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# Elementary Discrete Sets in Martin-Löf Type Theory

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**Abstract**

The concept of reducibility in the sense of computation is a central theme of computer science. Classic set theory, however, does not fully reflect this discrete notion. In Martin-Löf type theory, a set is viewed from a type-centric perspective; allowing more explicit structures to be considered. In this thesis we explore basic countable discrete sets from said type-centric perspective. While a keen focus on the notion of set is maintained, we also discuss the basic outline of the intuitionistic logic which the theory is based on.

# Acknowledgements

Out of the three Bachelor's theses that I have written during my three years at Uppsala University, this is by far the one that I have enjoyed working on the most, partly because of my excellent supervisor, Vera Koponen, with whom I have had several interesting and rewarding discussions. For this I am truly thankful.

# Contents

# Introduction

The utility of mathematics has always been implicitly enforced through its many applications. Even mathematical subfields such as number theory and the study of prime numbers, which was once seen as purely theoretical work, has found utility in a world of increasing dependency on computers.

With the development of modern set theory by Cantor and Dedekind [1] in the late nineteenth century, it has become somewhat customary to introduce students to the subject of mathematics – well, to be fair we should emphasize the prefix *university* to students here – through the outline of the frequently used sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$. With this knowledge students are able to see layers among the numbers that previously were an implicit notion. In essence, obviously 2 and $\frac{7}{18}$ have always been viewed as fundamentally different in some vague sense among most pre-university students, but through this explicit partitioning the difference is enforced.

Quite similar to the example of the common sets above, so too is the implicit notion of **type**. In this thesis we will outline this notion more explicitly and look at the ideas of Martin-Löf [2][3] with a keen focus on the ideas and concepts central to elementary discrete sets. We will maintain a view that is somewhat aligned with the central aspect of computer science, viz. computation, to illustrate the benefits of an explicit construction. The notion of a programming language, which is always typed in some sense[1], serves as a good illustration of what the apparent strengths of a type-centric view are.

This thesis is divided into two parts. In part one we will explore the foundations necessary to approach the second part, which covers sets and computation. Apart from (very) elementary knowledge of mathematical logic, no prior knowledge is assumed. Seeing as how this topic is constructive in its nature, we will not focus on proving lemmas and theorems, but rather illustrate the outline of the theory by asking philosophical questions and then explore suggested answers by Martin-Löf and Granström.

The main sources used are Granström [4][5], who approaches the theory in a very systematic and philosophical way, and Martin-Löf who has a more technical style. We will try to maintain a "golden mean", cf. Aristoteles, in

[1]Even a programming language with dynamic typing, e.g. Python, has underlying types. Cf. `a=12;type(a)` within CPython.

that we will approach the theory from a philosophical point of view but with a focus on the technical details.

Further, it should be noted that we restrict our investigation to propositional logic and that by “elementary” sets we refer to fundamental sets, e.g. $\mathbb{N}$ and similar countable collections. Martin-Löf type theory is an extensive theory and thus this thesis should not be viewed as anything but a rough introduction to a very interesting and philosophically pleasing theory.

## Part I
# Fundamentals

In this part we cover the fundamentals of the underlying theory required to approach the more interesting topics that are central to the notions of set and computations.

## 1 Intuitionism

This thesis focuses on Martin-Löf type theory, as made apparent by its title, which is also known as *intuitionistic type theory*. An apparent first question is thus what is meant by intuitionism.

To understand said concept we must first recall the basic ideas found in *classical logic*. We begin by introducing some elementary notation that will be reused throughout the thesis. Note, however, that in later sections the meaning may be altered in conjunction with us shifting focus from classical logic to its intuitionistic counterpart. We will pay careful attention to "re-define" concepts clearly as to avoid any confusion regarding notation.

In classic logic we concern ourselves with truth and the construction of classic propositions. By a classic proposition we mean a statement which we can evaluate to either *true* or *false*.

**Example 1** ((Classic) Proposition)**.** *"H is the ninth letter of the alphabet" is a (classic) proposition which (clearly) is false as "h" is the eighth letter of the alphabet.*

From a given classic proposition $p$ we can form new classic propositions through a few operators. These are given in the table below. Let $p$ and $q$ be classic propositions.

| **Symbol** | **Example** | **True when** |
|---|---|---|
| $\neg$ | $\neg p$ | $p$ false |
| $\vee$ | $p \vee q$ | at least one of $p$, $q$ true |
| $\wedge$ | $p \wedge q$ | both $p$, $q$ true |
| $\rightarrow$ | $p \rightarrow q$ | both $p$,$q$ true or $p$ false |
| $\bot$ | | "$p \wedge \neg p$" |

When we evaluate a statement, we first consider its composition. In the previous table, we noticed that the truth of a composite statement, viz. a statement which consists of smaller statements, depends on the evaluation of its parts in conjunction with the definition of the logical connective used to connect said substatements. Should our statement not be composite, e.g. only be a single atom, an evaluation depends on the valuation considered. By a valuation we mean an explicit assignment of boolean values to a set of atoms. However, we must also consider self-evident statements which are true by definition in that they are immediate. Such statements are known as **axioms**. One axiom is of particular interest to us, namely the law of the excluded middle (LEM). Further, some statements are true because of their semantic meaning. For instance, consider the statement $\phi \to \psi \Leftrightarrow \neg\phi \vee \psi$ which is true as a result of its semantic meaning rather than the evaluation of its parts.

It should be noted that axioms require a certain "leap of faith" in that our system relies on accepting certain laws to be valid and true on the sole basis of meta-argumentation. One can construct systems which do not have axioms, but instead rely on rules. However, one should note that while rules govern the continuency of truth through application, they must still be defined from a meta-context. As such, all systems are in some sense based on assumptions which are true by definition. We will consider rules in conjunction with axioms, which is a common construct. Further, note that the symbol $\perp$ is used to denote an impossibility, viz. there is an *inconsistency.*

**Notation 1.** *A derivation consists of two parts: the premisses and a conclusion. By premiss we mean previous conclusions or assumptions. Note that assumptions are treated in later sections of the thesis. A conclusion is a statement which follows from the premisses. We write*

$$\frac{P_1 \cdots P_n}{C}$$

*where $P_1$ to $P_n$ are the premisses from which we conclude $C$.*

It should be noted that the notation above is somewhat ambigious in that we will use it for rule definitions as well. However, rules will have a label on the right-hand side of the line separating the premisses and the conclusion. When no such label is present, the reader can assume that it is a derivation; viz. a conclusion obtained by applying a collection of rules on the premisses in a finite number of steps.

**Example 2.** *The conclusion of*

$$\frac{P_1 \wedge P_2 \qquad P_3 \wedge P_4}{(P_1 \wedge P_2) \wedge (P_3 \wedge P_4)}$$

*is based on the fact that if both premisses are true then, by definition of* $\wedge$*, it too must be true.*

**Definition 1** (LEM)**.** *In classical logic, we have the axiom*

$$\frac{}{\varphi \vee \neg\varphi}\ \mathit{LEM}$$

As can clearly be seen, an axiom does not have any premisses. With this in mind we can write out LEM as "for any proposition $\varphi$ it is always the case that it is either true or not true". Intuitively we might be tempted to agree with this statement. This is not strange as when Brouwer [6] first argued that it was flawed, most logicians disagreed with him. After all, when we only have two valid outcomes of any statement – *true* or *false* – then obviously one of them have to apply.

However, if we expand our view it becomes apparent that there is something *odd* about LEM. Let us illustrate this with a commonly used example, namely the following lemma:

**Lemma 1.**
*There are two irrational numbers* $a$ *and* $b$ *such that* $a^b$ *is a rational number.*

*Proof.* Let $a = b := \sqrt{2}$, then there are two possibilities:

1. $a^b$ is rational and we are done.

2. $a^b$ is irrational.

If (2), we let $a := \sqrt{2}^{\sqrt{2}}$ and $b := \sqrt{2}$. Since $a$ is irrational – by (2) which was true – it follows that $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$. As $2 \in \mathbb{Q}$, we are done. □

While we can certainly agree that the lemma is proven, viz. there are indeed two irrational numbers $a$ and $b$ such that $a^b \in \mathbb{Q}$, nowhere do we explicitly state which ones. There is thus an element of unsatisfactory reasoning in the proof, analogous to stating "it is not the case that there is not a lottery ticket which will win you 20 million dollars" and concluding that there is a lottery where one person will win 20 million dollars. That is, we approach the problem by considering its negation, e.g. it is not the case that there is not a way to choose irrational $a$ and $b$ so that $a^b \in \mathbb{Q}$.

In intuitionistic logic we reject LEM. This is done mainly for two reasons, the first being that we want to approach mathematics in a constructive way, e.g. actually find said numbers $a$ and $b$ explicitly. Secondly, LEM should, as Granström so brilliantly puts it, be seen as a principle of *being* rather than a law of *thought*. When reflecting upon the validity of LEM it is not uncommon to think along the lines of "either something is or it is not", viz. a focus on being. However, we do not wish to limit ourselves to such contexts, but rather concern ourselves with the notion of thought.

## 2 On assertions, propositions and proofs

In this section we will leave classical logic behind and discuss intuitionistic type theory. We will comply with Martin-Löfs view of his type theory as a foundation of mathematics in which symbolic logic, viz. logic using mathematical symbolism, is done. This view implicitly confines us to a place equally governed by mathematics and philosophy. That is, we are constructing mathematics by means which are philosophical in nature and mathematical in meaning.

