Pricing options on defaultable stocks

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Abstract

This work deals with important issue of pricing of options with default risk. We use results obtained by Carr and Linetsky in pricing European and American-type options. The key notion in this work is the risk-neutral survival probability which makes it possible for us to account for possibility of default.

To price American-type options, we will use Least Square Monte Carlo method, because it is easier to apply survival probability in repeated simulations.
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1.1 Monte Carlo simulations

In this work I use Monte Carlo simulations to calculate the price of options on underlying assets which can default. Monte Carlo simulations are based on repeated sampling in order to calculate the value of the option. In this work, implementation is based on simulation of stock paths in the new jump to default constant elasticity of variance model (CEV). Then we incorporate survival probability into the scheme.

1.2 Option types

The following type of options are considered:

- *European options* - which give the holder the right but not the obligation to exercise the option at fixed time, called maturity.

- *American options* - are the type of options which can be exercised at any time up to the time of maturity.
Jump to default CEV model

2.1 Bessel processes

Much of the theoretical results in this work are based on the properties of Bessel processes, so we give definition of these processes and some of their properties. Before that note that for every $\varepsilon \geq 0, x \geq 0$ the solution of the equation

$$R_t = x + \varepsilon t + 2 \int_0^t \sqrt{|R_s|} dB_s$$

is unique.

**Definition:** For every $\varepsilon \geq 0, x \geq 0$ the unique strong solution to the equation above is called the square of $\varepsilon$ dimensional Bessel process with starting point at $x$ and denoted as $BES^\varepsilon(x)$.

2.2 Application of Bessel processes to a jump to default CEV model.

We start by introducing some empirical findings that our model is based upon. One of these is "implied volatility skew". It is a name for the inverse relationship between implied volatility and an option's strike price. This relationship has been observed for both single name and index options since 1987 in (Dennis). Another interesting fact is the empirical evidence that suggest that realized stock volatility is negatively related to stock price. This is called a leverage effect and has been supported by empirical studies in (Christie).

When we model a diffusion process we design it so that it encompasses the above mentioned empirical findings. To fulfill the leverage effect and the implied volatility skew we assume a constant elasticity of variance (CEV). Also, in this model it is possible that the process will jump to zero. We assume that the instantaneous risk-neutral hazard rate of default is an increasing affine function of the instantaneous variance of the underlying stock. This new model is called the jump to default extended CEV or JDCEV process.

We assume market with no arbitrage and an equivalent martingale measure. The following is the stochastic differential equation (SDE) for the pre-default stock process $\{S_t, t \geq 0\}$.

$$dS_t = [r(t) - q(t) + \lambda(S_t)]S_t dt + \sigma(S_t, t)S_t dB_t, \quad S_0 > 0, \quad (1)$$
where $r(t) \geq 0$ the risk-free interest rate, $q(t) \geq 0$ is the dividend yield, $\sigma(S, t) \geq 0$ is the instantaneous stock volatility and $\lambda(S, t) \geq 0$ is the default intensity. To be consistent with the leverage effect and the implied volatility skew, we define the instantaneous volatility as follows

$$\sigma(S, t) = a(t)S^\beta,$$

where $\beta < 0$ is the volatility elasticity parameter and $a(t) > 0$ is the volatility scale parameter. Additionally, we assume that the default intensity is an affine function of the underlying stock:

$$\lambda(S, t) = b(t) + c\sigma^2(S, t) = b(t) + ca^2(t)S^{2\beta},$$

where $b(t) \geq 0$ and $c > 0$.

Before proceeding with the solution of SDE we make some definitions. The jump-to-default hazard process is defined in (Peter Carr) as follows:

$$\Lambda(t) = \begin{cases} \int_0^t \lambda(S(u), u)du, & t < T_0 \\ \infty, & t \geq T_0 \end{cases}$$

where $T_0 = \inf\{t \geq 0: S(t) = 0\}$. A random time of jump to default is the first time when $\Lambda$ is greater or equal to the random level $e \sim \exp(1)$:

$$\xi = \inf\{t \geq 0: \Lambda(t) \geq e\}.$$ 

And, finally the time of default is composed of two parts:

$$\xi = T_0 \wedge \xi.$$ 

In the above mentioned work, the authors consider three "building block claims" as a basis of more complex securities:

1) A European-style contingent claim with payoff $\Omega(S_T)$ at time $T$, given no default by $T$, and no recovery in case of default.

