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Three Systems of Orthogonal Polynomials and Associated Operators

John Musonda

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Department of Mathematics
Uppsala University

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John Musonda
Department of Mathematics, Uppsala University

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Abstract

In this report, three systems of polynomials, that are orthogonal systems for three different but related inner product spaces, are presented. Three basic operators that are related to the systems are described, and boundedness of two other operators on a few Hilbert spaces is proven.

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Contents

1 Introduction

More than a decade ago, Professor Sten Kaijser happened to discover two remarkable systems of orthogonal polynomials. The most interesting of the systems was in fact not a standard system, but it had some other useful properties. These discoveries led to a dissertation by Tsehaye K. Araaya [4, 5]. In March this year, Professor Lars Holst [7] presented a new way to calculate the Euler sum, $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$. His calculations inspired Professor Kaijser to calculate a third system of polynomials, a system that turned out to fill a gap related to the previous systems. In this report, we present these three systems of orthogonal polynomials, and discuss some operators related to them.

The weight function that is used in one of the first two systems is the function $\omega_1(x) = 1/(2\cosh\frac{\pi}{2}x)$, while for the third system we use the self convolution of this function, that is, $\omega_2 = \omega_1 * \omega_1$. The function, ω_1, has three interesting properties that make it useful as a weight function. The first is that it is the density function of a probability measure, and the second is that it is up to a dilation its own Fourier transform, that is, it is the Fourier transform of the function $1/\cosh t$. The third is that it is closely related to the Poisson kernel for a strip of width two. The second property makes it possible to interpret its moments as values at zero of successive derivatives, while the third can be used for direct computations of many integrals.

This report is organised as follows: In section (2), we present preliminaries needed to study and understand the work in the subsequent sections. This section has four subsections. In the first, some of the notation used throughout the report is explained. The second reviews those aspects of the theory of Hilbert spaces which are particularly relevant to our study, while the third reviews different aspects of the theory of orthogonal polynomials of one real variable. In the fourth subsection, we introduce the spaces that are of interest to our study. Our first system which we call the σ-system is presented in section (3), while our second system which we call the τ-system is presented in section (4). As aforementioned, these two systems were studied in Araaya papers [4, 5], and here we just take an overview of the results so that this report can be self contained. Also in section (4), we introduce three operators R, J and Q, which are related to the systems. The third system which we call the ρ-system is presented section (5), and we study this system in detail since it is a new addition filling a gap related to the previous systems. This system of orthogonal polynomials is obtained by applying the Gram-Schmidt procedure to the sequence $\{x^n\}_{n=0}^{\infty}$ on the real line with the ω_2-weighted L^2 inner product. It turns out that the system has a simple recurrence formula, so that the exponential generating function is easily computed. Using this the orthogonality is proven. In section (6) we discuss some useful connections between the systems, in terms of the operators. Finally in section (7), we present two operators, $T = R^{-1}$ and $S = JR^{-1}$, where J and R are the operators intoduced in section (4). Boundedness of these two operators on five Hilbert spaces (defined in subsection (2.4)) is proven.

2 Preliminaries

2.1 Some Notations

We use the Kronecker's delta: $\delta_{nm} = 0$ or 1, according as $n \neq m$, or $n = m$. The symbol $\mathbb{F}$ is used to denote the field of either real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. By $\text{Re}(z)$, $\text{Im}(z)$, $|z|$ and $\bar{z}$, we mean the real part, the imaginary part, the absolute and the conjugate complex value, respectively, of a complex number z. Closed intervals are denoted by $[a, b]$, open intervals by (a, b) and half-open intervals by $(a, b]$ or $[a, b)$.

We use $\mathbb{S}$ to denote the strip $\{z \in \mathbb{C} : -1 \leq \text{Im}(z) \leq 1\}$, $\partial\mathbb{S}$ for the boundary of the strip $\mathbb{S}$ and $\mathcal{P}$ for the Poisson kernel for the strip $\mathbb{S}$.

More notation will be introduced as we go on.

2.2 Elementary Theory of Hilbert Spaces

In this subsection, we review those aspects of the theory of separable Hilbert spaces which are particularly relevant to our study.

Definition 1. A normed linear space is a pair $(V, ||\cdot||)$ where V is a vector space over $\mathbb{F}$, and $||\cdot||$ is a function $||\cdot|| : V \to \mathbb{R}$ called a norm on V that satisfies the following conditions for all $x, y \in V$ and $\alpha \in \mathbb{F}$:

1. $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$.
2. $||\alpha x|| = |\alpha|\,||x||$.
3. $||x + y|| \leq ||x|| + ||y||$.

Definition 2. A bounded linear operator from a normed linear space $(V_1, ||\cdot||_1)$ to a normed linear space $(V_2, ||\cdot||_2)$ is a function L from V_1 to V_2 that satisfies the following for all $x, y \in V_1$ and $\alpha, \beta \in \mathbb{F}$:

1. $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$.
2. For some $M \geq 0, ||Lx||_2 \leq M||x||_1$.

The smallest such M is called the norm of L, written $||L||$. Thus,

$$||L|| = \sup_{||x||_1 \leq 1} ||Lx||_2.$$

If in the second condition equality holds with $M = 1$, then the operator L is called an isometry and the normed linear spaces $(V_1, ||\cdot||_1)$ and $(V_2, ||\cdot||_2)$ are said to be isometric. Isometric normed linear spaces can be regarded as the same as far as their normed linear space properties are concerned.

Definition 3. An inner product space is a pair $(V, \langle\cdot,\cdot\rangle)$ where V is a vector space over $\mathbb{F}$, and $\langle\cdot,\cdot\rangle$ is a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ called an inner product on V that satisfies the following four conditions for all $x, y, z \in V$ and $\alpha \in \mathbb{F}$:

1. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.
2. $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$.
3. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$.
4. $\langle\ x, y\rangle = \overline{\langle y, x\rangle}$.

Example 1. Let $C[a, b]$ denote the set of complex-valued continuous functions on the interval $[a, b]$. For $f, g \in C[a, b]$, define

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

Then $(C[a, b], \langle\cdot,\cdot\rangle)$ is an inner product space.

Given any inner product space V, we can define $||x|| = \sqrt{\langle x, x\rangle}$. This is, in fact, a norm on V, and to show this, we need what is known as the Schwarz inequality, that is, $|\langle x, y\rangle| \leq ||x||\,||y||$ for any two vectors $x, y \in V$ [9, lemma 4.2]. We formally present this result in the following proposition.

Proposition 1. *Every inner product space V is a normed linear space with the norm $||x|| = \sqrt{\langle x, x\rangle}$.*

Proof. We verify only the triangle inequality since the other properties follow immediately from definition (3). Let $x, y \in V$. Then,

$$\begin{aligned} ||x + y||^2 = \langle x + y, x + y\rangle &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= ||x||^2 + 2\,\mathrm{Re}\langle x, y\rangle + ||y||^2 \\ &\leq ||x||^2 + 2\,|\langle x, y\rangle| + ||y||^2 \\ &\leq ||x||^2 + 2\,||x||\,||y|| + ||y||^2, \quad \text{by Schwarz inequality} \\ &= (||x|| + ||y||)^2, \end{aligned}$$

which proves the triangle inequality. □

Definition 4. A complete inner product space is called a Hilbert space. (Complete here means that every Cauchy sequence converges.)

Example 2. Let $L^2[a,b]$ be the set of complex-valued measurable functions on a finite interval $[a,b]$ that satisfy $\int_a^b |f(x)|^2 dx < \infty$. For $f, g \in L^2[a,b]$ define

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

It can be shown that $L^2[a,b]$ equipped with this inner product is complete and therefore is a Hilbert space.

Definition 5. Let V be an inner product space. Two vectors $x, y \in V$ are said to be orthogonal if $\langle x, y\rangle = 0$. A sequence of vectors $\{x_n\}_{n=0}^{\infty}$ in V is called an orthogonal system if

$$\langle x_n, x_m\rangle = h_n \delta_{nm}. \tag{1}$$

The system is called orthonormal if $h_n = 1$.

Definition 6. A sequence of vectors $\{x_n\}_{n=0}^{\infty}$ in a Hilbert space H is complete if $\langle y, x_n\rangle = 0$ for all $n \geq 0$ implies that $y = 0$.

Definition 7. An orthonormal basis is a complete orthonormal system.

The following theorem is standard and can be found in many books, for example, in Reed and Simon [11].

Proposition 2. *Let $\{x_n\}_{n=0}^{\infty}$ be an orthonormal basis in a Hilbert space H. Then for each $y \in H$,*

$$y = \sum_{n=0}^{\infty} \langle y, x_n\rangle x_n \quad \textit{and} \quad ||y||^2 = \sum_{n=0}^{\infty} |\langle y, x_n\rangle|^2. \tag{2}$$

The equality in the first expression means that the sum on the right-hand side converges, regardless of order, to y.

Proof. See Reed and Simon [11, thm. II.6]. □

Corollary 1. *If $\{x_n\}_{n=0}^{\infty}$ is an orthogonal basis in a Hilbert space H then for each $y \in H$,*

$$y = \sum_{n=0}^{\infty} \frac{\langle y, x_n\rangle}{||x_n||^2} x_n \quad \textit{and} \quad \langle y, z\rangle = \sum_{n=0}^{\infty} \frac{\langle y, x_n\rangle\overline{\langle z, x_n\rangle}}{||x_n||^2}. \tag{3}$$

2.3 Elementary Theory of Orthogonal Polynomials

We review different aspects of the theory of orthogonal polynomials of one real variable. We require that the domain $X \subset \mathbb{R}$ of polynomials be measurable. X is most commonly either the infinte interval $(-\infty, \infty)$, a semi-infinite interval $[a, \infty)$ or a finite interval $[a, b]$. We also need a weight function described in the following definition.

Definition 8. Let $X \subset \mathbb{R}$ be a finite or infinite interval. A function w is called a polynomially bounded weight function if it satifies the following conditions:

1. w is everywhere nonnegative, integrable over X, and non-zero over a subset of X of positive measure, that is,

$$0 < \int_X w(x)dx < \infty.$$

2. For every $n \in \mathbb{N}$,

$$\int_X x^n w(x)dx < \infty.$$

   The quantity $\int_X x^n w(x)dx$ is often called the n^{th} moment of $w(x)$, and is symbolized by μ_n.

Now for a given polynomially bounded weight function w, let $L^2(w)$ denote the space of functions $f : X \to \mathbb{R}$ whose w-weighted squares have finite integral, that is,

$$f \in L^2(w) \iff \int_X f^2(x)w(x)dx < \infty. \tag{4}$$

It follows from condition (2) of definition (8) that all polynomials are included in the space $L^2(w)$.

Definition 9. Let $\{p_n\}_{n=0}^{\infty}$ be a system of polynomials in the space $L^2(w)$ described above, where the n^{th} polynomial p_n has degree n. Then $\{p_n\}_{n=0}^{\infty}$ is called an orthogonal system with respect to w if

$$\int_X p_n(x)p_m(x)w(x)dx = h_n\delta_{nm}. \tag{5}$$

The system is called orthonormal if $h_n = 1$.

More generally if μ is a monotonic non-decreasing function (usually called the distribution function), then we can write equation (5) in terms of the Stieltjes integral,

$$\int_X p_n(x)p_m(x)d\mu(x) = h_n\delta_{nm}. \tag{6}$$

which is reduced back to (5) in case μ is absolutely continuous, that is, if $d\mu(x) = w(x)dx$.

