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Applications of the Heath, Jarrow and Morton (HJM) model to energy markets

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APPLICATIONS OF HEATH, JARROW AND MORTON (HJM) MODEL TO ENERGY MARKETS

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In this thesis, we have used the NordPool exchange market data to calibrate the HJM model. PCA technique is used to analyze the market data and extract the major risk factor(s) which is an essential part to forecast future curves.

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1.0 INTRODUCTION

In the beginning of the nineties, many countries started deregulating their markets for electricity power. This phenomenon, created the power markets where the electricity energy is traded as commodity. One of the most important financial products of these markets are *futures* and *forward* contracts which are widely used for risk management purposes. The buyer of this type of financial instrument is promised the delivery of a pre-determined amount of electrical energy as a constant flow over the future period of time which is specified in the contract [9]. This delivery can be settled physically or financially. The buyer and writer of futures contract insure themselves against the possibility of fluctuating future prices through margin account. Similarly, financial derivatives are written on these futures and forward contracts and they are used widely to hedge the electricity risk. However, the valuation and estimation of these contracts is very difficult and still under discussion due to the lack of economical pricing concepts because the electricity is not a storable commodity.

The main objective of our thesis is to use the Nordic Electricity market data to calibrate the *Heath-Jarrow-Morton (HJM)* model. This thesis is based on the research article proposed by Broszkiewicz and Weron[4]. We review the multi-factor Heath-Jarrow-Morton (HJM) approach to model the pricing dynamics of forward contracts. We also perform empirical tests using market information of the Nordic monthly forward contracts and use the principal component analysis (PCA) to analyze the volatility structure of the forward curve.

The organization of this thesis is as followed:

- In the second chapter, fundamental definitions, theorems, bonds, interest rates and forward contracts are presented.
- We review the structure of Nordic power market in Chapter 3. The main reason of discussing Nordic power market is because we will later on use Nordic market data.
- In Chapter 4, we explain the principal component method and show that how it can be used for extracting key factors.
- We review the framework of Heath-Jarrow-Morton model in Chapter 5.
- In Chapter 6, we explain the description of the multi-factor Heath-Jarrow-Morton model for energy market [4] and describe the data sets and estimate the volatility functions by using the principal component analysis.

2.0 DEFINITIONS

The main propose of this chapter is to present some important definitions and theorems which we use later on. Most of these definitions and theorems are taken from Björk [3], Baxter [1] and Lamberton [10].

Filtration:

A filtration on a probability space $(\Omega, \mathcal{A}, P)$ is an increasing family$(\mathcal{F}_t)_{t\geq 0}$ of $\sigma-$algebras contained in $\mathcal{A}$. The σ-algebra $\mathcal{F}_t$ represents the information available at time t. The stochastic process X is called adapted to filtration $\mathcal{F}_t$ if for all $t \geq 0$, $X(t) \in \mathcal{F}_t$ which means that $X(t)$ is $\mathcal{F}_t$-measurable [10].

Wiener process:

A stochastic process $W = (W(t) : t \geq 0)$ is called $P-$Wiener process if the following conditions hold.

1. $W(0) = 0$.
2. The process W has the independent increments, that is if $0 \leq a < b \leq c < d$ then $W(d) - W(c)$ and $W(b) - W(a)$ are independent stochastic variables.
3. If $0 < s < t$,the stochastic variable $W(t) - W(s)$ has the Gaussian distribution $N(0, \sqrt{t-s})$.
4. W has continuous trajectories [3].

Martingale :

A stochastic process X is called an $\mathcal{F}_t-$martingale if the following conditions hold.

1. X is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

2. For all $t > 0$

$$E[|X(t)|] < \infty.$$

3. For all s and t with $s \leq t$, the following relationship holds

$$E[X(t) \mid \mathcal{F}_s] = X(s).$$

Absolutely continuous:

A probability measure Q is called absolutely continuous with respect to measure P if $Q(A) = 0$ whenever $P(A) = 0$ and it is denoted as $Q \ll P$ [10].

Theorem 1 (**Radon-Nikodym Theorem**). *A probability measure Q is absolutely continuous with respect to measure P if and only if there exists a non-negative random variable Z on $(\Omega, \mathcal{A})$ such that*

$$Q(A) = \int_A Z(w)dP(w) \ for \ all \ A \in \mathcal{A}.$$

Z is called Radon-Nikodym derivative *of the measure Q with respect to the measure P and is denoted by dQ/dP [10].*

Theorem 2 (**Girsanov's theorem**). *If $W(t)$ is a $P-$ Wiener process and $\gamma(t)$ is an process adapted to the filtration $\mathcal{F}_t$ which satisfy boundedness condition $E_P \exp\left(\frac{1}{2}\int_0^T \gamma(t)^2 dt\right) < \infty$, then there exists a measure Q such that*

1. *Q is equivalent to P.*
2. $\frac{dQ}{dP} = \exp\left(-\int_0^T \gamma(t)dW(t) - \frac{1}{2}\int_0^T \gamma(t)^2 dt\right)$.
3. $\widetilde{W(t)} = W(t) - \int_0^t \gamma(t)ds$ *is a Q-Wiener process [1].*

As a consequence of Girsanov's Theorem, one can derive one of the most important theoretical tools in financial mathematics, the so called First Fundamental Theorem of Asset Pricing (see e.g. [3]). This theorem can be stated informally as follows:

Theorem 3 (**The First Fundamental Theorem of Asset Pricing**). *Assume that a market model consists of the asset price processes* $S_0(t), S_1(t), ..., S_n(t)$*, where time* t *changes in the time interval* $[0, T]$*. Let* P *denote the underlying probability measure.* Under some technical assumptions *the model is arbitrage free if and only if there exists a probability measure* Q *which is equivalent to* P *and is such that each of the processes*

$$\frac{S_k(t)}{S_0(t)}, \quad k = 1, ..., n$$

is a martingale with respect to Q*. The measure* Q *is referred to as a* martingale measure *for this model, whereas the process* S_0 *is the chosen* numeraire.