2) A recovery payment of one dollar at the maturity date $T$ if default occurs by $T$;

3) A recovery payment of one dollar paid at the default time if default occurs by $T$.

They show that the valuation of these three "building blocks" reduces to computing risk-neutral expectation of the form
\[
E \left[ e^{-\int_t^T \lambda(S_u,u) du} \Omega(S_T) 1_{\{m^S_{t,T} > 0\}} \big| S_t = S \right].
\]  

(2)

This is calculated, and as a result an analytical solution is obtained, by using the theory of Bessel processes. We will denote a Bessel process of index \( \mu \) starting at \( x \) (\( Bes^\mu(x) \)) by \( R_t^\mu \).

According to Proposition 5.2, for any \( 0 \leq t < T \) the following holds:

\[
E \left[ \exp \left\{ -c \int_t^T a^2(u) S_u^2 du \right\} \Omega(S_T) 1_{\{m^S_{t,T} > 0\}} \big| S_t = S \right] = E_x^{(\mu)} \left[ \exp \left\{ -\frac{c}{\beta^2} \int_0^t \frac{du}{R_u^2} \Omega(e^{\int_t^T a(s) ds} (|\beta| R_t)^{\frac{3}{2}}) 1_{\{T_0^R > t\}} \right\} \right],
\]

(3)

where \( \tau(t,T) = \int_t^T a^2(u) e^{-2|\beta| \int_0^t a(s) ds} du \).

To get rid of the term \( \exp \left\{ -\frac{c}{\beta^2} \int_0^t \frac{du}{R_u^2} \right\} \) the following proposition from (Pitman) is used:

**Proposition:** Let \( P^\mu_t \) be the law of the Bessel process \( R_t^\mu \) started at \( x > 0 \) and let \( R_t \) be its canonical filtration. Then, for \( \nu \geq 0 \) and \( \mu \geq 0 \) the following absolute continuity relation holds

\[
P_x^\nu |_{R_t} = \left( \frac{R_t}{x} \right)^{\nu-\mu} \exp \left( -\frac{\nu^2 - \mu^2}{2} \int_0^t \frac{du}{R_u^2} \right) P_x^{(\mu)} |_{R_t},
\]

(4)

And for \( \nu < 0 \) and \( \mu \geq 0 \) the following absolute continuity relation holds before the first hitting time of zero, \( T_0^R \):

\[
P_x^\nu |_{R_t \cap T_0^R} = \left( \frac{R_t}{x} \right)^{\nu-\mu} \exp \left( -\frac{\nu^2 - \mu^2}{2} \int_0^t \frac{du}{R_u^2} \right) P_x^{(\mu)} |_{R_t},
\]

(5)

Consider the following Bessel process indexes

\[
\nu_+ = \nu + \frac{1}{|\beta|} = \frac{c + \frac{1}{2}}{|\beta|} > 0 \quad \text{and} \quad \nu = \frac{c - \frac{1}{2}}{|\beta|}.
\]

According to (5), we have

\[
P_x^\nu |_{R_t \cap T_0^R} = \left( \frac{R_t}{x} \right)^{\nu-\mu} \exp \left( -\frac{\nu^2 - \mu^2}{2} \int_0^t \frac{du}{R_u^2} \right) P_x^{(\mu)} |_{R_t},
\]

\[
P_x^{(\mu)} |_{R_t} = P_x^{\nu_+} |_{R_t \cap T_0^R} \left( \frac{R_t}{x} \right)^{-\nu_+ + \mu} \exp \left( \frac{\nu_+^2 - \mu^2}{2} \int_0^t \frac{du}{R_u^2} \right),
\]

\[
P_x^{(\mu)} |_{R_t} = P_x^{\nu} |_{R_t \cap T_0^R} \left( \frac{R_t}{x} \right)^{-\nu + \mu} \exp \left( \frac{\nu^2 - \mu^2}{2} \int_0^t \frac{du}{R_u^2} \right).
\]
\[ P_x^\nu \big|_{R_t \cap T_0} \exp \left( \frac{\nu^2 - \mu^2 - \nu^+_2 + \mu^2}{2} \int_0^t \frac{du}{R_u^2} \right) = \left( \frac{R_t}{x} \right)^{-\nu_+ + \nu - \mu} P_x^\nu \big|_{R_t} . \]  

(6)

