A Well-Posed Algorithm to Recover Implied Volatility

Yan Wang
Abstract

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Abstract: Implied volatility plays a very important role in financial sector. In the assets of trading market, everyone wants to know the implied volatility in the future. However, it is difficult to predict it. In this paper, we use a new well-posed algorithm to recover implied volatility under the Black-Scholes theoretical framework. I reproduce this algorithm at first, then prove its stability and give some examples to test. The results show that this algorithm can work and the error is small. We can use it in practice.
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Chapter 1 Introduction

An option is an agreement that gives the right to the holder to trade in the future at a precisely agreed price. There are many kinds of options. We will introduce two main options (call options and put options). A call option is a right to buy a special asset for an agreed amount at a specified time in the future. A put option is a right to sell a special asset for an agreed amount at a specified time in the future. In this paper, we will focus on European call option.

1.1 Volatility

In financial sector, volatility is an important concept. There are many applications of volatility but a few people really understand it. Volatility is one important parameter in Black-Scholes formula. It is sensitive to the changes in option price. For most people, it is understood from their intuition. Actually it is a measure of price changes in the value of financial products over a time period. Generally speaking, it is difficult for people to predict what the volatility will be in the future. However, the option markets exist and they “know” something about the volatility.

1.1.1 Types of volatility

There are many types of volatility, such as the seasonal volatility, the expect volatility, the realized volatility, the historical volatility, the implied volatility and so on. The realized volatility, the historical volatility and the implied volatility are the most useful and common volatilities. We will introduce them as follows.

Realized volatility

In order to apply majority of the financial models, being able to use the empirical data to measure the degree of variability of asset prices is necessary. Suppose that \( S_t \) denotes the price of an asset at time \( t \). The realized volatility of the asset in a period \( [t_1, t_2] \) based on the observations \( (S_0, S_1, \ldots, S_{n-1}, S_n) \) is defined as
\[ \sigma = \sqrt{\frac{252}{n-1} \sum_{i=1}^{n} r_i^2} \].

Here \( r_i = \ln \left( \frac{S_i}{S_{i-1}} \right) \) for \( i = 1, 2, 3, \ldots, n \), and 252 is an animalization factor corresponding to the typical number of trading days in a year.

**Historical volatility**

The formula for historical volatility is very similar as realized volatility and it is defined as follows

\[ \hat{\sigma} = \sqrt{\frac{252}{n-1} \sum_{i=1}^{n} (r_i - \bar{r})^2} \], where \( r_i = \ln \left( \frac{S_i}{S_{i-1}} \right) \), \( i = 1, 2, 3, \ldots, n \), and \( \bar{r} = \frac{1}{n} \sum_{i=1}^{n} r_i \) (it is the mean return).

Here 252 is an animalization factor corresponding to the typical number of trading days in a year.

If the returns are supposed to be drawn independently from the same probability distribution, and then \( \bar{r} \) is the sample mean. The historical volatility is simply the annualized sample standard deviation. We can see that \( \sum_{i=1}^{n} (r_i - \bar{r})^2 = \sum_{i=1}^{n} r_i^2 - n \bar{r}^2 \). And then we can get that \( \hat{\sigma}^2 = \bar{\sigma}^2 - \frac{252n \bar{r}}{n-1} \). It means that the realized volatility is equal to the historical volatility when the sample means approach to zero. Both types of the volatility can be used as predictors of the future volatility.

**Implied volatility**

Implied volatility is a very important concept in the field of modern finance. It is closely related to the financial derivatives such as options. The Implied volatility plays an important role whether in judging the futures market or in application of strategic investment options.
Implied volatility is the value that we get after taking the market price of options into the Black-Scholes model. The Black-Scholes model \[6\] gives the relationship between the option prices and the five basic parameters (underlying stock price \(S\), strike price \(K\), interest rate \(r\), maturity time \(T\), implied volatility \(\sigma\)). So the only unknown parameter (implied volatility \(\sigma\)) can be solved when take the first four basic parameters and the actual market price of options into the option pricing model. Implied volatility also can be regarded as an expectation of the actual market volatility. Therefore, implied volatility plays an increasingly important role in the option prices. It is not only assisted with how to measure the price changes in the whole period, but also provides the consistency of a future risk of the current market level to the traders or analysts.

In this paper, we mainly focus on the implied volatility.
Chapter 2 General Information about Calculating Volatility

2.1 Implied volatility estimation

The former researchers propose the two segmentation method [3] which is a relatively simple, fast and accurate iterative method.

The iterative method [3] formula is

\[ \sigma_0 = \sigma_L + (C_0 - C_L) \frac{\sigma_H - \sigma_L}{C_H - C_L}, \]

where \( \sigma_0 \) is the estimated volatility in the next iteration, \( \sigma_L \) is the lowest volatility estimate, \( \sigma_H \) is the highest volatility estimate, \( C_0 \) is the actual market price option, \( C_L \) is the low value of the option.

The two segmentation method is suitable for all types of the options contract. The implied volatility is the assumption of unbiased estimate volatility. It means that the volatility will be the same at the same maturity time. But in practical terms, when the strike price is different at the same maturity time, the implied volatility will also be different.

Some years later, a method called Vega weighted method [3] was introduced where Vega is the option price volatility of underlying assets for the sensitivity coefficient. This latest method is to construct a volatility matrix. Firstly, we calculate the implied volatility by using the market price of the options with the different strike price and the different maturity date. Secondly, it is to make the strike price standardized by dividing underlying asset at the strike price by the market price. Finally, it is to sort of the implied volatility order by the standardization of the strike price and expiration. Then, the volatility matrix is established. Based on this, we can calculate the volatility smile index matrix. By given maturity date, dividing the implied volatility corresponding to the different strike prices by the implied volatility corresponding to the parity option, then multiplying by 100, then we can get the smile index matrix.
(VSI). We can use it to estimate the future implied volatility.

2.2 Calculating implied volatility

In the previous part, it is mentioned that implied volatility can be solved by taking the first four basic parameters and the actual market price of options into the option pricing model. But actually, it is quite tough to solve the volatility by the deformation of the model. We often solve the volatility using some numerical methods. Like, Manaster [5] proposed the Newton-Raphson fast interior point search algorithm, but the estimate value was strongly dependent on the initial value; Liu Yang [15] put forward to use the optimal method to get the implied volatility based on the average options price. It is known that the interior point search algorithm needs the helps with the computer in order to get the approximation.

Due to the continuity in the Black-Scholes equation, the researchers brought forward to an easy algorithm which can be directly used by the market investors. Brenner [4] made the Taylor expansion to the standard normal distribution and it was given the formula of the implied volatility in the parity option. We can get \( \sigma_i^* \approx \sqrt{\frac{2\pi}{T-t}} \frac{C_i^*}{S_t} \), here \( C_i^* \) is the market value in the parity option. However, it cannot calculate the implied volatility when the option is in or out of the option price. Chance [7] improved the Brenner’s model. He thought that the error between the in (out) option price and the average option price was caused by the strike price and the volatility, which was \( \Delta C^*(K^*,\sigma^*) \). Here we make two order-Taylor expansions, and then we can get

\[
\Delta \sigma^* = -\frac{b + \sqrt{b^2 - 4aq}}{2a},
\]

where \( a = \frac{C^*_{/\sigma}}{2} \), \( b = C^*_{/\sigma} + C^*_{/\sigma} \Delta K \),

\[
\Delta \sigma^* = \sigma, \quad \Delta K^* = K - K^*.
\]
Here $\sigma^*$ is the estimated value according to the Brenner model.

Chance model just considered the effect of the strike price and make the maturity time as the constant. So we cannot get the implied volatility at different maturity time. It also maybe has some influence on the accuracy.

Kelly [8] put the volatility as an implicit function of the option market price, and then rose computing the implied volatility algorithm which was based on the implicit function.