This nature becomes apparent when we look at two central concepts of the theory discussed, namely that which is a proposition and that which is an assertion. Martin-Löf [3] states

> *What we combine by means of the logical operations and hold to be true are propositions. When we hold a proposition to be true, we make a judgement. In particular, the premisses and conclusions of a logical inference are judgements.*

We will denote that which Martin-Löf calls judgement as an assertion. In reasoning, a conclusion is an assertion based on previously known assertions which we, in agreement with the previous definition, will denote as **premisses**.

**Example 3** (Proposition vs. assertion)**.** *If A and B are propositions, e.g. "there are no numbers" and "7 does not exist", then "if A then B" is also* *a proposition. If "if A then B" is asserted to be true, then no assertion of either A or B has taken place.*

By the above example, it follows that when we work with propositions we are really performing inductive reasoning, viz. that $\theta$ is a proposition is a question of whether or not $\theta$ can be derived from valid rules and previous propositions. This means that should it follow that $\theta$ is indeed a proposition, then so must all of its parts (should they exist). In contrast to this is the notion of assertion, in which we simply state what *is* held to be true. As such, to assert that $\theta$ is true means that no assertion has been made on $\theta$:s possible parts (quite unlike how propositions work). It is useful to think of propositions as constructive "roads" that dictate how to reach truth, should the premisses be true. Assertions, on the other hand, are simply statements of what already lies in truth.

Recalling our intuitionistic view of things, we are forced to consider a proposition as something more generic than just a statement that can be evaluated to a truth value, reflected in the definition by Martin-Löf.

**Definition 2** (Proposition)**.** *A proposition is defined by laying down what counts as a proof of the proposition.*

**Definition 3** (Proposition truth)**.** *We say that a propostion is true if it has a proof.*

In essence a proposition becomes somewhat of a meta-concept in that it only conveys its "meaning" implicitly. As an example, a classical proposition "there is a natural number greater than ten" becomes "a proof of the existence of a natural number k which is greater than ten". The reason for this is that rather than defining our proposition explicitly (word-wise) we have to lay down what counts as a proof of it. In some sense these are equivalent, but the classical proposition focuses on what we want to conclude in a deduction whereas the intuitionistic counterpart simply states everything that is needed prior to the conclusion we want to reach. This becomes more apparent if we look at a sample derivation, consider:

| | | |
|---|---|---|
| 1 | | |
| 2 | ... | |
| 3 | ... | |
| 4 | $\varphi \land \phi$ | some rule (2,3) |

Our classic proposition $\varphi \wedge \phi$ simply conveys a statement which can be evaluated (truth-wise). So if we were to say that it is true no matter what valuation we consider, we would need to perform the derivation above to prove it or demonstrate by semantical ways that it is true. However, with the intuitionistic counterpart, we would have a statement of rows $(1, 4)$ since they constitute what a proof of $\varphi \wedge \phi$ is. Therefore, in some way, when we talk about propositions in the context of intuitionistic type theory, we are really quite explicit.

In essence, we focus on proof rather than truth. We can view proof as an intermediary for truth, but in a very restricted way. After all, if a proposition does not have a proof it does not mean that it is false. To hold something to be true which does not have a proof is a must to utilize axioms, as axioms are justified in a meta-context.

A direct consequence of the definition of truth when it comes to propositions is that the standard propositional connectives are viewed from a proof-centric perspective, summarized in the list below. Recall that since we must lay down what counts as a proof of a proposition to define it, we must also know that constitutes a proof. We do not make any judgement as to whether any of the proofs actually exist, we only state what would count as a proof.

- A proof of $\varphi \wedge \psi$ is a proof of $\varphi$ and a proof of $\psi$.
- A proof of $\varphi \vee \psi$ is a proof of $\varphi$ or a proof of $\psi$ and information about which one the proof is for.
- A proof of $\varphi \to \psi$ is a method which takes any proof of $\varphi$ into a proof of $\psi$.
- There is no proof of $\bot$.

Now that we know what it means for something to be a proposition, let us return to assertions. There are two important initial assertions to consider. The first, stating that $\varphi$ is a proposition, is written

$$\varphi : prop$$

and the second, written

$$\varphi \quad true$$

implicitly conveys that $\varphi$ is indeed a proposition, as it must be presupposed for the assertion to be meaningful. With this notation, in conjuncture with the form previously introduced as well as the definitions of the propositional connectives, we can conclude

$$\frac{\varphi : \mathit{prop} \qquad \psi : \mathit{prop}}{\varphi \land \psi : \mathit{prop}} \qquad \frac{\varphi : \mathit{prop} \qquad \psi : \mathit{prop}}{\varphi \lor \psi : \mathit{prop}}$$

$$\frac{\varphi : \mathit{prop} \qquad \psi : \mathit{prop}}{\varphi \to \psi : \mathit{prop}} \qquad \frac{}{\bot : \mathit{prop}}$$

Note that these inference rules only tell us what assertions can be made and should not be confused with proofs when determining if a proposition is true. For instance, the axiom $\vdash \bot : \mathit{prop}$ is not equivalent to $\vdash \bot\ \mathit{true}$ (recall that there is no proof for $\bot$!). Further, note that

$$\neg\varphi \equiv (\varphi \to \bot)$$

It should be pointed out that by "$a \equiv b$" we mean that $a$ is shorthand for $b$. Likewise, we treat equivalence as usual, namely

$$\varphi \Leftrightarrow \psi \equiv (\varphi \to \psi) \land (\psi \to \varphi) : \mathit{prop}$$

When we are dealing with more interesting deductions we will need to be able to refer to proofs directly. Granström refers to these as causes, but we shall call them proofs as it makes more sense to talk of proof objects than cause objects. A proof of $\varphi$ is written $c : \mathit{proof}(\varphi)$ and is a new kind of assertion. As defined on the previous page, we know that a proposition is true if it has a proof. Thus

$$\frac{c : \mathit{proof}(\varphi)}{\varphi \ \ \mathit{true}}$$

follows immediately. It also becomes apparent that an assertion on the form $A\ \ \mathit{true}$ on its own is somewhat flawed as it suppresses its proof, viz. we cannot evaluate its validity without any premisses. In general, we differentiate between **complete** and **incomplete** assertions; where the former does not suppress its proof.

An assertion will often have a number of presuppositions, i.e. assertions which must be known for it to make sense. Granström exemplifies this with the question "Do you still beat your wife?" which cannot be asserted unless the presupposition is fullfilled, e.g. "Have you *ever* beaten your wife?".

**Definition 4** (Well-formed inference rule)**.** *An inference rule is said to be well-formed if all presuppositions of the conclusion can be inferred from its premisses taken together with their respective presuppositions. Thus, to accept a premiss is to implicitly accept its presuppositions and their (possible) presuppositions recursively.*

While it is perfectly valid to consider inference rules where the conclusion is a presupposition of the premiss, e.g. $\varphi\ true \vdash \varphi : prop$, it is clearly useless. By the notation $P_1, \cdots, P_n \vdash C$, we mean that there is a derivation of $C$ from $P_1, \cdots, P_n$.

Now that we have an idea of what is meant by proposition, truth and assertion, it makes sense to look at the logical connectives in more detail. Most of the laws presented are essentially equivalent to their classical counterpart except that they are to be viewed in a different context.

# 3 Inference rules

In this section we outline some common inference rules. An inference rule is based on the notion that if its premisses are true, viz. have proofs (which in our context is equivalent to us asserting them as true), then from said proofs one must be able to construct a proof of the conclusion.

## 3.1 Conjunction

The first connective of the list is conjunction which typically has three laws, namely $\wedge i$, $\wedge e_1$ and $\wedge e_2$. It should be pointed out that by $i$ we mean *introduction* and by $e$ we mean *elimination*. Said laws are based on the notion that the proposition $\varphi \wedge \psi$ requires a proof of both $\varphi$ and $\psi$. We have

$$\frac{\varphi\ true \qquad \psi\ true}{\varphi \wedge \psi\ true}\wedge i \qquad \frac{\varphi \wedge \psi\ true}{\varphi\ true}\wedge e_1 \qquad \frac{\varphi \wedge \psi\ true}{\psi\ true}\wedge e_2$$

As can be seen, it follows (roughly) that an introduction rule allows us to build upon previous premisses whereas an elimination rule allows us to break a premiss apart.