Definition 10. If p is a polynomial of degree m and

$$p(x) = c_m x^m + c_{m-1}x^{m-1} + \cdots + c_2x^2 + c_1x + c_0, \tag{7}$$

then c_m is called the leading coefficient of p. If $c_m = 1$, we say that p is a monic polynomial.

A useful property of real orthogonal polynomials is that they obey a three-term recurrence relation as described in the next proposition [14].

Proposition 3. *For a weight function w described as in definition* (7)*, there exists a unique system of monic orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$. In particular, we can construct $\{p_n\}_{n=0}^{\infty}$ as follows:*

$$p_0(x) = 1, \quad p_1(x) = x - a_1 \quad \textit{with } a_1 = \frac{\int_X xw(x)dx}{\int_X w(x)dx}$$

and

$$p_{n+1}(x) = xp_n(x) - a_{n+1}p_n(x) - b_{n+1}p_{n-1}(x), \tag{8}$$

where

$$a_{n+1} = \frac{\int_X xp_n^2(x)w(x)dx}{\int_X p_n^2(x)w(x)dx} \quad \textit{and } b_{n+1} = \frac{\int_X xp_n(x)p_{n-1}(x)w(x)dx}{\int_X p_{n-1}^2(x)w(x)dx}.$$

Remark 1. If w is an even measure, then $a_{n+1} = 0$ since then its integrals with odd polynomials are all zero.

Proof. We begin by proving the existence of monic orthogonal polynomials. The first polynomial p_0 should be monic and of degree zero, and so,

$$p_0(x) = 1.$$

The next polynomial p_1 should be monic and of degree one. It should therefore take the form

$$p_1(x) = x - a_1,$$

and this orthogonal to p_1 implies that

$$0 = \langle p_1, p_0 \rangle = \int_X xw(x)dx - a_1 \int_X w(x)dx.$$

Since w is nonzero on X, it follows that

$$a_1 = \frac{\int_X xw(x)dx}{\int_X w(x)dx}.$$

a_{n+1} and b_{n+1} are found following the same procedure. To prove uniqueness of the sequence $\{p_n\}_{n=0}^{\infty}$ of monic orthogonal polynomials of degree n, assume that $\{q_n\}_{n=0}^{\infty}$ is another sequence of monic orthogonal polynomials of degree n. Then

$$\deg(p_{n+1} - q_{n+1}) \leq n,$$

and since p_{n+1} and q_{n+1} are orthogonal to any polynomial of degree n or less, we have

$$\langle p_{n+1}, p_{n+1} - q_{n+1} \rangle = 0 \quad \text{and} \quad \langle q_{n+1}, p_{n+1} - q_{n+1} \rangle = 0.$$

But this implies that

$$\langle p_{n+1} - q_{n+1}, p_{n+1} - q_{n+1} \rangle = 0,$$

and so, $p_{n+1} - q_{n+1} \equiv 0$ for all $n \geq 0$. □

2.4 Spaces of Interest

The following spaces are of particular interest to our study:

$$L^2(\omega_2), \quad L^2(\omega_1), \quad H^2(\mathbb{S},\mathcal{P}), \quad L^2(\mathbb{R}), \quad H^2(\mathbb{S}). \tag{9}$$

Other useful spaces are:

$$A_0(\mathbb{S}), \quad L^2_{\mathbb{R}}(\omega_2), \quad L^2_{\mathbb{R}}(\omega_1), \quad H^2_{\mathbb{R}}(\mathbb{S},\mathcal{P}), \quad L^2_{\mathbb{R}}(\mathbb{R}), \quad H^2_{\mathbb{R}}(\mathbb{S}). \tag{10}$$

In the above, and indeed throughout this paper, ω_1 denotes the weight function $1/(2\cosh\frac{\pi}{2}x)$ while ω_2 denote the self convolution of ω_1, that is, $\omega_2 = \omega_1 * \omega_1$. In fact, it can be shown that $\omega_2(x) = x/(2\sinh\frac{\pi}{2}x)$.

$L^2(\omega_1)$ denotes the Hilbert space of measurable functions on $\mathbb{R}$ that satisfy $\int_{-\infty}^{\infty} |f(x)|^2\omega_1(x)dx < \infty$ equipped with the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\omega_1(x)dx = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\frac{dx}{2\cosh\frac{\pi}{2}x}. \tag{11}$$

The Hilbert space $L^2(\omega_2)$ is like the space $L^2(\omega_1)$ but with the weight fuction ω_2 in place of ω_1.

$L^2(\mathbb{R})$ denotes the Hilbert space of measurable functions on $\mathbb{R}$ that satisfy $\int_{-\infty}^{\infty} |f(x)|^2dx < \infty$ equipped with the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx. \tag{12}$$

$H^2(\mathbb{S},\mathcal{P})$ denotes the Hilbert space of analytic functions on $\mathbb{S}$ that satisfy $\int_{\partial\mathbb{S}} |f(z)|^2d\mathcal{P}(z) < \infty$ equipped with the inner product

$$\begin{aligned}
\langle f, g\rangle &= \int_{\partial\mathbb{S}} f(z)\overline{g(z)}d\mathcal{P}(z)\\
&= \int_{-\infty}^{\infty} f(x+i)\overline{g(x+i)}\left(\frac{\omega_1(x)}{2}\right)dx + \int_{-\infty}^{\infty} f(x-i)\overline{g(x-i)}\left(\frac{\omega_1(x)}{2}\right)dx\\
&= \int_{-\infty}^{\infty} \frac{f(x+i)\overline{g(x+i)} + f(x-i)\overline{g(x-i)}}{2}\omega_1(x)dx\\
&= \int_{-\infty}^{\infty} \frac{f(x+i)\overline{g(x+i)} + f(x-i)\overline{g(x-i)}}{2}\frac{dx}{2\cosh\frac{\pi}{2}x}.
\end{aligned} \tag{13}$$

The Hilbert space $H^2(\mathbb{S})$ is like the space $H^2(\mathbb{S},\mathcal{P})$ but without any weight function.

$A_0(\mathbb{S})$ is the space of functions f that are analytic in $\mathbb{S}$, continuous on $\partial\mathbb{S}$ and $f(x+iy) \to 0$ as $|x| \to \infty$.

The spaces $L^2_{\mathbb{R}}(\omega_2)$, $L^2_{\mathbb{R}}(\omega_1)$ and $L^2_{\mathbb{R}}(\mathbb{R})$ are like the corresponding spaces but restricted to real-valued functions. For the spaces, $H^2_{\mathbb{R}}(\mathbb{S},\mathcal{P})$ and $H^2_{\mathbb{R}}(\mathbb{S})$, we talk of real-valued functions on the real axis.

3 The τ-System

In this section, we present our first system of orthogonal polynomials which we call the τ-system. This system was studied in Araaya's paper [4], and it was found that it has a simple recurrence relation

$$\tau_{-1} = 0, \ \tau_0 = 1 \text{ and } \ \tau_{n+1}(x) = x\tau_n(x) - n^2\tau_{n-1}(x).$$

The first few polynomials for this system are shown below.

$$\begin{aligned} \tau_0 &= 1 \\ \tau_1 &= x \\ \tau_2 &= x^2 - 1 \\ \tau_3 &= x^3 - 5x \\ \tau_4 &= x^4 - 14x^2 + 9 \\ &\vdots \end{aligned}$$

The weight function for this system is $\omega_1(x) = 1/(2\cosh\frac{\pi}{2}x)$, and as such, we start by looking at two interesting properties of this function that make it useful for this purpose.

Proposition 4. *The function ω_1 is a probability density function.*

Proof. This follows from the integration,

$$\int_{-\infty}^{\infty} \omega_1(x)dx = \int_{-\infty}^{\infty} \frac{dx}{2\cosh\frac{\pi}{2}x} = \left[\frac{1}{\pi}\arctan(\sinh\frac{\pi}{2}x)\right]_{-\infty}^{\infty} = 1.$$

□

The following property makes it possible to interpret the moments of ω_1 as values at zero of successive derivatives.

Proposition 5. *The function ω_1 is up to a dilation its own Fourier transform. In particular, it is a Fourier transform of $1/\cosh t$, that is,*

$$\omega_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ixt}dt}{\cosh t}$$

Proof. Using the Fourier inversion theorem, we can write

$$\hat{\omega}_1(t) = \int_{-\infty}^{\infty} e^{ixt}\omega_1(x)dx = \int_{-\infty}^{\infty} \frac{e^{(it+\frac{\pi}{2})x}}{e^{\pi x}+1}dx$$

and show that $\hat{\omega}_1(t) = 1/\cosh t$. For the complete proof, see similar calculations in lemma (1). □

We now present the main results for this section. The calculations in proving these results are crucial for proving the main results for the other two systems.

Theorem 1. *Let the system* $\{\tau_n\}_{n=0}^{\infty}$ *be given by the recurrence relation*

$$\tau_{-1} = 0,\ \tau_0 = 1 \text{ and } \tau_{n+1}(x) = x\tau_n(x) - n^2\tau_{n-1}(x). \tag{14}$$

Then

1. *The function* τ_n *is a monic polynomial of degree* n *for* $n \geq 0$.
2. *The exponential generating function*[1], $G_\tau(x,s) = \sum_{n=0}^{\infty} \frac{\tau_n(x)}{n!} s^n$, *is given by the function*
$$G\tau(x,s) = \frac{e^{x \arctan s}}{\sqrt{1+s^2}}.$$
3. *The polynomials* $\{\frac{\tau_n}{n!}\}_{n=0}^{\infty}$ *are an orthonormal basis in the Hilbert space* $L^2(\omega_1)$.

As aforementioned, the calculations for this proof are similar to those for the other two systems, and since will shall provide a complete proof for the system of section (5), we omit this proof. Instead, we provide some tools needed to do this proof and these will also be needed in section (5).