There is also another method called the implied volatility surface model [2], which is, the volatility is the implicit function of the maturity time and the strike price. In this model, it uses the parity option $(S = K^* e^{-r(T-t)} )$. It is believed that the deviation between the parity option and non-parity option is aroused by the volatility, strike price and the maturity time, that is $\Delta C^*_t(K^*, \sigma^*, T^*)$. Firstly, we take the two-order Taylor expansion to $\Delta C^*_t$ and then we can get

$$
\Delta C^*_t = \frac{\partial C^*}{\partial t} \Delta t^* + \frac{\partial C^*}{\partial K^*} (\Delta K^*) + \frac{\partial C^*}{\partial \sigma^*} (\Delta \sigma^*) + \frac{1}{2} \frac{\partial^2 C^*}{\partial K^*^2} (\Delta K^*)^2 + \frac{1}{2} \frac{\partial^2 C^*}{\partial \sigma^*^2} (\Delta \sigma^*)^2 \quad \text{[2.2.1]},
$$

where $\Delta t^* = T - T^*$, $\Delta \sigma^* = \sigma_t - \sigma_t^*$,

$$
C^*_t = rK^* e^{-r(T-t)} N(-d'_1) + \frac{\sigma^* K^*}{2\sqrt{T^* - t}} e^{-r(T-t)} n(d'_1),
$$

$$
d'_1 = \frac{\sigma^*}{2} \sqrt{T^* - t},
$$

$$
C^*_K = \frac{\partial C^*}{\partial K^*} = -e^{-r(T-t)} N(-d'_1),
$$

$$
C^*_{kk} = \frac{\partial^2 C^*}{\partial K^*^2} = \frac{e^{-r(T-t)}}{\sigma^* K^* \sqrt{T^* - t}} n(d'_1),
$$
\[ C_\sigma^* = \frac{\partial C^*}{\partial \sigma^*} = K^* \sqrt{T^* - t} e^{-r(T^* - t)} n(d_1^*), \]

\[ C_{\sigma\sigma}^* = \frac{\partial^2 C^*}{\partial \sigma^* \partial \sigma^*} = -\frac{\sigma^* K^*(T - t)^{3/2} e^{-r(T^* - t)}}{4} n(d_1^*), \]

\[ C_{\sigma X}^* = \frac{\partial^2 C^*}{\partial \sigma^* \partial K^*} = \frac{\sqrt{T^* - t} e^{-r(T^* - t)}}{2} n(d_1^*), \]

\[ C_{XX}^* = \frac{\partial^2 C^*}{\partial K^* \partial T} = re^{-r(T^* - t)} N(d_2^*) - \frac{(r - \sigma^*/\sqrt{4}(T^* t)}{\sigma^* \sqrt{T^* - t}} e^{-r(T^* - t)} n(d_2^*), \]

\[ C_{XT}^* = \frac{\partial^2 C^*}{\partial \sigma^* \partial T} = \frac{K^*}{2\sqrt{T^* - t}} e^{-r(T^* - t)} n(d_1^*) - \frac{K^*(r + \sigma^*/\sqrt{4}\sqrt{T^* - t})}{2} e^{-r(T^* - t)} n(d_1^*). \]

Substitute them into 【2.2.1】 , we can get that:

\[ a(\Delta \sigma^*_T)^2 + b \Delta \sigma^*_T + q = 0, \]

where \( a = C_{\sigma \sigma}^*/2, \)

\[ b = C_{\sigma K}^* + C_{\sigma T}^* \Delta K^* + C_{\sigma T}^* \Delta \tau^*, \]

\[ q = C_{K}^* (\Delta K) + \frac{C_{K K}^* (\Delta K)^2}{2} + C_{K \tau}^* \Delta K \Delta \tau^* - \Delta C^*. \]

By using the surface model, the numerical calculation shows that it has improved the estimated precision of volatility. At the same time, the implied volatility surface model can give the characters of volatility sneer and term structure of volatility.

In this paper, we focus on how to use the Black-Scholes theoretical framework, and obtain the information from the options market in the sense of risk-neutral measure to recovering underlying asset price movement. That is to get the implied volatility of the underlying asset price.
Chapter 3 The Original Problem

3.1 The implied volatility as a constant

Black-Scholes model leads an important role in option pricing and it is widely applied both in the theory and in practice. The volatility $\sigma$ is the important part in the Black-Scholes. One of the assumptions of Black-Scholes is that we always treat the volatility $\sigma$ as the constant. According to the Black-Scholes formula, we can express the option value as $V = V(S, t; \sigma, K, T)$. From the option market, we can know when $t = t_0$, $S = S_0$, $K = K_0$, $T = T_0$ and the option value is $V_0$.

We take all of them into the Black-Scholes formula; then derive an equation of $\sigma$:

$$V_0 = V(S_0, t_0; \sigma, K_0, T_0). \quad \text{【3.1】}$$

Since $\frac{\partial V}{\partial \sigma} > 0$, the implied volatility $\sigma = \sigma_0$ is the only solution in the equation 【3.1】. Thus, from the price of the options with the strike price $K_0$ and maturity time $T_0$, we can derived the implied volatility as $\sigma = \sigma_0$.

Based on the Black-Scholes formula assumption, when the implied volatility $\sigma$ is a constant, the implied volatility $\sigma_0$ should not be related to the strike price $K_0$ and the maturity time $T_0$. It seems that, according to the Black-Scholes pricing theory, the same assets with the same maturity time and different strike prices should have the same volatility. However, the market and the empirical test show that the volatility is not the constant. The volatility $\sigma$ is the function of $K, T$. it can be expressed as $\sigma = \sigma(K, T)$.

3.2 Volatility smile and volatility skew

From the market research, we can know that the price of the assets with the same maturity time is more different from the agreement price, and then the volatility $\sigma$
will become larger. As the shape looks like smiling, it is known as “the smile volatility”. For “the smile volatility” phenomenon, so many scholars have done some significant researches in previous. Heston\(^{[17]}\) pointed that the reason of the “the smile volatility” may be the non-continuous process of the stock price change in 1993. In 2003, Xiaorong Zhang\(^{[9]}\) said that the main cause of “the smile volatility” was the assets price process assumptions, and the factor of the market mechanism which brought the additional risk and hedging costs to the option sellers. For a given maturity time \(T\), when \(t = t_0\), the asset price is \(S_0\), the implied volatility \(\sigma\) varies with the strike price \(K\). “The smile volatility” shows in picture

![Smile](image)

The implied volatility curve of the stock option also can appear skewed; we call it “the fake smile”. “The fake smile” is related to the expectation of calibration of the future asset price movements. “The fake smile” graph is following

![Skew](image)

With a given strike price \(K\), the implied volatility \(\sigma\) changes with the maturity time \(T\). It just has one way and we show it as follows
These graphs show that treating the implied volatility $\sigma$ as a constant cannot reflect the reality problem. In one word, there is a certain error in assuming the implied volatility as a constant.

### 3.3 Some reasonable assumptions and a modified model

The reasonable assumption should be that the implied volatility $\sigma$ is the function of the time $T$ and the asset price $S$, that is, in the sense of the risk-neutral measure, we change the random process into the model
\[
dS(t) = S(t)(r-q)dt + S(t)\sigma(t,S(t))dW(t).
\]

Then we can get the corresponding Black-Scholes equation
\[
\frac{\partial V(t,S)}{\partial t} + \frac{1}{2} \sigma^2(t,S)S^2 \frac{\partial^2 V(t,S)}{\partial S^2} + (r-q)S \frac{\partial V(t,S)}{\partial S} - rV(t,S) = 0.
\]

It can be proved as follows

Original Black-Scholes equation
\[
dB(t) = rB(t)dt, \quad [3.3.1]
\]
\[
dS(t) = S(t)\alpha(t,S(t))dt + S(t)\sigma(t,S(t))dW(t). \quad [3.3.2]
\]

We suppose that trading in the market and their price has the form
\[
\pi(t) = V(t,S(t)), 0 \leq t \leq T. \quad [3.3.3]
\]

According to the Black-Scholes equation, we can get the form of its price
\[
\frac{\partial V(t,S)}{\partial t} + rS \frac{\partial V(t,S)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V(t,S)}{\partial S^2} = rV(t,S).
\]

After changing it, we have that
\[
dS(t) = S(t)(r-q)dt + S(t)\sigma(t,S(t))dW(t).
\]

So now, we can get the equation of the Black-Scholes
\[
\frac{\partial V(t,S)}{\partial t} + \frac{1}{2} \sigma^2 (S(t))^2 \frac{\partial^2 V(t,S)}{\partial S^2} + (r-q)S \frac{\partial V(t,S)}{\partial S} - rV(t,S) = 0.
\]

Here we use the call option \( V(S,T) = (S - K)^+ \).