**Notation 2** (Dual direction inference)**.** *To show that an inference rule is valid in both directions, Martin-Löf [3] proposes the use of a double line, e.g.*

$$\frac{\varphi \wedge \psi \ \textit{true}}{\psi \wedge \varphi \ \textit{true}}$$

*which follows from*

$$\frac{\dfrac{\varphi \wedge \psi \ \textit{true}}{\psi \ \textit{true}} \wedge e_2 \qquad \dfrac{\varphi \wedge \psi \ \textit{true}}{\varphi \ \textit{true}} \wedge e_1}{\psi \wedge \varphi \ \textit{true}} \wedge i$$

*and*

$$\frac{\dfrac{\psi \wedge \varphi \ \textit{true}}{\varphi \ \textit{true}} \wedge e_2 \qquad \dfrac{\psi \wedge \varphi \ \textit{true}}{\psi \ \textit{true}} \wedge e_1}{\varphi \wedge \psi \ \textit{true}} \wedge i$$

*We will comply with this notation suggestion.*

## 3.2 Disjunction

Recall that a proof of a disjunction is a proof of one of its parts together with information regarding which part the proof is for. Typically a proof of a disjunction $\varphi \vee \psi$ is denoted $i(c : \textit{proof}(\varphi))$ or $j(d : \textit{proof}(\psi))$, viz. we let $i$ or $j$ dictate which formula the proof is for. We will denote premisses which are only required to make the inference rules well-formed by writing them in parentheses. From classical logic we know that disjunction introduction is on the form $\phi \vdash \phi \vee \psi$ for any $\psi$ and like conjunction elimination there are two cases to consider, namely

$$\frac{\varphi \ \textit{true} \qquad (\psi : \textit{prop})}{\varphi \vee \psi \ \textit{true}} \vee i_1 \qquad\qquad \frac{(\varphi : \textit{prop}) \qquad \psi \ \textit{true}}{\varphi \vee \psi \ \textit{true}} \vee i_2$$

That is, we must know that the term "introduced" in the disjunction introduction is indeed a proposition. Disjunction elimination looks like we would expect:

$$\frac{\varphi \vee \psi \ \textit{true} \qquad \varphi \to \theta \ \textit{true} \qquad \psi \to \theta \ \textit{true}}{\theta \ \textit{true}} \vee e \ .$$

That is, given that both propositions imply the same thing and as we know that a proof of the disjunction guarantees that at least of one them applies, the conclusion follows immediately. Note that we do not consider $\varphi \vdash \theta$ at this point (which is required for the general version of the rule), as assumptions will be treated later.

## 3.3 Implication

Implication is very straightforward. We know that a proof of an implication $\varphi \to \psi$ is a method which takes any proof of $\varphi$ into a proof of $\psi$. Thus, should we have a proof of $\varphi$ as a premiss along with a method $\varphi \to \psi$ (which takes any proof of $\varphi$ into a proof of $\psi$), all that remains is to apply the proof to the method. We get

$$\frac{\varphi \to \psi \ \mathit{true} \qquad \varphi \ \mathit{true}}{\psi \ \mathit{true}} \to e$$

Implication introduction requires us to make assumptions, which is covered later. However, the law looks exactly as we can expect.

## 3.4 Laws involving $\bot$

We have previously stated that $\bot$ has no proof and that it is required of us to avoid implicit reasoning in intuitionistic type theory, i.e. infering $\varphi$ from $\neg\neg\varphi$[2]. With this in mind, one might question the utility of using $\bot$ in the usual sense. However, it turns out that most of the usual rules still apply but that we need to justify them with a reasoning that is somewhat different.

A useful classical law is that $\bot \vdash \varphi$ for any $\varphi$, i.e. from absurdity any conclusion may follow. There is an inherent truth in this view as it perfectly matches our definitional view of $\bot$ as having no proof. That is, if absurdity had a proof, then everything would follow. As such we cannot speak of $\bot$ introduction, as that would rely on a proof of absurdity. This means that even if we have inconsistent premisses, we cannot conclude $\bot$ since that would be equivalent to laying down what counts as a proof of $\bot$. Such a thing cannot exist by definition. We can, however, define $\bot e$ as

[2] In fact, while $\varphi \to \neg\neg\varphi$ in intuitionistic logic, $\varphi$ and $\neg\neg\varphi$ are not equivalent.

$$\frac{\bot\ \mathit{true} \qquad (\varphi : \mathit{prop})}{\varphi\ \mathit{true}} \bot e$$

It is justified by noting that if $\bot$ *true* then there exists $c$ so that $c : \mathit{proof}(\bot)$. A proof of $\varphi$ must thus be given for each of the possible forms of $c$, but by definition $\bot$ has no proof, so $c$ cannot exist. Therefore there are no possible forms to consider. It is sufficient to do nothing at all and the result follows.

Further, let us look at negation elimination. In classical logic we have $\{\varphi, \neg\varphi\} \vdash \bot$. Can we justify this rule in an intuitionistic context? That is, is

$$\frac{\varphi\ \mathit{true} \qquad \neg\varphi\ \mathit{true}}{\bot\ \mathit{true}} \neg e$$

valid and well-formed? The answer is *yes* and becomes rather obvious when we recall that $\neg\varphi \equiv \varphi \rightarrow \bot$. By rewriting the negation we can apply $\rightarrow e$ in accordance with

$$\frac{\varphi\ \mathit{true} \qquad \varphi \rightarrow \bot\ \mathit{true}}{\bot\ \mathit{true}} \rightarrow e$$

resulting in $\neg e$ being a perfectly valid law.

# 4 Equality

So far we have only looked at very restricted reasoning, e.g. how truth is handled in conjunction with the logical connectives $\wedge, \vee, \rightarrow$ and $\neg$. As we extend our consideration to more useful and thus more complex notions, we will need to be careful. It has already been stated that we will discuss the notion of set in the second part of this thesis and in that context it is already apparent that objects will need to be considered. Further, the very idea of *type* as a kind of specification that enables us to group things (in some vague sense at this point) implies that comparison must be handled explicitly. In this section we will discuss equality as it is central to any comparison and will be an integral part of our approach to sets.

First, it must be made clear that equality is usually treated in a vague and implicit way in mathematics. When we write $2 + 5 = 8 - 1$ we mean that if we were to complete the computations we would end up with $7 = 7$, which is true due to reflexivity. This relies on the notion that we can create alternate forms for expressions, viz. 7 and $9-2$. Likewise, when we write $a = 93$ we mean that $a$ is an alternate form of 93, so if we have a computation on the form $3 + 19 + 93$ it is perfectly legal to replace the occurrence of 93 with $a$. This type of equality is called *definitional equality* and it is what we use when we define things in "ordinary" mathematics. Note that if we consider the *process* of computation, $2 + 7$ and $10 - 1$ are different. However, we will focus on definitional equality in this thesis as it is central to defining sets.

**Notation 3** (Equality)**.** *A generic statement such as $a = b$ is flawed when looked at from a type-centric perspective in that nowhere do we convey the types of $a$ and $b$. It is our desire to be explicit so we must convey this information. We write*

$$a = b : P$$

*for some predicate $P$ and we say that $P$ is a logical category (type). This presupposes that $a : P$ and $b : P$.*

**Definition 5** (Definitional equality)**.** *In Martin-Löf type theory every logical category comes equipped with a definitional equality. Such a relation must satisfy the following criteria (where $=_D$ is the definitional equality):*

1. *All objects are equal to themselves,*
2. *Equals can be substituted for equals giving equal results,*
3. *Reflexivity ($a =_D b \Leftrightarrow b =_D a$),*
4. *Transitivity ($a =_D b \wedge b =_D c \Rightarrow a =_D c$)*

**Example 4** (Natural numbers)**.** *Let us illustrate the concepts discussed so far by looking at Peanos construction of the natural numbers, which we (for simplicity) will denote $\mathbb{N}$. There are two inital inference rules to consider, namely*

$$\frac{}{0 : \mathbb{N}} \qquad\qquad \frac{n : \mathbb{N}}{S(n) : \mathbb{N}}$$

*That is, "there is a natural number 0" and "every natural number has a successor", where $S(n)$ denotes the successor of $n$. Obviously we want to form ordinary propositions involving numbers, such as $a < b : prop$. Further,* *we want to be able to assert any such proposition to be true and to do so we need two inference rules, totaling three (one deals with the formation of the proposition):*

$$\frac{a : \mathbb{N} \qquad b : \mathbb{N}}{a < b : prop} \qquad \frac{a : \mathbb{N}}{a < S(a) \quad true} \qquad \frac{a < b \quad true}{a < S(b) \quad true}$$

*Returning to the notion of definitional equality, we continue our work with* $\mathbb{N}$ *by looking at equality between natural numbers. For instance, according to point* $(3)$ *of the definition of definitional equality, we have the assertion* $0 = 0 : \mathbb{N}$ *and by point* $(2)$ *we construct the inference rule*

$$\frac{a = b : \mathbb{N}}{S(a) = S(b) : \mathbb{N}}$$

*To allow us to conclude that* $S(0) + S(S(0)) = S(S(S(0)))$ *we need to make some inference rules that deal with addition. Addition is typically defined recursively through*

$$\begin{cases} a + 0 & = & a : \mathbb{N} \\ a + S(b) & = & S(a + b) : \mathbb{N} \end{cases}$$

*From this view of addition in* $\mathbb{N}$*, three inference rules follow immediately*

$$\frac{a : \mathbb{N} \qquad b : \mathbb{N}}{a + b : \mathbb{N}} \qquad \frac{a : \mathbb{N}}{a + 0 = a : \mathbb{N}} \qquad \frac{a : \mathbb{N} \qquad b : \mathbb{N}}{a + S(b) = S(a + b) : \mathbb{N}}$$

*With addition handled we can easily define multiplication recursively as*

$$\begin{cases} a \times 0 & = & 0 : \mathbb{N} \\ a \times S(b) & = & (a \times b) + a : \mathbb{N} \end{cases}$$

*and similarly three inference rules follow trivially*

$$\frac{a : \mathbb{N} \qquad b : \mathbb{N}}{a \times b : \mathbb{N}} \qquad \frac{a : \mathbb{N}}{a \times 0 = 0 : \mathbb{N}} \qquad \frac{a : \mathbb{N} \qquad b : \mathbb{N}}{a \times S(b) = (a \times b) + a : \mathbb{N}}$$

In ordinary mathematics some equalities are, however, not definitional. For instance, to prove that addition is commutative by induction we give the proof for two terms to be equal, viz. it has to be expressed by a proposition. Recall that a proposition is *true* only if we have a proof of it.