Note that
\[ \nu - \nu_+ = \frac{c - \frac{1}{2} - c - \frac{1}{2}}{|\beta|} = - \frac{1}{|\beta|} \]

\[ \nu^2 - \nu^+_2 = c^2 - c + \frac{1}{4} - c^2 - c - \frac{1}{4} = - \frac{2c}{|\beta|^2} . \]

Plugging it in (6) we get
\[ P_x^\nu \big|_{R_t \cap T_0} \exp \left( - \frac{c}{|\beta|^2} \int_0^t \frac{du}{R_u^2} \right) = \left( \frac{R_t}{x} \right)^{\frac{1}{|\beta|}} P_x^\nu \big|_{R_t} . \]  

(7)

Applying this to (3) we obtain the following equality:
\[ E \left[ \exp \left\{ \exp -c \int_t^T a^2(u) S_u^{2\beta} du \right\} \Omega(S_T) I_{\{m^s_{t,T} > 0\}} \right] = E_x^{(\nu)} \left[ \left( \frac{R_t}{x} \right)^{\frac{1}{|\beta|}} \Omega(e^{\int_t^T a(s) ds}) \left( |\beta| R_t \right)^{\frac{1}{|\beta|}} \right] . \]  

(8)

Thus, the problem has been reduced to computation of the expected value of a function of the standard Bessel process. From (Göing) we have the following expression for the density of \( R_t \):
\[ P_x^\nu (R_t \in dy) = p^\nu(\tau; x; y) dy = \frac{y}{\tau} \left( \frac{\nu}{x} \right)^\nu \exp \left( - \frac{x^2 + y^2}{2\tau} \right) I_{(\frac{xy}{\tau})} dy , \]

where
\[ I_{\nu}(z) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + \nu + 1)} \left( \frac{z}{2} \right)^{\nu + 2n} \]

is the Bessel function of the third kind of index \( \nu \).

We use the fact that Bessel process density can be expressed in terms of the non-central chi-squares density. For this, note that the non-central chi-squares distribution with \( \delta \) degrees of freedom and non-centrality parameter \( \alpha > 0 \) has the density
\[ f_{x^2}(x; \delta, \alpha) = \frac{1}{2} e^{-\frac{(x + \alpha)^\delta}{2}} \left( \frac{x}{\alpha} \right)^\nu I_{\nu} \left( \sqrt{\alpha x} \right) 1_{(x > 0)} , \]

and we arrive to the following expression for the Bessel process distribution function:
Finally, to calculate (8) we will need an expression for moments of chi-square distribution.

**Lemma 5.1** (from (Peter Carr)): Let $X$ be a $X^2(\delta, \alpha)$ random variable, $\nu = \frac{\delta}{2} - 1$, $p > -\nu - 1$ and $k > 0$. The p-th moments and truncated p-th moments are given by

$$
M(p; \delta, \alpha) = E_{X^2(\delta, \alpha)}[X^p] = 2^p e^{-\alpha / 2} \frac{\Gamma(p + \nu + 1)}{\Gamma(v + 1)} F_1 \left( p + \nu + v + 1, \nu + 1, \frac{\alpha}{2} \right), \quad (9)
$$

$$
\Phi^+(p, k; \delta, \alpha) = E_{X^2(\delta, \alpha)}[X^p 1_{X > k}] = 2^p e^{-\alpha / 2} \sum_{n=0}^{\infty} \left( \frac{\alpha}{2} \right)^n \frac{\Gamma(p + \nu + n + 1, \frac{k}{2})}{n! \Gamma(\nu + 1 + n)}, \quad (10)
$$

$$
\Phi^-(p, k; \delta, \alpha) = E_{X^2(\delta, \alpha)}[X^p 1_{X \leq k}] = 2^p e^{-\alpha / 2} \sum_{n=0}^{\infty} \left( \frac{\alpha}{2} \right)^n \frac{\nu(p + \nu + n + 1, \frac{k}{2})}{n! \Gamma(\nu + 1 + n)}, \quad (11)
$$

The formulas (9) - (11) enables us to calculate the expected value of (8) thus enabling us to derive formulas for more complex securities.
3.1 Numerical results