In order to get \( 3.3.6 \), we can apply Ito’s lemma to \( 3.3.3 \) and \( 3.3.5 \) given
\[
d\pi(t) = (r-q)\pi(t)dt + \sigma\pi(t)dW(t),
\]
where \((r-q)\pi = \frac{V_r + (r-q)SV_s + \frac{1}{2} \sigma^2 S^2 V_{ss}}{V}, \sigma = \frac{\sigma SF_S}{F}.\)

The function \( \alpha, \sigma, V, V_r, V_S, V_{ss} \) are evaluated. We build a portfolio from the stock and the derivative. If \((u_S, u_\pi)\) denotes the relative portfolio and \( V \) denotes the value process, then
\[
dV = V \left\{ u_S \left[(r-q)dt + \sigma dW\right] + u_\pi (r-q)\pi dt + \sigma_\pi dW \right\}.
\]

So, if we choose \( u_S \) and \( u_\pi \), that \( u_S \sigma + u_\pi \sigma = 0 \).

Then we can make the \( dW \) term vanish. Knowing \( 3.3.9 \) and \( u_S + u_\pi \equiv 1 \),
we can see that \( u_S = \frac{\sigma_\pi}{\sigma_\pi - \sigma}, u_\pi = \frac{-\sigma}{\sigma_\pi - \sigma}. \)

At last, we can get that \( u_S = \frac{SV_S}{SV_S - V}, u_\pi = \frac{-V}{SV_S - V}. \)

So now we can get \( 3.3.6 \).

For \( 3.3.6 \), it is impossible to get the explicit expression. The only way to solve it is to imply the numerical methods.
3.4 Original problem

After changing the model of the underlying asset, we question that “how can we determine the volatility of an underlying asset price from its option price quotes in the options market? Mathematically, \( t = t_0, S = S_0 \), we know \( V(S_0, t_0; \sigma, K, T) = V_{k,l} \)
(k=1,...,m, l=1,...,n), and how to get the volatility \( \sigma = \sigma(S, t) \)?”

From the call-put parity, we know that we will get the same volatility \( \sigma = \sigma(S, t) \) by using the call option or the put option. In this paper, we take the call option as the example. Here we show the problem \( P \) at first.

Problem P Let \( V = V(S, t; \sigma, K, T) \) be the price of the call option, and it satisfies the equation:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 (S, t) S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} - rV = 0 \quad (0 \leq S < \infty, 0 \leq t < T),
\]

\[
V(S, T) = (S - K)^+, \quad (0 \leq S < \infty).
\]

Suppose at \( t = t^*, (0 \leq t^* < T) \), \( S = S^* \), we know that

\[
V(S^*, t^*; \sigma, K, T) = F(K, T), \quad (0 < K < \infty, T_1 \leq T \leq T_2).
\]

Now it is needed to find the volatility \( \sigma = \sigma(S, t), (0 \leq S < \infty, T_1 \leq t \leq T_2) \).
Chapter 4 Dupire’s Equation

4.1 Dupire’s formula

In this paper, we use the Dupire’s Method to get the volatility $\sigma$. Assume that the European call option price is $V = V(S, t; K, T)$.

Based on the Black-Scholes formula

$$V(S, t; K, T) = e^{-r(T-t)} \int_0^\infty p(S, t; \xi, T)(\xi - K)^+ d\xi,$$

where $p(S, t; \xi, T)$ is the transition probability density of stochastic process $S$.

Let $T > 0$ and $K > 0$, the price at time 0 of a European call option expiring at time $T$ with the strike price $K$ is

$$V(T, K) = e^{-rT} \int_K^\infty (S - K) p(T, S) dS.$$

And here we can calculate that

$$\frac{\partial V}{\partial K} = e^{-rT} \frac{\partial}{\partial K} \left( \int_K^\infty S p dS - K \int_K^\infty p dS \right) = -e^{-rT} \int_K^\infty p(T, S) dS,$$  \hspace{1cm} [4.1.1]

$$\frac{\partial^2 V}{\partial K^2} = e^{-rT} p(T, K).$$  \hspace{1cm} [4.1.2]

So we can know that $p(S, t; \xi, T) = e^{r(T-t)} \frac{\partial^2 V}{\partial K^2}$.  \hspace{1cm} [4.1.3]

Now, in order to get the volatility $\sigma$, we use the formula

$$V(T, K) = e^{-rT} \int_K^\infty (S - K) p(T, S) dS,$$

$$\frac{\partial V}{\partial T} = -re^{-rT} \int_K^\infty (S - K) p(T, S) dS + e^{-rT} \int_K^\infty (S - K) \frac{\partial p(T, S)}{\partial T} dS.$$  \hspace{1cm} [4.1.4]

In [4.1.4], we take the $I_0 = \int_K^\infty (S - K) \frac{\partial p(T, S)}{\partial T} dS,$  \hspace{1cm} [4.1.5]

we can get that the equation [4.1.4] then it can be expressed as

$$\frac{\partial V}{\partial T} = -rV + e^{-rT} I_0.$$  \hspace{1cm} [4.1.6]

For [4.1.2], we use the Kolmogorov’s Equation and we can get the formula:

$$I_0 = -\int_K^\infty (S - K)(r - q) \frac{\partial}{\partial S} (Sp(T, S)) dS.$$
\[ + \frac{1}{2} \int_{k}^{\infty} (S - K) \frac{\partial^2}{\partial S^2} (\sigma^2(S,T)S^2) p(T,S) \, dS. \]  

Here we take \( I_1 = -\int_{k}^{\infty} (S - K)(r - q) \frac{\partial}{\partial S} (Sp(T,S)) \, dS , \)
and that 
\[ I_2 = \frac{1}{2} \int_{k}^{\infty} (S - K) \frac{\partial^2}{\partial S^2} (\sigma^2(S,T)S^2) p(T,S) \, dS . \]

We can get \( I_1 \) and \( I_2 \) using integration by parts

\[ I_1 = -\int_{k}^{\infty} (S - K)(r - q) \frac{\partial}{\partial S} (Sp(T,S)) \, dS \]
\[ = -[(S - K)(r - q)Sp(T,S)]_{k}^{\infty} + (r - q)\int_{k}^{\infty} Sp(T,S) \, dS \]
\[ = (r - q)\int_{k}^{\infty} (S - K)p(T,S) \, dS + rK\int_{k}^{\infty} p(T,S) \, dS , \]

where we have to assume that \( \lim_{S \to \infty} p(T,S)S^2 = 0 \). So we can get that 
\( (r - q)\int_{k}^{\infty} (S - K)p(T,S) \, dS = -(r - q)V e^{rT} \). And then we can get the expression of the 
\( I_1, \quad I_1 = (r - q)V e^{rT} + rK\int_{k}^{\infty} p(T,S) \, dS . \)  

【4.1.7】

Now we will compute the \( I_2 \)

\[ I_2 = \frac{1}{2} \int_{k}^{\infty} (S - K) \frac{\partial^2}{\partial S^2} (\sigma^2(S,T)S^2) p(T,S) \, dS \]
\[ = \frac{1}{2} \left[ (S - K) \frac{\partial}{\partial S} (\sigma^2(S,T)S^2 p(T,S)) \right]_{k}^{\infty} - \frac{1}{2} \int_{k}^{\infty} \frac{\partial}{\partial S} \left[ \sigma^2(S,T)S^2 p(T,S) \right] \, dS \]
\[ = \frac{1}{2} \sigma^2(K,T)K^2 p(T,K) . \]  

【4.1.8】

We take the 【4.1.8】and 【4.1.9】 into the 【4.1.7】 we can get the \( I_0 \) at first, then we take the \( I_0 \) into the equation 【4.1.6】 , we can get that

\[ \frac{\partial V}{\partial T} = -rV + e^{-rT} \left[ (r - q)V e^{rT} + rK\int_{k}^{\infty} p(T,S) \, dS + \frac{1}{2} \sigma^2(K,T)K^2 p(T,K) \right] . \]  

【4.1.10】

Here we plug 【4.1.1】 and 【4.1.3】 into 【4.1.10】 , we can get that:

\[ \frac{\partial V}{\partial T} = -rV + (r - q)V - (r - q)K \frac{\partial V}{\partial K} + \frac{1}{2} \sigma^2(K,T)K^2 \frac{\partial^2 V}{\partial K^2} . \]

Then we get that the Dupire’s Equation which is
\[
\begin{aligned}
\frac{\partial V}{\partial T} &= \frac{1}{2} \sigma^2(K,T) K^2 \frac{\partial^2 V}{\partial K^2} - (r-q)K \frac{\partial V}{\partial K} - qV, (0 \leq K < \infty, t \leq T) \\
V|_{t,t} &= (S-K)^+, (0 \leq K < \infty)
\end{aligned}
\]

Then we solve the equation (4.11), we can get the volatility \( \sigma \)

\[
\sigma(K,T) = \sqrt{\frac{\frac{\partial V}{\partial T} + (r-q)K \frac{\partial V}{\partial K} + qV}{\frac{1}{2} K^2 \frac{\partial^2 V}{\partial K^2}}}.
\]

Therefore, at \( t = t^*, (0 \leq t^* < T_1) \), \( S = S^* \), we know that

\[
V(S^*, t^*; \sigma, K, T) = F(K, T), \quad (0 < K < \infty, T_1 \leq T \leq T_2).
\]

Then we can easily calculate the \( \sigma(K, T) \) by (4.1.12). From the equation (4.1.12) we can see that to get \( \sigma(K, T) \), we must compute the derivatives \( F_{KK}, F_T \) at first. But a small error in \( F \) can result in a big changes in its derivatives, especially in its second derivatives. So the result is overly sensitive and the algorithm is unstable. Then, we can say that the algorithm to calculate \( \sigma(K, T) \) is ill-posed.