There is a difference between definitional equality, which is a complete assertion, and propositional equality, which is a form of proposition. So, when we wish to state that two numbers are propositionally equal we write $a\ eq\ b$. Said term is a proposition, viz.

$$\frac{a : \mathbb{N} \qquad b : \mathbb{N}}{a\ eq\ b : prop}$$

For a proposition to be true, it needs to have a proof. So should we have as a premise that two natural numbers are definitionally equal, we have a proof of them being propositionally equal as well. Thus, it follows that

$$\frac{a = b : \mathbb{N}}{a\ eq\ b\ \ true}$$

## Part II
# Discrete sets and computations

In this part we will approach the notion of discrete set and look at computations which are central to computer science. In essence, it turns out that through our utility of types we explicitly specify behaviour which is closely related to how we *work* with objects. That is, by defining behaviour implicitly through inference rules it is a natural step to consider computation. Granström [4] has done extensive work in this area but due to the restrictions of this thesis we will be unable to fully explore this interesting field.

## 5 Set – Introduction

In this section we outline what it means for something to be a set in intuitionistic type theory. This task requires us to begin from a philosophical context and then constructively expand our view of sets.

First, when we say that $A$ is a set, what do we actually mean? Moschovakis [7] compares Cantors description

> *By a set we are to understand any collection into a whole of definite and separate objects of our intuition or our thought.*

with Euclids definition of a point

*A point is that which has no parts.*

Neither of these definitions would classify as explicitly rigorous in a mathematical sense, but rather illustrates that the concepts are – in some sense – irreducible.

The standard definition allows us to conclude that a set consists of smaller objects – elements – which in turn define the set. Yet, when we write $a \in A$, do not the properties of $A$ reflect on $a$? Indeed, there is a duality at play: the whole is defined by its parts, and the part is but a portion of the whole.

When approaching the notion of set, Martin-Löf [3] asks three fundamental questions:

- What is a set?
- What is it that we must know in order to have the right to assert something to be a set?
- What does an assertion of the form "A is a set" mean?

Returning to the definition of Cantor, we are compelled to assume that a set is defined by its elements. In our intuitionistic view, we would mainly concern ourselves with how said elements are formed. Recall that the axioms for the natural numbers

$$\frac{}{0 \; \varepsilon \; \mathbb{N}} \qquad\qquad \frac{a \; \varepsilon \; \mathbb{N}}{S(a) \; \varepsilon \; \mathbb{N}}$$

govern the "generation" of the set itself. Here we use Granströms notation, where $\varepsilon$ is used in conjunction with canonical sets and elements (":" is used for the noncanonical counterparts – see definition below). However, as Martin-Löf points out, there is a very apparent weakness to this approach. For instance, the number $101^{102^{103}}$ is clearly an element of $\mathbb{N}$ but is in all practical sense unobtainable with the axioms given. We are thus forced to distinguish between elements which we directly can see are a result of the rules and those which we cannot.

**Definition 6** (Canonical)**.** *A term or expression is said to be canonical if it refers to something directly, viz. there is no intermediary.*

**Definition 7** (Noncanonical)**.** *Similarly, we say that a term or expression is noncanonical if it refers to something through some intermediary, viz. refers to an object through another expression or term.*

**Example 5** (Canonical vs. noncanonical)**.** *Granström exemplifies the distinction between canonical and noncanonical by the example of* Paris *and* the capital of France *respectively.*

As illustrated in figure 1, it can be helpful to picture canonical elements and their sets as layered circles. In the middle we have the basic construct of $\mathbb{N}$ in accordance with Peano. Outside we find 2, which is really just a representation of $S(S(0))$. Therefore it is a noncanonical element of the inner circle. In

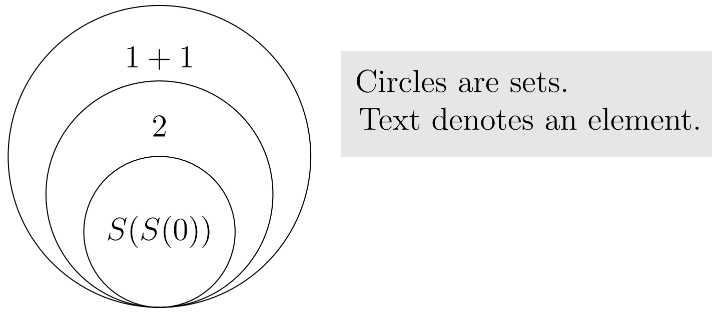


Figure 1: Canonical/Noncanonical relationship.

a similar fashion, the expression $1+1$ is a noncanonical element of the circle containing 2. When we extend our view of these notions into whole sets, it will become apparent that it is possible to continue forever in this fashion outwards from our "core" of $\mathbb{N}$ (or whatever fundamental set we choose).

As we certainly wish to work with noncanonical elements in many applications, we must determine what it means for two of them to be equal. Any such definition relies on equality between canonical elements. Martin-Löf [3] writes

> *A set A is defined by prescribing how a canonical element of A is formed as well as how two equal canonical elements of A are formed.*

This defines what it means to make an assertion of the form "A is a set". We thus add rules for equality on $\mathbb{N}$ accordingly

$$\frac{}{0 = 0 \;\varepsilon\; \mathbb{N}} \qquad\qquad \frac{a = b \;\varepsilon\; \mathbb{N}}{S(a) = S(b) \;\varepsilon\; \mathbb{N}}$$

The equality between canonical elements must be an equivalence relation, viz. transitive, reflexive and symmetric. Granström chooses to instead require it to be reflexive and cancellable from the right, i.e. that two elements which equal a third element are equal to each other. However, as these sets of criteria are equivalent it is mainly a matter of taste.

Now that we have defined what it means for two canonical elements to be equal, a natural extension would be to ask the same regarding set equality.

**Definition 8** (Set equality)**.** *Two sets A and B are said to be equal (we use Granströms notation for consistency, viz. el(A), see Definition 9) iff*

$$\frac{a \ \varepsilon \ \mathrm{el}(A)}{a \ \varepsilon \ \mathrm{el}(B)} \quad \textit{and} \quad \frac{a = b \ \varepsilon \ \mathrm{el}(A)}{a = b \ \varepsilon \ \mathrm{el}(B)}$$

Further, to assert a proposition $a \ \varepsilon \ A$ presupposes that $A$ is a set and thus also how its canonical elements are formed. This process outlines Martin-Löfs [3] definition of what it means for something to be an element of a set. He writes

> *An element a of a set A is a method (or program) which, when executed, yields a canonical element of A as result.*

This definition assumes the notion of method as primitive and that execution is carried out in such a fashion that it is what we denote as "lazy evalution". That is, the computation of an element $a$ of a set $A$ will terminate with a value $b$ as soon as the outermost form of $b$ is a canonical element of $A$.

**Example 6** (Lazy evaluation)**.** $2+2 \ : \ \mathbb{N}$ *terminates with the value* $S(2+1)$ *as* $S(2+1) \ : \ \mathbb{N}$*. Recursively we can easily assert that result until we reach* $S(S(S(S(0))))$*, which, according to our definition of* $\mathbb{N}$*, is a canonical element.*

Another added benefit of this definition of what it means to be an element in a set, is that it implicitly defines what it means for two noncanonical elements to be equal. Martin-Löf [3] writes

> *Two arbitrary elements a, b of a set A are equal if, when executed, a and b yield equal canonical elements of A as result.*

Granström chooses to define the concept of set in a somewhat more explicit sense. Looking at his definition we note that most of what we have discussed is included in the definition itself. Further, as we shall see, his approach is – in a sense – more satisfying from a philosophic point of view.

**Definition 9** (Set according to Granström)**.** *That $A$ is a set, written $A \varepsilon$ set, means four things:*

1. *It is defined (i.e. we can algorithmically determine) when $a$ is an element of $A$, written $a \ \varepsilon \ el(A)$.*

2. *It is defined when two elements $a$ and $b$ of $A$ are definitionally equal, written $a = b \ \varepsilon \ el(A)$.*

3. *That an element of $A$ is always equal to itself.*

4. *That two elements that equal a third element in $A$ are equal to one another.*

By this definition, the meaning of an assertion such as $a \ \varepsilon \ el(A)$ depends on the definition of $A$. In some way this is to be expected, as the statement presupposes that $A \ \varepsilon \ set$.

Further, to distinguish between *canonical* and *noncanonical* sets (covered later) and elements, we write $a : el(A)$ to denote that $a$ is a noncanonical element of $A$. The epsilon, $\varepsilon$, will similarly be used when dealing with canonical sets and elements. By $el(A)$, Granström refers to the elements of a particular set $A$. He makes a point of distinguishing between that which is a *universal concept*, viz. an idea which has not been restricted by any realisation, and that which is a set. This also means that if $M$ is a set, it cannot be universal but should rather be seen as an object. Further, we cannot say that an element is an $M$ (cf. 3 is a natural number), but only that something is an *element* of $M$ (as our definition of set implies that it is something more than just its elements, cf. definition above). We will adhere to this notation, viz. $el(A)$, as it enforces explicity.