2.1 FIXED-INCOME MARKETS

In this section, our primary objective will be to explain some common definitions of bonds and interest rates which we will use later on in development and calibration of *Heath-Jarrow-Morton*(HJM) model with market data.

Definitions

1. A T-maturity *zero-coupon bond* is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t, T)$, and $P(T, T) = 1$ for all T.
2. The *time to maturity* $T - t$ is the amount of time(in years/months) from the present time t to the maturity time $T > t$.
3. *Forward rates* are characterized by three time instants, namely time t at which the rate is considered, its future time interval S and T, with $t \leq S \leq T$. We made a contract at time t, guaranteeing an interest rate payment over the future interval $[S, T]$, such an interest rate is called *forward rate.*
4. The *instantaneous forward interest rate* prevailing at time t for the maturity $T > t$ is denoted by $f(t, T)$ and is defined as

$$f(t, T) = -\partial \ln P(t, T)/\partial T,$$

above equation can be written as:

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right).$$

The instantaneous forward interest rate $f(t,T)$ is a forward rate at time t whose maturity is very close to its expiry T.

5. *The instantaneous short interest rate* at time t is defined by $r(t) = f(t,t)$.
6. *Forward price:* The price $F(t,T)$ agreed upon at time t is the price of underlying asset to be used at delivery time T.

Forward contracts:

Forward contracts are an important class of exchange traded derivatives and forward contracts are applied to wide range of commodities like gold, oil, electricity, etc. Options on forward contracts are known as forward options.

A *forward contract* on a commodity is an agreement entered into at time t between two parties to buy/sell a specified amount of the said commodity at time T ,for the forward price $F(t,T)$ agreed upon at time t. The buyer in the contract is said to take *long position* and the seller is said to take *short position.* Let $F(t,T)$ is a forward price, T is a maturity time and the market price of underlying commodity at time T is denoted by $K(T)$ then forward contract payoff $V(T)$ at maturity is:

$$\begin{aligned} V(T) &= K(T) - F(t,T) \quad \text{(Long position)} \\ V(T) &= F(t,T) - K(T) \quad \text{(Short position)} \end{aligned}$$

Example: Suppose that forward price is 50 Euros. If the market price of the commodity turns out to be 55 Euros on the delivery time T then holder of a long position will buy the commodity for 50 Euros and can sell it immediately for 55 Euros by making the profit of 5 Euros. On the other side, the seller of the contract will have to sell the commodity for 50 Euros, with lost of 5 Euros.

3.0 THE NORDIC ELECTRICITY MARKET

Electricity is not a storable commodity which is why electricity has been called a flow commodity. Due to its non-storability, it has a very strong effect on the infrastructure and the organization of electricity market as compared to other commodities markets. Deregulated electricity markets have a mechanism to balance the supply and demand between the costumers and suppliers, where electricity is traded through standardized contracts. These contracts guarantee the delivery of some amount of electricity for a specified or a decided time period. Some of these contracts are settled through physical delivery and others are settled financially.

In this chapter, we will discuss the Nordic power market and it's structure. The primary reason of discussing Nordic power market is because we will later on use Nordic market data to calibrate the *Heath-Jarrow-Morton* (HJM) model. The explanation in this chapter follows the research articles Benth [2] and Koekebakker [9].

3.1 HISTORY OF NORDIC POWER EXCHANGE

Norway was the first country in the Nordic region that deregulated its markets for electricity in 1991. 'Statnett Marked AS' was established as an independent company in 1993. Total volume in the first operating year was 18.4 TWH at a value of NOK 1.55 billion[1]. In 1995, Sweden joined this exchange. It was the first world multinational electricity exchange. The exchange was renamed as Nord Pool ASA. Then Finland and Denmark joined the

[1] www.nordpoolspot.com

Nordic exchange market in 1998. EL-EX exchange for electricity of Finland entered into an agreement to present Nord Pool in 1998 and Elbas was launched as the separate market for balance adjustment between Finland and Sweden in 1998. The Nordic electricity market is non-mandatory market where both physical power and financial contracts are traded.

3.2 THE PHYSICAL MARKET

Physical electricity contracts mean the contracts with actual fulfillment of consumption or production of electricity. Since the capacity of producing the electricity is restricted, the supply and demand must be balanced. These markets are supervised by transmission system operators (TSO)[2]. Moreover the players in these markets are restricted to consumption and production facilities. The physical contracts are usually organized in two different types of markets, namely the real time and day ahead market. They are also called explicit and implicit markets. These are known as two-settlement systems.

3.2.1 The real time market

System operator organized the real time markets for short term upward and downward regulation. Auction specifies the both time period and load for consumption and production. Bids are submitted to TSO and bids can be posted or changed close to maturity time with accordance to agreement rules. In real time market, bids are for upward regulation and downward regulation and both supply-demand bids are posted with starting prices and volumes. TSO list the bids for each hour according to the price and the merit order is used for each hour to balance the power system. Upward regulation is applied in order to resolve the grid power deficit and downward regulation is applied for grid power surplus.