Lemma 1. *If* $Re(\alpha) < \frac{\pi}{2}$, *then*

$$\int_{-\infty}^{\infty} e^{\alpha x}\omega_1(x)dx = \frac{1}{\cos\alpha}.$$

Proof. The complex-valued function $\omega_1(z) = 1/(2\cosh\frac{\pi}{2}z)$ has a simple pole $z = i$, and so we consider a rectangular contour with vertices $(-R,0), (R,0), (R,2i)$ and $(-R,2i)$, that is, a contour containing the simple pole. Call this contour C. Then, by the residue theorem, we have

$$\oint_C e^{\alpha z}\omega_1(z)dz = 2\pi i \cdot \mathrm{Res}(i) = 2\pi i \left(\left. \frac{e^{\alpha z}}{\pi \sinh\frac{\pi}{2}z} \right|_{z=i} \right) = 2e^{\alpha i}. \tag{15}$$

Now

$$e^{\alpha z}\omega_1(z) = e^{\alpha z}\frac{1}{2\cosh\frac{\pi}{2}z} = e^{\alpha z}\frac{2e^{\frac{\pi}{2}z}}{2(e^{\pi z}+1)} = \frac{e^{(\alpha+\frac{\pi}{2})z}}{e^{\pi z}+1},$$

and so, we can integrate around the contour C as follows:

$$\begin{aligned}\oint_C e^{\alpha z}\omega_1(z)dz &= \int_{I_1} + \cdots + \int_{I_4} \\ &= \int_{-R}^{R} \frac{e^{(\alpha+\frac{\pi}{2})x}}{e^{\pi x}+1}dx + i\int_0^2 \frac{e^{(\alpha+\frac{\pi}{2})(R+iy)}}{e^{\pi(R+iy)}+1}dy \\ &\quad - \int_{-R}^{R} \frac{e^{(\alpha+\frac{\pi}{2})(x+2i)}}{e^{\pi(x+2i)}+1}dx - i\int_0^2 \frac{e^{(\alpha+\frac{\pi}{2})(-R+iy)}}{e^{\pi(-R+iy)}+1}dy.\end{aligned}$$

[1] The exponential generating function of a squence $\{a_n\}$ is defined as $G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

Alone the side I_2, we have

$$|e^{\alpha z}\omega_1(z)| = \left|\frac{e^{(\alpha+\frac{\pi}{2})(R+iy)}}{e^{\pi(R+iy)}+1}\right| \leq \frac{e^{(\alpha+\frac{\pi}{2})R}}{e^{\pi R}-1} = \frac{e^{-\frac{\pi}{2}R}e^{\alpha R}}{1-e^{-\pi R}}$$

so that by Darboux inequality,

$$\left|\int_{I_2} e^{\alpha z}\omega_1(z)dz\right| \leq \frac{2e^{-\frac{\pi}{2}R}e^{\alpha R}}{1-e^{-\pi R}} \to 0 \text{ as } R \to \infty.$$

Similarly, the integral alone I_4 vanish as $R \to \infty$.

Thus, taking $R \to \infty$ and combining with (15), we have

$$\begin{aligned}
2e^{\alpha i} &= \lim_{R\to\infty} \oint_C e^{\alpha z}\omega_1(z)dz \\
&= \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{(\alpha+\frac{\pi}{2})x}}{e^{\pi x}+1}dx - \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{(\alpha+\frac{\pi}{2})(x+2i)}}{e^{\pi(x+2i)}+1}dx \\
&= \lim_{R\to\infty} \left(1+e^{i2\alpha}\right) \int_{-R}^{R} \frac{e^{(\alpha+\frac{\pi}{2})x}}{e^{\pi x}+1}dx \\
&= \left(1+e^{i2\alpha}\right) \int_{-\infty}^{\infty} \frac{e^{(\alpha+\frac{\pi}{2})x}}{e^{\pi x}+1}dx
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{(\alpha+\frac{\pi}{2})x}}{e^{\pi x}+1}dx &= \frac{2e^{\alpha i}}{1+e^{i2\alpha}} \\
&= \frac{1}{\cos\alpha}.
\end{aligned}$$

□

Lemma 2. *The following identity holds:*

$$\cos(\alpha+\beta) = \frac{1-\tan\alpha\tan\beta}{\sqrt{1+\tan^2\alpha}\sqrt{1+\tan^2\beta}}.$$

Proof. It is a well known fact that $\cos^2 x + \sin^2 x = 1$. Dividing through by $\cos^2 x$ gives $1 + \tan^2 x = \sec^2 x$. Thus,

$$\begin{aligned}
\frac{1-\tan\alpha\tan\beta}{\sqrt{1+\tan^2\alpha}\sqrt{1+\tan^2\beta}} &= \frac{1-\tan\alpha\tan\beta}{\sec\alpha\sec\beta} \\
&= \cos\alpha\cos\beta(1-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}) \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
&= \cos(\alpha+\beta).
\end{aligned}$$

□

4 The σ-System and Some Useful Operators

Like the τ-system, this system was studied in Araaya's paper [4], and it was found that it has a simple recurrence relation

$$\sigma_{-1} = 0,\ \sigma_0 = 1 \text{ and } \sigma_{n+1}(x) = x\sigma_n(x) - n(n-1)\sigma_{n-1}(x).$$

The first few polynomials for this system are shown below.

$$\begin{aligned}
\sigma_0 &= 1 \\
\sigma_1 &= x \\
\sigma_2 &= x^2 \\
\sigma_3 &= x^3 - 2x \\
\sigma_4 &= x^4 - 8x^2 \\
&\vdots
\end{aligned}$$

The first two properties of the function $\omega_1(x) = 1/(2\cosh\frac{\pi}{2}x)$ were discussed in section (3). The third useful property is that, it is closely related to the Poisson kernel for a strip of width two in the manner of the following proposition.

Proposition 6. *Let the function f be continuous and harmonic in the strip $\mathbb{S} = \{z \in \mathbb{C} : -1 \leq \mathrm{Im}(z) \leq 1\}$, and suppose further that $|f(z)| < Ce^{a|z|}$ for some $a \in [0, \frac{\pi}{2})$. Then*

$$\begin{aligned}
f(0) &= \int_{-\infty}^{\infty} f(x+i)\frac{dx}{4\cosh\frac{\pi}{2}x} + \int_{-\infty}^{\infty} f(x-i)\frac{dx}{4\cosh\frac{\pi}{2}x} \\
&= \int_{-\infty}^{\infty} \frac{f(x+i)+f(x-i)}{2}\frac{dx}{2\cosh\frac{\pi}{2}x} \\
&= \int_{-\infty}^{\infty} \frac{f(x+i)+f(x-i)}{2}\omega_1(x)dx.
\end{aligned}$$

Proof. This is simply the Poisson integral. □

In the preceding proposition, we used the operator,

$$Rf(x) = \frac{1}{2}(f(x+i) + f(x-i)), \tag{16}$$

which is densely defined in $L^2(\omega_1)$. For symmetry, we also consider the operator,

$$Jf(x) = \frac{1}{2i}(f(x+i) - f(x-i)). \tag{17}$$

It is clear from the definition of these two operators that

$$(R \pm iJ)f(x) = f(x \pm i). \tag{18}$$

In the next section, we shall see that multipying the ρ-system by x gives a relation to this system, and this is the reason to define the third operator,

$$Qf(x) = xf(x). \tag{19}$$

The notation for this operator is inspired by analogies with quantum mechanics, an analogy which seems natural in the light of the following easily verified relations between the operators.

Proposition 7. *The operators R, J and Q satisfy the following relations:*

$$RQ - QR = -J \tag{20}$$

$$JQ - QJ = R \tag{21}$$

$$RJ - JR = 0 \tag{22}$$

$$R^2 + J^2 = I \tag{23}$$

where I is the identity operator.

Proof. Use the definition of the operators involved. □

We now present the main results for this section which describe an orthogonal basis for the space $H^2(\mathbb{S}, \mathcal{P})$ where $\mathcal{P}$ is the Poisson measure for 0.

Theorem 2. *Let the system $\{\sigma_n\}_{n=0}^{\infty}$ be given by the recurrence relation.*

$$\sigma_{-1} = 0,\ \sigma_0 = 1 \text{ and } \sigma_{n+1}(x) = x\sigma_n(x) - n(n-1)\sigma_{n-1}(x). \tag{24}$$

Then

1. *The function σ_n is a monic polynomial of degree n for $n \geq 0$.*
2. *The exponential generating function, $G_\sigma(x, s) = \sum \frac{\sigma_n(x)}{n!} s^n$, is given by the function*
$$G_\sigma(x, s) = e^{x \arctan s}.$$
3. *The norm of the polynomial $\frac{\sigma_n}{n!}$ is 1 for $n = 0$ and $\sqrt{2}$ for $n \geq 1$.*
4. *The polynomials $\{\frac{\sigma_n}{n!}\}_{n=0}^{\infty}$ are an orthogonal basis in the Hilbert space $H^2(\mathbb{S}, \mathcal{P})$.*

Proof. See similar calculations in the proof of theorem (3) in the next section. □

5 The ρ-System

We study this system in detail since it is a new addition, filling a gap related to the previous systems. In fact, it is the main motivation behind this thesis. Unlike the two previous systems, the weight function for this system is $\omega_2 = \omega_1 * \omega_1$, the self convolution of $\omega_1(x) = 1/(2\cosh\frac{\pi}{2}x)$. By the convolution theorem and proposition (5) , the Fourier transform $\hat{\omega}_2$ of ω_2 is given by $\hat{\omega}_2(t) = \hat{\omega}_1(t)\cdot\hat{\omega}_1(t) = 1/\cosh^2 t$. Abramowitz [1] gives the Maclaurin series expansion

$$\begin{aligned}\frac{1}{\cosh^2 t} &= \left(\sum_{n=0}^{\infty}\frac{E_{2n}t^{2n}}{(2n)!}\right)^2\\ &= \left(1-\frac{t^2}{2}+\frac{5t^4}{24}-\frac{61t^6}{720}+\frac{1385t^8}{40320}+\cdots\right)^2\\ &= 1-t^2+\frac{2t^4}{3}-\frac{17t^6}{45}+\cdots \end{aligned}\tag{25}$$

where E_n is the n^{th} Euler number[2].

Now using the Fourier inversion theorem, $\hat{\omega}_2(t) = \int_{-\infty}^{\infty} e^{ixt}\omega_2(x)dx$, we derive the n^{th} derivative of $\hat{\omega}_2$ evaluated at zero as follows:

$$\begin{aligned}
\hat{\omega}_2(t) &= \int_{-\infty}^{\infty} e^{ixt}\omega_2(x)dx, & \hat{\omega}_2(0) &= \int_{-\infty}^{\infty}\omega_2(x)dx\\
\hat{\omega}_2'(t) &= \int_{-\infty}^{\infty} ixe^{ixt}\omega_2(x)dx, & \hat{\omega}_2'(0) &= \int_{-\infty}^{\infty} ix\omega_2(x)dx\\
\hat{\omega}_2''(t) &= \int_{-\infty}^{\infty} (ix)^2e^{ixt}\omega_2(x)dx, & \hat{\omega}_2''(0) &= \int_{-\infty}^{\infty} (ix)^2\omega_2(x)dx\\
&\vdots & &\vdots\\
\hat{\omega}_2^{(n)}(t) &= \int_{-\infty}^{\infty} (ix)^ne^{ixt}\omega_2(x)dx, & \hat{\omega}_2^{(n)}(0) &= \int_{-\infty}^{\infty} (ix)^n\omega_2(x)dx
\end{aligned}$$

Since $\hat{\omega}_2$ is an even function, it is orthogonal to all odd polynomials. Thus all odd derivatives vanish, and we can rewrite the expression for the n^{th} derivative of $\hat{\omega}_2$ evaluated at zero as

$$\int_{-\infty}^{\infty} x^{2n}\omega_2(x)dx = (-i)^{2n}\hat{\omega}_2^{(2n)}(0) = (-i)^{2n}\left(\frac{d}{dt}\right)^{2n}\left(\frac{1}{\cosh^2 t}\right)\Bigg|_{t=0}, \tag{26}$$

[2] The first few Euler numbers are: 1, -1, 5, -61, 1385, -50521 with alternating signs. For the explicit definition and formula, see [1, p. 804].

which is then used together with equation (25) to find the moments as follows:

$$
\begin{aligned}
n &= 0, \qquad \int_{-\infty}^{\infty} \omega_2(x)dx = (-i)^0 \hat{\omega}_2(0) = 1 \\
n &= 1, \qquad \int_{-\infty}^{\infty} x^2\omega_2(x)dx = (-i)^2 \hat{\omega}_2''(0) = (-i)^2 (-2! \times 1) = 2 \\
n &= 2, \qquad \int_{-\infty}^{\infty} x^4\omega_2(x)dx = (-i)^4 \hat{\omega}_2^{(4)}(0) = (-i)^4 \left(4! \times \frac{2}{3}\right) = 16 \\
n &= 3, \qquad \int_{-\infty}^{\infty} x^6\omega_2(x)dx = (-i)^6 \hat{\omega}_2^{(6)}(0) = (-i)^6 \left(-6! \times \frac{17}{45}\right) = 272 \\
&\vdots
\end{aligned}
$$