From the definition of a canonical set, two inference rules become apparent immediately: that any element is equal to itself and that two elements that equal a third are equal. We have

$$\frac{a \ \varepsilon \ el(A)}{\overline{a = a \ \varepsilon \ el(A)}} D_1 \qquad\qquad \frac{a = c \ \varepsilon \ el(A) \qquad b = c \ \varepsilon \ el(A)}{a = b \ \varepsilon \ el(A)} D_2$$

The inference rules that we label with the prefix $D$ are immediate and meaning determining. However, when doing actual work we might feel compelled to create what is known as *mediate inference rules*. These are essentially macros, or theorems if you will, that are used as an abbreviation of a longer schematic demonstration. Let us look at an example, namely reflexivity of definitional equality on a particular set $A$.

$$\frac{a = b \ \varepsilon \ el(A)}{b = a \ \varepsilon \ el(A)} \, M_1$$

This rule is really only an abbreviation of the demonstration

$$\frac{\dfrac{b \ \varepsilon \ el(A)}{b = b \ \varepsilon \ el(A)} \, D_1 \qquad a = b \ \varepsilon \ el(A)}{b = a \ \varepsilon \ el(A)} \, D_2$$

where $b \ \varepsilon \ el(A)$ is a presupposition of $a = b \ \varepsilon \ el(A)$.

Whenever we create a set of rules, we obviously want to be *restrictive* to ensure a low level of complexity. However, this does not mean that we can eliminate immediate meaning determining rules in favor of mediate ones. This is due to the fact that mediate rules build on the meaning conveyed by the immediate counterpart, as illustrated by the previous example.

Granström also labels some inference rules as *justified*, in that they require some further explanation (cf. meaning determining rules which are evident from the meaning of the terms involved). For instance, that equality between sets is cancellable from the right hand side

$$\frac{A = C \ \varepsilon \ set \qquad B = C \ \varepsilon \ set}{A = B \ \varepsilon \ set} \, J_1$$

would need to be explained. This is done by looking at the four criteria Granström lists in his set definition and seeing that they apply. The difference between a mediate inference rule and a justified inference rule is that the former is "justified" in the language of intuitionistic type theory and the latter requires us to reason in a meta-language.

# 6 Building finite canonical sets

So far we have mostly concerned ourselves with what it means for something to be a set or an element of a set. We have also concluded what it is for two elements of some set to be equal and how to justify equality between sets. This is all well and good but to do some real work in intuitionistic type theory we need to be able to build sets as well.

In classic set theory there are many ways to build a set. For instance, we can write

$$\{x \in \mathbb{R} \mid x + 9 > 3\}$$

to obtain the set of all reals which are greater than $-6$. However, when we work within intuitionistic type theory we have a definition of what it means for something to be a set that is more explicit than the standard definition by Cantor. We will thus begin by considering how to build a finite set. To illustrate this more clearly, we reuse the example of the equivalence classes modulo $n$ of $\mathbb{N}$, viz. $\mathbb{N}_1 \cdots \mathbb{N}_n$, used by Martin-Löf [3].

**Example 7** (Finite set construction)**.** *We will consider a particular set modulo $n$, namely $n = 2$. As a first step we must assert that $\mathbb{N}_2$ is indeed a set and we write*

$$\mathbb{N}_2 \;\varepsilon\; \text{set}$$

*Next, we need to populate the set. Recall that a set is defined in part by how its elements are formed. In this particular case we have only two elements to worry about and we write*

$$\begin{aligned} 1 \quad & \varepsilon\ \text{el}(\mathbb{N}_2) \\ 0 \quad & \varepsilon\ \text{el}(\mathbb{N}_2) \end{aligned}$$

*Finally, we need to create a definitional equality. This too is easy as there are only two possible equalities to deal with, namely*

$$\begin{aligned} 1 = 1 \quad & \varepsilon\ \text{el}(\mathbb{N}_2) \\ 0 = 0 \quad & \varepsilon\ \text{el}(\mathbb{N}_2) \end{aligned}$$

*By definition, the equality is reflexive. That it is also cancellable from the right hand side is easy to see. Therefore we have ensured that our definitional equality is indeed an equivalence relation.*

*At this point we feel secure that we have complied with all the criteria of the set definition. However, there is something rather odd with the creation of* $\mathbb{N}_2$ *in that we first stated that* $\mathbb{N}_2 \ \varepsilon$ set*. Does this statement not presuppose all the steps we later performed*[3]*? Indeed it does. Therefore we should view this entire process as one unit, viz. all steps together. In such a view, it becomes apparent that the first step is valid because* something has been done *(namely, we have formed elements and defined the definitional equality). Granström compares the situation to how one writes down a theorem before proving it, yet it is the proof that defines the theorem as a theorem.*

**Remark 1** (The empty set)**.** *We should mention the empty set,* $\emptyset$*, as it is of importance to any set theory. To define the empty set in intuitionistic type theory is to state that there is a set with no elements. By definition a set is defined by how its elements are formed and as there are no elements to consider, nothing needs to be done in that regard. A similar argument applies to definitional equality. As such we simply write*

$$\emptyset \ \varepsilon \ set$$

*noting that* $\emptyset = \mathbb{N}_0$*.*

# 7 Basic canonical set forming operations

Sometimes we wish to form sets from pre-existing ones. In classical set theory this can be done in many ways. For instance, consider

$$\{2, 3, 5, 7, 11, 13\} \cup \{2, 7, 17, 19\}$$

We may consider a similar situation in intuitionistic type theory as well, with the distinction that we must use a disjoint union. The reason for this is analogous to how $\vee$ is treated, namely that a proof of a disjunction consists

[3]Actually, it is also true that $1 \ \varepsilon \ el(\mathbb{N}_2)$ presupposes that $\mathbb{N}_2 \ \varepsilon \ set$ and $0 = 0 \ \varepsilon \ el(\mathbb{N}_2)$ presupposes that $0 \ \varepsilon \ el(\mathbb{N}_2)$.

in part of information regarding origin. The same principle applies here but not for a reason of deduction (well, explicitly at least) but rather as a means for allowing equality to apply. Recall that a proof of the proposition $A \vee B$ is either $i(a : \mathit{proof}(A))$ or $j(b : \mathit{proof}(B))$. A similar approach is useful in this scenario as well. The inference rules are

$$\frac{A \ \varepsilon \ \mathit{set} \qquad B \ \varepsilon \ \mathit{set}}{A + B \ \varepsilon \ \mathit{set}} \, R \qquad \frac{a \ \varepsilon \ \mathit{el}(A) \qquad (B \ \varepsilon \ \mathit{set})}{i(a) \ \varepsilon \ \mathit{el}(A + B)} \, D_3$$

$$\frac{(A \ \varepsilon \ \mathit{set}) \qquad b \ \varepsilon \ \mathit{el}(B)}{j(b) \ \varepsilon \ \mathit{el}(A + B)} \, D_4 \qquad \frac{a = b \ \varepsilon \ \mathit{el}(A) \qquad (B \ \varepsilon \ \mathit{set})}{i(a) = i(b) \ \varepsilon \ \mathit{el}(A + B)} \, D_5$$

$$\frac{(A \ \varepsilon \ \mathit{set}) \qquad a = b \ \varepsilon \ \mathit{el}(B)}{j(a) = j(b) \ \varepsilon \ \mathit{el}(A + B)} \, D_6$$

That is, any element of any of the sets of the disjoint union is also an element of the union, along with the information on which set it came from. Likewise, any two elements that were equal in any of the sets of the disjoint union are equal in the union as well.

**Remark 2** (Tuples)**.** *Cartesian products are also of interest but are, in general, a lot more intuitive as far as their inference rules go. As any element of a product* $A \times B$ *is a tuple which implicitly maintains information (in conjunction with the type) on the origin of its components, equality becomes trivial. That is,* $(a, b) = (c, d) \ \varepsilon \ \mathrm{el}(A \times B)$ *if and only if* $a = c \ \varepsilon \ \mathrm{el}(A)$ *and* $b = d \ \varepsilon \ \mathrm{el}(B)$*.*

# 8 Computation

We have now outlined what a canonical set is and what is meant by a canonical and a noncanonical element of said canonical set. A natural step is thus to consider the notion of noncanonical set. To do so, however, we must recall how we distinguished between canonical and noncanonical elements on the basis of whether or not we directly could determine if they followed from the inference rules. Further, we concluded that every noncanonical element $e$ of a set $A$ has a canonical *value* $a$ in $A$. By this we mean that there is some method of *computing* $a$ from $e$. However, this view is quite dependent on what we mean by computing. Granström defines three characteristics which outline the concept:

1. Finiteness
2. Exactness
3. Typing

First, it is apparent that when we say that a particular function is *computable*, we mean that it is *possible in principle* to compute it in finite time. This claim is thus of a hypothetical nature, not unlike the task of classifying an arbitrary noncanonical element as a member of some set. When we say that it is possible in principle, we are referring to the notion that any such endeavour will terminate in the sense that an output will be produced. A consequence of this view is that the work being performed by a partially recursive function [8] cannot be viewed as computation in the general case.

Granström partitions the notion of exactness, to be understood as analogous to the exactness found in mathematics, into two parts which we will denote as *explicit foundation* and *static concept*. By explicit foundation we refer to the strictness and detail found within the definitions and proofs which are a basis to a sound mathematical environment. When we say static concept, we mean that the ideas – to be understood from a meta-level – that permeate the work being done are timeless and changeless in that they convey what is (which, incidentally is not to change).

It is from these two characteristics that exactness follows. When we refrain from exactness it becomes impossible to outline a computation, even of hypothetical nature. As an example, say we wanted to compute who was "the nicest man in Sweden". It would not be possible as several presuppositions are undefined and there is a large portion of subjectiveness involved, viz. exactness is lacking.

