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U.U.D.M. Project Report 2012:2

A bar construction in Morse-Witten homology

Emilia Lundberg

Examensarbete i matematik, 30 hp
Handledare och examinator: Tobias Ekholm
Januari 2012

Department of Mathematics
Uppsala University

Contents

1 Introduction

In a classical paper, [2], Adams introduced a cobar construction for chain complexes and used it to express the homology of the based loop space of a given space in terms of its singular chain complex. In the present paper we present a related construction in Morse theory. Starting from a Morse function we introduce a A_∞-structure on the Morse-Witten complex of the function by counting perturbed Morse flow trees. This leads to a Hochschild differential on the associated bar complex. Inspired by relations between the Fukaya category and the symplectic homology of cotangent bundles (or more general Weinstein manifolds), see [6] and [3], we expect that the Hochschild homology of the bar complex associated to the Morse-Witten complex on a manifold is closely related to the homology of the free loop space of the manifold. We verify that the two are isomorphic for spheres of any dimension and for products of spheres by direct calculation.

2 Preliminaries

Let X be a compact smooth Riemannian n-manifold without boundary. Let g be a Riemannian metric on X and $f : X \to \mathbb{R}$ a Morse function, with one critical point of index 0 and one of index n. Also assume that (f, g) is Morse-Smale (as is the case for (f, g) in an open dense subset).

A function is called *Morse* if the Hessian is non-singular at every critical point. The *index* of a critical point of a Morse function is the dimension of a maximal subspace of the tangent space at the critical point where the Hessian is negative definite. In order to discuss the Morse-Smale condition on (f, g) we need the following definitions.

Definition 2.1. For $x \in X$ a *(gradient) flow line* is a smooth map $\gamma_x : \mathbb{R} \to X$ s.t.

$$\begin{aligned} \frac{d}{dt}\gamma_x &= -\nabla f \circ \gamma_x, \\ \gamma_x(0) &= x. \end{aligned}$$

A *broken* flow line consist of several flow lines joined at critical points.

Definition 2.2. Let a be a critical point of the Morse function f. Define the *unstable* (or *descending*) manifold of a as

$$W^u(a) = \{x \in X | \lim_{t \to -\infty} \gamma_x(t) = a\}$$

and the *stable* (or *ascending*) manifold of a as

$$W^s(a) = \{x \in X | \lim_{t \to +\infty} \gamma_x(t) = a\}.$$

Then

$$\begin{aligned} \dim W^u(a) &= \lambda_a, \\ \dim W^s(a) &= n - \lambda_a, \end{aligned}$$

where λ_a is the index of a.

Denote by $\mathcal{M}(a,b)$ the moduli space of flow lines from a to b modulo translation in the source with $\mathbb{R}$, or equivalently the space of rigid flow lines from a to b. Note that

$$\mathcal{M}(a,b) = W^u(a) \cap W^s(b)/\mathbb{R}.$$

The pair (f,g) is called *Morse-Smale* if the stable and unstable manifolds intersect transversally. Thus, since we assume (f,g) to be Morse-Smale then

$$\dim \mathcal{M}(a,b) = \lambda_a - \lambda_b - 1.$$

The Morse differential ∂^M below is defined by counting the number of points in the moduli space $\mathcal{M}(a,b)$ for a and b such that $\lambda_a - \lambda_b = 1$. To do this we need the moduli space to be compact. The following proposition and corollary are essentially Proposition 2.35 and Corollary 2.36 in [5] and gives the compactness result. In both cases we assume that (f,g) satisfies the Morse-Smale condition.

Proposition 2.3. *Every sequence $(\gamma) \in \mathcal{M}(a,b)$ has a subsequence (γ_n) converging to a flow line in $\mathcal{M}(a,b)$ or to a broken flow line in*

$$\mathcal{M}(a,x_1) \times \mathcal{M}(x_1,x_2) \times \cdots \times \mathcal{M}(x_i,b)$$

where $x_1,\ldots,x_i$, $1 \leq i \leq \lambda_a - \lambda_b - 1$, are critical points, i.e. (γ_n) converges to a broken flow line with breaks at $x_1,\ldots,x_i$.

Corollary 2.4. *If $\lambda_a - \lambda_b = 1$, then $\mathcal{M}(a,b)$ is a finite set of rigid flow lines.*

Remark 2.5. We obtain a compactification $\overline{\mathcal{M}}(a,b)$ of $\mathcal{M}(a,b)$ by adding appropriate broken flow lines. E.g. for a and b such that $\lambda_a - \lambda_b = 2$ we have that

$$\partial\overline{\mathcal{M}}(a,b) = \bigcup_{\substack{x \\ \lambda_x = \lambda_a - 1}} \mathcal{M}(a,x) \times \mathcal{M}(x,b).$$

Define the Morse-Witten complex $CM_*(X,f)$ as the $\mathbb{Z}_2$-vector space generated by critical points of f and graded by the indices of the critical points. The Morse differential ∂^M is defined by:

$$\partial^M(a) = \sum_{\substack{b \\ \lambda_b=\lambda_a-1}} \#\mathcal{M}(a,b)b,$$

for $\#\mathcal{M}(a,b) \in \mathbb{Z}_2$, the number of rigid flow lines from a to b modulo 2.

Theorem 2.6. $\partial^M \circ \partial^M = 0$.