3.2.2 The day-ahead market

[2] In Nordic market the TSOs are Statnett (Norway), Svenska Kraftnatt (Sweden), Fingrid (Finland), Elkraft System AS (Eastern Denmark) and Eltra (Western Denmark)

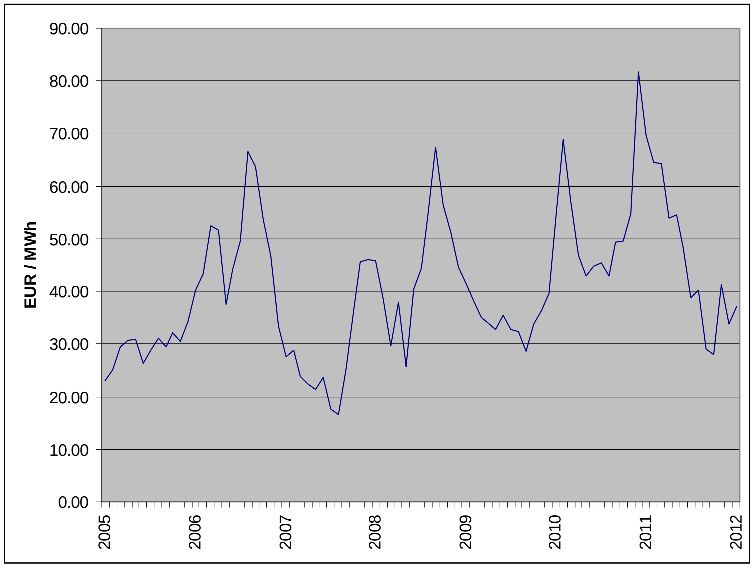

Figure 1: Time series of average monthly spot prices from Nord Pool in period 2005-2012

In the Nordic market, the day-ahead market is known as Elspot and it is organized by Nord Pool Spot. In Elspot, hourly electricity contracts are traded daily with the physical delivery in next day's 24 hour period. In Nord Pool's spot market, players from Sweden, Finland, Denmark, Estonia and Norway trade hourly contracts for each hour of next 24 hours of coming day. In morning, they submit the bids for buying and selling of the electricity contracts for the different hours of the next day. Once the market is closed for the bids at noon of each day, the day-ahead price is given for next day. The day-ahead price is called the system price and it is common in all Nordic countries. Due to capacity and cross border constraints, the Nordic market is geographically divided into several bidding areas or trade zones. Traders must make the bids according to where their production and consumption is physically connected to Nordic grid.

3.3 ELECTRICITY FINANCIAL MARKET

The rules for trading electricity financial contracts vary among different power exchanges. There is no physical delivery for financial contracts. Technical conditions such as capacity of the supply and access to grid are not taken into consideration in these contracts. Buyers and sellers of these contracts use to manage the risk associated with physical market prices. Nordic electricity financial contracts are traded through NASDAQ OMX Commodities. The system price which is calculated by "Nord Pool Spot" is used as the reference price for the electricity financial market.

Forward/ Future contracts which are traded at Nord Pool are written on the average of the hourly system price over the specified delivery period and during the delivery period contracts are settled in cash against the system price. Since the contracts are settled against the Nord Pool system price and all contracts are base load contracts, their underlying amount of electricity energy is determined by

$$DP * 24 * Mwh.$$

Here DP denotes the length of the delivery period in days. From the above equation, we can compare the contracts with different delivery periods. Prices of these contracts are listed in EURO for 1 MWh during every hour of energy delivered as a constant flow of electricity during the delivery period.

Nord Pool offers forward and futures financial electricity contracts. In the beginning of 2005, Nord Pool made changes to its product structure. The futures contracts were first reduced to 8-9 weeks then down to 6 weeks and they introduced forward contracts up to four year of maturity. Nord Pool has daily and weekly futures contracts and monthly, quarterly and yearly forward contracts. Nord Pool has listed 6 weekly futures contracts and each week only weekly contract is unlisted and the new one is introduced which is traded for next 6 weeks. These futures contracts are settled through daily mark to market settlement. They have also listed 6 monthly forward contracts on continuous rolling base, quarterly forward contracts Q1, Q2, Q3 and Q4 which are traded for two years and a yearly forward contract that is traded for 4 years.

4.0 PRINCIPAL COMPONENT ANALYSIS

Trading and pricing of complex option and portfolios depend on the accurate choice of a model. The modeling of volatilities and forecasting them plays very important role while solving different types of pricing models like Black-Scholes pricing formula.

To predict the future volatilities, we need to extract market risk factors. Many financial markets count on a high degree of correlation between risk factors on similar type of contracts with different maturities. A popular technique to compute the market risk factors is principal component analysis which was invented in 1901 by Karl Pearson. Now it is mostly used in dimension reduction, data exploratory analysis and for predicting models. With principal component analysis, we can decompose a symmetric matrix into eigenvector and eigenvalues matrices and draw out the key factors, thereby reduce the dimensionality of the system.

4.1 PRINCIPAL COMPONENT ANALYSIS METHOD

PCA method is based upon the analysis of eigenvalues and eigenvectors of $V = X^{'}X$, where $X^{'}X$ is the symmetric matrix of covariances between the columns of X. Each principal component is the linear combination of these columns, where as weights are chosen in such a way that it explain the first principal component covers the largest amount of variation of and so on.

A symmetric matrix can be factored in to $A = Q\Lambda Q^{'}$ with the orthonormal eigenvectors in Q and the eigenvalues in Λ with properties $Q \times Q^{'} = I$ and $Q^{'} = Q^{-1}$. Consider a data matrix $X = [X_1, ..., X_k]$, where each element of X_i denotes the ith column of X. PCA finds the $k \times k$ weight factor (orthogonal) matrix with set of eigenvectors of the covariance matrix V, such that

$$V = W \Lambda W^{'}.$$

Here Λ is the $k \times k$ diagonal matrix of eigenvalues of V. The columns of $W = (w_{ij})\ for\ i, j = 1, ..., k$, are orders according to the size of corresponding eigenvalue so that nth column of W, denoted as $w_n = (w_{1n}, .., w_{kn})^{'}$ is the $k \times 1$ eigenvector corresponding to the eigenvalue λ_n, where the columns of Λ are ordered so that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$. All eigenvalues are real because the covariance matrix is always symmetric. These eigenvalues can be distinct or identical.