We can now use proposition (3) to construct a unique system of monic orthogonal polynomials $\{\rho_n\}_{n=0}^{\infty}$. Set $\rho_0(x) = 1$ and since this has an even power of x, it is orthogonal to all odd polynomials and in particular to $\rho_1(x) = x$. To find the third polynomial, set $\rho_2(x) = x^2 + a$ and this orthogonal to ρ_0 implies that

$$0 = \int_{-\infty}^{\infty} (x^2 + a)\omega_2(x)dx = \int_{-\infty}^{\infty} x^2\omega_2(x)dx + a\int_{-\infty}^{\infty} \omega_2(x)dx = 2 + a.$$

Thus $a = -2$. To find the fourth polynomial, set $\rho_3(x) = x^3 + bx$ and this orthogonal to ρ_1 implies that

$$0 = \int_{-\infty}^{\infty} (x^3 + bx)x\omega_2(x)dx = \int_{-\infty}^{\infty} x^4\omega_2(x)dx + b\int_{-\infty}^{\infty} x^2\omega_2(x)dx = 16 + 2b.$$

Thus $b = -8$. To find the fifth polynomial, set $\rho_4(x) = x^4 + cx^2 + d$ and this orthogonal to ρ_0 implies that

$$
\begin{aligned}
0 &= \int_{-\infty}^{\infty} (x^4 + cx^2 + d)\omega_2(x)dx \\
&= \int_{-\infty}^{\infty} x^4\omega_2(x)dx + c\int_{-\infty}^{\infty} x^2\omega_2(x)dx + d\int_{-\infty}^{\infty} \omega_2(x)dx \\
&= 16 + 2c + d. \qquad (27)
\end{aligned}
$$

ρ_4 should also be orthogonal to ρ_2, and so,

$$
\begin{aligned}
0 &= \int_{-\infty}^{\infty} (x^4 + cx^2 + d)(x^2 - 2)\omega_2(x)dx \\
&= \int_{-\infty}^{\infty} (x^6 + cx^4 + dx^2)\omega_2(x)dx \\
&= \int_{-\infty}^{\infty} x^6\omega_2(x)dx + c\int_{-\infty}^{\infty} x^4\omega_2(x)dx + d\int_{-\infty}^{\infty} x^2\omega_2(x)dx \\
&= 272 + 16c + 2d \\
&= 272 + 16c + 2(-16 - 2c), \quad \text{by (27)} \\
&= 240 + 12c.
\end{aligned}
$$

Thus, $c = -20$ and $d = 24$. The rest of the ρ-polynomials are obtained following the same procedure, and we have

$$\begin{aligned}
\rho_0(x) &= 1 \\
\rho_1(x) &= x \\
\rho_2(x) &= x^2 - 2 \\
\rho_3(x) &= x^3 - 8x \\
\rho_4(x) &= x^4 - 20x^2 + 24 \\
&\vdots
\end{aligned}$$

We now establish the relationship between these polynomials. Setting $\rho_{-1} = 0$, we note that

$$\begin{aligned}
\rho_1(x) &= x\rho_0(x) - \rho_{-1}(x) & 0 &= 0 \times 1 \\
\rho_2(x) &= x\rho_1(x) - 2\rho_0(x) & 2 &= 1 \times 2 \\
\rho_3(x) &= x\rho_2(x) - 6\rho_1(x) & 6 &= 2 \times 3 \\
\rho_4(x) &= x\rho_3(x) - 12\rho_2(x) & 12 &= 3 \times 4 \\
&\vdots & &\vdots
\end{aligned}$$

where the second column shows the pattern of the coefficients of the second terms on the right hand side of the polynomial equations. This pattern of the coefficients motivates us to define the system of polynomials $\{\rho_n\}_{n=0}^{\infty}$ by the recurrence relation

$$\rho_{-1} = 0, \ \rho_0 = 1, \text{ and } \rho_{n+1}(x) = x\rho_n(x) - n(n+1)\rho_{n-1}(x),$$

which we will later use to compute the exponential generating function for proving orthogonality of our system.

Before proceeding further, we present two lemmas that will be useful in proving the main results of this section.

Lemma 3. *If the function f is integrable on $(-\infty, \infty)$ and*

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{ixt}dt \equiv 0,$$

then $f = 0$ almost everywhere.

Proof. See Andrews, Askey and Roy [3, thm. 6.5.1]. □

Lemma 4. *If $Re(\alpha) < \frac{\pi}{2}$, then*

$$\int_{-\infty}^{\infty} e^{\alpha x}\omega_2(x)dx = \left(\frac{1}{\cos\alpha}\right)^2. \tag{28}$$

Proof. Bearing in mind that $\omega_2 = \omega_1 * \omega_1$, a self convolution, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{\alpha x}\omega_2(x)dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\alpha x}\omega_1(x-y)\omega_1(y)dydx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\alpha(t+y)}\omega_1(t)\omega_1(y)dtdy \quad \text{if we let } x-y=t \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{\alpha t}\omega_1(t)dt\right) e^{\alpha y}\omega_1(y)dy \\
&= \int_{-\infty}^{\infty} e^{\alpha t}\omega_1(t)dt \int_{-\infty}^{\infty} e^{\alpha y}\omega_1(y)dy \\
&= \left(\int_{-\infty}^{\infty} e^{\alpha x}\omega_1(x)dx\right)^2 \quad \text{if we let } t=y=x \\
&= \left(\frac{1}{\cos\alpha}\right)^2, \qquad \text{by lemma (1)} .
\end{aligned}$$

□

Lemma 5. *For* $|x| < 1$*,*

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n.$$

Proof. Differentiate the geometric series, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, with respect to x, that is,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n.$$

□

We now have all the necessary definitions and lemmas needed to present and prove the main results of this section.

Theorem 3. *Let the system* $\{\rho_n\}_{n=0}^{\infty}$ *be given by the recurrence relation*

$$\rho_{-1} = 0, \;\; \rho_0 = 1 \;\; \textit{and} \;\; \rho_{n+1}(x) = x\rho_n(x) - n(n+1)\rho_{n-1}(x). \tag{29}$$

Then

1. *The function* ρ_n *is a monic polynomial of degree* n *for* $n \geq 0$*.*
2. *The exponential generating function,* $G_\rho(x,s) = \sum_{n=0}^{\infty} \frac{\rho_n(x)}{n!}s^n$*, is given by the function*
$$G\rho(x,s) = \frac{e^{x\arctan s}}{1+s^2}.$$
3. *The sequence of polynomials* $\{\frac{\rho_n}{n!}\}_{n=0}^{\infty}$ *is an orthogonal basis in the Hilbert space* $L^2(\omega_2)$*.*

Proof. (1) follows immediately from the definition of the recurrence relation. To prove (2), we multiply the recurrence by $s^n/n!$ and sum over n so that

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty}[\rho_{n+1}(x) - x\rho_n(x) + n(n+1)\rho_{n-1}(x)]\frac{s^n}{n!} \\
&= \sum_{n=0}^{\infty}\rho_{n+1}(x)\frac{s^n}{n!} - x\sum_{n=0}^{\infty}\rho_n(x)\frac{s^n}{n!} + \sum_{n=1}^{\infty}n(n+1)\rho_{n-1}(x)\frac{s^n}{n!} \\
&= G'_\rho(x,s) - xG_\rho(x,s) + \sum_{n=0}^{\infty}(n+1)(n+2)\rho_n(x)\frac{s^{n+1}}{(n+1)!} \\
&= G'_\rho(x,s) - xG_\rho(x,s) + 2s\sum_{n=0}^{\infty}\rho_n(x)\frac{s^n}{n!} + \sum_{n=1}^{\infty}n\rho_n(x)\frac{s^{n+1}}{n!} \\
&= G'_\rho(x,s) - xG_\rho(x,s) + 2sG_\rho(x,s) + \sum_{n=0}^{\infty}(n+1)\rho_{n+1}(x)\frac{s^{n+2}}{(n+1)n!} \\
&= G'_\rho(x,s) - xG_\rho(x,s) + 2sG_\rho(x,s) + s^2G'_\rho(x,s) \\
&= (1+s^2)G'_\rho(x,s) + (2s-x)G_\rho(x,s).
\end{aligned}
$$

Thus,

$$G'_\rho(x,s) + \frac{2s-x}{1+s^2}G_\rho(x,s) = 0. \tag{30}$$

This is a first-order linear differential equation where all derivatives are with respect to s, holding x fixed. The integrating factor is

$$
\begin{aligned}
\exp\left(\int \frac{2s-x}{1+s^2}ds\right) &= \exp\left(\int \frac{2s}{1+s^2}ds - \int \frac{x}{1+s^2}ds\right) \\
&= \exp(\ln(1+s^2) - x\arctan s)) \\
&= (1+s^2)e^{-x\arctan s}.
\end{aligned}
$$

Multiplying both sides of equation (30) by this factor gives

$$\frac{d}{ds}\left((1+s^2)e^{-x\arctan s}G_\rho(x,s)\right) = 0$$

which implies that

$$G_\rho(x,s) = c\frac{e^{x\arctan s}}{1+s^2}.$$

Now since $G_\rho(x,s) = \sum_{n=0}^{\infty}\frac{\rho_n(x)}{n!}s^n$, it implies that $G_\rho(x,0) = 1$. Thus c=1 and (2) follows.