We must point out that \( F(K, T) \) is given on a set of discrete points \( \{(K_i, T_i)\} \) \( (k = 1, \ldots, m, l = 1, \ldots n) \). But in the domain \( (0 \leq K < \infty, T_1 \leq T \leq T_2) \), \( F(K, T) \) is got from the discrete points using interpolation or extrapolation technique and it will occur the error. This will also make the value of \( \sigma(K, T) \) lose its distortion. So Dupire’s method cannot easily be applied to practical problems.

### 4.2 Duality problem

Although Dupire’s method is difficult to use in practice, we can make some remainds on it. Firstly, we must see the significance of the Dupire’s Method. That is, we can make the implied volatility determination become a parabolic equation of the “terminal” problem. This is a typical partial differential equation’s inverse problem and we can find a well-posed algorithm for it through many ways.
Using the Dupire’s dual method, we can make the original problem P into the following inverse problem of parabolic equation \((q = 0)\).

**Question D:** Suppose that at \(t = 0, S = S_0\), we know that all the maturity time are in the domain \([T_1,T_2]\), the price of the European option asset \(V_0(K,T)\) with the strike price \(K \in (0, \infty)\), we need to find the implied volatility \(\sigma(K,T)\):

\[
\hat{\sigma}(K,T) = \begin{cases} 
\sigma_0(K), & (0 \leq T \leq T_1, 0 < K < \infty) \\
\sigma(K,T), & (T_1 \leq T \leq T_2, 0 < K < \infty)
\end{cases}
\] 【4.2.1】

subject to \(V(K,T;S_0,0) = V_0(K,T), (0 < K < \infty, T_1 \leq T \leq T_2)\). 【4.2.2】

It satisfies the equation:

\[
\begin{cases}
\frac{\partial V}{\partial T} = \frac{1}{2} K^2 \hat{\sigma}^2(K,T) \frac{\partial^2 V}{\partial K^2} - rK \frac{\partial V}{\partial K}, (0 \leq T \leq T_2, 0 < K < \infty).
\end{cases}
\] 【4.2.3】

\(V|_{T=0} = (S_0 - K)^+, (0 < K < \infty)\)

**Problem D** can be divided into two problems:

**Problem D₁:** Given \(V(K,T_1)\), find \(\sigma_0(K)\).

Subject to \(V(K,T_1) = V(K,T_1;S_0,0;\sigma_0(K))\).

**Problem D₂:** Given \(V(K,T), (T_1 \leq T \leq T_2)\) and \(\sigma_0(K)\), find \(\sigma(K,T)\).

Subject to \(V(K,T) = V(K,T;S_0,0;\hat{\sigma}(K,T))\).

\(\hat{\sigma}(K,T) = \sigma_0(K), (\forall T \in [0,T_1])\).
In order to simplify the problem $D$, we need to set and change the variables

$$
y = \ln \frac{K}{S_0}, \quad \tau = T,\quad [4.2.4]
$$

$$
v^*(y, \tau) = \frac{1}{S_0} V_0(S_0 e^y, \tau),\quad [4.2.5]
$$

$$
v(y, \tau) = \frac{1}{S_0} V(S_0 e^y, \tau),\quad [4.2.6]
$$

$$
a(y, \tau) = \begin{cases} 
a_0(y), (y \in \mathcal{R}, 0 \leq \tau \leq T_1) \\
 a(y, \tau), (y \in \mathcal{R}, T_1 \leq \tau \leq T_2).
\end{cases} \quad [4.2.7]
$$

And we must point out that

$$
a_0(y) = \frac{1}{2} \sigma_0^2(S_0 e^y),\quad [4.2.8]
$$

$$
a(y, \tau) = \frac{1}{2} \sigma^2(S_0 e^y, \tau).\quad [4.2.9]
$$

Now we will get the new problems in new variables:

$P_1$: Find $a_0(y)$ in the domain $\{y \in \mathcal{R}, 0 \leq \tau \leq T_1\}$ where $v(y, \tau; a_0)$ at time $\tau = T_1$,

$v(y, T_1; a_0) = v^*(y, T_1)$ satisfies the equations

$$
\begin{align*}
\frac{\partial v}{\partial \tau} &= a_0(y) \left( \frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} \right) - r \frac{\partial v}{\partial y} \\
v(y, 0) &= (1 - e^y)^+. \quad [4.2.10]
\end{align*}
$$
Question $P_1$ is a typical terminal problem, which is, we know the solution in last time $\tau = T_1$, and then we need to find the first factor $a_0(y)$.

$P_2$: Find $a(y, \tau)$ in the domain $(y \in \Re, T_1 \leq \tau \leq T_2)$, where $a(y, T_1) = a_0(y)$ and $v(y, \tau; a) = v^*(y, \tau)$ satisfies the equations

\[
\left\{\begin{array}{l}
\frac{\partial v}{\partial \tau} = a(y, \tau)\left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}\right) - r \frac{\partial v}{\partial y} \\
v(y, T_1) = v(y, T; a_0)
\end{array}\right. \quad \text{【4.2.11】}
\]

The question $P_2$ is different from the question $P_1$. For this problem, the given value $v^*(y, \tau)$ is in the whole domain and it needs to find the first factor $a(y, \tau)$ which is the function of $y$ and $\tau$.

Question $P_1$ and $P_2$ have the same difficulties, that is, the ill-posed in the inverse problem. It means that $a_0(y)$ and $a(y, \tau)$ have no continuous dependence on the given function $v^*(y, T_1)$ and $v^*(y, \tau)$. In another word, $a_0(y)$ and $a(y, \tau)$ are extremely sensitive on the given function $v^*(y, T_1)$ and $v^*(y, \tau)$. 
Chapter 5 The Regularization Method

5.1 Regularization idea

To solve the ill-posed problem, we take the regularization method in this paper which is proposed by A. N. Tikhonov in 1950s. Its main idea is as follows.

Let $U$, $F$ be given metric spaces, $A$ is an operator defined on $F$, that is $A: F \to U$.

The original problem: Given $V_0 \in U$, find $\sigma_0 \in F$, satisfied $A(\sigma_0) = V_0$. 【5.1.1】

There are two possibilities

1. We know that $AF \subset U$, but $AF \neq U$, according to the given $V_0 \in U$, 【5.1.1】may be have no solution in $F$.

2. According to the given $V_0$, 【5.1.1】may be have solution $\sigma_0$, but it is unstable.

That is, $\sigma_0$ has no continuous dependence on $V_0$. We can say that the small change in $V_0$ in $U$ may lead to a big change $\sigma_0$ in $F$.

The main idea of the regularization method is to use a cluster of well-posed problems to take place of the original problem. Although it is just an approximation solution to the original problem 【5.1.1】，the process of getting the approximation solution is quite stable. We can achieve the approximation solution in computing and use this solution to instead of the original true solution. We call the well-posed problems as the “regularization problem”. It is often from the operator $A$ and takes the parameter $\alpha$ to implement it.

Regularization Problem: Given $V_0 \in U$, find $\sigma_\alpha \in F$ ($\alpha > 0$),

satisfied $J_\alpha(\sigma_\alpha) = \min_{\sigma \in F} J_\alpha(\sigma)$. 【5.1.2】

And we define $J_\alpha(\sigma) = \rho^U(A\sigma, V_0) + \alpha R(\sigma)$, 【5.1.3】

where $\rho^U(\cdot, \cdot)$---the distance in the space $U$,
\[ \alpha \mathfrak{R}(\sigma) \] --- the regularization operator,

\[ \alpha \] --- the regularization factor.