Formulas (10) and (11) pose certain problems for calculations. To avoid these problems we will employ Monte Carlo method for computation of option prices (in this case, for simplicity, call option). In this case, to factor in the probability of default by jump to zero as opposed to simple diffusion process, we use the risk neutral survival probability. This probability has the following form

\[ Q(S, t; T) = E \left[ e^{-\int_t^T \lambda(S_u) du} 1_{[S_u > 0]} \big| S_t = S \right] \]  

(12)

and after applying Lemma 5.1 we arrive at the following formula

\[ Q(S, t; T) = e^{-\int_t^T b(u) du} \left( \frac{x^2}{\tau} \right)^{1/2|\beta|} M \left( -\frac{1}{2|\beta|}; \delta_+, \frac{x^2}{\tau} \right), \]  

(13)

where \( \delta_+ = 2(v_+ + 1) \) and \( \tau \) is defined as in the equation (3).

3.1 Valuation of European options

The survival probability is applied as follows: In each of the simulations we compare the expression above with a uniformly distributed random level. If the survival probability is larger than this random level, then we accept the current path \( S_t \) (i.e., default does not occur), otherwise we consider the current path equal to zero (i.e., default occurred).

For convenience we will write the SDE (1) in a more compact way:

\[ dS_t = a(t, S_t) dt + b(t, S_t) dB_t \]

With this in mind, and using Euler’s discretization we can formulate the Monte-Carlo method for CEV and JDCEV models as follows:
Algorithm 1

\[ \Delta t = \frac{T}{M} \]

for \( j=1, \ldots, N \)

\[ t_i = t_{i-1} + \Delta t \]

\[ \Delta w = u_n \sqrt{\Delta t}, \quad u_n \in N(0,1) \]

\[ \Delta S_i = S_{i-1} + a(t_{i-1}, S_{i-1}) \Delta t + b(t_{i-1}, S_{i-1}) \Delta w \]

end

Compute the value \( V_T(j) = (S_T - K)^+ \).

end

Compute \( E(V_T) = \frac{1}{N} \sum_{j=1}^{N} V_T(j) \).

Compute the discounted value \( V_0 = e^{-rT} E(V_T) \).
Algorithm 2 (when default is possible)

\[ \Delta t = \frac{T}{M} \]

for \( j = 1, \ldots, N \)

\[
\text{for } i = 1, \ldots, M \\
\quad t_i = t_{i-1} + \Delta t \\
\quad \Delta w = u_n \sqrt{\Delta t}, \quad u_n \in N(0,1) \\
\quad \Delta S_i = S_{i-1} + a(t_{i-1}, S_{i-1}) \Delta t + b(t_{i-1}, S_{i-1}) \Delta w \\
\]

Calculate risk-neutral survival probability in (13): \( Q_i = Q(s_j, t; T) \)

end

Generate normally distributed sample \( e_n \in E(0,1), n = 1, \ldots, N \).

if \( Q_j > e_n \)

Compute the value \( V_T(j) = (S_T - K)^+ \).

end

Compute \( E(V_T) = \frac{1}{N} \sum_{j=1}^{N} V_T(j) \).

Compute the discounted value \( V_0 = e^{-rT} E(V_T) \).
3.1.1 Parameter sensitivity

The following parameters are considered in this Monte Carlo simulation: $\sigma = 0.2$, $\beta = -0.2$, $r = 0.1$, $T = 0.5$, $K = 16$, $S_0 = 12$. Also, for convenience, we use the following values for the functions which form the basis for JDCEV: $q(t) = 0$, $a(t) = 1$, $c(t) = 1$, $r(t) = r$. We simulate the price process using Euler discretization.

![Fig.1](image)

In Fig.1 we model the price process as we vary the initial stock price. As one can note, there is a considerable difference in option prices between these two cases. This is caused by the fact that in the second case we take into account the possibility of default and so, some of the paths in our simulation we accept as zeros.

Figures 2, 3 and 4 show us how the price process behave with respect to variations in stock process with different volatility scale parameter and with respect to strike price. It is important to remember that the volatility in this model is not a constant but a power function of the form $\sigma(S, t) = aS^\beta$. It is easy to show that the picture roughly corresponds to the standard CEV process\(^1\) with constant coefficients, when we adjust the parameters of our current model accordingly.

\(^1\)The same Monte Carlo method can be used with minor changes; The main, and the most important difference would be the absence of the risk-neutral probability of survival, because in standard CEV model default occurs through diffusion to zero.
Fig. 2

Fig. 3
There is a visible positive relationship between the Call prices and volatility scale parameter $\alpha$. This is due to the fact that there is a positive relationship between the default intensity and survival probability as one can see in the formulas (12) and (13).