The nth principal component is defined by:

$$P_n = w_{1n} X_1 + w_{2n} X_2 + ... + w_{kn} X_k.$$

In matrix notation:

$$P_n = X w_n.$$

The matrix $T \times k$ of principal components is :

$$P = XW. \tag{4.1}$$

The sum of k eigenvalues is equal to the sum of the k diagonal entries

$$\lambda_1 + \lambda_2 + ... + \lambda_k = a_{11} + a_{22} + ... + a_{kk},$$

where $a_{11}, a_{22}, ...a_{kk}$ is a diagonal entries of matrix V. Right hand side of the above equation is known as the trace of matrix V. The trace of matrix V is a total variance because $Cov(x_n, x_n) = var(x_n, x_n),\ for\ n = 1, .., k$ and $var(x_1, x_1) + var(x_2, x_2) + ... + var(x_k, x_k) = total\ variance\ of\ data$. So it can also be written as

$$\lambda_1 + \lambda_2 + ... + \lambda_k = total\ variance\ of\ data$$

Thus total variance of data is equal to the sum of all eigenvalues of the covariance matrix. Thus, the proportional variance explained by the first n principal components is:

$$\frac{\sum_{i=1}^{n} \lambda_i}{\omega}$$

Here ω is the sum of the all eigenvalues of covariance matrix.

The eigenvalue λ_1 that corresponds to the first principal component so P_1 explains the largest proportion of the total variance in X. Next we have P_2, which explains the second largest total proportion and so on. Since $W^{'} = W^{-1}$ so equation (4.1) can be written as

$$X = PW^{'}$$

That is:

$$X_i = w_{i1}P_1 + w_{i2}P_2 + .. + w_{ik}P_k \; where \; i = 1, ..., k. \tag{4.2}$$

Above equation (4.2) is the principal component representation of the original factors. And one can choose the components according to the requirement. By doing this, it will reduce the dimension of the system.

4.1.1 Implementation

In this part, We are going to discuss about steps needed to perform principal component analysis. We are also going to discuss some algorithms on how to solve the eigenvalues problems.

1. First we have to calculate the mean of sample data. Consider $T \times k$ data matrix $X = [X_1, ..., X_k]$, the mean of each column k of the data matrix X can be evaluated by this formula

$$\overline{X_n} = \sum_{i=1}^{T} x_{in}/T \quad for \; n = 1, 2, ..., k.$$

Now that we have the mean of each column of the matrix X, we can define the mean of X as $\overline{X} = [\overline{X_1}, \overline{X_2}, .., \overline{X_k}]$. Next step is to subtract mean of each column from the each element of that specified column, this will give us the mean zero modification of X:

$$\widetilde{X} = \begin{bmatrix} x_{11} - \overline{X_1} & . & . & x_{kk} - \overline{X_k} \\ . & . & . & . \\ . & . & . & . \\ x_{1T} - \overline{X_1} & . & . & x_{kT} - \overline{X_k} \end{bmatrix}$$

2. Now we have to calculate the covariance matrix of the $\widetilde{X}$. The covariance matrix for a set of data with k dimensions is

$$cov^{k\times k} = \frac{1}{T-1}\widetilde{X'}\widetilde{X}$$

   Here $cov^{k\times k}$ is the matrix with k rows and k columns.

3. Calculate the eigenvectors and eigenvalues of the covariance matrix.

   We have calculated the eigenvalues by MATLAB command *eig*. If covariance matrix is huge then using this command is insufficient and it will lose the accuracy. Then one has to use some other method.

4. Once we have found the eigenvalues then sort them highest to lowest. Now, we can decide which component we want to ignore. Ignoring the eigenvalue with lowest value, it will not lose too much information. We choose the first n highest eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$.

Now final step is to choose components which we want to keep in the data. Take the eigenvectors and multiply with the original data.

4.2 APPLICATION OF PCA

Let us consider an application of PCA to energy forward prices curves (Electricity forward contract prices). PCA facilitates us to gain more knowledge and understanding of the way in which forward price curve changes. By looking on the historical forward curves, we can deduce and identify the movements of principal components and their contribution to total variance of the data. Figure 1 is a graph of daily forward prices from December 1, 2005 to April 1, 2010 for 1-month, 2-month, 3-month, 4-month, 5-month and 6-month forward contract maturities.

As the Figure (2) shows, forward price curves of different maturities move in tandem with each other. The data matrix X consists of 1108×6 rows and columns and there is total of 1108 days of data for each 6 different maturities forward contracts. By decomposing covariance matrix V, we get eigenvalues Λ and set of eigenvectors, as follows,

$$\Lambda = [923.973, 80.374, 11.72, 1.679, 1.213, 1.1439].$$

And the covariance matrix V,

$$V = \begin{bmatrix} & 1-mo & 2-mo & 3-mo & 4-mo & 5-mo & 6-mo \\ 1-mo & 165.575 & 162.339 & 154.753 & 144.556 & 127.882 & 106.624 \\ 2-mo & 162.339 & 168.627 & 167.326 & 160.745 & 144.603 & 120.9817 \\ 3-mo & 154.753 & 167.326 & 177.306 & 175.591 & 162.202 & 137.896 \\ 4-mo & 144.556 & 160.745 & 175.591 & 183.263 & 174.359 & 153.015 \\ 5-mo & 127.882 & 144.603 & 162.202 & 174.359 & 174.217 & 157.657 \\ 6-mo & 106.624 & 120.9817 & 137.896 & 153.015 & 157.657 & 151.119 \end{bmatrix}$$