Proof.

$$\begin{aligned}
\partial^M \circ \partial^M(a) &= \sum_{\lambda_b=\lambda_a-1} \#\mathcal{M}(a,b)\partial^M(b) \\
&= \sum_b \sum_{\lambda_c=\lambda_b-1} \#\mathcal{M}(a,b)\#\mathcal{M}(b,c)c \\
&= \sum_b \sum_c \#\mathcal{M}(a,b) \times \mathcal{M}(b,c)c \\
&= \sum_{\lambda_c=\lambda_a-2} \#\partial\overline{\mathcal{M}}(a,c)c,
\end{aligned}$$

which is zero since $\overline{\mathcal{M}}(a,c)$ is a 1-dimensional compact manifold and hence the boundary components sum to zero modulo two. □

The corresponding homology $HM_*(X)$ is isomorphic to the singular homology of X and so it is independent of the choice of both the Morse function and the metric.

3 Intersection product and higher products

In this section we will define higher products (similar to the Morse differential) which maps an arbitrary (finite) number of critical points to a linear combination of critical points. For this purpose we want to count objects called *(perturbed Morse) flow trees* in analogy to counting flow lines in the Morse differential. The counterpart of the Morse-Smale condition is that all relevant *pertubed* unstable and stable manifolds of the Morse function f have to intersect transversally. We define the flow trees using a perturbation scheme inspired by Abouzaid [1] and we will sketch the arguments in the first part of this section.

In the second part, we discuss transversality and compactness. We define the higher products leading to an A_∞-structure in the last part.

3.1 Perturbed Morse flow trees

For $k \geq 2$ define $\mathcal{T}_k$ to be the space of strips with $k-1$ slits, see Figure 1, where we make the following identification

$$Strip(r_1, \ldots, r_{k-1}) \sim Strip(r_1 + s, \ldots, r_{k-1} + s),$$

for $s \in \mathbb{R}$. Hence we may put $r_1 = 0$. We call the punctures at $-\infty$ *inputs* and the one at $+\infty$ *output*. Define $\Delta \in \mathcal{T}_1$ to be an infinite strip, i.e. with no slits.

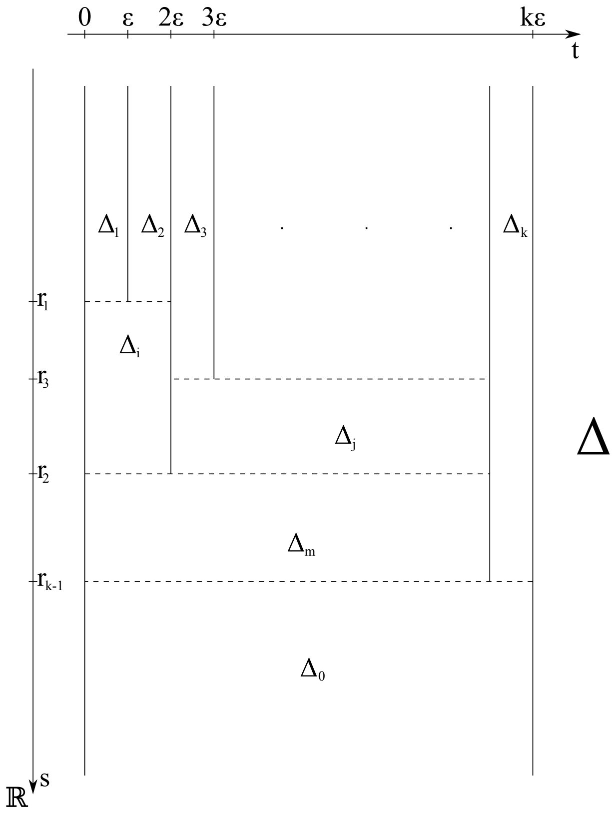

Figure 1: A strip $\Delta \in \mathcal{T}_k$.

Denote the infinite segments of $\Delta \in \mathcal{T}_k$ by Δ_i for $i = 0, \ldots, k$ as in Figure 1 and the finite parts by Δ_i for $i = k+1, \ldots$ up to whatever number is needed to label all parts. From this definition it is clear that $\mathcal{T}_k$ is diffeomorphic to $\mathbb{R}^{k-2}$ for $k \geq 2$.

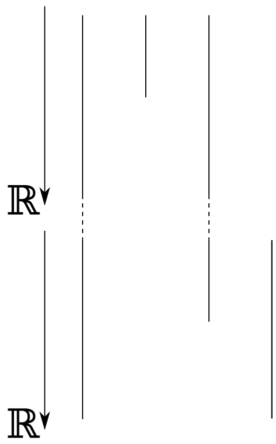

Figure 2: Example of a broken strip.

By allowing broken strips we obtain a compactification $\overline{\mathcal{T}}_k$. These broken strips can be viewed as the limit of a sequence in $\mathcal{T}_k$ where at least one $|r_i - r_{i+1}|$ tends to infinity, an example is illustrated in Figure 2. Further, the boundary of $\overline{\mathcal{T}}_k$ is the set of all broken strips with k inputs:

$$\partial\overline{\mathcal{T}}_k = \bigcup_{i=2}^{k-2} \bigcup_{\substack{k_1,\ldots,k_i \geq 2 \\ k=\sum_{j=1}^{i} k_j - (i-1)}} \mathcal{T}_{k_1} \times_{k_2} \mathcal{T}_{k_2} \times_{k_3} \cdots \times_{k_i} \mathcal{T}_{k_i},$$

where $\mathcal{T}_p \times_q \mathcal{T}_q = \mathcal{T}_p \times \mathcal{T}_q \times \{1, \ldots, q\}$ is the set of broken strips where the output of the elements of $\mathcal{T}_p$ is attached to one of the inputs of elements of $\mathcal{T}_q$, the input is decided by the elements of $\{1, \ldots, q\}$. For example the broken strip in Figure 2 is an element of $\mathcal{T}_2 \times_2 \mathcal{T}_2$ and more specifically of $\mathcal{T}_2 \times \mathcal{T}_2 \times \{1\}$.