Finally, we have the concept of typing. It is a similar observation to that of finiteness in that we expect output. However, we do not expect this output to be of *any* type, but rather of a particular type. By this we mean that while the explicitness of the *value* of an arbitrary output is unknown until obtained, we always have explicit knowledge of what *type* of value to expect. For instance, running the Ackermann function – known for having an *extreme* output growth [8] – with some very large input would take a *very* long time to complete, but we can be sure that said output will lie in the natural numbers.

The concept of typing becomes even more apparent when we consider what it would mean to have an unvalued computation. It would be analogous to not knowing what is being done, in the sense that the work would not have a purpose. So how could we be sure that anything is being done at all? Even more so, how could this work ever terminate? These questions alone convey the absurdity of the concept discussed.

**Definition 10** (Computation)**.** *A computation is a finite and exact mode of procedure by which an expression refers to an object of a certain type.*

We write $a \Rightarrow b \; \varepsilon \; el(A)$ to denote that computing the term $a$ gives the value $b$, which is a canonical element of $A$. When dealing with functions, we write $f[a] \Rightarrow k \; \varepsilon \; el(A)$ to denote that the function $f$ computes the canonical element $k$ of $A$ from input $a$. To distinguish between canonical and noncanonical input we write $f[a]$ and $f(a)$ respectively.

# 9 Noncanonical sets

Now that we have an explicit understanding of what is meant by computation in conjunction with relevant notation, we may approach the notion of a noncanonical set.

**Definition 11** (Noncanonical set)**.** *That $A$ is a noncanonical set, denoted $A : set$, means that the value of $A$ is a canonical set $B$. An assertion of this has the form $A \Rightarrow B \; \varepsilon \; set$.*

**Remark 3** (Exactness of noncanonical sets)**.** *It should be stressed that exactness – as outlined in the discussion on computation – requires a noncanonical set to have a singular value. That is, if $A \Rightarrow B \; \varepsilon \; set$, it is illegal to also have $A \Rightarrow C \; \varepsilon \; set$ where $B \neq C \; \varepsilon \; set$. Note that "$\neq$" should be interpreted on a meta-level.*

**Definition 12** (Noncanonical set equality)**.** *By $A = B : set$ we mean that the values of $A$ and $B$ are equal, viz.*

$$\frac{A = B : set \qquad A \Rightarrow C \; \varepsilon \; set \qquad B \Rightarrow D \; \varepsilon \; set}{C = D \; \varepsilon \; set} \; D_7$$

As can be expected, the equality relation between noncanonical sets is a equivalence relation. Hence all "common" rules apply, e.g.

$$\frac{A = C : set \qquad B = C : set}{A = B : set} J_2$$

Further, we must consider noncanonical elements of noncanonical sets. Such a construct requires dual computation, both in the sense of the element and that of the set.

**Definition 13** (Noncanonical element of a noncanonical set)**.** *An element* $a : \mathrm{el}(A)$ *of a noncanonical set* $A$ *has a value* $b$ *which is a canonical element of the value of* $A$*, say* $B$*. We write*

$$\mathrm{el}(A) : a \Rightarrow b \;\varepsilon\; \mathrm{el}(B)$$

*Examples of this will follow.*

**Definition 14** (Equality of noncanonical elements of noncanonical sets)**.** *Two noncanonical elements* $a$ *and* $b$ *of a noncanonical set* $A$ *are said to be equal, written* $a = b : \mathrm{el}(A)$*, if their values are equal canonical elements of the value of* $A$*. We have*

$$\frac{a = b : \mathrm{el}(A) \qquad \mathrm{el}(A) : a \Rightarrow c \;\varepsilon\; \mathrm{el}(C) \qquad \mathrm{el}(A) : b \Rightarrow d \;\varepsilon\; \mathrm{el}(C)}{c = d \;\varepsilon\; \mathrm{el}(C)} D_8$$

Again, this is an equivalence relation so we are free to reason with noncanonical elements of noncanonical sets in the way we are used to. An easy way to imagine the constructs we are dealing with is to compare them with pointers found in many programming languages. That is, a noncanonical element of a noncanonical set is essentially a pointer to a pointer.

To build a noncanonical set we need to introduce it, define its value, populate it, define computation rules and define equality.

**Example 8** (Building a noncanonical set with noncanonical elements)**.**
*We construct a noncanonical set* $\mathcal{N}$ *with value* $\mathbb{N}$*. Several steps are involved in this process, namely:*

$$\mathcal{N} : set \tag{1}$$

$$\mathcal{N} \Rightarrow \mathbb{N} \; \varepsilon \; set \tag{2}$$

$$0 : \mathrm{el}(\mathcal{N}) \tag{3}$$

$$\frac{n : \mathrm{el}(\mathcal{N})}{S(n) : \mathrm{el}(\mathcal{N})} \tag{4}$$

$$\mathrm{el}(\mathcal{N}) : 0 \Rightarrow 0 \; \varepsilon \; \mathrm{el}(\mathbb{N}) \tag{5}$$

$$\frac{\mathrm{el}(\mathcal{N}) : n \Rightarrow m \;\; \varepsilon \; \mathrm{el}(\mathbb{N})}{\mathrm{el}(\mathcal{N}) : S(n) \Rightarrow S(m) \;\; \varepsilon \; \mathrm{el}(\mathbb{N})} \tag{6}$$

$$\frac{n = m : \mathrm{el}(\mathcal{N})}{S(n) = S(m) : \mathrm{el}(\mathcal{N})} \tag{7}$$

*In* (1) *we assert that we have a noncanonical set, which we then refine in that we define its value in* (2)*. Populating the set with noncanonical elements is done in the usual sense in* (3) *and* (4)*, but we need to define their values as well. This is done in* (5) *and* (6)*. Finally we define equality in the traditional way* (7)*.*

Let us consider a less artificial example. Say we wanted to construct the set $\{a + b \mid a, b \in \mathbb{N}\}$ where $\mathbb{N}$ is Peanos construction of the natural numbers.

$$\mathcal{X} : set \tag{1}$$

$$\mathcal{X} \Rightarrow \mathbb{N} \; \varepsilon \; set \tag{2}$$

$$0 + 0 : el(\mathcal{X}) \tag{3}$$

$$\frac{a + b : el(\mathcal{X})}{S(a) + b : el(\mathcal{X})} \qquad (4a)$$

$$\frac{a + b : el(\mathcal{X})}{a + S(b) : el(\mathcal{X})} \qquad (4b)$$

The computational rules are rather apparent:

$$el(\mathcal{X}) : 0 + 0 \Rightarrow 0 \ \varepsilon \ el(\mathbb{N}) \qquad (5)$$

$$\frac{el(\mathcal{X}) : 0 + b \Rightarrow c \ \varepsilon \ el(\mathbb{N})}{el(\mathcal{X}) : 0 + S(b) \Rightarrow S(c) \ \varepsilon \ el(\mathbb{N})} \qquad (6a)$$

$$\frac{el(\mathcal{X}) : a + b \Rightarrow c \ \varepsilon \ el(\mathbb{N})}{el(\mathcal{X}) : S(a) + b \Rightarrow S(c) \ \varepsilon \ el(\mathbb{N})} \qquad (6b)$$

Equality requires us to define four rules apart from the trivial one. Essentially we ensure that equal elements have the same value in the underlying canonical set. The reason for this is that we must respect the requirements of the definitional equality, e.g. that substituting equals for equals yields the same result.

$$0 + 0 = 0 + 0 : el(\mathcal{X}) \qquad (7a)$$

$$\frac{a + b = c + d : el(\mathcal{X})}{S(a) + b = S(c) + d : el(\mathcal{X})} \qquad (7b)$$

$$\frac{a + b = c + d : el(\mathcal{X})}{S(a) + b = c + S(d) : el(\mathcal{X})} \qquad (7c)$$

$$\frac{a + b = c + d : el(\mathcal{X})}{a + S(b) = S(c) + d : el(\mathcal{X})} \qquad (7d)$$

$$\frac{a + b = c + d : el(\mathcal{X})}{a + S(b) = c + S(d) : el(\mathcal{X})} \qquad (7e)$$

Finally, we can conclude that two equal elements of $\mathcal{X}$ must be equal once evaluated. This is ensured by our rules governing equality, but let us be explicit:

$$\frac{a+b=c+d : el(\mathcal{X}) \quad el(\mathcal{X}) : a+b \Rightarrow e \;\varepsilon\; el(\mathbb{N}) \quad el(\mathcal{X}) : c+d \Rightarrow f \;\varepsilon\; el(\mathbb{N})}{e = f \;\varepsilon\; el(\mathbb{N})}$$

**Remark 4** (Integers)**.** *We have mainly looked at* $\mathbb{N}$ *and its finite congruence classes modulo* $n$ *when examplifying the various concepts covered. A valid question would obviously be how to extend the notions illustrated through said examples to a discrete set that is somewhat less obvious in its construction. Let us therefore consider* $\mathbb{Z}$*.*

*As* $\mathbb{N} \subsetneq \mathbb{Z}$ *our primary concern is how to deal with negative numbers. The main problem here is that with* $\mathbb{N}$ *we had a nice and easy "starting position" to begin our inductive element generation from, i.e.* $0$*, but with* $\mathbb{Z}$ *things are a bit more complicated. A good approach is to consider* $\mathbb{Z}$ *in terms of* $\mathbb{N} \times \mathbb{N}$*. If we let each integer be represented as a pair* $(a, b)$ *we can make an interpretation according to* $(a, b) = a - b = c \in \mathbb{Z}$*. That is, if* $a > b$ *then we have a negative number. Equality can be defined in accordance with* $(a, b) = (c, d) \Leftrightarrow_{\mathbb{Z}} a + d = b + c$*. Note that this construct is an example of a noncanonical set.*

# 10 Function objects

Clearly, noncanonical sets and elements are closely related to the notion of a function. Typically, in a set context, we view functions as a means of mapping or transforming elements of one set into another. This obviously applies to our context as well, seeing as how we want to be able to perform similar operations and reasoning. However, we shall view a function as an independent object which we may apply to input.