And the eigenvectors corresponding to the first three largest eigenvalues and hence to the

first three principal components:

$$\widetilde{W} = \begin{bmatrix} 1-mo & 0.3812 & -0.5871 & -0.5264 \\ 2-mo & 0.4094 & -0.3993 & 0.0378 \\ 3-mo & 0.4321 & -0.1195 & 0.4953 \\ 4-mo & 0.4393 & 0.1686 & 0.3990 \\ 5-mo & 0.4165 & 0.40233 & 0.0040 \\ 6-mo & 0.3657 & 0.5395 & -0.5628 \end{bmatrix} \tag{4.3}$$

As we see in figure (3), the first eigenvalue, $\lambda_1 = 923.973$, is much larger than others which indicates that system is highly correlated. The matrix W contains eigenvectors of V and the first three eigenvectors are listed in table (4.3). Figure (4) is a plot of the first three eigenvectors and we take first three principal components because they explain 99 percent of total variation. The first eigenvalue explains 90 percent of total variation and it attributes a parallel shift to term-structure. This factor explains a change in the overall level of prices

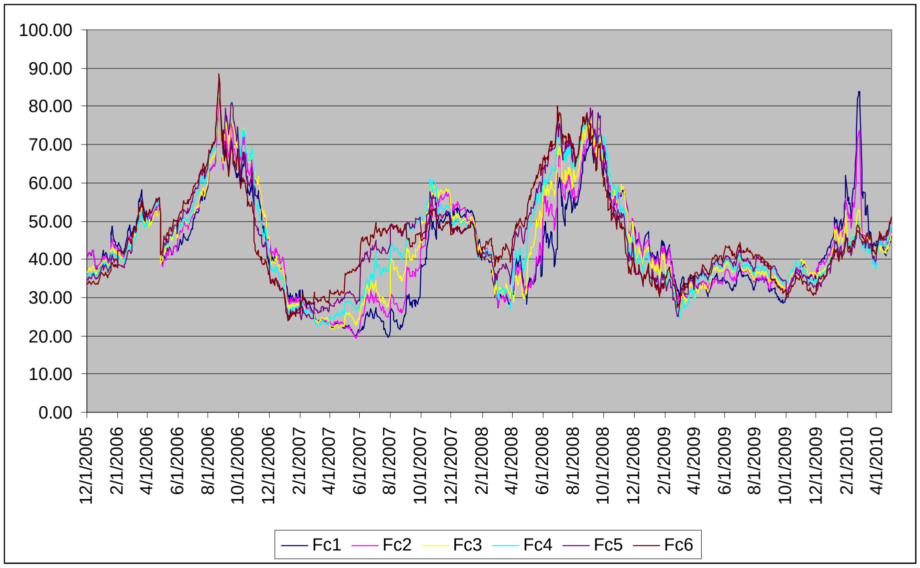

Figure 2: Forward Prices of monthly forward contracts from 1-12-2005 to 1-4-2010

and this first principal component is called the "tend" component of term-structure. The second principal component is called "tilt" component and contributes 7.8790 percent of total variation. The second component is increasing over time and indicates a slope shape. It can be interpreted as a change in overall price level of term-structure. The third principal component is positive for medium-term prices and negative for both short-term and long-term prices. This component explains the convexity of term-structure.

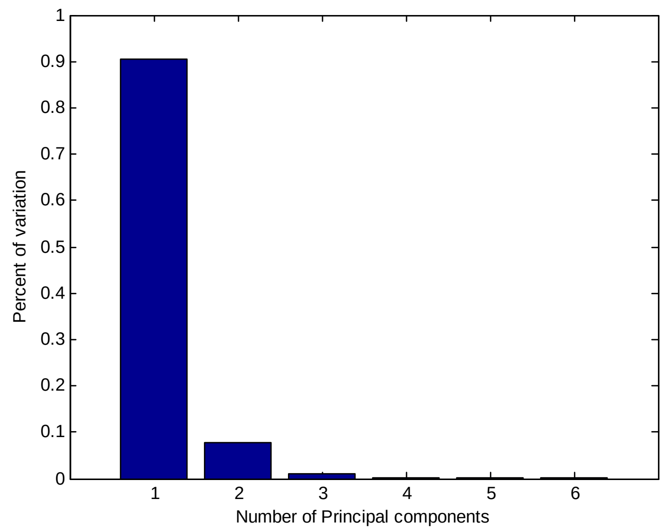

Figure 3: This Bar chart explains the proportion of variation explained by first 6 principal components

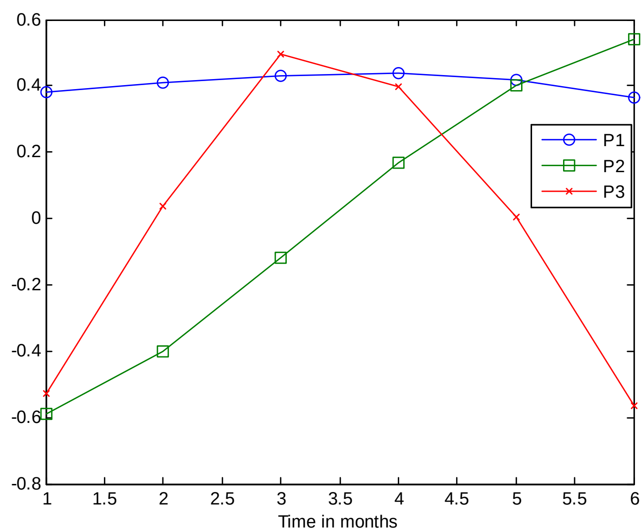

Figure 4: Plot of Principal component EigenVectors

5.0 FORWARD RATE MODEL

In this chapter, we will give the introduction to the framework of *Heath-Jarrow-Morton* (HJM) model. Furthermore, the usage and implementation of the procedure and estimation of parameters are discussed.