Define *gluing maps* for strips as

$$\mathcal{T}_{k_1} \times \mathcal{T}_{k_2} \stackrel{\#_\rho}{\to} \mathcal{T}_k, \tag{1}$$

for $k = k_1 + k_2 - 1 \geq 1$ and $\rho \in (0, \infty)$ and such that $T_{k_1} \#_\rho T_{k_2} \to (T_{k_1}, T_{k_2})$ when $\rho \to \infty$. A gluing is illustrated below in Figure 3. For $k_i = 1$ the gluing is just a projection map. By the definition of gluing and by the discussion above about broken strips it should be understood that the strips are independent of the width.

Now, we want to define perturbation vector fields on X depending on the strips such that the perturbation data is compatible with the perturbation data on the boundary. Meaning, a (possibly broken) strip in $\overline{\mathcal{T}}_k$ will have some perturbation data from $\overline{\mathcal{T}}_k$ and some data from $\mathcal{T}_{k_1}$ wherein the unbroken strip lies and these has to agree.

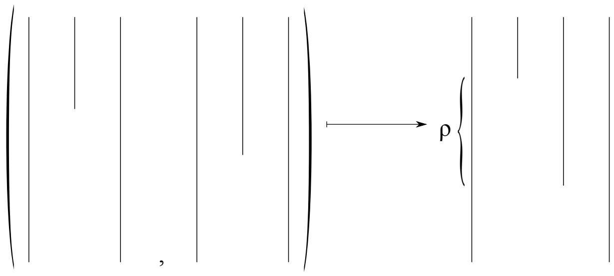

Figure 3: Gluing of strips.

For the strip with one input we want no perturbation data. For $\Delta \in \mathcal{T}_k$, $k > 1$, we define a perturbation datum to be a choice of smooth maps

$$V_j : \mathcal{T}_k \times \Delta_i \times X \to TX,$$

vanishing away from a bounded subset of Δ_i and such that $V_j(\Delta, \tau, \cdot)$ is a smooth vector field on X for all $\tau \in \Delta_i$. This is defined inductively, asserting the compability condition, by the following procedure:

For the strip, $\Delta \in \mathcal{T}_2$, with two inputs a perturbation datum is a choice as described above with $j = i = 0, 1, 2$ such that V_1 and V_2 vanish on $(-\infty, -1)$ in Δ_1 and Δ_2 respectively and V_0 vanish on $(1, \infty)$ in Δ_0.

For $k > 2$ a perturbation datum for a strip near the boundary of $\mathcal{T}_k$ is defined by the elements of $\partial\overline{\mathcal{T}}_k$ and the gluing maps (1). For the strip $\Delta \in \mathcal{T}_k$ where the slits are all leveled a perturbation datum is a choice as above with $j = i = 0, \dots, k$. For all other strips in $\mathcal{T}_k$ we also have a choice of one or two vector fields on the finite parts Δ_i, one in each end, such that they are compatible with the two cases above.

Remark 3.1. It is natural to choose the vector fields satisfying a balancing condition at each r_j so that the sum of all vector fields associated to r_j is zero.

Now we can define the (perturbed Morse) flow trees; maps $\mathcal{T}_k \ni \Delta \to X$ which are counted in the higher products.

Definition 3.2. Let $\Delta \in \mathcal{T}_k$ be a strip with perturbation data $\{V_i\}_{i=0}^{l}$, $k \leq l \leq 3k - 4$. A *perturbed Morse flow tree* with k inputs given by the critical points $a_1, \dots, a_k$ and output given by the critical point b of f is a continuous map

$$\Phi : \Delta \to X,$$

whose restriction to every Δ_j is a smooth map

$$\phi^j : \Delta_j \to X,$$

such that for each t

$$\frac{d}{ds}\phi^j_t = -\nabla f \circ \phi^j_t + V_i((t, \cdot), \phi^j_t),$$

for $j = 0, \dots, k$ and where V_i vanish on $\{(t, s)\}$ such that $s \in (-\infty, r_j - 1)$ for $j = 1, \dots, k$ and $s \in (\max r_i + 1, \infty)$ for $j = 0$.

For $j > k$ we could have two perturbation vector fields on each Δ_j, so for each t we have

$$\frac{d}{ds}\phi_t^j = -\nabla f \circ \phi_t^j + V_i((t,\cdot),\phi_t^j) + V_{i+1}((t,\cdot),\phi_t^j),$$

where V_i and V_{i+1} vanish on $\{(t,s)\}$ such that $s \in (r_m+1, r_k)$ and $s \in (r_m, r_k-1)$ respectively. Also, the ϕ^j's should match at every r_i and the following should hold:

$$\begin{aligned}\lim_{s\to-\infty} \Phi_t(s) &= a_i, \quad t \in \Delta_i,\\ \lim_{s\to+\infty} \Phi_t(s) &= b.\end{aligned}$$

For $k = 1$ we define a perturbed Morse flow tree to be a gradient flow line.

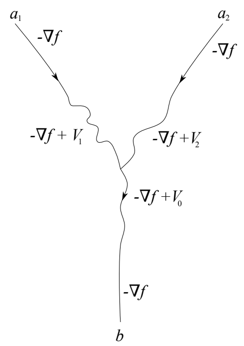

Figure 4: A pertubed Morse flow tree with inputs a_1, a_2 and output b.

A *broken* flow tree is, in similarity to a broken flow line, an object that consists of several flow trees joined at critical points, an example is illustrated in Figure 5.

For $\mathbf{a}_k := a_1 \otimes \cdots \otimes a_k$, let $\mathcal{T}(\mathbf{a}_k, b)$ be the moduli space of unparametrized perturbed Morse flow trees with inputs given by $\mathbf{a}_k$ and output b. Note that $\mathcal{T}(a,b) = \mathcal{M}(a,b)$. Next, we will study this moduli space more carefully, but before moving on let us define the F-degree of $\mathbf{a}_k$.