The regularization method is that solving the well-posed problem \([5.1.2]\) by using the solution \( \sigma_{\alpha} \) instead of \( \sigma_0 \). The \( \alpha \) can be any number. When \( \alpha = 0 \), \( \sigma_{\alpha} = \sigma_0 \) we can get the true solution, but the algorithm is unstable. When \( \alpha \) becomes bigger, \( \sigma_{\alpha} \) will get more far from the true solution \( \sigma_0 \). So in practice, we try our best to take the \( \alpha \) smaller then we can make the process more stable.

### 5.2 The regularized version of the original problem

Now we go back to the problem \( P_1 \) and \( P_2 \). Firstly we take the regularization method into the problem \( P_1 \) and we call it problem \( Q_0 \).

Problem \( Q_0 \): At \( \tau = T \), find \( \tilde{a}_0(y) \in \mathcal{F} \), which satisfies

\[ J_{\alpha}^0(a_0) = \min_{a_0 \in \mathcal{F}} J_{\alpha}^0(a_0), \]

where

\[ J_{\alpha}^0(a_0) = \frac{1}{2} \int_{\mathbb{R}} \left[ v(y,T;\tilde{a}_0) - v^*(y,T) \right]^2 dy + \frac{\alpha}{2} \int_{\mathbb{R}} \left| \frac{da_0}{dy} \right|^2 dy, \]

\( v(y,\tau;\tilde{a}_0) \) is the solution to the Cauchy problem \( P_1 \) \([4.2.10]\),

\[ F = \{ a_0(y) \mid 0 < a_2 \leq a_0(y) \leq a_1, \int_{\mathbb{R}} \left| \frac{da_0}{dy} \right|^2 dy < \infty \}. \]

We take \( a_1 \) and \( a_2 \) as the given positive constant. We often call \( F \) as admissible set. This set make the Cauchy problem \([4.2.9]\) to have the exactly solution. \( J_{\alpha}^0(a) \) is called the cost function. \( a_0 = a_0(y) \) is called control variable. \( \tilde{a}_0(y) \) is named as optimal control or minimizer. We call this variation problem \( Q_0 \) as the optimal control problem.

There exists a minimizer \( \tilde{a}_0(y) \in F \) in variation problem \( Q_0 \). We can prove it by using the theory of the partial differential equations \([10][11]\). The proof process is as follows.
Proof: let \((V_n, a_n)\) be a minimizing sequence. Since \(J(a_n) \leq C\) and due to the special structure of \(J\), we infer that
\[
\| \nabla a_n \|_{L^2(\mathbb{R})} \leq C \quad (C \text{ is dependent of } n).
\]

And now we use the imbedding theorem we can get that
\[
\| a_n \|_{C^{1/2}(\mathbb{R})} \leq C.
\]

Thus, we know that
\[
\| V_n(y, \tau) \|_{C^{0,1/4}(Q)} \leq C, \quad (C \text{ is dependent of } n)
\]
\[
\| V_n(y, \tau) \|_{C^{0,1/4}(w)} \leq C, \text{ for any } w \subset Q,
\]
here \(Q = \mathbb{R} \times [0, \tau^+]\).

Thence we can choose a subsequence of \(a_n\) and \(V_n\), again denoted by \(a_n\) and \(V_n\), such that
\[
a_n(y) \to a_0(y) \in C^\alpha(\mathbb{R}), \text{ uniformly in } C^\alpha(\mathbb{R}) \quad (0 \leq \alpha < \frac{1}{2}),
\]
\[
V_n(y, \tau) \to V_0(y, \tau), \text{ uniformly in } C^{\alpha, \alpha/2}(\mathbb{R}) \cap C^{2+\alpha, \alpha/2}(Q).
\]

We can check easily that \((a_0(y), V_0(y, \tau))\) satisfy these two equations
\[
LV = V_y - a(y)(V_{yy} - V_y) + (r - q)V_y = 0, \quad y \in \mathbb{R}, \tau \in (0, \tau^+),
\]
\[
V(y, 0) = (1 - e^y)^+, \quad y \in \mathbb{R}.
\]

By the Lebesgue control convergence theorem and the weak semi continuity of the \(L^2\) norm we get that
\[
J(a_0) \leq \liminf_{n \to \infty} J(a_n) = \min_{a \in F} J(a) = J(a_0).
\]

Hence, we can know that \(J(a_0) = \min_{a \in F} J(a_0)\).

So we can know that there exists a minimizer \(a_0(y) \in F\) in variation problem \(Q_0\)

In order to get the solution of \(P_2\), we can make the time range \([T_1, T_2]\) discrete.

We set \(T_i = \tau_0 < \tau_1 < \ldots < \tau_N = T_2\),
$$h = \frac{1}{N}(T_2 - T_1),$$
$$\tau_n = T_1 + nh, \quad (n = 0,1,...N).$$

From the problem $Q_0$, when $\tau = \tau_0$ we can get the $a_0(y)$. And then we use the induction method, then we can get $a_n(y) = a(y, \tau_n)$ when $\tau = \tau_n$.

Problem $Q_n$: Suppose we know $a_0(y),...,a_{n-1}(y)$ and $v(y, \tau; a_k)$ $(k = 1,...n-1)$ which is the solution to the equation
$$\frac{\partial v}{\partial \tau} = a_k(y)\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y}\right) - r \frac{\partial v}{\partial y}, \quad (y \in \mathbb{R}, x_{k-1} \leq \tau \leq \tau_k), \quad \text{【5.2.1】}$$
$$v(y, \tau_{k-1}) = v(y, \tau_{k-1}; a_{k-1}), \quad (y \in \mathbb{R}). \quad \text{【5.2.2】}$$

Find $a_n(y) \in F$ in the domain $\{y \in \mathbb{R}, \tau_{n-1} \leq \tau \leq \tau_n\}$, which satisfies $J_n^a(a_n) = \min_{a_n \in F} J_n^a(a_n)$.

We define
$$J_n^a(a_n) = \frac{1}{2h} \int_{\xi} |v(y, \tau_n; a_n) - v^*(y, \tau_n)|^2 \, dx + \frac{\alpha}{2} \left(\frac{1}{h} \int_{\xi} |a_n(y) - a_{n-1}(y)|^2 \, dy + \int_{\xi} \left|\frac{da_n}{dy}\right|^2 \, dy\right)$$
where $v(y, \tau_n; a_n)$ is the solution to the equations 【5.2.1】 and 【5.2.2】.

We also can prove that there exists a minimizer $a_n(y) \in F$ in variation problem $Q_n$\textsuperscript{[13][14]}.

Now we consider when $h \rightarrow 0$, how the $\{a_n^-(y)\}$ will show. We fix $h$, and define the functions $a^h(y, \tau)$ and $v^h(y, \tau)$:

$$a^h(y, \tau) = \begin{cases} a_n(y), & \tau = \tau_n, (n = 0,1,...,N) \\ \frac{\tau - \tau_{n-1}}{h} a_n(y) + \frac{\tau_n - \tau}{h} a_{n-1}(y), & \tau_{n-1} \leq \tau \leq \tau_n, (n = 1,...N) \end{cases},$$
$$v^h(y, \tau) = v(y, \tau; a_n), \tau_{n-1} \leq \tau \leq \tau_n, (n = 1,...N).$$
We can know that when $h \to 0, \{a^h(y, \tau), v^h(y, \tau)\}$ converges to the limit function $\{a(y, \tau), v(y, \tau)\}$. Certain estimates of $a^h(y, \tau)$ are constructed which are uniformly bounded and it is independent of the mesh parameter $h$. So we can take the limit for the approximate sequence $a(y, \tau)$ as $h \to 0$. 
Chapter 6 The Stability of the Volatility Function \[1\]

Our main reference in this chapter is [1]. In this part, we need to check whether the volatility functions \( \tilde{a}_0(y) \) and \( a(y, \tau) \) is stable or not. Firstly, we establish a necessary condition for the minimizer \( \tilde{a}(y) \).