To prove (3), we first show that

$$\int_{-\infty}^{\infty} G_\rho(x,s)\overline{G_\rho(x,t)}\omega_2(x)dx = \frac{1}{(1-s\bar{t})^2}.$$

Now

$$G_\rho(x,s)\overline{G_\rho(x,t)} = \frac{e^{x\arctan s}}{1+s^2}\frac{e^{x\arctan \bar{t}}}{1+\bar{t}^2} = \frac{1}{(1+s^2)(1+\bar{t}^2)}e^{x(\arctan s+\arctan \bar{t})}.$$

Set $u = \frac{1}{(1+s^2)(1+\bar{t}^2)}$, $\alpha = \arctan s$, $\beta = \arctan \bar{t}$ and assume that $Re(\alpha+\beta) < \frac{\pi}{2}$. Then we have

$$\begin{aligned}
\int_{-\infty}^{\infty} G_\rho(x,s)\overline{G_\rho(x,t)}\omega_2(x)dx &= u\int_{-\infty}^{\infty} e^{(\alpha+\beta)x}\omega_2(x)dx \\
&= u\left(\frac{1}{\cos(\alpha+\beta)}\right)^2, \quad \text{by lemma (4)} \\
&= u\left(\frac{\sqrt{1+\tan^2\alpha}\sqrt{1+\tan^2\beta}}{1-\tan\alpha\tan\beta}\right)^2, \quad \text{by lemma (2)} \\
&= u\left(\frac{\sqrt{1+s^2}\sqrt{1+\bar{t}^2}}{1-s\bar{t}}\right)^2 \\
&= u\cdot\frac{(1+s^2)(1+\bar{t}^2)}{(1-s\bar{t})^2} \\
&= \frac{1}{(1-s\bar{t})^2}. \qquad (31)
\end{aligned}$$

Next, by lemma (5) we see that this implies that

$$\int_{-\infty}^{\infty} G_\rho(x,s)\overline{G_\rho(x,t)}\omega_2(x)dx = \sum_{n=0}^{\infty}(n+1)(s\bar{t})^n. \qquad (32)$$

But using the definition, $G_\rho(x,s) = \sum_{n=0}^{\infty}\frac{\rho_n(x)}{n!}s^n$, gives

$$\begin{aligned}
\int_{-\infty}^{\infty} G_\rho(x,s)\overline{G_\rho(x,t)}\omega_2(x)dx &= \int_{-\infty}^{\infty}\left(\sum_{n=0}^{\infty}\frac{\rho_n(x)}{n!}s^n\right)\left(\sum_{n=0}^{\infty}\frac{\rho_k(x)}{k!}\bar{t}^k\right)\omega_2(x)dx \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}s^n\bar{t}^k\int_{-\infty}^{\infty}\frac{\rho_n(x)\rho_k(x)}{n!k!}\omega_2(x)dx. \qquad (33)
\end{aligned}$$

It therefore follows from (32) and (33) that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}s^n\bar{t}^k\int_{-\infty}^{\infty}\frac{\rho_n(x)\rho_k(x)}{n!k!}\omega_2(x)dx = \sum_{n=0}^{\infty}(n+1)(s\bar{t})^n.$$

Comparing the coefficients of the powers of s and $\bar{t}$ proves orthogonality, that is,

$$\langle\frac{\rho_n(x)}{n!},\frac{\rho_k(x)}{k!}\rangle = (n+1)\delta_{nk} \qquad (34)$$

To show that this system of polynomials $\{\frac{\rho_n}{n!}\}_{n=0}^{\infty}$ is a basis in the Hilbert space $L^2(\omega_2)$, we need to show that it is complete. But since the span of $\{\frac{\rho_n}{n!}\}_{n=0}^{\infty}$ is the space of all polynomials, it suffices to show density of the system $\{x^n\}_{n=0}^{\infty}$. Let $\langle f, x^n\rangle = 0$ for some $f \in L^2(\omega_2)$ and all $n \geq 0$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)e^{itx}\omega_2(x)dx &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \int_{-\infty}^{\infty} f(x)x^n\omega_2(x)dx \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(it)^n}{n!} \int_{-\infty}^{\infty} f(x)x^n\omega_2(x)dx \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(it)^n}{n!} \cdot 0 \\
&= 0.
\end{aligned}$$

By Lemma (3), $f\omega_2 = 0$ almost everywhere. But $\omega_2 \neq 0$ and so $f = 0$ almost everywhere which by definition (6) implies that $\{x^n\}_{n=0}^{\infty}$ is dense in $L^2(\omega_2)$. Therefore, the system $\{\frac{\rho_n}{n!}\}_{n=0}^{\infty}$ is complete, and in particular, it is an orthogonal basis in the Hilbert space $L^2(\omega_2)$. □

6 Some Connections Between the Systems

Having presented the three systems of polynomials in the previous sections, we can now discuss some useful connections between them, in terms of the operators R, J and Q. To start with, let us write a few terms for each system. By definition, $\sigma_{-1} = \tau_{-1} = \rho_{-1} = 0$, $\sigma_0 = \tau_0 = \rho_0 = 1$, and $\sigma_{n+1}(x) = x\sigma_n(x) - n(n-1)\sigma_{n-1}(x)$, $\tau_{n+1}(x) = x\tau_n(x) - n^2\tau_{n-1}(x)$ and $\rho_{n+1}(x) = x\rho_n(x) - n(n+1)\rho_{n-1}(x)$. We thus have

σ	τ	ρ
$\sigma_0 = 1$	$\tau_0 = 1$	$\rho_0 = 1$
$\sigma_1 = x$	$\tau_1 = x$	$\rho_1 = x$
$\sigma_2 = x^2$	$\tau_2 = x^2 - 1$	$\rho_2 = x^2 - 2$
$\sigma_3 = x^3 - 2x$	$\tau_3 = x^3 - 5x$	$\rho_3 = x^3 - 8x$
$\sigma_4 = x^4 - 8x^2$	$\tau_4 = x^4 - 14x^2 + 9$	$\rho_4(x) = x^4 - 20x^2 + 24$
$\vdots$	$\vdots$	$\vdots$

Our three operators are defined by $Rf(x) = \frac{1}{2}(f(x+i)+f(x-i))$, $Qf(x) = xf(x)$ and $Jf(x) = \frac{1}{2i}(f(x+i) - f(x-i))$. Comparing columns 1 and 3, we see that $x\rho_n = \sigma_{n+1}$ which by definition of Q implies that $Q\rho_n = \sigma_{n+1}$. In what follows below, we check the operations of R, J and Q on the three systems of polynomials. We start with the operator R. On the first column, we have

$$
\begin{aligned}
R\sigma_0 &= \frac{\sigma_0(x+i)+\sigma_0(x-i)}{2} = \frac{1+1}{2} = 1 \\
R\sigma_1 &= \frac{(x+i)+(x-i)}{2} = \frac{2x}{2} = x \\
R\sigma_2 &= \frac{(x+i)^2+(x-i)^2}{2} = \frac{2x^2-2}{2} = x^2-1 \\
R\sigma_3 &= \frac{(x+i)^3-2(x+i)+(x-i)^3-2(x-i)}{2} = x^3-5x \\
&\vdots
\end{aligned}
$$

This indicates that the operation of R on column 1 gives column 2. We can therefore claim that $R\sigma_n = \tau_n$ which we will prove later. On column 2, we have

$$
\begin{aligned}
R\tau_0 &= 1 \\
R\tau_1 &= x \\
R\tau_2 &= \frac{(x+i)^2-1+(x-i)^2-1}{2} = \frac{2x^2-4}{2} = x^2-2 \\
R\tau_3 &= \frac{(x+i)^3-5(x+i)+(x-i)^3-5(x-i)}{2} = x^3-8x \\
&\vdots
\end{aligned}
$$

This indicates that the operation of R on column 2 gives column 3. We can therefore claim that $R\tau_n = \rho_n$ which we will prove later. We now turn to the operator J. On column 1, we have

$$
\begin{aligned}
J\sigma_0 &= \frac{\tau_0(x+i)-\tau_0(x-i)}{2i} = \frac{1-1}{2i} = 0 \\
J\sigma_1 &= \frac{(x+i)-(x-i)}{2i} = \frac{2i}{2i} = 1 \\
J\sigma_2 &= \frac{(x+i)^2-(x-i)^2}{2i} = \frac{4xi}{2i} = 2x \\
J\sigma_3 &= \frac{(x+i)^3-2(x+i)-(x-i)^3+2(x-i)}{2i} = 3x^2-3 = 3(x^2-1) \\
&\vdots
\end{aligned}
$$

From this, we can claim that $J\sigma_n = n\tau_{n-1}$ which will be proved later. On column 2, we have

$$
\begin{aligned}
J\tau_0 &= 0\\
J\tau_1 &= 1\\
J\tau_2 &= \frac{(x+i)^2-1-(x-i)^2+1}{2i} = \frac{4xi}{2i} = 2x\\
J\tau_3 &= \frac{(x+i)^3-5(x+i)-(x-i)^3+5(x-i)}{2i} = 3x^2-6 = 3(x^2-3)\\
&\vdots
\end{aligned}
$$

From this, we can claim that $J\tau_n = n\rho_{n-1}$ which will be proved later.

We can now state the main results of this section.

Theorem 4. *The following connections between the three systems of orthogonal polynomials* $\{\sigma_n\}$*,* $\{\tau_n\}$ *and* $\{\rho_n\}$ *hold:*

$$R\sigma_n = \tau_n \tag{35}$$

$$J\sigma_n = n\tau_{n-1} \tag{36}$$

$$R\tau_n = \rho_n \tag{37}$$

$$J\tau_n = n\rho_{n-1} \tag{38}$$

$$Q\rho_n = \sigma_{n+1} \tag{39}$$

Proof. We shall prove only (37) and (38) since the proofs for the rest follow the same procedure. The idea of the proof is that, given (37), we prove by induction (38), and viceversa.
We start with (37). For $n = 0$, the statement is true since we have

$$R\tau_0(x) = \frac{\tau_0(x+i)+\tau_0(x-i)}{2} = \frac{1+1}{2} = 1 = \rho_0(x).$$

Now assume that both (37) and (38) hold for all τ_k, $k \le n$, then

$$
\begin{aligned}
R\tau_{n+1}(x) &= R[x\tau_n(x) - n^2\tau_{n-1}(x)], \quad \text{by recurrence relation}\\
&= \frac{(x+i)\tau_n(x+i)+(x-i)\tau_n(x-i)}{2} - n^2\frac{\tau_{n-1}(x+i)+\tau_{n-1}(x-i)}{2}\\
&= x\frac{\tau_n(x+i)+\tau_n(x-i)}{2} + i\frac{\tau_n(x+i)-\tau_n(x-i)}{2} - n^2R\tau_{n-1}(x)\\
&= xR\tau_n(x) - \frac{\tau_n(x+i)-\tau_n(x-i)}{2i} - n^2R\tau_{n-1}\\
&= xR\tau_n(x) - J\tau_n(x) - n^2R\tau_{n-1}\\
&= xR\tau_n(x) - n\rho_{n-1}(x) - n^2R\tau_{n-1}, \quad \text{by (38) assumption}\\
&= x\rho_n(x) - n\rho_{n-1}(x) - n^2\rho_{n-1}(x), \quad \text{by induction assumption}\\
&= x\rho_n(x) - n(n+1)\rho_{n-1}(x)\\
&= \rho_{n+1}(x).
\end{aligned}
$$

Therefore, since the statement is also true for $n+1$, it follows by induction that it is true for all integers $n \geq 0$.
The proof for (38) follows the same procedure, and as such, we omit it. □

We now introduce some notations related to the three systems of polynomials. Denote the polynomials $\frac{\sigma_n}{n!}, \frac{\tau_n}{n!}, \frac{\rho_n}{n!}$ by $\tilde{\sigma}_n, \tilde{\tau}_n, \tilde{\rho}_n$ respectively. It follows from Theorems (1), (2) and (3) that the systems $\{\tilde{\sigma}_n\}_{n=0}^{\infty}$, $\{\tilde{\tau}_n\}_{n=0}^{\infty}$ and $\{\tilde{\rho}_n\}_{n=0}^{\infty}$ are orthogonal bases for the Hilbert spaces $H^2(\mathbb{S}, \mathcal{P})$, $L^2(\omega_1)$ and $L^2(\omega_2)$ respectively. In fact, the system $\{\tilde{\tau}_n\}_{n=0}^{\infty}$ is orthonormal. In what follows, we look at some consequences of the relations in Theorem (4).