We see the similar picture in the case of put option.

Fig. 4

Fig. 5
Table 1 ($\beta = -0.8$)

<table>
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<tr>
<th>K</th>
<th>Put (defaultable)</th>
<th>Put</th>
</tr>
</thead>
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</table>

Table 2 ($\beta = -0.5$)

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<tr>
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<td>4,23818106</td>
</tr>
</tbody>
</table>

Parameters: $S=20$, $q=0$, $r=0.05$, $q=0$, $T=1$ year.

As evident from Tables 1 and 2, the smaller the $\beta$ the closer the value of the option on defaultable stock process is to the option on non-defaultable stock. This is caused by the behavior of the survival probability (12). For example, for the above parameters we have the following picture of the risk-neutral survival probability:
It is clear that there is positive relationship between $\beta$ and default probability, which explains why the difference between option value on defaultable and non-defaultable stock is bigger with smaller volatility elasticity parameter $\beta$.

### 3.2 Valuation of American options

American type options differ from European types in that they can be exercised at any time during its life. Consequently, option price depends on exercise strategy. This makes Monte Carlo simulations, which are very simple to implement in a European type options case, a little less trivial.

There are several methods for pricing American options. By doing time discretization of Black-Scholes formula we get a linear complimentary problem. One of the methods of solution is projected successive over relaxation (PSOR) method. As PSOR is inefficient for some cases of space discretization, other method called operator splitting(OS) is applied to solve the problem. The main idea of this method is splitting of operator into separate fractional time steps in the discretization (Toivanen). We won't go into this method, as it can be problematic to integrate the risk-neutral survival probability in this scheme.

For numerical calculations we will use least squares Monte Carlo simulations with Laguerre polynomials as set of basis functions. In this regression method American-type option is approximated by using a Bermuda-type option. This is Least Square Monte Carlo method by Longstaff and Schwartz (Schwartz). We start by simulating stock paths and then, do backward iterations. At each step of iterations we perform a least square approximation of the continuation function.

Longstaff and Schwartz method is based on the assumption that the continuation value can be expressed as a linear combination of basis functions:

$$\hat{C}_i(s) = E[V_{i+1}(S_{i+1})|S_i = s] = \sum_{j=1}^{m} \beta_{ij} L_j(s),$$

where $\beta_{ij}$ are constants, and $L_j(s)$ is the Rodriguez representations of Laguerre polynomials:

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x}x^n).$$

The Longstaff and Schwartz method algorithm as presented in (Korn):
Algorithm 3

1. Choose $k$ basic functions $L_1, ..., L_k$.

2. Generate the $N$ underlying geometric Brownian paths of the stock prices at each time step: $\{S(t_1)_j, ..., S(t_m)_j\}$.

3. Generate normally distributed sample $e_n \in E(0,1), n=1, ..., N$.

4. Set the terminal value for each path: $\hat{V}_{m,j} = e^{-rt} f(S(T)_j), j = 1, ..., N$.

5. Continue backward in time and at each time $t_i, i = m - 1, ..., 1$:
   - Using linear least square method compute $\beta_i$.
   - Compute $\hat{C}_i$ using $\beta_i$.
   - Compute the risk-neutral survival probability $Q_i = Q(S_j, t_i, T)$.
   - Decide to exercise or hold the option. Set for $j = 1, ..., N$
     \[
     \hat{V}_{ij} = \begin{cases} 
     e^{-rt_i} f(S(t_i)_j), & \text{if } e^{-rt_i} f(S(t_i)_j) \geq \hat{C}_i(S(t_i)_j, i) \text{ and } Q_i > e_n \\
     \hat{V}_{i+1,j}, & \text{else} 
     \end{cases}
     \]

6. Compute $\hat{V}_0 = \frac{1}{N} (\hat{V}_{11} + \cdots + \hat{V}_{1N})$. 
3.2.1 Parameter sensitivity

As evident from Tables 3 and 4, lower $\beta$ results in bigger difference between the options on defaultable and non-defaultable option. This is consistent with the European type option cases. The reason is, as a part of numerical method, we add the time points when the risk-neutral survival probability is smaller than a random level, to the set of time points in which we accept the stock price as equal to zero.