Set \( \tilde{a}_0(y) \) is a minimizer, and \( F \) is a convex set, thus for an arbitrary given \( h(y) \in F \),
\[
a_\lambda(y) = (1-\lambda) \tilde{a}_0 + \lambda h \quad (\lambda \in [0,1]) .
\]

And here we define the function \( j(\lambda) \)
\[
j(\lambda) = J^a_0((1-\lambda) \tilde{a}_0 + \lambda h),
\]
then we can know that
\[
j(\lambda) = \frac{1}{2} \int_{\mathbb{R}} \left[ v(y, T_1; (1-\lambda) \tilde{a}_0 + \lambda h) - v^*(y, T_1) \right]^2 dy + \alpha \int_{\mathbb{R}} \left[ \frac{d}{dy} a_\lambda \right]^2 dy .
\]

Now we attain the minimum at \( \lambda = 0 \), thus
\[
j'(0) = \left[ \frac{d}{d\lambda} \int_{\mathbb{R}} \left[ v_\lambda(y, T_1) - v^*(y, T_1) \right]^2 dy + \alpha \frac{d}{d\lambda} \int_{\mathbb{R}} \left[ \frac{d}{dy} ((1-\lambda) \tilde{a}_0 + \lambda h) \right]^2 dy \right]_{\lambda=0} . \tag{6.1}
\]

We know that \( j'(0) > 0 \),

and here \( v_\lambda(y, \tau) \) which is satisfied the equation
\[
\begin{align*}
\frac{\partial v_\lambda}{\partial \tau} &= a_\lambda(y) \left[ \frac{\partial^2 v_\lambda}{\partial y^2} - \frac{\partial v_\lambda}{\partial y} \right] - r \frac{\partial v_\lambda}{\partial y} . \tag{6.2}
\end{align*}
\]

Here we make \( \xi_\lambda(y, \tau) = \frac{dy_\lambda}{d\lambda} \) and calculate [6.2], we can get that
\[
\begin{align*}
\frac{\partial \xi_\lambda}{\partial \tau} &= a_\lambda(y) \left[ \frac{\partial^2 \xi_\lambda}{\partial y^2} - \frac{\partial \xi_\lambda}{\partial y} \right] - r \frac{\partial \xi_\lambda}{\partial y} + \left[ \frac{\partial^2 v_\lambda}{\partial y^2} - \frac{\partial v_\lambda}{\partial y} \right] (h-a_0) . \tag{6.3}
\end{align*}
\]

So we can express [6.1] as
\[
\int_{\mathbb{R}} [v(y, T_1) - v^*(y, T_1)] \xi_\lambda(y, T_1) dy + \alpha \int_{\mathbb{R}} \frac{d}{dy} a_\lambda dy (h-a_0) dy \geq 0 , \quad \forall h \in F . \tag{6.4}
\]
Here $\tilde{v}(y, \tau)$ is the solution to the problem $P_1$. And $a(y) = \tilde{a}_0(y), \xi_0(y, \tau)$ is the solution to 【6.3】.

When $\lambda = 0$, we have the following equation

$$
L \xi_0 = \frac{\partial \xi_0}{\partial \tau} - a_0(y) \left[ \frac{\partial^2 \xi_0}{\partial y^2} - \frac{\partial \xi_0}{\partial y} \right] + r \frac{\partial \xi_0}{\partial y} = \left[ \frac{\partial^2 \tilde{v}}{\partial y^2} - \frac{\partial \tilde{v}}{\partial y} \right] (\hat{a} - a_0) .
$$

【6.5】

$\xi_0(y, 0) = 0$

Let $\tilde{\varphi}(y, \tau)$ be the solution to the ad-join problem of the problem 【6.3】 i.e

$$
\left\{ \begin{array}{l}
\tilde{L} \tilde{\varphi} = 0, \ (y \in \mathbb{R}, 0 \leq \tau < T_1) \\
\tilde{\varphi}(y, T_1) = v(y, T_1) - \tilde{v}^*(y, T_1), \ (y \in \mathbb{R})
\end{array} \right.
$$

【6.6】

where the differential operator $\tilde{L}$ is the ad-join operator of $\hat{L}$ and

$$
\tilde{L} \tilde{\varphi} = -\frac{\partial \tilde{\varphi}}{\partial \tau} - \frac{\partial^2 (a_0 \tilde{\varphi})}{\partial y^2} - \frac{\partial (a_0 \tilde{\varphi})}{\partial y} - r \frac{\partial \tilde{\varphi}}{\partial y}.
$$

From the Green formula,

$$
\int_{\mathbb{R}}^T \int_{\mathbb{R}} (\varphi \tilde{L} \xi_0 - \xi_0 \tilde{L} \varphi) dyd\tau = \int_{\mathbb{R}}^T \int_{\mathbb{R}} \left[ \frac{\partial (\varphi \xi_0)}{\partial \tau} - \frac{\partial (a_0 \varphi \xi_0)}{\partial y} + \frac{\partial (\xi_0 \varphi)}{\partial y} (a_0 \varphi) + \frac{\partial (\xi_0 a_0 \varphi)}{\partial y} + r \frac{\partial (\xi_0 \varphi)}{\partial y} \right] dyd\tau
$$

$$
= \left[ (\varphi \xi_0)_{\tau=T_1} - (\varphi \xi_0)_{\tau=0} \right] dy.
$$

From the equations 【6.5】 and 【6.6】 ,we can derive that

$$
\int_{\mathbb{R}}^T \int_{\mathbb{R}} (\tilde{v}^2 \tilde{v} - \frac{\partial \tilde{v}}{\partial y}) (\hat{a} - a_0) dyd\tau = \int_{\mathbb{R}} \xi_0(y, T_1) \left[ v(y, T_1) - v^*(y, T_1) \right] dy .
$$

【6.7】

Now we take 【6.7】 into 【6.4】 , we can get that

$$
\int_{\mathbb{R}}^T \int_{\mathbb{R}} (\tilde{v}^2 \tilde{v} - \frac{\partial \tilde{v}}{\partial y}) (\hat{a} - a_0) dyd\tau + a \int_{\mathbb{R}} \frac{d}{dx} \frac{d}{dx} (\hat{a} - a_0) dy \geq 0. \forall h \in F .
$$

【6.8】
Now we let  \( f(y; v, \phi) = \frac{1}{\alpha} \int_0^t \phi(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}) \, d\tau \), \[6.9\]
then we can see that the equation \[6.1.8\] can change into the form of
\[
\int \left[ \frac{d}{dx} a_0^\ast \frac{d}{dx} (h - a_0^\ast) + f(y; v, \phi)(h - a_0^\ast) \right] \, dy \geq 0, \forall h \in F.
\]

We know that \( h(y) \) is arbitrary, so the equation above is equal to
\[
-\frac{d^2 a_0^\ast}{dy^2} + f(y; v, \phi) = 0, \quad a_2 < a_0^\ast(y) < a_1, \quad \[6.10\]
\[
-\frac{d^2 a_0^\ast}{dy^2} + f(y; v, \phi) \leq 0, \quad a_0^\ast(y) = a_1, \quad \[6.11\]
\[
-\frac{d^2 a_0^\ast}{dy^2} + f(y; v, \phi) \geq 0, \quad a_0^\ast(y) = a_2. \quad \[6.12\]
\]
These equations are the double obstacles elliptic variation inequality problem.
So now we finish proving the necessary condition of the variation problem \( Q_0 \).
Then, we start to talk about the sufficient condition of the variation problem \( Q_0 \).
If \( a_0^\ast(y) \in F \) is the minimizer of the variation problem \( Q_0 \), there exists a triplet
\[
\left\{ a_0^\ast(y), v(y, \tau), \phi(y, \tau) \right\},
\]
which is a solution to the following PDE problem in the domain \( \{ y \in \mathbb{R}, 0 \leq \tau \leq T \} \)

\[
\begin{align*}
\frac{\partial v}{\partial \tau} & - a_0^\ast(y) \frac{\partial^2 v}{\partial y^2} + a_0^\ast(y) \frac{\partial v}{\partial y} + r \frac{\partial v}{\partial y} = 0, \\
\vspace{-0.5cm}
\end{align*}
\]
\[6.13\]
\[
\begin{align*}
\vspace{-0.5cm}
\phi(y, 0) = (1 - e^y)^r
\end{align*}
\]
\[
\begin{align*}
\frac{\partial \phi}{\partial \tau} - \frac{\partial^2 \phi}{\partial y^2} (a_0^\ast(y) \phi) - \frac{\partial}{\partial y} (a_0^\ast(y) \phi) - r \frac{\partial \phi}{\partial y} = 0, \\
\phi(y, T_1) = v(y, T_1) - v^*(y, T_1)
\end{align*}
\]
\[6.14\]
And the equation \[6.10\] \[6.11\] \[6.12\], here
\[ v^*(y, \tau) = \frac{1}{S_0} V_t(S_0 e^y, \tau), \]

\[ f(y; v, \varphi) = \frac{1}{\alpha} \int_0^\tau \varphi \left( \frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} \right) d\tau. \]

From these equations, we can see that \([6.13]\) is a Cauchy problem to a forward parabolic equation; \([6.14]\) is a Cauchy problem to a backward parabolic equation; \([6.10]\) \([6.11]\) \([6.12]\) are a variation inequality to a second order ODE. So this is a series of forward-backward parabolic equations coupled with an elliptic variation inequality. Here we will prove the implied volatility \(a_0(y)\) uniqueness \([10]\)[12].