Corollary 2. *The following connections between the three systems of orthogonal polynomials $\{\tilde{\sigma}_n\}$, $\{\tilde{\tau}_n\}$ and $\{\tilde{\rho}_n\}$ hold:*

$$R\tilde{\sigma}_n = \tilde{\tau}_n \tag{40}$$
$$J\tilde{\sigma}_n = \tilde{\tau}_{n-1} \tag{41}$$
$$R\tilde{\tau}_n = \tilde{\rho}_n \tag{42}$$
$$J\tilde{\tau}_n = \tilde{\rho}_{n-1} \tag{43}$$
$$Q\tilde{\rho}_n = (n+1)\tilde{\sigma}_{n+1} \tag{44}$$

Proof. Divide the equations in Theorem (4) through by $n!$. For instance, we have for relation (43),

$$J\tau_n = n\rho_{n-1}$$
$$\frac{J\tau_n}{n!} = \frac{n\rho_{n-1}}{n!} \Rightarrow J\tilde{\tau}_n = \frac{\rho_{n-1}}{(n-1)!} \Rightarrow J\tilde{\tau}_n = \tilde{\rho}_{n-1}.$$

□

Corollary 3. *Let the operators K, L, M, A, B and C be defined as follows: $K = RRQ$, $L = QRR$, $M = RQR$, $A = RQJ$, $B = QJR$ and $C = RJQ$. Then the following relations hold:*

$$K^n(\rho_0) = \rho_n \tag{45}$$
$$L^n(\sigma_0) = \sigma_n \tag{46}$$
$$M^n(\tau_0) = \tau_n \tag{47}$$
$$A(\tau_n) = n\tau_n \tag{48}$$
$$B(\sigma_n) = n\sigma_n \tag{49}$$
$$C(\rho_n) = n\rho_n \tag{50}$$

Proof. We prove only (45) and (50). The proofs for the rest follow the same procedure.
For (45), we proceed by induction. For $n = 0$, the statement is trivially true.

Now assume that it is true for some integer $n \geq 0$, then

$$\begin{aligned} K^{n+1}(\rho_0) &= KK^n(\rho_0) \\ &= K\rho_n, \quad \text{by induction assumption} \\ &= RRQ\rho_n \\ &= \rho_{n+1}, \quad \text{by Theorem (4).} \end{aligned}$$

Therefore, since the statement is also true for $n+1$, it follows by induction that it is true for all integers $n \geq 0$.
For (50), we use the definition of C and the relations in Theorem (4),

$$\begin{aligned} C(\rho_n) &= RJQ(\rho_n) \\ &= RJ\sigma_{n+1} \\ &= Rn\tau_n \\ &= n\rho_n. \end{aligned}$$

□

Corollary 4. *The following relations hold:*

$$\tilde{\tau}_n(x \pm i) = \tilde{\rho}_n(x) \pm i\tilde{\rho}_{n-1}(x) \tag{51}$$

$$\tilde{\sigma}_n(x \pm i) = \tilde{\tau}_n(x) \pm i\tilde{\tau}_{n-1}(x) \tag{52}$$

Proof. We prove only (51) since the proof for (52) follows the same procedure. From corollary (2), $\tilde{\rho}_n = R\tilde{\tau}_n$ and $\tilde{\rho}_{n-1} = J\tilde{\tau}_n$. Thus,

$$\begin{aligned} \tilde{\rho}_n(x) \pm i\tilde{\rho}_{n-1}(x) &= R\tilde{\tau}_n(x) \pm iJ\tilde{\tau}_n(x) \\ &= (R \pm iJ)\tilde{\tau}_n(x) \\ &= \tilde{\tau}_n(x \pm i), \quad \text{by relation (18).} \end{aligned}$$

□

7 Two Bounded Operators

In this section, we study two more operators, namely $T = R^{-1}$ and $S = JR^{-1}$, where J and R are the operators that where defined and presented in section (4). It is clear from the connections in corollary (2) that

$$T\tilde{\rho}_n = \tilde{\tau}_n \tag{53}$$

$$T\tilde{\tau}_n = \tilde{\sigma}_n \tag{54}$$

$$S\tilde{\tau}_n = \tilde{\tau}_{n-1} \tag{55}$$

$$S\tilde{\rho}_n = \tilde{\rho}_{n-1} \tag{56}$$

Also by relation (18),

$$\begin{aligned} Tf(x \pm i) &= (R + iJ)Tf(x) \\ &= RTf(x) + iJTf(x) \\ &= f(x) + iSf(x). \end{aligned} \tag{57}$$

The integral representations of these two operators, S and J, were developed and presented in Araaya's paper [5]. For the operator T, we have

$$Tf = \frac{1}{2\cosh\frac{\pi}{2}x} * f, \tag{58}$$

and for the operator S, we have

$$Sf = -\frac{1}{2\sinh\frac{\pi}{2}x} * f, \tag{59}$$

where in both cases $*$ denotes convolution. Using the convolution theorem, the Fourier transforms for T and S were shown to be

$$\widehat{Tf}(t) = \operatorname{sech} t \hat{f}(t) \tag{60}$$

and

$$\widehat{Sf}(t) = -i \tanh t \hat{f}(t) \tag{61}$$

respectively. We shall also make use of what is known as the Plancherel theorem which states that $||\hat{f}|| = ||f||$ for any $f \in L^2(\mathbb{R})$. See [13, thm. 9.13].

Proposition 8. *For the operator T, we have the following:*

1. *T is linear and bounded from $L^2(\omega_2)$ to $L^2(\omega_1)$.*
2. *If $L^2_0(\omega_1) = \{f \in L^2(\omega_1) : \langle f, 1\rangle = 0\}$ and $H^2_0(\mathbb{S}, \mathcal{P}) = \{f \in H^2(\mathbb{S}, \mathcal{P}) : f(0) = 0\}$, then $T/\sqrt{2}$ is a unitary operator from $L^2_0(\omega_1)$ onto $H^2_0(\mathbb{S}, \mathcal{P})$.*

Remark 2. Let $f \in L^2(\omega_1)$ and $b_n = \langle f, \tilde{\tau}_n\rangle$. Then the operator $U : L^2(\omega_1) \to H^2(\mathbb{S}, \mathcal{P})$ defined by $Uf = b_0 + \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty} b_n\tilde{\sigma}_n$ is unitary.

Proof. Since all other properties are clear, we prove only boundedness.

1. Let $f \in L^2(\omega_2)$ and $a_n = \langle f, \tilde{\rho}_n \rangle$. By Theorem (3), the system $\{\tilde{\rho}_n\}_{n=0}^{\infty}$ is an orthogonal basis in $L^2(\omega_2)$ with norm $\sqrt{n+1}$, and so, by proposition (2),

$$f = \sum_{n=0}^{\infty} a_n \tilde{\rho}_n \quad \text{and} \quad ||f||^2_{L^2(\omega_2)} = \sum_{n=0}^{\infty} (n+1)|a_n|^2.$$

By relation (53),

$$Tf = \sum_{n=0}^{\infty} a_n \tilde{\tau}_n.$$

Since by Theorem (1) the system $\{\tilde{\tau}_n\}_{n=0}^{\infty}$ is an orthonormal basis in $L^2(\omega_1)$, we have

$$\begin{aligned} ||Tf||^2_{L^2(w)} &= \sum_{n=0}^{\infty} |a_n|^2 \\ &\leq \sum_{n=0}^{\infty} (n+1)|a_n|^2 \\ &= ||f||^2_{L^2(\omega_2)}, \end{aligned}$$

which proves boundedness of T from $L^2(\omega_2)$ to $L^2(w)$ with norm 1.

2. Let $f \in L_0^2(\omega_1)$ and $b_n = \langle f, \tilde{\tau}_n \rangle$. Then $b_0 = \langle f, \tilde{\tau}_0 \rangle = \langle f, 1 \rangle = 0$, and since by Theorem (1) the system $\{\tilde{\tau}_n\}_{n=0}^{\infty}$ is an orthonormal basis in $L^2(\omega_1)$, we have

$$f = \sum_{n=1}^{\infty} b_n \tilde{\tau}_n \quad \text{and} \quad ||f||^2_{L_0^2(\omega_1)} = \sum_{n=1}^{\infty} |b_n|^2.$$

By relation (54),

$$Tf = \sum_{n=1}^{\infty} b_n \tilde{\sigma}_n.$$

Since by Theorem (2) the system $\{\tilde{\sigma}_n\}_{n=0}^{\infty}$ is an orthogonal basis in $H^2(\mathbb{S}, \mathcal{P})$ with norm 1 for $n = 0$ and $\sqrt{2}$ for $n \geq 1$, we have

$$||\frac{1}{\sqrt{2}} Tf||^2_{H_0^2(S,\mathcal{P})} = \sum_{n=1}^{\infty} |b_n|^2 = ||f||^2_{L^2(\omega_1)}.$$

This proves that $T/\sqrt{2}$ is an isometry.

□

Before proceeding further, we present two lemmas that will be useful in proving the main results of this section. The proof of the next lemma depends on Cauchy's theorem [2, thm. 1.4.2] which says that if two different paths connect the same two points, and a function is holomorphic everywhere in between the two paths, then the two path integrals of the function will be the same.

Lemma 6. *If $f \in H^2(\mathbb{S})$ then $\hat{f} \in L^2(\mathbb{R}, \cosh 2t\, dt)$ where $\hat{f}$ is the Fourier transform of an analytic function f. Furthermore, $||f||^2_{H^2(\mathbb{S})} = ||\hat{f}||^2_{L^2(\mathbb{R}, \cosh 2t\, dt)}$.*

Proof. Recall from subsection (2.4) that analytic functions in the Hilbert space $H^2(\mathbb{S})$ have the norm

$$||f||_{H^2(\mathbb{S})} = \int_{-\infty}^{\infty} \frac{|f(x+i)|^2 + |f(x-i)|^2}{2} dx.$$

For $f(x+i)$, we have the Fourier transform

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x+i)e^{-ixt}dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x+i)e^{-i(x+i)t}e^{t}dx \\ &= e^{t}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ixt}dx, \quad \text{by Cauchy's theorem} \\ &= e^{t}\hat{f}(t). \end{aligned}$$

Similarly for $f(x-i)$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-i)e^{-ixt}dx = e^{-t}\hat{f}(t).$$

It therefore follows by the Plancherel theorem that

$$\begin{aligned} ||f||_{H^2(\mathbb{S})} &= \int_{-\infty}^{\infty} \frac{|f(x+i)|^2 + |f(x-i)|^2}{2} dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|e^{t}\hat{f}(t)|^2 + |e^{-t}\hat{f}(t)|^2}{2} dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(t)|^2 \left(\frac{e^{2t}+e^{-2t}}{2}\right) dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(t)|^2 \cosh 2t\, dt \\ &= ||\hat{f}||_{L^2(\mathbb{R}, \cosh 2t\, dt)}. \end{aligned}$$

□

The proof of the next lemma depends on the Hadamard three-lines theorem [13, thm. 12.8] which for our particular case says that if $f \in A_0(\mathbb{S})$ and if $M(n) = \max |f(x+in)|$ then $M(0) \leq (M(-1)M(1))^{\frac{1}{2}}$.