Definition 3.3. For a critical point a of the Morse function f the *F-degree* of a is

$$\deg^F(a) = \lambda_a - n + 1,$$

where $n = \dim X$ and λ_a is the index of a. The F-degree will also be denoted by λ^F.

3.2 Compactness and transversality of $\mathcal{T}(\mathbf{a}_k, b)$

In similarity to the Morse differential we want to define the higher products by counting the elements of $\mathcal{T}(\mathbf{a}_k, b)$. By the following compactness and transversality results we may conclude that it is finite in dimension zero. We begin with compactness:

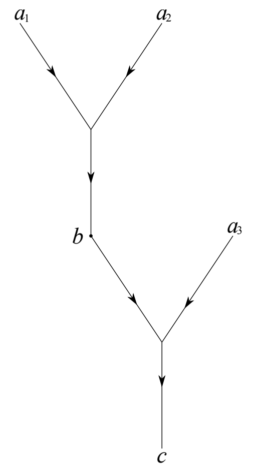

Figure 5: Example of a broken flow tree.

Theorem 3.4 (Compactness)**.** *Any sequence of flow trees in $\mathcal{T}(\mathbf{a}_k, b)$ has a subsequence that converges to a flow tree in $\mathcal{T}(\mathbf{a}_k, b)$ or to a broken flow tree. Hence, we obtain a compactification, $\overline{\mathcal{T}}(\mathbf{a}_k, b)$, with boundary given by adding appropriate broken flow trees.*

Proof. Assume that we have a sequence of flow trees (Φ) in $\mathcal{T}(\mathbf{a}_k, b)$. By the map $\pi : \mathcal{T}(\mathbf{a}_k, b) \to \mathcal{T}_k$, which maps a flow tree to its domain with related perturbation data, we obtain a sequence of strips $(\pi(\Phi))$ in $\mathcal{T}_k$. This sequence have a converging subsequence $(\pi(\Phi)_i)$, from which we obtain a subsequence of (Φ), namely $(\Phi_j) := (\pi^{-1}(\pi(\Phi)_i) \cap (\Phi))$.

If, for all $j > M$ for some $M > 0$, Φ_j is outside some neighbourhood U_x of all critical points x of f then $\Phi_j \to \Psi$ for some $\Psi \in \mathcal{T}(\mathbf{a}_k, b)$, Figure 6. This is clear since we can choose the perturbation vector fields so that $-\nabla f + V \neq 0$ whenever $-\nabla f \neq 0$.

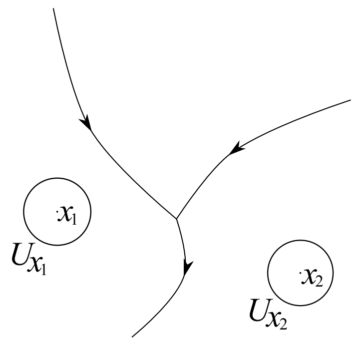

Figure 6: For $j > M$, all Φ_j's are outside some neighbourhood U_x of all critical points x of f.

If Φ_j pass through some U_x of all $j > M$ for some $M > 0$, see Figure 7, then

$$\frac{d}{ds}\Phi_j(\tau) = -\nabla f \circ \Phi_j(\tau) \text{ for all } (\tau) \text{ s.t. } \Phi_j(\tau) \in U_x$$

since the strip $\pi(\Phi)_i$ can be viewed as a gluing of $(T_1, T_2) \in \mathcal{T}_{k_1} \times \mathcal{T}_{k_2}$ for $k_1, k_2 \geq 1, k_1 + k_2 - 1 = k$ and so on a subset of the finite segment between T_1 and T_2 on $\pi(\Phi)_i$ the perturbation vector fields are zero. So in U_x the sequence (Φ_j) looks like a sequence of flow *lines* and so

$$\Phi_j \to (\varphi, \psi),$$

where φ and ψ are perturbed Morse flow trees on T_1 and T_2 respectively, endowed with perturbation data from $\pi(\Phi)_i$.

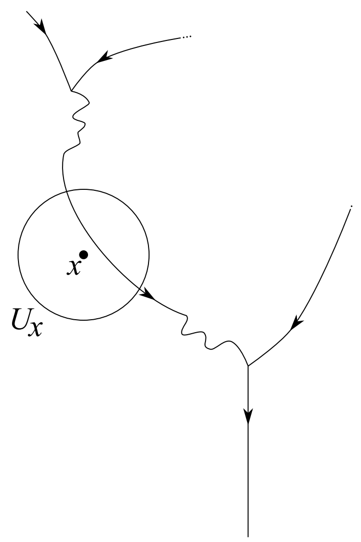

Figure 7: For $j > M$, all Φ_j's pass through some neighbourhood U_x of a critical point x of f.

The flow tree which is the limit of (Φ_j) can have more than one break but by Proposition 2.3 the number of breaks is finite. □

The following transversality theorem follows from standard arguments and so the proof will be sketched.

Theorem 3.5 (Transversality)**.** *For a generic choice of perturbation data* $\mathcal{T}(\mathbf{a}_k, b)$ *is a manifold of dimension* $\deg^F(\mathbf{a}_k) - \deg^F(b) - 1$*.*

Sketch of proof. Let the perturbed unstable/stable manifold of a critical point x of f, denoted $\widetilde{W}^{u/s}(x)$, be the unstable/stable manifold of x using the perturbed gradient flow instead of the original one. Note that for $k > 1$ the flow lines of the perturbed gradient flow are not infinite in both ends. However, they have the same dimension. We want to prove that all relevant perturbed unstable/stable manifolds intersect transversally.

For $k = 1$ this is clear since (f, g) is assumed to be Morse-Smale and we have no perturbation data on the strip without slits.

