, η_2) -admissible characteristic hyperplane at g . All (η_1, η_2) -admissible characteristic directions at g are denote by $A_{\eta_1, \eta_2}(g)$.

Recall that if we consider $\Pi^\nu(g)$, $\nu = (a, b, c) \in \mathbb{S}^{2n+1}$, $c \neq 0$, then the point $\hat{g} = (\hat{z}, \hat{t}) \in \Pi$ with $\hat{z} = (-2b/c, 2a/c)$ is characteristic. A straightforward calculation shows that

$$|z + c^{-1}(a, b)^\perp| = d(g, \hat{g}). \quad (9.15)$$

I.e., $|z + c^{-1}(a, b)^\perp|$ is the distance from g to the characteristic point $\hat{g}$ and through Definition 9.1 we impose a restriction on this distance.

Using admissible hyperplanes the following notions of flatness can be introduced.

Definition 9.2 Let $\Omega \subset \mathbb{H}^n$ be a domain, $\eta_1, \eta_2 \in (0, \infty)$, $\eta_1 < \eta_2$. Let $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Then $\partial\Omega \cap B(g_0, r_0)$ is said to be uniformly (δ, r_0) -approximable by (η_1, η_2) -admissible characteristic hyperplanes the Heisenberg group, in the sense of the gauge Hausdorff distance, provided there exists, whenever $g \in \partial\Omega \cap B(g_0, r_0)$ and $0 < r < r_0$, a hyperplane $\Pi = \Pi^\nu = \Pi^\nu(g, r)$, with $\nu \in A_{\eta_1, \eta_2}(g)$, such that

$$D(\partial\Omega \cap B(g, r), \Pi \cap B(g, r)) \leq \delta r.$$

Definition 9.3 Let $\Omega \subset \mathbb{H}^n$ be a domain, $\eta_1, \eta_2 \in (0, \infty)$, $\eta_1 < \eta_2$. Let $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Then $\Omega \cap B(g_0, r_0)$ and $\partial\Omega \cap B(g_0, r_0)$ are said to be (δ, r_0) -flat in $\mathbb{H}^n$, with respect to (η_1, η_2) -admissible characteristic hyperplanes, provided $\partial\Omega \cap B(g_0, r_0)$ is uniformly (δ, r_0) -approximable (η_1, η_2) -admissible characteristic hyperplanes in the Heisenberg group, in the sense of gauge Hausdorff distance, as defined in Definition 2.2 and provided (2.7) hold whenever $0 < r < r_0$, $g \in \partial\Omega \cap B(g_0, r_0)$.

Definition 9.4 Let $\Omega \subset \mathbb{H}^n$ be a domain, $\eta_1, \eta_2 \in (0, \infty)$, $\eta_1 < \eta_2$. Let $r_0 > 0$, $\delta \in (0, 1/8)$ and let $g_0 \in \partial\Omega$. Assume that $\partial\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$ with respect to (η_1, η_2) -admissible characteristic hyperplanes. Let $g \in \partial\Omega \cap B(g_0, r_0/2)$, $0 < r < r_0/2$. We say that $\partial\Omega \cap B(g, r)$ is flat in $\mathbb{H}^n$, with respect to (η_1, η_2) -admissible characteristic hyperplanes, with vanishing constant, if for each $\epsilon > 0$, there exists $\tilde{r} = \tilde{r}(\epsilon) > 0$ with the following property. If $g' \in \partial\Omega \cap B(g, r)$ and $0 < \rho < \tilde{r}$, then there exists a (η_1, η_2) -admissible characteristic hyperplane $\tilde{\Pi} = \tilde{\Pi}^\nu = \Pi^\nu(g, \rho)$, with $\nu \in \mathcal{S}$, such that

$$D(\partial\Omega \cap B(g', \rho), \tilde{\Pi} \cap B(g', \rho)) \leq \epsilon \rho.$$

We here simply remark that it is a good ambition to work out the program of this paper in a context assuming that $\partial\Omega \cap B(g_0, r_0)$ is (δ, r_0) -flat in $\mathbb{H}^n$ with respect to (η_1, η_2) -admissible characteristic hyperplanes. Indeed, we claim that the assumption concerning approximation by non-characteristic hyperplanes can be weakened to domains which again are well-approximated by more general hyperplanes, approximating hyperplanes which in this case are allowed to contain characteristic points. However, in that case we define the class of flat domains by imposing a (uniform) condition/restriction concerning the location of the characteristic point on any of the approximating hyperplanes, a condition/restriction which enable us to establish estimates.

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