Table 3 ($\beta = -0.5$)

<table>
<thead>
<tr>
<th>$K$</th>
<th>Put(default)</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>0,11153476</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>0,12710312</td>
<td>0,11153476</td>
</tr>
<tr>
<td>35</td>
<td>4,97005468</td>
<td>4,97061257</td>
</tr>
<tr>
<td>40</td>
<td>9,96857552</td>
<td>9,96151396</td>
</tr>
<tr>
<td>45</td>
<td>14,9636761</td>
<td>14,9632046</td>
</tr>
<tr>
<td>50</td>
<td>19,9544929</td>
<td>19,9548011</td>
</tr>
<tr>
<td>55</td>
<td>24,9452399</td>
<td>24,9506628</td>
</tr>
<tr>
<td>60</td>
<td>29,9385832</td>
<td>29,9459733</td>
</tr>
<tr>
<td>65</td>
<td>34,9375425</td>
<td>34,9447761</td>
</tr>
<tr>
<td>70</td>
<td>39,9391133</td>
<td>39,9394238</td>
</tr>
</tbody>
</table>

Table 4($\beta = -0.2$)

<table>
<thead>
<tr>
<th>$K$</th>
<th>Put(default)</th>
<th>Put</th>
</tr>
</thead>
<tbody>
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<td>25</td>
<td>4,92638962</td>
<td>4,95357142</td>
</tr>
<tr>
<td>30</td>
<td>9,93429036</td>
<td>9,95427846</td>
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<td>19,9193856</td>
<td>19,9396028</td>
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<tr>
<td>45</td>
<td>24,919995</td>
<td>24,9331697</td>
</tr>
<tr>
<td>50</td>
<td>29,9245166</td>
<td>29,9215411</td>
</tr>
<tr>
<td>55</td>
<td>34,8940074</td>
<td>34,9265439</td>
</tr>
<tr>
<td>60</td>
<td>39,8803758</td>
<td>39,9177479</td>
</tr>
<tr>
<td>65</td>
<td>44,8960086</td>
<td>44,9192177</td>
</tr>
<tr>
<td>70</td>
<td>49,8789401</td>
<td>49,9058121</td>
</tr>
</tbody>
</table>

Again we use the above mentioned values for parameters: $S=20, q=0, r=0.05, q=0, T=1$ year, $a=0.2, c=1$. 

These plots show American put option price as a function of volatility scale parameter. Predictably, this relationship is positive.
The parameters in above simulations: K=50, r=0.1, c=1, b=0, T=0.5, a=0.2 (volatility scale parameter). We can see a clear pattern in Fig. 9-10. Decrease in $\beta$ leads to a shrinking of the continuation region. Remember that $\beta < 0$ is a volatility elasticity parameter, and as such, this relationship is quit predictable. This is because of the inverse relationship between the price and the volatility, which has been shown in empirical studies (Christie).
3.2.2 Early exercise curve

We define the value of American put option in our case as follows:

$$P(S, t, T) = \sup_{t \leq \tau \leq T} E_{t, S} \left[ e^{-\int_t^\tau r(s) \, ds} (K - S_\tau)^+ \right],$$

where $\tau$ is the exercise time, and the continuation region is defined as

$$\Omega = \{(s, t) | P(s, t, T) > (K - s)^+\}.$$  

Because the price function is continuous, it is obvious that at some point it hits the straight line of pay-off function at some $S_f(t)$. This is called an early exercise curve.

$$K=60, r=0.1, c=1, b=0, T=1, a=0.2, \beta = -0.5$$

Note that, the early exercise curve is not convex, as opposed to the geometric Brownian motion case. We did not have time to verify this, so it might be an interesting topic of future research.
4.1 Framework

We can conclude that the framework proposed by P.Carr and Linetsky offers a relatively easy way to price different types of financial instruments with a positive probability of default. In this thesis, we focus on American and European-type options, but as described in (Peter Carr), this can be extended to all kinds of different securities.

4.2 Future research

It can be interesting to find out how this new model affects the hedging, as portfolio may consists of risky assets. In this case, using methods other than Monte Carlo would be more appropriate considering the problems associated with multiple simulations to calculate the greeks.

Another interesting nuance is, because American-type options can be exercised at any time before the maturity, the holder might choose, in the face of probable default to alleviate the threat of losing more money, to exercise the option before default happens, and losing less. It would be interesting to conduct further research into this scenario.
Bibliography