For $k = 2$ we consider $\mathcal{T}(a_1 \otimes a_2, b)$ with perturbation data V_0, V_1, V_2. We may interpret the moduli space of trees as

$$\mathcal{T}(a_1 \otimes a_2, b) = \widetilde{W}^u(a_1) \cap \widetilde{W}^u(a_2) \cap \widetilde{W}^s(b)$$

by sending each flow tree in $\mathcal{T}(a_1 \otimes a_2, b)$ to its intersection point. Now, by a transversality theorem by Hirsch [4] we have that every neighbourhood of the inclusion

$$i : \widetilde{W}^u(a_1) \to X$$

in $C^\infty(\widetilde{W}^u(a_1), X)$ contains an embedding which is transverse to $\widetilde{W}^u(a_2)$. In our case this means that since we *chose* all V_j's we can change it slightely so that we get a transverse intersection, $i \pitchfork \widetilde{W}^u(a_2)$, i.e. $\widetilde{W}^u(a_1) \pitchfork \widetilde{W}^u(a_2)$. By the same argument we obtain the transverse intersection

$$\widetilde{W}^u(a_1) \pitchfork \widetilde{W}^u(a_2) \pitchfork \widetilde{W}^s(b) = \mathcal{T}(a_1 \otimes a_2, b),$$

proving the theorem for $k = 2$.

For $k > 2$ we have three cases. First, we consider a tree near the boundary, i.e. the case where the domain have no slits that are leveled. Here we obtain the result inductively by looking at appropriate boundary components. E.g. for $k = 3$ consider the boundary element (Φ_1, Φ_2) with two inputs each and with perturbation data $\{V_1, V_2, V_4\}$ and $\{V_5, V_3, V_0\}$ respectively, see Figure 8, note that $V_1 = V_5$, $V_2 = V_3$ and $V_4 = V_0$ since Φ_1 and Φ_2 have the same domain. We allready have transversality on each unbroken tree and by gluing them together (keeping the perturbation data) we obtain transversality near the boundary as well.

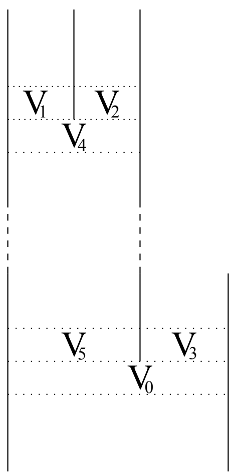

Figure 8: The domain for the boundary element (Φ_1, Φ_2) of $\overline{\mathcal{T}}(a_1 \otimes a_2 \otimes a_3, b)$ with the perturbation vector fields marked.

Second, we consider the case where all slits are leveled. Now we achieve the transversality result by possibly changeing the perturbation data as before.

Last, we have to consider the case where some of the slits are leveled and some are not. Again the transversality result is achieved inductively.

The dimension. Note that for $k = 1$ we are done since $\mathcal{T}(a,b) = \mathcal{M}(a,b)$. For $k > 1$ we first note that

$$\dim \mathcal{T}(a_1 \otimes a_2, b) = \lambda_{a_1} + \lambda_{a_2} - n - \lambda_b = \deg^F(a_1 \otimes a_2) - \deg^F(b) - 1,$$

since it is a transverse intersection of $\widetilde{W}^u(a_1), \widetilde{W}^u(a_2)$ and $\widetilde{W}^s(b)$. For a tree in $\mathcal{T}(\mathbf{a}_k, b)$ each vertex represents a transverse intersection of perturbed unstable manifolds of the inputs except the last one which is an intersection with the perturbed stable manifold of the output. Also, at each vertex we flow out from the intersection, giving $+1$ in the dimension. Hence:

$$\begin{aligned} \dim \mathcal{T}(\mathbf{a}_k, b) &= \sum_{i=1}^{k} \lambda_{a_i} - \underbrace{(k-1)n}_{\text{vertices}} + \underbrace{(k-2)}_{\text{extra 1's}} + \underbrace{(n-\lambda_b) - n}_{\text{last intersection}} \\ &= \deg^F(\mathbf{a}_k) - \deg^F(b) - 1. \end{aligned}$$

□

Remark 3.6. When $\dim \mathcal{T}(\mathbf{a}_k, b) = 1$ a broken flow tree in the boundary only have one breaking point, i.e. the boundary of $\overline{\mathcal{T}}(\mathbf{a}_k, b)$ is

$$\bigcup_{i,c} \overline{\mathcal{T}}(a_i, c) \times \overline{\mathcal{T}}(a_1 \otimes \cdots a_{i-1} \otimes c \otimes a_{i+1} \otimes \cdots a_k, b) \cup \bigcup_{c} \overline{\mathcal{T}}(\mathbf{a}_k, c) \times \overline{\mathcal{T}}(c, b)$$
$$\bigcup_{i,m,c} \overline{\mathcal{T}}(a_i \otimes \cdots \otimes a_{i+m}, c) \times \overline{\mathcal{T}}(a_1 \otimes \cdots a_{i-1} \otimes c \otimes a_{i+m+1} \otimes \cdots a_k, b).$$

3.3 Higher products

Now, we define the k'th higher product which in turn defines the Hochschild differential on the bar complex. We define the k'th higher product as follows:

$$\begin{aligned} m_k : CM_{l_1} \otimes \cdots \otimes CM_{l_k} &\longrightarrow CM_{\deg^F(\mathbf{a}_k)-1} \\ a_1 \otimes \cdots \otimes a_k &\longmapsto \sum_{\substack{b \text{ s.t.} \\ \deg^F(b) = \deg^F(\mathbf{a}_k)-1}} \#\mathcal{T}(\mathbf{a}_k, b) b, \end{aligned}$$

where $\#\mathcal{T}(\mathbf{a}_k, b) \in \mathbb{Z}_2$.