5.1 HEATH, JARROW AND MORTON FORWARD RATE MODEL

The forward rate model was introduced by Heath, Jarrow and Morton [7]. They developed the model for forward rate curves. It was a major breakthrough in the pricing of fixed income products. They derived an arbitrage free model for stochastic evolution of the entire term structure of interest rates, where the forward rates are determined by their instantaneous volatility structures. Heath, Jarrow and Morton choose the entire forward rate curve in their model which makes it a non-Markov process.

They assumed that for a fixed maturity $T \geq 0$ and the instantaneous forward rates $f(t,T)$ evolve under a measure $\mathbf{P}$ are given by:

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{N} \sigma_i(t,T)dW_i(t), \tag{5.1}$$

or we can also write the above equation as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t), \tag{5.2}$$

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s), \tag{5.3}$$

where $f(0,T)$ is the instantaneous forward curve at time $t = 0$, $W = (W_1, W_2, ..., W_N)$ is N-dimensional $\mathbf{P}$-Wiener process and $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ are adapted processes to the filtration $\mathcal{F}_t$ generate by $W(t)$. Integrals of these adapted processes are finite in the sense that $\int_0^T |\alpha(u,T)|\, du + \int_0^T |\sigma(u,T)|^2\, du < \infty$, and volatility function σ has a finite expectation $E\int_0^T \left|\int_0^u \sigma(t,u) dW(t)\right| du$, where $|\cdot|$ is denoted as Euclidean norm. As we know from the definition of short-term interest rate satisfies $r(t) = f(t,t)$. We can write down the integral equation for the short rate which is

$$r(t) = f(t,t) = f(0,t) + \int_0^t \sigma(s,t) dW(s) + \int_0^t \alpha(s,t) ds. \tag{5.4}$$

Our tradeable asset will be T-maturity bond $P(t,T)$. The relationship between forward rate and bond price is

$$P(t,T) = \exp\left(-\int_t^T f(t,u) du\right) = \exp(Y(t,T)),$$

where $Y(t,T) = -\int_t^T f(t,u) du$. By using Ito's formula, the dynamics of bond price is given by,

$$dP(t,T) = d\exp(Y(t,T)) = \exp(Y(t,T)) dY + \frac{1}{2}\exp(Y(t,T))(dY)^2,$$

$$dP(t,T) = P(t,T)(dY(t,T) + \frac{1}{2}(dY(t,T))^2), \tag{5.5}$$

and using equation (5.2)

$$dY(t,T) = r(t)dt - \left(\int_t^T \alpha(t,u) du\right) dt + \Upsilon(t,T) dW(t),$$

where $\Upsilon(t,T)$ is defined as the integral $-\int_t^T \sigma(t,u) du$ and $(dY(t,T))^2 = |\Upsilon(t,T)|^2\, dt$. The solution of above equation is non-trivial, see details [7]. Now putting these values in equation (5.5)

$$dP(t,T) = P(t,T)\left\{\left(r(t) - \int_t^T \alpha(t,u) du + \frac{1}{2}|\Upsilon(t,T)|^2\right) dt + \Upsilon(t,T) dW(t)\right\}. \tag{5.6}$$

Let us now introduce the saving account as additional tradeable security. The saving account B is a stochastic process satisfying the following SDE

$$dB(t) = r(t)B(t)dt,\; B(0) = 1. \tag{5.7}$$

Putting the value of $r(t) = f(t,t)$ in equation (5.7) and integrating it,

$$\begin{aligned} B(t) &= \exp\left(\int_0^t f(u,u)du\right), \\ &= \exp\left(\int\limits_0^t \left(f(0,u) + \int_0^u \sigma(s,u)dW(s) + \int_0^u \alpha(s,u)ds\right) du\right), \\ &= \frac{1}{P(0,t)} \exp\left(\int_0^t \left(\int_s^t \sigma(s,u)du\right) dW(s) + \int_0^t \left(\int_s^t \alpha(s,u)du\right) ds\right). \end{aligned}$$

We used the condition that volatility function σ has a finite expectation and the integrals $\int_0^t \left(\int_0^u \sigma(s,u)dW(s)\right) du$ can be interchanged to $\int_0^t \left(\int_s^t \sigma(s,u)du\right) dW(s)$ [1, 7].

5.2 HJM DRIFT CONDITION

According to *First Fundamental Theorem,* a market model does not have arbitrage opportunities if and only if there exists the probability measure Q equivalent to probability measure $\mathbf{P}$, such that the discounted bond process $Z(t,T) = B(t)^{-1}P(t,T)$ is a $Q-$martingale [10]. We will find the dynamics for bond prices and forward rates under the probability measure Q and also show that the dynamics of $\alpha(t,T)$ is determined by $\sigma(t,T)$.

Let us define discounted bond price process and fix one particular maturity T,

$$Z(t,T) = B(t)^{-1}P(t,T),\ 0 \leq t \leq T.$$

By using Ito's formula, the differential equation of the discounted bond process is given by,

$$\begin{aligned} d(Z(t,T)) &= d\left(\frac{P(t,T)}{B(t)}\right), \\ &= d(\frac{1}{B(t)})P(t,T) + \frac{1}{B(t)} d(P(t,T)), \end{aligned}$$

using equation (6.1) and (5.7) :

$$\begin{aligned} d(Z(t,T)) &= \frac{P(t,T)}{B(t)} \left(\left(r(t) - \left(\int_t^T \alpha(t,u)du\right) + \frac{1}{2}\left|\Upsilon(t,T)\right|^2\right) dt + \Upsilon(t,T)dW(t) - r(t)dt\right), \\ &= \frac{P(t,T)}{B(t)} \left(\Upsilon(t,T)dW(t) + \left(\frac{1}{2}\left|\Upsilon(t,T)\right|^2 - \int_t^T \alpha(t,u)du\right) dt\right). \end{aligned}$$

Here $\Upsilon(t,T)$ is defined as the integral $-\int_t^T \sigma(t,u)du$ and $W(t)$ is $N-$dimensional $\mathbf{P}$-Wiener process.