**Proof:** Suppose \(a_1(y), a_2(y)\) be the two minimizes of the problem

\[ J_a(\sigma_a) = \min_{\sigma \in \mathcal{F}} J_a(\sigma). \]

When \(h = a_2\), we take \(\sigma_a = a_1\). When \(h = a_1\), we use \(\sigma_a = a_2\) in the equation \([6.8]\).

Then we have the following equations

\[ \int_0^\tau \left[ \varphi_1 \left( V_{1y} - V_{1y} \right)(a_2 - a_1) dy + \alpha \int \nabla a_1 \nabla (a_2 - a_1) dy \right] \geq 0, \quad [6.15] \]

\[ \int_0^\tau \left[ \varphi_2 \left( V_{2y} - V_{2y} \right)(a_1 - a_2) dy + \alpha \int \nabla a_2 \nabla (a_1 - a_2) dy \right] \geq 0, \quad [6.16] \]

where \(\{V_i, \varphi_i\} (i = 1, 2)\) are the solutions of the equations \([6.13]\) and \([6.14]\). All of them are with \(\tilde{a}_i = a_i (i = 1, 2)\).

From the equations \([6.15]\) and \([6.16]\), we can get that

\[ \alpha \int \nabla (a_1 - a_2)^2 dy \leq \int_0^\tau \left[ \varphi_2 \left( V_{2y} - V_{2y} \right)(a_1 - a_2) dy + \alpha \int \varphi_1 \left( V_{1y} - V_{1y} \right)(a_2 - a_1) dy \right] d\tau \]

\[ \leq \int_0^\tau a_2 \varphi_2 \left( \frac{a_2}{a_1} - 1 + \frac{a_1}{a_2} - 1 \right) \left( V_{1y} - V_{1y} \right)(a_1 - a_2) dy d\tau \]

\[ + \int_0^\tau (a_1 \varphi_1 - a_2 \varphi_2) \left( \frac{a_1}{a_2} - 1 \right) \left( V_{1y} - V_{1y} \right)(a_2 - a_1) dy d\tau \]

\[ + \int_0^\tau a_2 \varphi_2 (\frac{a_1}{a_2} - 1) (D_y^2 - D_y) (V_2 - V_1)(a_1 - a_2) dy d\tau \]
\[
\leq \int_0^t \left( -\Phi \frac{1}{a_i} A \right) \left( V_{1yy} - V_{1y} \right) dy \, d\tau + \int_0^t \int_{\mathbb{R}} \phi_2 A^2 \left( V_{1yy} - V_{1y} \right) dy \, d\tau + \int_0^t \int_{\mathbb{R}} -\phi_2 A \left( V_{yy} - V_y \right) dy \, d\tau .
\]

From the assumption, there exist a point \( y_0 \in [-\infty, +\infty] \) and it satisfies that
\[
A(y_0) = a_1(y_0) - a_2(y_0) = 0.
\]

Here we can prove that the implied volatility \( \tilde{a}_0(y) \) uniqueness.

Now we look into the other variation problem \( Q_n \). The limit function \( a(y, \tau) \) is determined when \( h \to 0 \), \( \{a^h(y, \tau), v^h(y, \tau)\} \) converges to the limit function \( \{a(y, \tau), v(y, \tau)\} \).

Suppose \( a(y, \tau) \) is the volatility determined by the limit function. And it exist a function \( \{a(y, \tau), v(y, \tau)\} \) in the domain \( \{y \in \Re, T_1 \leq \tau \leq T_2\} \) which is the solution of the following partial differential equations:

\[
\frac{\partial v}{\partial \tau} - a(y, \tau) \frac{\partial^2 v}{\partial y^2} + a(y, \tau) \frac{\partial v}{\partial y} + r \frac{\partial v}{\partial y} = 0, \quad [6.17]
\]

\[
\frac{\partial a}{\partial \tau} - \frac{\partial^2 a}{\partial y^2} + \frac{1}{2\alpha} \left[ v - v^*(y, \tau) \right] \frac{\partial^2 v}{\partial y^2} = 0, \quad a_2 < a(y, \tau) < a_1, \quad [6.18]
\]

\[
\frac{\partial a}{\partial \tau} - \frac{\partial^2 a}{\partial y^2} + \frac{1}{2\alpha} \left[ v - v^*(y, \tau) \right] \frac{\partial^2 v}{\partial y^2} \leq 0, \quad a(y, \tau) = a_1, \quad [6.19]
\]

\[
\frac{\partial a}{\partial \tau} - \frac{\partial^2 a}{\partial y^2} + \frac{1}{2\alpha} \left[ v - v^*(y, \tau) \right] \frac{\partial^2 v}{\partial y^2} \geq 0, \quad a(y, \tau) = a_2, \quad [6.20]
\]

\[
v(y, T_1) = \tilde{v}(y, T_1), \quad [6.21]
\]

\[
a(y, T_1) = \tilde{a}(y), \quad [6.22]
\]

We find out that the \( \tilde{v}(y, \tau) \) and \( \tilde{a}_0(y) \) are the solutions to the coupled equations \([6.10] [6.11] [6.12]\). And the equation \([6.17] [6.18] [6.19] [6.20] [6.21]\) \([6.22]\) are a Cauchy problem to the parabolic equations and variation inequalities of
parabolic type coupled nonlinear parabolic equations. It is a non-linear developing equation.
Chapter 7 Calculation of the Implied Volatility

In order to get the implied volatility $\sigma$ in the time domain $[0,T_2]$, we must divide it into two parts. Firstly we must get $\tilde{a}_0(y)$ from the problem $Q_0$, and then use the $\tilde{a}_0(y)$ as the initial value to solve the series of the variation problem $Q_n$ in the time domain $[T_1,T_2]$. Or we can directly solve the Euler equations $[6.17]$ $[6.18]$ $[6.19]$ $[6.20]$ $[6.21]$ $[6.22]$. Now we can get the value $a(y,\tau)$.

There are two ways to get the numerical solution to the problem $Q_0$. One is starting from the variation problem $Q_0$, discrete it, and make it into optimization problem which has a convex constraint, find its approximate solution at last. The other is starting from the necessary condition of the variation problem, it means that find the triplet $\left\{a(y),\tilde{v}(y,\tau),\tilde{\phi}(y,\tau)\right\}$, that is satisfied the forward-backward partial differential equation problem coupled with an elliptic variation inequality $[6.12]$ $[6.13]$ $[6.14]$ $[6.10]$ $[6.11]$.

In this paper, we focus on using the second method, that is solving the problem $[6.12]$ $[6.13]$ $[6.14]$ $[6.10]$ $[6.11]$.

7.1 Calculating the option price

Suppose that when $t=0, S=S^*$, and we can get the different strike price $K$ at the same maturity time, the option price $V(S^*,0;K)=F(K)$. Then we define the function that:

$$v^*(y) = \frac{1}{S} F(S^* e^y),$$

here $y = \ln \frac{K}{S^*}$.

In this paper, we use the implied volatility $\sigma$ as a constant or the smile as examples.
For testing, the initial implied volatility $\sigma = 0.4$. At the same time, assuming $S = S^* = 1$, $K_{\text{max}} = 10$, $r = 0.2$, $T_1 = 1$, $T_2 = 5$. We divide the strike price $K$, the time $[0, T_1]$ and $[T_1, T_2]$ into $N = 30$ steps and $M_1 = 300$, $M_2 = 1000$, respectively.

As we know, when the implied volatility is constant, we can use the Black-Scholes formula directly. The Black-Scholes formula is following

$$V(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

here $d_1 = \frac{\ln \left(\frac{S}{K} + \frac{r}{2}N(T-t)\right)}{\sigma \sqrt{T-t}},$

$$d_2 = d_1 - \sigma \sqrt{T-t}.$$

We take all the parameters into the Black-Scholes formula, we can get the option price $F(K)$. We can show $F(K)$ in the picture.