Note that $m_1 = \partial^M$ and that the higher products gives an A_∞-structure on the Morse-Witten complex.

4 The bar complex $F(CM_*)$

For convenience put $\bigotimes_k C = (C)^k$. Define

$$F(CM_*(X,f)) := \bigoplus_{k \geq 1} (CM_*(X,f))^k,$$

denoted only by $F(X)$ if the situation permits it. We grade $F(CM_*)$ by $\deg^F$ from Definition 3.3. Also, we obtain a product,

$$\mu : F(CM_*) \otimes F(CM_*) \to F(CM_*)$$

from the natural isomorphism

$$(CM_*)^k \otimes (CM_*)^l \to (CM_*)^{k+l}.$$

Now, we define a differential $D : F(CM_*) \to F(CM_*)$ of degree -1 as follows:

$$\begin{aligned} D(\mathbf{a}_k) &= \sum_{j=1}^{k} \sum_{i=1}^{k-(j-1)} a_1 \otimes \cdots \otimes m_j(a_i \otimes \cdots \otimes a_{i+j-1}) \otimes \cdots \otimes a_k \\ &+ \sum_{j=2}^{k} \sum_{i=0}^{j-2} m_j(a_{k-i} \otimes \cdots \otimes a_k \otimes a_1 \cdots \otimes a_{j+i-1}) \otimes \cdots a_{k-i-1}. \end{aligned}$$

Theorem 4.1. $D^2 = 0$.

Proof. For $k = 1$ we are done, since $m_1 = \partial^M$. For $k > 1$ we interpret $\mathbf{a}_k$ to be a word written on a circle oriented clockwise with a_1 marked. Then D may be interpreted as the sum of all different ways of attaching trees to the circle with inputs given by subwords. The result of attaching a tree is a circle where the inputs of the tree are replaced by the output. If one of the inputs is marked then so is the output.

Then, D^2 is the procedure of summing over all possible ways of attaching two trees in this manner. There are two ways of attaching two trees to the circle. One way is that first we attach a tree with j inputs and output b and then a tree with i inputs, all different from b, and output c. These terms are of the form $\cdots \otimes m_i(\cdot) \otimes \cdots \otimes m_j(\cdot) \otimes \cdots$. They come in pairs since we also have the term in $D^2(\mathbf{a}_k)$ that attach the trees in the opposite order and so they sum to zero modulo two. The other way of attaching two trees is that first we attach a tree with j inputs and output b and then we attach a tree with i inputs where one of the inputs is b and with output c. These terms also sum to zero modulo two since broken trees with given inputs and output is the boundary of a 1-dimensional compact manifold. More precisely, consider $a_l \otimes \cdots \otimes a_{l+p}$, where $a_i \in \{a_1, \ldots, a_k\}$ for $i = l, \ldots, l+p$. Then these terms in $D^2(\mathbf{a}_k)$ are:

$$\begin{aligned} & \cdots \otimes \sum_{\substack{i+j=p+2 \\ 1 \le i,j \le p+1}} \sum_{q \ge l} m_i(a_l \otimes \cdots \otimes m_j(a_q \otimes \cdots \otimes a_{q+j-1}) \cdots a_{l+p}) \otimes \cdots \\ = & \cdots \otimes \sum_{\substack{c \\ dim \mathcal{T}=1}} \# \partial \overline{\mathcal{T}}(a_l \otimes \cdots \otimes a_{l+p}, c) c \otimes \cdots \end{aligned}$$

Which is zero modulo two since $\overline{\mathcal{T}}$ is a compact 1-dimensional manifold. (For a more detailed calculation of the last equality see Appendix, Proposition A.1.) □

We denote the corresponding homology by $HF_*(X)$. In general it is defined as a direct limit.

5 Computations for the n-sphere and products of spheres

As discussed in the introduction, inspired by relations between the Fukaya category and the symplectic homology of the cotangent bundle ([6], [3]) we expect that $HF_*(X)$ is closely related to the singular homology of the free loop space of X. In this section we will verify that they are isomorphic for $X = S^n$, the n-sphere, and $X = S^p \times S^q$, for $p, q \geq 2$, by direct calculation.

The abbreviation xy will be used for $x \otimes y$, and sometimes $\underbrace{x \otimes \cdots \otimes x}_{k}$ will be denoted x^k.

5.1 The n-sphere

Let $f : S^n \to \mathbb{R}$ be the height function, i.e. if we embed the n-sphere in $\mathbb{R}^{n+1}$ such that it is tangent to the hyperplane $\mathbb{R}^n$, Figure 9, f is the height over the hyperplane. Hence, f has two critical points which we denote a and b, where $\lambda_a = n$ and $\lambda_b = 0$, i.e.

$$\begin{aligned} \lambda_a^F &= 1 \\ \lambda_b^F &= -(n-1). \end{aligned}$$

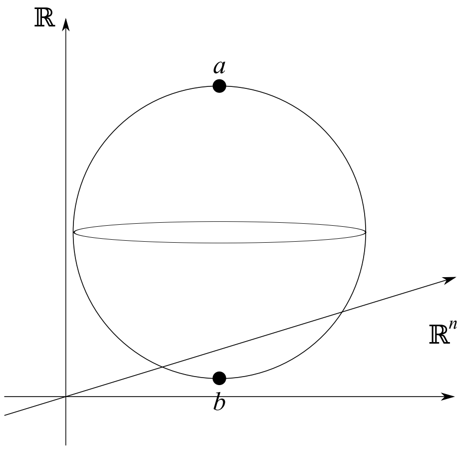

Figure 9: The n-sphere embedded in $\mathbb{R}^{n+1}$ with the critical points marked.