We write $app(f, a)$ to apply a function object $f$ to the argument $a$ and $app[f, a]$ when $f$ and $a$ are canonical objects.

**Definition 15** (Function set)**.** *Let* $A$ *and* $B$ *be sets, then* $f : A \to B$ *means that if* $a \;\varepsilon\; \mathrm{el}(A)$ *then* $\mathrm{app}[f, a] : \mathrm{el}(B)$*. Note that* $A$ *must be canonical and* $B$ *noncanonical for this to make sense. We have*

$$\frac{A\ \varepsilon\ \text{set} \qquad B : \text{set}}{A \to B\ \varepsilon\ \text{set}}$$

**Definition 16** (Canonical function)**.** *That $f$ is a canonical function from $A$ to $B$ means that if $a\ \varepsilon\ \text{el}(A)$ then $\text{app}[f,a]$ is a noncanonical element of $B$ and if $a = b\ \varepsilon\ \text{el}(A)$ then $\text{app}[f,b] = \text{app}[f,a] : \text{el}(B)$. That is*

$$\frac{f\ \varepsilon\ \text{el}(A \to B) \qquad a\ \varepsilon\ \text{el}(A)}{\text{app}[f,a] : \text{el}(B)}\ D_9$$

$$\frac{f\ \varepsilon\ \text{el}(A \to B) \qquad a = b\ \varepsilon\ \text{el}(A)}{\text{app}[f,a] = \text{app}[f,b] : \text{el}(B)}\ D_{10}$$

That $f$ and $g$ are equal canonical elements of $el(A \to B)$ means that we can expect the same value from their computation, viz.

$$\frac{f = g\ \varepsilon\ el(A \to B) \qquad a\ \varepsilon\ el(A)}{app[f,a] = app[g,a] : el(B)}\ D_{11}$$

Further, we expect similar output from equal input

$$\frac{f = g\ \varepsilon\ el(A \to B) \qquad a = b\ \varepsilon\ el(A)}{app[f,a] = app[g,b] : el(B)}\ M_2$$

which is a mediary of

$$\frac{\dfrac{f = g\ \varepsilon\ el(A \to B) \qquad a\ \varepsilon\ el(A)}{app[f,a] = app[g,a] : el(B)}\ D_{11} \qquad \dfrac{g\ \varepsilon\ el(A \to B) \qquad a = b\ \varepsilon\ el(A)}{app[g,a] = app[g,b] : el(B)}\ D_{10}}{app[f,a] = app[g,b] : el(B)}$$

We may extend our work to also include noncanonical function objects and noncanonical elements. That is, we simply let $A : \text{set}$ as a premiss of the set formation, viz.

$$\frac{A : \text{set} \qquad B : \text{set}}{A \to B : \text{set}}\ \text{R}$$

The corresponding computation rule is

$$\frac{A \Rightarrow C\ \varepsilon\ \text{set}}{A \to B \Rightarrow C \to B\ \varepsilon\ \text{set}}\ C_1$$

Note that in the form $A \to B$ we do not evaluate $B$, viz. it is eager in $A$ but lazy in $B$. Noncanonical application is governed according to

$$\frac{f : el(A \to B) \qquad a : el(A)}{app(f, a) : el(B)} \text{R}$$

So step-wise, a computation would look like

$$\begin{array}{rcl} el(A) : a & \Rightarrow & c \ \varepsilon \ el(C) \\ el(A \to B) : f & \Rightarrow & g \ \varepsilon \ el(C \to B) \\ el(B) : app[g, c] & \Rightarrow & d \ \varepsilon \ el(D) \\ \hline el(B) : app(f, a) & \Rightarrow & d \ \varepsilon \ el(D) \end{array}$$

That is, first we eagerly evaluate $a$ to $c$, allowing us to evaluate the value of $f$ as the canonical element $g$ of $C \to B$. Note that $g$ does *not* expect an input of type $A$ – as it must accept $c$ – and is thus of type $C \to B$ rather than $A \to B$. We perform the application $app[g, c] : el(B)$ which has the value $d \ \varepsilon \ el(D)$. Since $app[g, c] : el(B)$ is really the result of $app(f, a) : el(B)$ the result follows.

**Remark 5** (Notation ambiguity)**.** *When we write* $\text{app}(f, a)$ *we are not explicitly conveying the type of $f$ and $a$. Since it is possible for $f$ and $a$ to be polymorphic (i.e. essentially to be of a generic type*[4]*), the result of the application computation is not known unless said types are stated. The principle of exactness is thus broken unless we by our notation assume that no ambiguities are possible, viz. all terms involved are clearly defined in an explicit content.*

# 11 Families of sets

**Definition 17** (Set valued function)**.** *A function $F$ is called a set valued function on a canonical set $A$, written $F \ \varepsilon \ \text{fam}(A)$, if an application of $F$ on a canonical element $a$ of $A$ yields a noncanonical set. Further, if two canonical elements of $A$ are equal, then so is the application of $F$ on both elements. We have*

[4] In computer science, polymorphism is a technique utilized to provide a common interface for several different types. So if $f$ can accept a set of possible input types, we do not explicitly know what computation is being performed unless the exact type of its input is given.

$$\frac{F \ \varepsilon \ \text{fam}(A) \qquad a \ \varepsilon \ \text{el}(A)}{\text{app}[F,a] : \text{set}} D_{11}$$

$$\frac{F \ \varepsilon \ \text{fam}(A) \qquad a = b \ \varepsilon \ \text{el}(A)}{\text{app}[F,a] = \text{app}[F,b] : \text{set}} D_{12}$$

**Definition 18** (Set valued function equality)**.** *$F$ and $G$ are said to be equal set valued functions on the canonical set $A$, written $F = G \ \varepsilon \ \text{fam}(A)$, if $a \ \varepsilon \ \text{el}(A)$ implies $\text{app}[F,a] = [G,a] : \text{set}$.*

A set valued function on a canonical set $A = \{a_1, a_2, \cdots\}$ is called a *family of sets* over $A$ because by repeated application on the canonical elements of $A$ we obtain a set of noncanonical sets, viz.

$$\{ \ app[F,a_1], \ app[F,a_2], \ \cdots \}$$

With the above definition, there are two important justified inference rules in particular to consider, namely

$$\frac{A = B \ \varepsilon \ \text{set} \qquad F \ \varepsilon \ fam(A)}{F \ \varepsilon \ fam(B)} J_3$$

$$\frac{A = B \ \varepsilon \ \text{set} \qquad F = G \ \varepsilon \ fam(A)}{F = G \ \varepsilon \ fam(B)} J_4$$

If $A = B \ \varepsilon$ set and $F \ \varepsilon \ fam(A)$ then we know that for all $a \ \varepsilon \ el(A)$ we have $app[F,a] : \text{set}$. Further, since $A = B \ \varepsilon$ set, it follows that $a \ \varepsilon \ el(A) \dashv\vdash a \ \varepsilon \ el(B)$ and $a = b \ \varepsilon \ el(A) \dashv\vdash a = b \ \varepsilon \ el(B)$. By Martin-Löfs definition of element equality, we can easily see that $app[F,a] = app[F,b] : \text{set}$. Since all elements of $A$ are in $B$ and equality is preserved, the process of applying $F$ on $a_1, a_2, \cdots$ in $A$ is equivalent to doing so for $b_1, b_2, \cdots$ in $B$.

The second justified inference rule mentioned is based on a similar notion, with the added condition of $app[F,a] = app[G,a] : \text{set}$. So if $F$ applied on $a_1, a_2, \cdots$ in $A$ yield the same sets as application of $G$, the addition of $A = B \ \varepsilon$ set allows us to draw the conclusion.

There are two natural set forming operations on a family of sets: the *sum* and the *product*.