![Option Value Calculated by Given Implied Volatility F](image)

### 7.2 Calculating the local implied volatility

Now we use the option price $F(K)$ which is got form the part 7.1. Due to the strict monotonic of the option price $F(K)$, we can know that the Black-Scholes equation has the only solution $\sigma = \sigma_0(K)$.
(1) At first, we can directly use the defined function
\[ a_0(y) = \frac{1}{2} \sigma_0^2 (S^* e^y), \]
We can get that \( a_0 = \frac{1}{2} \times 0.4^2 = 0.08. \)

(2) Now we start to solve the Cauchy problem \([6.13]\]
\[
\begin{aligned}
\frac{\partial \bar{v}}{\partial \tau} - a_0(y) \frac{\partial^2 \bar{v}}{\partial y^2} + a_0(y) \frac{\partial \bar{v}}{\partial y} + r \frac{\partial \bar{v}}{\partial y} &= 0,
\end{aligned}
\]
where we take \( \bar{a}_0 = a_0, \) we can get the solution \( \bar{v}_0(y, \tau) \) now.

(3) We start to get the \( \varphi_0 \) from the Cauchy problem to the backward parabolic equation \([6.14]\]
\[
\begin{aligned}
\frac{\partial \bar{\varphi}}{\partial \tau} - \frac{\partial^2 \bar{\varphi}}{\partial y^2} (a_0(y) \bar{\varphi}) - \frac{\partial}{\partial y} (a_0(y) \bar{\varphi}) - r \frac{\partial \bar{\varphi}}{\partial y} &= 0,
\end{aligned}
\]
which \( \bar{a}_0 = a_0, \) \( \bar{\varphi}(y, T_1) = \bar{v}(y, T_1) - \bar{v}^*(y, T_1) \), we can get \( \bar{v}^*(y) \) it from the 7.1.

(4) From the definition of the \( f(y; v, \varphi) = \frac{1}{\alpha} \int_0^{T_1} \dot{\varphi}_0(y, \tau) \left[ \frac{\partial^2 v_0}{\partial y^2} - \frac{\partial v_0}{\partial y} \right] d\tau, \)
we can get \( f(y; v_0, \varphi_0) = \frac{1}{\alpha} \int_0^{T_1} \dot{\varphi}_0(y, \tau) \left[ \frac{\partial^2 v_0}{\partial y^2} - \frac{\partial v_0}{\partial y} \right] d\tau. \)

(5) Now we start to solve the variation inequality equation
\[
- \frac{d^2 \bar{a}_0}{dy^2} + f(y; \bar{v}, \bar{\varphi}) = 0, \quad a_2 < \bar{a}_0(y) < a_1,
\]
\[
- \frac{d^2 \bar{a}_0}{dy^2} + f(y; \bar{v}, \bar{\varphi}) \leq 0, \quad \bar{a}_0(y) = a_1,
\]
\[
- \frac{d^2 \bar{a}_0}{dy^2} + f(y; \bar{v}, \bar{\varphi}) \geq 0, \quad \bar{a}_0(y) = a_2,
\]
where the \( f(y; \bar{v}, \bar{\varphi}) \) is defined in the step (4), therefore we can get the solution \( a = a_1(y) (y \in \mathbb{R}) \). Then we use the induction process in order to get the sequence
$a = a_n(y)$.

(6) At last, we get all of them back to the original variables and compute the local implied volatility $\sigma_n(K)$.

$$\sigma_n(K) = \sqrt{2a_n(y)} = \sqrt{2a_n(\ln \frac{K}{S})}.$$  

In the time domain $[T_i, T_j]$, we use the same method as in $[0, T_i]$. But we must point out that the initial value is at the time $t = T_i$.

We also show the local implied volatility $\sigma_n$ in the picture.

7.3 The error in the implied volatility

We can express the error term between the original implied volatility and the local implied volatility as $\sigma_{err} = \sigma - \sigma_{local}$. We can see it in the picture clearly.

At the implied volatility $\sigma = 0.4$, the picture of the error is following.
From the figure, for the price part, we can see that the error of local implied volatility is large when the strike price is more different from the initial option price. That is, in the cases, deeply in the money and out of the money, the local volatility needs to be used. For the maturity data part, the local implied volatility is more accurate than the original one when the maturity data is small. And when it is large, the error will become smaller.

7.4 The error in the option price

We can also show the option price error. It is the error between the option price which is derived by the given volatility and the option price which is calculated by the local volatility.

Using the implied volatility $\sigma = 0.4$, and we can get the local implied volatility. Then we put it back into the Black-Scholes formula and we can get another option price. Now we can show it in picture.
At last, we will show the option price error pictures.

When the implied volatility $\sigma = 0.4$

The figure means that in the case at the money, when the maturity time is small, the local implied volatility is almost the same with the original one. However, with the maturity time increasing, the difference will appear and get larger and larger. At the case out of money, the value of the option for using the given implied volatility and the local implied volatility will be the same because the value of the option will always be zero. At the case in the price, when the maturity time is small, the error of the option value will be large so that the local implied volatility should be used in this situation.
7.5 The implied volatility is a function of the stock price

The same assumptions as below but the implied volatility $\sigma$ is not a constant. We use the smile curve to take the place of it. The original volatility can show in the picture:

Now we should apply the Black-Scholes equation to get the option price $F(K)$. The Black-Scholes equation is as follows

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

We also give the picture of $F(K)$.

And then we use the same method as below and get the local implied volatility $\sigma_n$ can be showed as follows.
We must take it into the Black-Scholes Equation and we can get the option price which is under the local implied volatility. We can get the option price under this local implied volatility.
The error of the implied volatility is showing.

The error of the option price is following.

The result is similar with the $\sigma = 0.4$ above. Local volatility should be used in the case in the price with small maturity data and in the case at the money with large maturity time.
Chapter 8 Conclusion

As we know that the implied volatility $\sigma$ has been playing an important role both in judging the futures market and application of strategic investment options. So the analysts need to know that how the implied volatility $\sigma$ varies. In fact, we cannot be prophets to forecast what $\sigma$ will be in the future. But we can be interpreters and translate all the information for the option markets into the volatility $\sigma$ of the underlying asset. The only thing for us is that we need to take the Today’s observed market option pricing into the Black-Scholes equations and solve it by the numerical methods.

There are some many papers about how to get the implied volatility $\sigma$. Manaster proposed the Newton-Raphson fast interior point search algorithm, and we know that the estimate value is relied on the initial value strongly. Then Liu Yang \cite{15} took the research by using the optimal method to get the implied volatility, this method was based on the average option price. In Liu’s study, he used the interior point search algorithm which needed the computer iteratively. Brenner \cite{4} brought the Taylor expansion to the standard normal distribution. It also can make the formula of the implied volatility from the parity option. But the Brenner’s model had some problems and then Chance \cite{7} took the effect of the Brenner’s model and made some changes. Chance started to consider the effect of the strike price and assumed the maturity time as a constant. But we cannot get the accuracy implied volatility at any different maturity time through this method. Kelly \cite{8} focused on rising computing the implied volatility algorithm which was putting the implicit function of the option market price. Kelly’s method considered to change the algorithm. There occurred another method which is better, called implied volatility surface model \cite{2}. In the implied volatility surface model, it made the implied volatility as the function of the maturity time and the strike price. The implied volatility surface model improved the accurate estimated volatility. It can also give the characters of volatility sneer and the term structure of
the implied volatility.

In this paper, we use a new method to calculate the implied volatility \(^1\). The numerical methods what we introduced in this paper is stable. On one hand, we prove the stability of this method. On the other hand, we show the method of how to calculate the implied volatility in this paper. The best way to check whether this method is useful is to see the error term of the implied volatility. Now we will show the error of the implied volatility.

The picture of the error is following.

From above, we can see that the error is very small. We can say that the algorithm we proposed in this paper is stable. So we can use this numerical algorithm in nowadays option markets. We just need to put all the information into this method and then use the numerical method to calculate the implied volatility. At last, we can get the implied local volatility \( \sigma_t = \sigma(S_t, t) \). Using this method, the implied volatility \( \sigma_t = \sigma(S_t, t) \) will be recovered based on Today’s knowledge.
References


[10]. Lishang Jiang, Youshan Tao, Identifying the Volatility of Underlying


[17]. Heston, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Option, 1993.
Appendix

The calculation Results

When the implied volatility is 0.4, we can get the local volatility data:

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