On $F(CM_*(S^n, f))$ the only non-zero products are:

$$\begin{aligned} m_2(aa) &= a \\ m_2(ab) &= b = m_2(ba). \end{aligned}$$

This is easy to see since we can not have more than one a except the one above (or the flow tree will not be rigid) and then more than one b will result in a failure in dimension.

Then the differential is reduced to:

$$D(x_1 \cdots x_l) = m_2(x_1x_2) \cdots x_l + \cdots + x_1 \cdots m_2(x_{l-1}x_l) + m_2(x_lx_1) \cdots x_{l-1}$$

for $x_i = a, b$ and $l \geq 1$. First note that monomials consisting of only b's are certainly in the kernel of D. Adding a's such that the number of sequential a's are all odd numbers, also lie in the kernel since the terms in $D(x_1 \cdots x_l)$ will cancel each other or be equal to zero. If there is more than one a the monomial must start with a b otherwise terms will not cancel properly.

To hit something with D our only choice is to add an a, since we want to add $+1$ to the degree and since the only non-zero product is m_2 we can only add one element to the monomial. Hence, the only objects we have left of the above are $b \cdots b$ and $ab \cdots b$. Furthermore, it is easy to see that a is left in the homology as well.

The monomials where we in at least one place add an even number of sequential a's do not directly lie in the kernel but may be complemented by some monomials to do exactly that, but these are hit by D. For example, $b^p a^k b^q$ where k is an even number can be complemented by $ab^p a^{k-1} b^q + b^{p+1} a^k b^{q-1}$, i.e. $D(b^p a^k b^q + ab^p a^{k-1} b^q + b^{p+1} a^k b^{q-1}) = 0$. However,

$$D(ab^p a^k b^q) = b^p a^k b^q + ab^p a^{k-1} b^q + b^{p+1} a^k b^{q-1}.$$

Since,

$$\begin{aligned} deg^F(a) &= 1 \\ deg^F(\underbrace{b \cdots b}_{k}) &= -k(n-1) \\ deg^F(a\underbrace{b \cdots b}_{k}) &= 1 - k(n-1) \end{aligned}$$

in homology we have $\mathbb{Z}_2$ in degrees 1, $-(k(n-1)-1)$ and $-k(n-1)$, for all $k \geq 1$. E.g. for $n = 2$ this means that

$$HF_i(S^2) \cong \begin{cases} \mathbb{Z}_2, & i = 1 \\ \mathbb{Z}_2, & i = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & i < 0 \\ 0, & \text{otherwise.} \end{cases}$$

By the table on page 21 of [7] and the Universal Coefficients Theorem $HF_*(S^n)$ is isomorphic to the singular homology of the free loop space of the n-sphere, by a shift and negation in degree, namely $H_i \cong HF_{-i+1}$.

5.2 Products of spheres

For the product space $S^p \times S^q$ for $p, q \geq 2$ we use a similar embedding as for the sphere, namely the one where we embed the product space in $\mathbb{R}^{p+q+1}$ such that it is tangent to the hyperplane $\mathbb{R}^{p+q}$ and slightly tilted, Figure 10. We let f be the height over this hyperplane. Then, f has four critical points which we denote by a, c_1, c_2 and b, with index $p+q, p, q$ and 0 respectively, i.e.

$$\begin{aligned} \lambda_a^F &= 1 \\ \lambda_{c_1}^F &= -q+1 \\ \lambda_{c_2}^F &= -p+1 \\ \lambda_b^F &= -p-q+1. \end{aligned}$$

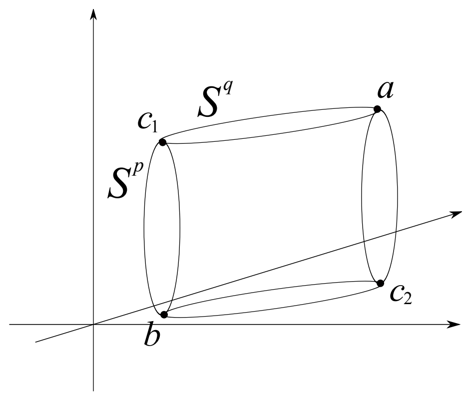

Figure 10: $S^p \times S^q$ embedded in $\mathbb{R}^{p+q+1}$ with the critical points marked.

Again, the only non-zero product is m_2, which is:

$$\begin{aligned} m_2(aa) = &\quad a \\ m_2(ab) = &\quad b \quad = m_2(ba) \\ m_2(ac_1) = &\quad c_1 \quad = m_2(c_1a) \\ m_2(ac_2) = &\quad c_2 \quad = m_2(c_2a) \\ m_2(c_1c_2) = &\quad b \quad = m_2(c_2c_1). \end{aligned}$$

In the kernel of D we have the same objects as in the S^n case. In addition we have monomials including c_1 and c_2. To hit something with D we now have two choices; adding an a or subtracting a b *and* adding c_i and c_j, $i \neq j$.

First, consider the monomials $ab\cdots b, b\cdots b \in \ker D$. Note that

$$D(c_1c_2\underbrace{b\cdots b}_{k-1}) = \underbrace{b\cdots b}_{k}, \qquad \forall k \geq 2$$

and

$$D(a\underbrace{b\cdots b}_{k-1}c_1c_2 + c_2\underbrace{b\cdots b}_{k-2}c_1c_2c_1 + c_2c_2c_1\underbrace{b\cdots b}_{k-3}c_1b) = a\underbrace{b\cdots b}_{k} \qquad \forall k \geq 3.$$

Hence, the ones that are left in homology in this case are b in degree $-p-q+1$, abb in degree $-2p-2q+3$, ab in degree $-p-q+2$ and a in degree 1.