**Definition 19** (Sum operation)**.** *Let* $A\ \varepsilon$ set*,* $a_i\ \varepsilon$ el$(A)$ *and* $F$ *be a set valued function. We consider the (possibly) infinite sum of sets*

$$\text{app}[F, a_1] + \text{app}[F, a_2] + \text{app}[F, a_3] + \cdots$$

*A good representation of a canonical element of this sum is a pair of two elements,* $(a_i, b)$*. The first part, i.e.* $a_i$*, denotes which* $a_i\ \varepsilon$ el$(A)$ *was used in* app$[F, a_i]$ *to obtain the second element* $b$*, which is also a canonical element. We denote the sum of a family of sets* $F$ *over the canonical set* $A$ *by* $\Sigma(A, F)$*. The formation rule is apparent by the definition, i.e.*

$$\frac{A\ \varepsilon\ \text{set} \qquad F\ \varepsilon\ \text{fam}(A)}{\Sigma(A, F)\ \varepsilon\ \text{set}}\ R$$

*So is the rule determining the meaning of a canonical element of the sum*

$$\frac{a\ \varepsilon\ \text{el}(A) \qquad (F\ \varepsilon\ \text{fam}(A)) \qquad \text{app}[F, a] \Rightarrow B\ \varepsilon\ \text{set} \qquad b\ \varepsilon\ \text{el}(B)}{(a, b)\ \varepsilon\ \text{el}(\Sigma(A, F))}\ D_{13}$$

*That two canonical elements of* $\Sigma(A, F)$ *are equal, viz.* $(a, b) = (c, d)\ \varepsilon\ \text{el}(\Sigma(A, F))$*, means that* $a = c\ \varepsilon$ el$(A)$ *and* $b = d : \text{el}(B)$[5]*.*

**Definition 20** (Product operation)**.** *Let* $A\ \varepsilon$ set*,* $a_i\ \varepsilon$ el$(A)$ *and* $F\ \varepsilon$ fam$(A)$ *and consider the (possibly) infinite product of sets*

$$\text{app}[F, a_1] \times \text{app}[F, a_2] \times \text{app}[F, a_3] \times \cdots$$

*We denote the product of a family of sets* $F$ *over the canonical set* $A$ *by* $\Pi(A, F)$*. A canonical element of* $\Pi(A, F)$ *must consist of an element of* app$[F, a_1]$*, an element of* app$[F, a_2]$ *and so forth. That is, if* $a\ \varepsilon$ el$(A)$ *then an element of* $\Pi(A, F)$ *consists of an element of* app$[F, a]$*. To refer to single elements of* app$[F, a]$ *we write* app$[f, a]$*, viz.*

$$\frac{f\ \varepsilon\ \text{el}(\Pi(A, F)) \qquad a\ \varepsilon\ \text{el}(A)}{\text{app}[f, a] : \text{el}(\text{app}[F, a])}\ D_{14}$$

*That is, applying a canonical element of the product of the set valued function* $F$ *on a canonical element* $a$ *in* $A$*, yields a noncanonical element of the set generated by* app$[F, a]$*.*

[5] It should be pointed out for clarity that it obviously also means that *app*$[F, a] \Rightarrow B\ \varepsilon$ set and that $F\ \varepsilon$ *fam*$(A)$.

As we can expect, equality is respected by the above definition. If we have an equality between two canonical elements of a set $A$ and apply $f \ \varepsilon \ el(\Pi(A,F))$ on them, the equality is preserved in the noncanonical applications. By definition, this obviously results in the value being equal as well. To summarize

$$\frac{f \ \varepsilon \ el(\Pi(A,F)) \qquad a = b \ \varepsilon \ el(A)}{app[f,a] = app[f,b] : el(app[F,b])} \, D_{15}$$

Similar to the equality being preserved when we have as a premiss equality among the canonical input, equality is preserved when we work with equal canonical elements of the product. That is

$$\frac{f = g \ \varepsilon \ el(\Pi(A,F)) \qquad a \ \varepsilon \ el(A)}{app[f,a] = app[g,a] : el(app[F,a])} \, D_{16}$$

# 12 Context

We have now outlined the basic notions of what it means to be a discrete set in Martin-Löfs type theory. Further, we have discussed noncanonical elements and sets, which are closely related to the notion of function. By looking at function objects and applying them to input, we have seen that new sets and families of sets can be formed. In this last section of this thesis we introduce the notion of context, first in the sense of "ordinary" intuitionistic type theory and then by expanding it further. Mainly we will note that by looking at context, we can generalize our previous results as it allows us to consider assumptions.

When we write general rules we are concerned with **context**, viz. we would write

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \wedge \psi} \wedge i$$

to emphasize that our premisses are but conclusions from premisses themselves. Similarily, we write

$$\frac{\varphi \ true \ (\Gamma) \qquad \psi \ true \ (\Delta)}{\varphi \wedge \psi \ true \ (\Gamma, \Delta)} \wedge i$$

It is important to stress that the context is *hypothetical*, i.e. by $\psi \;\; true\ (\Delta)$ we mean $\Delta \rightarrow \psi\ true$. That is, we have (where $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$)

$$\cfrac{\cfrac{\varphi\ true\ (\gamma_1\ true, \cdots, \gamma_n\ true)}{\varphi\ true\ (\Gamma)}}{\Gamma \rightarrow \varphi\ true}$$

To make an assumption, which is obviously of hypothetical nature, is thus closely related to having a method which takes a proof of the context into a proof of the assumption. Such a method must "make sense", so if we utilize projection we ensure a level of validity, viz.

$$\frac{\gamma_1 : true \cdots \gamma_n : true}{\gamma_i\ true\ (\Gamma)}$$

We have $\gamma_i\ true\ (\Gamma) \equiv \Gamma \rightarrow \gamma_i\ true \equiv (\gamma_1\ true, \cdots, \gamma_n\ true) \rightarrow \gamma_i\ true$ so the restriction $1 \leq i \leq n$ is required. Further, we want to be able to add extra assumptions at any time (cf. the example above with dual context: $\Gamma$ and $\Delta$). To do so is equivalent to weakening the argument, as we rely on assumptions rather than what is. No surprise then, that the rule governing assumption introduction is known as the *weakening rule*:

$$\frac{\varphi\ true\ \ (\Gamma) \qquad \psi : prop}{\varphi\ true\ \ (\Gamma, \psi\ true)}$$

Further, given an assertion $\varphi\ true$ with a context $(\Gamma, \psi\ true)$, we may want to be able to show that under the "partial" context $\Gamma$ we have an assertion of an implication, viz. $\psi \rightarrow \varphi\ true\ \ (\Gamma)$. To justify this, we need to consider the proof of the conclusion. It is a method which takes as input proofs $\gamma$ for the assumptions in $\Gamma$ and outputs a method which takes a proof $p : proof(\psi)$ into $q : proof(\varphi)$. Since our conclusion is $\psi \rightarrow \varphi\ true\ \ (\Gamma) \equiv \Gamma \rightarrow (\psi \rightarrow \varphi\ true)$ we implicitly have the method described. Thus

$$\frac{\varphi\ true\ \ (\Gamma, \psi\ true)}{\psi \rightarrow \varphi\ true\ \ (\Gamma)}$$

It is quite useful to consider a context as a collection of assumptions. That way we can extend our reasoning to be situational, i.e. to only be valid given a certain set of conditions; namely that the assumptions are true. We will use a notation that is quite analogous to that of regular context, viz. we write $a : el(A)\ (b : el(B))$ if $a$ is a noncanonical element of $A$ under the assumption that $b$ is a noncanonical element of $B$.

**Definition 21** (Context)**.** *A collection of assumptions is said to be a context. An assertion of this form is written* $\Gamma$ : context*. A context is defined recursively to either be*

1. *Empty, denoted* ( )*, or*
2. *the extension, denoted* $(\Delta, x : \mathrm{el}(A))$*, of a previously defined context* $\Delta$ *with a variable* $x$ *declared to be an element of a set* $A$ *defined in* $\Delta$*.*

Obviously we may make assertions on sets under context as well, e.g. $A : \mathit{set}\,(\Gamma)$. An assertion on this form is to be understood in terms of assignment of values to the sets declared in $\Gamma$. We write $b|(x \leftarrow a) : \mathit{el}(B)$ if $b$ is a noncanonical element of $B$ if $a$ is assigned to $x$ ($x$, $B$, $a$ as defined previously)[6].

**Definition 22** (Canonical assignment)**.** *A canonical assignment* $\gamma$ *for the context* $\Gamma$*, denoted* $\gamma\ \varepsilon\ \mathrm{ass}(\Gamma)$*, is either*

1. *the empty assignment,* ( )*, or*
2. *on the form* $(\delta, x \leftarrow a)$*, where* $\delta$ *is a canonical assignment for* $\Delta$ *and* $a$ *is a canonical element of the set which is the value of* $B|\delta$ *(where* $\Gamma$ *has the form* $(\Delta, x : \mathrm{el}(B))$*).*

Whenever $A$ is a set in the context $\Gamma$ and $\gamma\ \varepsilon\ \mathit{ass}(\Gamma)$, $A|\gamma$ is a noncanonical set. Further, if $\gamma = \delta\ \varepsilon\ \mathit{ass}(\Gamma)$ we have $A|\gamma = A|\delta : \mathrm{set}$. We can also speak of set equality under a context, viz.

$$\frac{B = C : \mathit{set}\ (\Gamma) \qquad \gamma\ \varepsilon\ \mathit{ass}(\Gamma)}{B|\gamma = C|\gamma : \mathrm{set}}$$

and

$$\frac{B = C : \mathit{set}\ (\Gamma) \qquad \gamma = \delta\ \varepsilon\ \mathit{ass}(\Gamma)}{B|\gamma = C|\delta : \mathrm{set}}$$

Finally, we can extend this notion to also concern elements. That is, we can state that $b$ is an element of the set $B$ in the context $\Gamma$, which we write as

$$b : \mathit{el}(B)\ (\Gamma)$$

This means that given $\gamma\ \varepsilon\ \mathit{ass}(\Gamma)$ we have $b|\gamma : \mathit{el}(B|\gamma)$. Further, if $\gamma$ and $\delta$ are equal assignments in $\Gamma$ then $b|\gamma$ and $b|\delta$ are equal noncanonical elements of the set $B|\delta$.

[6]Further, if $a = c\ \varepsilon\ \mathit{el}(A)$ and $b : \mathit{el}(B)\ (x : \mathit{el}(A))$ then $b|(x \leftarrow a) = b|(x \leftarrow c) : \mathit{el}(B)$.

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