We have to find a change of measure drift $\gamma(t)$, which makes $Z(t,T)$ driftless. The change of measure drift is

$$\gamma(t) = \frac{1}{2}\Upsilon(t,T) - \frac{1}{\Upsilon(t,T)}\int_t^T \alpha(t,u)du.$$

We can apply Girsanov's theorem to change probability measure $\mathbf{P}$ to probability measure Q where $\widetilde{W}(t) = \widetilde{W}(t) = W(t) + \int_0^t \gamma(s)ds$ is $Q-$Wiener process. This will give us the SDE of discounted bond under measure Q,

$$dZ(t,T) = Z(t,T)\Upsilon(t,T)d\widetilde{W}(t).$$

Here $\widetilde{W}$ is N-dimensional $Q-$Wiener process and discounted bond $Z(t,T)$ is Q-martingale under the probability measure Q because it is driftless.

The process $\gamma(t)$ is also called the market price of risk. As we assume that there is no *arbitrage opportunity* in the market, the market price of risk should be same for all bonds and $\gamma(t)$ must be independent of T.

$$\int_t^T \alpha(t,u)du = \frac{1}{2}\left|\Upsilon(t,T)\right|^2 - \Upsilon(t,T)\gamma(t), \;\; t \leq T,$$

differentiating with respect to T, we have

$$\alpha(t,T) = \sigma(t,T)(\gamma(t) - \Upsilon(t,T)).$$

Heath, Jarrow and Morton proved that in order for a unique equivalent martingale to exist, the drift can not be arbitrarily chosen. It must be depend upon the volatility σ . Heath, Jarrow and Morton proved this famous result called drift condition.

Theorem 4 (HJM drift condition)**.** *Assume that market is arbitrage free and forward rates are given by equation (5.1) then there exists a process $\gamma(t)$ such that*

$$\alpha(t,T) = \sigma(t,T)(\gamma(t) - \Upsilon(t,T)'),$$

holds for all $T \geq 0$ and $t \leq T$ [3].

Theorem 5. *Under the probability measure Q, the processes $\alpha(t,T)$ and $\sigma(t,T)$ have to satisfy following equation for all t and $t \leq T$.*

$$\alpha(t,T) = \sigma(t,T)\Upsilon(t,T)^{'}. \tag{5.8}$$

Using the above equation (5.8), the dynamics of the HJM-Model under the measure Q became

$$df(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)^{'} ds dt + \sigma(t,T) d\widetilde{W}(t), \tag{5.9}$$

where $\widetilde{W}$ is a N-dimensional Q-Wiener process. The above equation can be also written as

$$df(t,T) = \sum_{i=1}^{N} \sigma_i(t,T)\int_t^T \sigma_i(t,s)^{'} ds dt + \sum_{i=1}^{N} \sigma_i(t,T) d\widetilde{W}_i(t).$$

Integrating the above equation from 0 to t, we get

$$f(t,T) = f(0,T) + \left(\sum_{i=1}^{N}\int_0^t \sigma_i(s,T)\int_s^T \sigma_i(s,u)^{'} du\right) ds + \sum_{i=1}^{N}\int_0^t \sigma_i(s,T) d\widetilde{W}_i(s).$$

The dynamics of bond prices under the probability measure Q is follows as:

$$dP(t,T) = P(t,T)\left[r(t)dt - \sum_{i=1}^{N}\int_t^T \sigma_i(t,u) du d\widetilde{W}_i(t)\right].$$

Example(Ho-Lee model): Lets suppose the process σ is a deterministic constant and we can write $\sigma(t,T) = \sigma =$ constant. Then $\alpha(t,T) = \sigma\int_t^T \sigma ds = \sigma^2(T-t)$ and using in forward rate dynamics,

$$\begin{aligned} df(t,T) &= \sigma^2(T-t)dt + \sigma d\widetilde{W}(t), \\ f(t,T) &= f(0,T) + \sigma^2 t(T - \frac{t}{2}) + \sigma\widetilde{W}(t). \end{aligned}$$

And the dynamics of short-interest rate is given by:

$$r(t) = f(t,t) = f(0,t) + \sigma^2\frac{t^2}{2} + \sigma\widetilde{W}(t).$$

The bond prices in term of forward rates are given by:

$$
\begin{aligned}
P(t,T) &= \exp\left(-\int_t^T f(t,u)du\right), \\
&= \exp\left(-\int_t^T f(0,u)du - \sigma^2 \int_t^T t(u-\frac{t}{2})du - \sigma\widetilde{W}(t)(T-t)\right),
\end{aligned}
$$

by Using $-\int_t^T f(0,u)du = \ln P(0,T) - \ln P(0,t)$ in above equation, we can write bond prices as:

$$
P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(-\sigma^2 tT\frac{(T-t)}{2} - \sigma\widetilde{W}(t)(T-t)\right).
$$

Here $\widetilde{W}$ is a 1-dimensional Q-Wiener process [10].

6.0 MULTI-FACTOR HJM MODEL FOR ENERGY MARKET

In this Chapter, we give the description of *Multi-Factor Heath-Jarrow-Morton* model for energy markets [4], presented by Ewa Broszkiewicz and Aleksander Weron. In their paper, they used the toolkit of interest rate theory (HJM model) and derived the explicit option pricing formula, analyzed the risk factors by using principle component analysis and calibrated the theoretical model to the empirical electricity market.