Lemma 7. *If $f \in H^2(\mathbb{S})$ then $||f||_0 \leq (||f||_{+1} \cdot ||f||_{-1})^{\frac{1}{2}}$ where we have used the notation $||f||_n^2 = \int_{-\infty}^{\infty} |f(x+in)|^2 dx$.*

Proof. Define the convolution $F = f * \tilde{f} \in A_0(\mathbb{S})$ where $\tilde{f}(x) = \overline{f(-x)}$, that is,

$$F(z) = \int_{-\infty}^{\infty} f(z-x)\tilde{f}(x)dx.$$

Then

$$F(0) = \int_{-\infty}^{\infty} |f(-x)|^2 dx = ||f||_0^2.$$

Recall from subsection (2.4) that $A_0(\mathbb{S})$ is the space of functions that are analytic in $\mathbb{S}$, continuous on $\partial\mathbb{S}$ and $f(x+iy) \to 0$ as $|x| \to \infty$. Thus $f(x+in)$ attains a maximum, say, $F(n) = \max_{x\in\mathbb{R}} |f(x+in)|$. Then by the Hadamard three-lines theorem [13, thm. 12.8],

$$F(0) \leq (F(+1)F(-1))^{\frac{1}{2}},$$

and by the Schwarz inequality,

$$F(+1) \leq ||f||_{+1} \cdot ||f||_0 \quad \text{and} \quad F(-1) \leq ||f||_{-1} \cdot ||f||_0.$$

Therefore,

$$\begin{aligned} ||f||_0^2 = F(0) &\leq (F(+1)F(-1))^{\frac{1}{2}} \\ &\leq (||f||_{+1} \cdot ||f||_0 \cdot ||f||_{-1} \cdot ||f||_0)^{\frac{1}{2}} \\ &\leq (||f||_{+1} \cdot ||f||_{-1})^{\frac{1}{2}} \, ||f||_0 \end{aligned}$$

so that

$$||f||_0 \leq (||f||_{+1} \cdot ||f||_{-1})^{\frac{1}{2}}$$

□

Lemma 8. *For all $x \geq 0$,*

$$\frac{\sqrt{x^2+1}}{2\cosh\frac{\pi}{2}x} \leq \frac{\pi}{2}\frac{x}{2\sinh\frac{\pi}{2}x}$$

Proof. This is equivalent to proving the inequality

$$\tanh\frac{\pi}{2}x \leq \frac{\pi}{2}\frac{x}{\sqrt{1+x^2}}.$$

For all $x \geq 0$, define

$$f(x) = \frac{\pi}{2}\frac{x}{\sqrt{1+x^2}} - \tanh\frac{\pi}{2}x.$$

Then,

$$f(0) = 0 \quad \text{and} \quad f'(x) = \frac{\pi}{2}\left(\frac{1}{(1+x^2)^{3/2}} - \operatorname{sech}^2\frac{\pi}{2}x\right).$$

We need to show that $f' > 0$ for all $x > 0$. This is equivalent to proving the inequality

$$\begin{aligned}\left(\frac{1}{(1+x^2)^{3/2}} - \operatorname{sech}^2\frac{\pi}{2}x\right) &> 0\\ \cosh^2\frac{\pi}{2}x &> (1+x^2)^{3/2}\\ \cosh^4\frac{\pi}{2}x &> (1+x^2)^3 \qquad (62)\end{aligned}$$

Inequality (62) can be proved using Maclaurin series expansion, that is,

$$(\cosh\frac{\pi x}{2})^4 > (1+\frac{\pi^2 x^2}{8})^4 > (1+x^2)^4 > (1+x^2)^3.$$

We have thus showed that $f'(x) > 0$ for all $x > 0$, and since $f(0) = 0$, it follows that for all $x \geq 0$,

$$\begin{aligned}f(x) &\geq 0\\ \frac{\pi}{2}\frac{x}{\sqrt{1+x^2}} - \tanh\frac{\pi}{2}x &\geq 0\\ \tanh\frac{\pi}{2}x &\leq \frac{\pi}{2}\frac{x}{\sqrt{1+x^2}}\\ \frac{\sqrt{x^2+1}}{2\cosh\frac{\pi}{2}x} &\leq \frac{\pi}{2}\frac{x}{2\sinh\frac{\pi}{2}x}.\end{aligned}$$

□

We now have all the necessary definitions and lemmas needed to present and prove the main results of this section. In fact, they are the final results for this project thesis, and they will be presented in two separate theorems, one for the operator T and the other for the operator S.

Theorem 5. *The operator S is linear and bounded on the following Hilbert spaces with norm 1:*

1. $L^2(\omega_2)$
2. $L^2(\omega_1)$
3. $L^2(\mathbb{R})$
4. $H^2(\mathbb{S}, \mathcal{P})$
5. $H^2(\mathbb{S})$

Proof. Since linearity follows immediately from the fact that S is a convolution, we shall prove only boundedness.

1. Let $\tilde{\tilde{\rho}}_n = \frac{\tilde{\rho}_n}{\sqrt{n+1}}$, $f \in L^2(\omega_2)$ and $a_n = \langle f, \tilde{\tilde{\rho}}_n \rangle$. Since by equation (34) $\sqrt{n+1}$ is the norm of the polynomial $\tilde{\rho}_n$ in the Hilbert space $L^2(\omega_2)$, it follows from Theorem (3) that $\{\tilde{\tilde{\rho}}_n\}_{n=0}^{\infty}$ is an orthonormal basis in $L^2(\omega_2)$. Thus by proposition (2),

$$f = \sum_{n=0}^{\infty} a_n \tilde{\tilde{\rho}}_n \quad \text{and} \quad ||f||^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

By relation (56),

$$\begin{aligned} Sf &= \sum_{n=1}^{\infty} a_n \left(\frac{\tilde{\rho}_{n-1}}{\sqrt{n+1}} \right) \\ &= \sum_{n=0}^{\infty} a_{n+1} \left(\frac{\tilde{\rho}_n}{\sqrt{n+2}} \right) \\ &= \sum_{n=0}^{\infty} a_{n+1} \left(\sqrt{\frac{n+1}{n+2}} \right) \left(\frac{\tilde{\rho}_n}{\sqrt{n+1}} \right) \\ &= \sum_{n=0}^{\infty} \left(\sqrt{\frac{n+1}{n+2}} \right) a_{n+1} \tilde{\tilde{\rho}}_n. \end{aligned}$$

Thus,

$$||Sf||^2 = \sum_{n=0}^{\infty} \left(\frac{n+1}{n+2} \right) |a_{n+1}|^2 = \sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right) |a_n|^2 \leq \sum_{n=1}^{\infty} |a_n|^2 \leq ||f||^2,$$

which proves boundedness of S on $L^2(\omega_2)$ with norm 1.

2. Let $f \in L^2(\omega_1)$ and $b_n = \langle f, \tilde{\tau}_n \rangle$. By Theorem (1), the system $\{\tilde{\tau}_n\}_{n=0}^{\infty}$ is an orthonormal basis in $L^2(\omega_1)$ so that by proposition (2),

$$f = \sum_{n=0}^{\infty} b_n \tilde{\tau}_n \quad \text{and} \quad ||f||^2 = \sum_{n=0}^{\infty} |b_n|^2.$$

By relation (55),

$$Sf = \sum_{n=1}^{\infty} b_n \tilde{\tau}_{n-1} = \sum_{n=0}^{\infty} b_{n+1} \tilde{\tau}_n.$$

Thus,

$$||Sf||^2 = \sum_{n=0}^{\infty} |b_{n+1}|^2 = \sum_{n=1}^{\infty} |b_n|^2 \leq \sum_{n=0}^{\infty} |b_n|^2 = ||f||^2,$$

which boundedness of S on $L^2(\omega_1)$ with norm 1.

3. Let $f \in L^2(\mathbb{R})$. Using the Plancherel theorem, we have

$$\begin{aligned}
||Sf||^2 = ||\widehat{Sf}||^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{Sf}(t)|^2\, dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tanh t \hat{f}(t)|^2\, dt, \quad \text{by (61)} \\
&\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(t)|^2\, dt, \quad \text{since } |\tanh t| \leq 1 \\
&= ||f||^2
\end{aligned}$$

which proves boundedness of S on $L^2(\mathbb{R})$ with norm 1.

4. Let $\tilde{\tilde{\sigma}}_n = 1$ if $n = 0$, and $\tilde{\tilde{\sigma}}_n = \frac{\tilde{\sigma}_n}{\sqrt{2}}$ for all $n \geq 1$. Then by Theorem (2), the system $\{\tilde{\tilde{\sigma}}_n\}_{n=0}^{\infty}$ is an orthonormal basis in $H^2(\mathbb{S}, \mathcal{P})$. Let $f \in H^2(\mathbb{S}, \mathcal{P})$ and $c_n = \langle f, \tilde{\tilde{\sigma}}_n \rangle$. By proposition (2),

$$f = \sum_{n=0}^{\infty} c_n \tilde{\tilde{\sigma}}_n \quad \text{and} \quad ||f||^2 = \sum_{n=0}^{\infty} |c_n|^2.$$

Since $RJ = JR$ by proposition (7), it follows that $S = JR^{-1} = R^{-1}J$. Thus $S\tilde{\sigma}_n = \tilde{\sigma}_{n-1}$ by corollary (2) so that

$$\begin{aligned}
Sf &= \sum_{n=1}^{\infty} c_n \left(\frac{\tilde{\sigma}_{n-1}}{\sqrt{2}}\right) \\
&= \sum_{n=0}^{\infty} c_{n+1} \left(\frac{\tilde{\sigma}_n}{\sqrt{2}}\right) \\
&= \frac{c_1}{\sqrt{2}} + \sum_{n=1}^{\infty} c_{n+1} \tilde{\tilde{\sigma}}_n.
\end{aligned}$$

Thus,

$$||Sf||^2 = \frac{|c_1|^2}{2} + \sum_{n=1}^{\infty} |c_{n+1}|^2 = \frac{|c_1|^2}{2} + \sum_{n=2}^{\infty} |c_n|^2 \leq \sum_{n=1}^{\infty} |c_n|^2 \leq ||f||^2,$$

which proves boundedness of S on $H^2(\mathbb{S}, \mathcal{P})$ with norm 1.

5. Let $f \in H^2(\mathbb{S})$. Then by Lemma (6),

$$\begin{aligned}
||Sf||^2_{H^2(\mathbb{S})} &= ||\widehat{Sf}||^2_{L^2(\mathbb{R},\cosh 2t\,dt)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{Sf}(t)|^2 \cosh 2t\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tanh t\hat{f}(t)|^2 \cosh 2t\,dt, \quad \text{by (61)} \\
&\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(t)|^2 \cosh 2t\,dt, \quad \text{since } |\tanh t| \leq 1 \\
&= ||\hat{f}||^2_{L^2(\mathbb{R},\cosh 2t\,dt)} \\
&= ||f||^2_{H^2(\mathbb{S})},
\end{aligned}$$

which proves boundedness of S on $H^2(\mathbb{S})$ with norm 1.

□

Theorem 6. *The operator T is linear and bounded on the following Hilbert spaces:*

1. *$L^2(\omega_2)$ with norm less than or equal to $\sqrt{\pi}$.*
2. *$L^2(\omega_1)$ with norm less than or equal to 2.*
3. *$L^2(\mathbb{R})$ with norm 1.*
4. *$H^2(\mathbb{S})$ with norm 1.*

Proof. Since linearity follows immediately from the fact that S is a convolution, we shall prove only boundedness.