Now, consider the c-words, i.e. monomials only consisting of c_1's and/or c_2's. A word of the form $c_i\cdots c_ic_j\cdots c_j$ may be complemented to lie in the kernel of D in a rather nice way. The complement is the sum of different words that begins with c_i and with the rest permutated in all different ways. Let $(x_1\cdots x_k)$ denote the complemented element for $x_i = a, c_1, c_2, b$. Then:

$$(\underbrace{c_i\cdots c_i}_{I}\underbrace{c_j\cdots c_j}_{J}) := \sum_{\substack{\#\{q_p=i\}=I-1 \\ \#\{q_p=j\}=J}} c_ic_{q_1}\cdots c_{q_{I+J-1}}.$$

It is easy to see that this holds even when I or J is equal to zero, then there is only one term. We are not able to hit any of them with D since our only choice is to add an a to the monomial. Hence, in homology we have the following c-words:

$$\begin{aligned}(c_1 c_1^I c_2^J) &= (c_1^{I+1} c_2^J)\\ (c_2 c_1^I c_2^J) &= (c_2^{J+1} c_1^I)\end{aligned}$$

for all $I, J \geq 0$. Also note that $\deg^F(c_1 c_1^I c_2^J) = \deg^F(c_2 c_1^{I+1} c_2^{J-1})$ for $I \geq 0$ and $J \geq 1$. Left is $\deg^F(c_1^I) = I(-q+1), I \geq 1$ and $\deg^F(c_2^J) = J(-p+1), J \geq 1$.

Adding a's and/or b's to the c-words gives us $(ac_1^I c_2^J)$ and $(bc_1^I c_2^J)$ in homology for all $I, J \geq 0$, where

$$(xc_1^I c_2^J) = x(c_1^I c_2^J) + x(c_2^J c_1^I) = (xc_2^J c_1^I).$$

First consider adding only b's. If we add only one b to a word of the form $c^I c^J$ then the complement will be equal to $(bc^I c^J)$ in homology. If more than one b is added we proceed as follows; one b can be "moved" using $c_i c_j$, $i \neq j$, cancelling the extra terms in each step and so that in the last step we obtain a term where we have two sequential b's. Then we hit this last term with D since

$$D(c_i c_i c_j b) = c_i bb.$$

For example, $D(c_1 b c_1 b c_1) = 0$ but we can "move" the first b and then get rid of that term:

$$\begin{aligned}D(c_1 c_1 c_2 c_1 b c_1) &= c_1 b c_1 b c_1 + c_1 c_1 b b c_1, \text{ and}\\ D(c_1 c_1 c_1 c_2 b c_1) &= c_1 c_1 b b c_1.\end{aligned}$$

In a similar way as before and possibly in combination with the result above words in the kernel that contain a's can be hit by D except for $(ac^I c^J)$. Also, in homology $[abb] = [(bc_1 c_2)]$ and $[ab] = [c_1 c_2 + c_2 c_1]$. So, in homology we have the complemented c-words that begins with one of the four critical points.

The conclusion is that in homology we have $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ in degrees

- $-Iq - Jp + I + J + 1$ for all $I, J \geq 1$

and $\mathbb{Z}_2$ in degrees

- $I(-q+1), I(-p+1)$ for all $I \geq 1$ and
- $1 + I(-q+1) + J(-p+1)$ for all $I, J \geq 0$ and
- $-p - q + 1 + I(-q+1) + J(-p+1)$ for all $I, J \geq 0$.

By the previous isomorphism for spheres and Künneth's theorem this is isomorphic to the singular homology of the free loop space of the product space of spheres, by the same shift and negation as before.

A Calculations

Proposition A.1.

$$\begin{aligned} & \cdots \otimes \sum_{\substack{i+j=p+2 \\ 1 \le i,j \le p+1}} \sum_{q \ge l} m_i(a_l \otimes \cdots \otimes m_j(a_q \otimes \cdots \otimes a_{q+j-1}) \cdots a_{l+p}) \otimes \cdots \\ = \quad & \cdots \otimes \sum_{\substack{c \\ dim\mathcal{T}=1}} \#\partial\overline{\mathcal{T}}(a_l \otimes \cdots \otimes a_{l+p}, c)c \otimes \cdots \end{aligned}$$

Proof.

$$\begin{aligned} & m_i(a_l \cdots m_j(\underbrace{a_q \cdots a_{q+j-1}}_{\alpha}) \cdots a_{l+p}) \\ = \quad & m_i(a_l \cdots a_{q-1} \otimes \sum_{\substack{b \\ dim\mathcal{T}(\alpha,b)=0}} \#\mathcal{T}(\alpha,b)b \otimes a_{q+j} \cdots a_{l+p}) \\ = \quad & \sum_{\substack{b \\ dim\mathcal{T}(\alpha,b)=0}} \#\mathcal{T}(\alpha,b) \cdot m_i(\underbrace{a_l \cdots b \cdots a_{l+p}}_{\beta}) \\ = \quad & \sum_{\substack{b \\ dim\mathcal{T}(\alpha,b)=0}} \#\mathcal{T}(\alpha,b) \cdot \sum_{\substack{c \\ dim\mathcal{T}(\beta,c)=0}} \#\mathcal{T}(\beta,c)c \end{aligned}$$

Sum over all i, j and q and get the following:

$$\sum_{\substack{\alpha,\beta,b,c \\ dim\mathcal{T}(\alpha,b)=dim\mathcal{T}(\beta,c)=0}} \#\bigcup \mathcal{T}(\alpha,b) \times \mathcal{T}(\beta,c) \cdot c = \sum_{\substack{c \\ dim\mathcal{T}=1}} \#\partial\overline{\mathcal{T}}(a_l \cdots a_{l+p}, c) \cdot c$$

□

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