1. We first show that if $f \in L^2_{\mathbb{R}}(\omega_2)$ and $\psi = \sqrt{\omega_2}$, then $Tf\psi \in H^2(\mathbb{S})$. In particular, we show that

$$||Tf\psi||^2_{H^2(\mathbb{S})} = \int_{-\infty}^{\infty} \frac{|Tf(x+i)\psi(x+i)|^2 + |Tf(x-i)\psi(x-i)|^2}{2} dx$$

is finite. Now,

$$|\psi(x \pm i)|^2 = \left|\frac{x \pm i}{2\sinh\frac{\pi}{2}(x \pm i)}\right| = \left|\frac{x \pm i}{\pm i\,2\cosh\frac{\pi}{2}x}\right| = \frac{\sqrt{x^2+1}}{2\cosh\frac{\pi}{2}x}, \tag{63}$$

and by relation (57),

$$|Tf(x \pm i)|^2 = |f(x) + iSf(x)|^2 = |f(x)|^2 + |Sf(x)|^2. \tag{64}$$

Thus by (63) and (64),

$$\begin{aligned}
|Tf(x+i)\psi(x+i)|^2 &= |Tf(x-i)\psi(x-i)|^2 && (65)\\
&= \left(|f(x)|^2 + |Sf(x)|^2\right)\frac{\sqrt{x^2+1}}{2\cosh\frac{\pi}{2}x}. && (66)
\end{aligned}$$

Therefore,

$$\begin{aligned}
||Tf\psi||^2_{H^2(\mathbb{S})} &= \int_{-\infty}^{\infty} \frac{|Tf(x+i)\psi(x+i)|^2 + |Tf(x-i)\psi(x-i)|^2}{2} dx \\
&= \int_{-\infty}^{\infty} |Tf(x+i)\psi(x+i)|^2 dx, \quad \text{by (65)} \\
&= \int_{-\infty}^{\infty} \left(|f(x)|^2 + |Sf(x)|^2\right) \frac{\sqrt{x^2+1}}{2\cosh\frac{\pi}{2}x} dx, \quad \text{by (66)} \\
&\leq \frac{\pi}{2} \int_{-\infty}^{\infty} \left(|f(x)|^2 + |Sf(x)|^2\right) \frac{x}{2\sinh\frac{\pi}{2}x} dx, \quad \text{by lemma (8)} \\
&= \frac{\pi}{2} \int_{-\infty}^{\infty} |f(x)|^2 \omega_2 dx + \frac{\pi}{2} \int_{-\infty}^{\infty} |Sf(x)|^2 \omega_2 dx \\
&= \frac{\pi}{2} ||f||^2_{L^2_{\mathbb{R}}(\omega_2)} + \frac{\pi}{2} ||Sf||^2_{L^2_{\mathbb{R}}(\omega_2)} \\
&\leq \frac{\pi}{2} ||f||^2_{L^2_{\mathbb{R}}(\omega_2)} + \frac{\pi}{2} ||f||^2_{L^2_{\mathbb{R}}(\omega_2)}, \quad \text{by Theorem (5) part(1)} \\
&= \pi \, ||f||^2_{L^2_{\mathbb{R}}(\omega_2)}, \qquad (67)
\end{aligned}$$

which proves that $Tf\psi \in H^2(\mathbb{S})$ for any $f \in L^2_{\mathbb{R}}(\omega_2)$.

Next, we show that T is bounded on $L^2_{\mathbb{R}}(\omega_2)$. Let $f \in L^2_{\mathbb{R}}(\omega_2)$, then

$$\begin{aligned}
||Tf||^2_{L^2_{\mathbb{R}}(\omega_2)} &= \int_{-\infty}^{\infty} |Tf(x)|^2 \omega_2(x) dx \\
&= \int_{-\infty}^{\infty} |Tf(x)\sqrt{\omega_2(x)}|^2 dx \\
&= \int_{-\infty}^{\infty} |Tf(x)\psi(x)|^2 dx \\
&\leq \left(\int_{-\infty}^{\infty} |Tf(x+i)\psi(x+i)|^2 dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |Tf(x-i)\psi(x-i)|^2 dx\right)^{\frac{1}{2}} \\
&= \int_{-\infty}^{\infty} |Tf(x+i)\psi(x+i)|^2 dx, \quad \text{by (65)} \\
&\leq \pi \, ||f||^2_{L^2_{\mathbb{R}}(\omega_2)}, \quad \text{by (67)}
\end{aligned}$$

where the first inequality follows by Lemma (7) since $Tf\psi \in H^2(\mathbb{S})$.

We have shown that T is bounded on $L^2_{\mathbb{R}}(\omega_2)$ with norm $\sqrt{\pi}$. Now since every analytic function g can be written as $g = f + i\,h$ for $f, h \in L^2_{\mathbb{R}}(\omega_2)$, it follows that T is bounded on $L(\omega_2)$ with norm $\sqrt{\pi}$.

Remark 3. Since T and S map functions that are real on the real line to functions that also have this property and therefore $T(f+ih) = Tf + iTh$, the same bounds hold for complex-valued functions as for real-valued.

2. We first show that if $f \in L^2_{\mathbb{R}}(\omega_1)$ and $\varphi = 1/(2\cosh\frac{\pi}{4}x)$, then $Tf\varphi \in H^2(\mathbb{S})$. In particular, we show that

$$||Tf\varphi||^2_{H^2(\mathbb{S})} = \int_{-\infty}^{\infty} \frac{|Tf(x+i)\varphi(x+i)|^2 + |Tf(x-i)\varphi(x-i)|^2}{2} dx$$

is finite. Now,

$$\begin{aligned}\left|2\cosh\frac{\pi}{4}(x\pm i)\right|^2 &= \left|\sqrt{2}\cosh\frac{\pi}{4}x \pm i\sqrt{2}\sinh\frac{\pi}{4}x\right|^2 \\ &= \left(\sqrt{2}\cosh\frac{\pi}{4}x\right)^2 + \left(\sqrt{2}\sinh\frac{\pi}{4}x\right)^2 \\ &= 2\cosh\frac{\pi}{2}x,\end{aligned}$$

and so,

$$|\varphi(x\pm i)|^2 = \frac{1}{2\cosh\frac{\pi}{2}x}. \tag{68}$$

By relation (57),

$$|Tf(x\pm i)|^2 = |f(x) + iSf(x)|^2 = |f(x)|^2 + |Sf(x)|^2. \tag{69}$$

Thus by (68) and (69),

$$\begin{aligned}|Tf(x+i)\varphi(x+i)|^2 &= |Tf(x-i)\varphi(x-i)|^2 &(70)\\ &= \left(|f(x)|^2 + |Sf(x)|^2\right)\frac{1}{2\cosh\frac{\pi}{2}x}. &(71)\end{aligned}$$

Therefore,

$$\begin{aligned}||Tf\varphi||^2_{H^2(\mathbb{S})} &= \int_{-\infty}^{\infty} \frac{|Tf(x+i)\varphi(x+i)|^2 + |Tf(x-i)\varphi(x-i)|^2}{2} dx \\ &= \int_{-\infty}^{\infty} |Tf(x+i)\varphi(x+i)|^2 dx, \quad \text{by (70)} \\ &= \int_{-\infty}^{\infty} \left(|f(x)|^2 + |Sf(x)|^2\right)\frac{dx}{2\cosh\frac{\pi}{2}x}, \quad \text{by (71)} \\ &= \int_{-\infty}^{\infty} |f(x)|^2\omega_1 dx + \int_{-\infty}^{\infty} |Sf(x)|^2\omega_1 dx \\ &= ||f||^2_{L^2_{\mathbb{R}}(\omega_1)} + ||Sf||^2_{L^2_{\mathbb{R}}(\omega_1)} \\ &\le ||f||^2_{L^2_{\mathbb{R}}(\omega_1)} + ||f||^2_{L^2_{\mathbb{R}}(\omega_1)}, \quad \text{by Theorem (5) part(2)} \\ &= 2||f||^2_{L^2_{\mathbb{R}}(\omega_1)}, \end{aligned} \tag{72}$$

which proves that $Tf\varphi \in H^2(\mathbb{S})$ for any $f \in L^2_{\mathbb{R}}(\omega_1)$.

Next, we show that T is bounded on $L^2_{\mathbb{R}}(\omega_1)$. Let $f \in L^2_{\mathbb{R}}(\omega_1)$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} |Tf(x)\psi(x)|^2 dx &= \int_{-\infty}^{\infty} \frac{|Tf(x)|^2}{\left(2\cosh\frac{\pi}{4}x\right)^2} dx \\
&= \int_{-\infty}^{\infty} \frac{|Tf(x)|^2}{2\left(\cosh\frac{\pi}{2}x + 1\right)} dx \\
&\geq \int_{-\infty}^{\infty} \frac{|Tf(x)|^2}{2\left(\cosh\frac{\pi}{2}x + \cosh\frac{\pi}{2}x\right)} dx, \text{ since } \cosh\frac{\pi}{2}x \geq 1 \\
&= \int_{-\infty}^{\infty} \frac{|Tf(x)|^2}{4\cosh\frac{\pi}{2}x} dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} |Tf(x)|^2 \omega_1(x) dx
\end{aligned}$$

so that

$$\begin{aligned}
||Tf||^2_{L^2_{\mathbb{R}}(\omega_1)} &= \int_{-\infty}^{\infty} |Tf(x)|^2 \omega_1(x) dx \\
&\leq 2\int_{-\infty}^{\infty} |Tf(x)\varphi(x)|^2 dx \\
&\leq 2\left(\int_{-\infty}^{\infty} |Tf(x+i)\varphi(x+i)|^2 dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |Tf(x-i)\varphi(x-i)|^2 dx\right)^{\frac{1}{2}} \\
&= 2\int_{-\infty}^{\infty} |Tf(x+i)\varphi(x+i)|^2 dx, \quad \text{by } (70) \\
&\leq 4||f||^2_{L^2_{\mathbb{R}}(\omega_1)}, \quad \text{by } (72)
\end{aligned}$$

where the second inequality follows by the Lemma (7) since $Tf\varphi \in H^2(\mathbb{S})$.

We have shown that T is bounded on $L^2_{\mathbb{R}}(\omega_1)$ with norm 2. Now since every analytic function g can be written as $g = f + i\,h$ for $f, h \in L^2_{\mathbb{R}}(\omega_1)$, it follows by Remark (3) that T is bounded on $L(\omega_1)$ with norm 2.

3. Let $f \in L^2(\mathbb{R})$. Using the Plancherel theorem, we have

$$\begin{aligned}
||Tf||^2 = ||\widehat{Tf}||^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{Tf}(t)|^2\, dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\operatorname{sech} t \hat{f}(t)|^2\, dt, \quad \text{by } (60) \\
&\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(t)|^2\, dt, \quad \text{since } |\operatorname{sech} t| \leq 1 \\
&= ||f||^2
\end{aligned}$$

which proves boundedness of T on $L^2(\mathbb{R})$ with norm 1.

4. Let $f \in H^2(\mathbb{S})$. Then by Lemma (6),

$$\begin{aligned}
||Tf||^2_{H^2(\mathbb{S})} &= ||\widehat{Tf}||^2_{L^2(\mathbb{R},\cosh 2t\,dt)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{Tf}(t)|^2 \cosh 2t\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\operatorname{sech} t \hat{f}(t)|^2 \cosh 2t\,dt, \quad \text{by (60)} \\
&\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(t)|^2 \cosh 2t\,dt, \quad \text{since } |\operatorname{sech} t| \leq 1 \\
&= ||\hat{f}||^2_{L^2(\mathbb{R},\cosh 2t\,dt)} \\
&= ||f||^2_{H^2(\mathbb{S})},
\end{aligned}$$

which proves boundedness of T on $H^2(\mathbb{S})$ with norm 1.

□

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