6.1 INTEREST RATE FORMULATION

Suppose that the time span of our energy market model is $[0, T^*]$ and that no arbitrage opportunities exists. The model can be viewed as a currency market with two currencies. The "domestic currency" has the MWh (Mega Watt hour) as the unit, and the Euro € plays the role of the "foreign currency". A fixed interest rate $r > 0$ is used in the Euro market. The "domestic" price at time $t \in [0, T^*]$ of 1 MWh to be delivered at time $T \in [t, T^*)$ (or more precisely in a very short time interval $[T, T + \Delta T]$ will be denoted by $p(t, T)$. Note that $p(t, T)$ behaves like the price process for zero coupon bonds. In particular $p(T, T) = 1$. The "exchange rate" between MWh and € is

$$\frac{N(t)}{e^{rt}}.$$

Thus this is the reciprocal of the Euro price of 1 MWh at time t for immediate delivery (or more precisely in a very short time interval $[t, t+\Delta t]$). Equivalently, one can buy $e^{-rt}N(t)$ MWh for 1 €.

By the First Fundamental Theorem of Asset pricing there exists a measure Q [3], for which the discounted process $Z(t,T) = p(t,T)N^{-1}(t)$ is the $Q-$martingale and power forward maturing at time T is given by $p(t,T) = N(t)E_Q(N^{-1}(T) \mid \mathcal{F}_t)$, where for all $t \in [0,T]$. Let $W(s), s \in [0,T]$ be a $p-$dimensional $Q-$Wiener process and according to Heath-Jarrow-Morton model [1,11], suppose that the forward rate dynamics $f(t,T)$ is given by a stochastic differential equation

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{p} \sigma_i(t,T)dW_i(t), \tag{6.1}$$

where $W(s)$, $s \in [0,T^*]$ is a $p-$dimensional $Q-$Wiener process, whereas $\alpha(\cdot,T)$ and $\sigma(\cdot,T)$ are adapted processes to filtration $\mathcal{F}_t$, satisfying

$$\int_0^t \alpha(u,t)du < \infty, \quad \int_0^t |\sigma(u,t)|^2 du < \infty \text{ for all } t \in [0,T].$$

The price process of forward contact is given by:

$$p(t,T) = \exp\left(-\int_t^T f(t,s)ds\right). \tag{6.2}$$

We define a second discounting process

$$\Lambda = \exp(-\int_0^t f(u,u)du). \tag{6.3}$$

By using this process Λ as a new numeraire, according to the First Fundamental Theorem of Asset pricing we can change from Q *measure* to a new equivalent probability martingale measure $\widetilde{Q}$. In fact

$$d\widetilde{Q} = \frac{N(0)\Lambda(T)}{N(T)\Lambda(0)}dQ,$$

and with respect to $\widetilde{Q}$, the discounted electricity forward prices defined as

$$\widetilde{p}(t,T) = p(t,T)\Lambda^{-1}(t) \quad t \in [0,T]. \tag{6.4}$$

and the discounted saving bank account defined as $\widetilde{N}(t) = N(t)\Lambda^{-1}(t)$ are a $\widetilde{Q}$ *martingales.* We will find the dynamics of discounted electricity forward prices and discounted saving bank account under the measure $\widetilde{Q}$. The forward rates $f(t,T)$ is given by:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \sum_{i=1}^{p} \int_0^t \sigma_i(s,T)dW_i(s), \tag{6.5}$$

where $W(t) = [W_1(t), ..., W_p(t)]$ is a $p-$dimensional $\widetilde{Q}-$Wiener process. By using Ito's formula, the differential equation of the discounted electricity forward prices are given as

$$\begin{aligned} d\widetilde{p}(t,T) &= d(p(t,T)\Lambda^{-1}(t)), \\ &= d\left(\frac{1}{\Lambda(t)}\right)p(t,T) + \frac{1}{\Lambda(t)}d\left(p(t,T)\right), \end{aligned}$$

by using the equation (6.2), (6.3) and (6.5), we have

$$d\widetilde{p}(t,T) = \widetilde{p}(t,T)\left(b(t,T) + \frac{1}{2}\sum_{i=1}^{p} s_i\,|(t,T)|^2\right)dt + \widetilde{p}(t,T)\sum_{i=1}^{p} s_i(t,T)dW_i(t). \tag{6.6}$$

Here $b(t,T) = -\int_t^T \alpha(t,u)du$ and $s_i(t,T) = -\int_t^T \sigma_i(t,u)du$.

From equation (6.4), we know that $\widetilde{p}(t,T)$ is a $\widetilde{Q}$ *martingale*. We summarize that $b(t,T) + \frac{1}{2}\sum_{i=1}^{p}|s_i(t,T)|^2 = 0$, and the process $\widetilde{p}(t,T)$ finally can be described as

$$d\widetilde{p}(t,T) = \widetilde{p}(t,T)\sum_{i=1}^{p} s_i(t,T)dW_i(t), \text{ where } s_i(t,T) = -\int_t^T \sigma_i(t,u)du. \tag{6.7}$$

The dynamics of discounted saving bank account can be described [4].

$$d\widetilde{N}(t) = \widetilde{N}(t)\sum_{i=1}^{p} \upsilon_i(t)dV_i(t) = \widetilde{N}(t)\sum_{i=1}^{p} \upsilon_i(t)(\rho dW_{i,1}(t) + \sqrt{1-\rho^2}dW_{i,2}(t)). \tag{6.8}$$

Here parameter υ is chosen to represent the deterministic volatility and $W_{i,1}(t) = (W_{i,1}(t),...,W_{p,1}(t))$ for all $t \in [0,T]$ and $W_{i,2}(t) = (W_{i,2}(t), ..., W_{p,2}(t))$ are independent p-dimensional Wiener processes and $\rho \in [-1,1]$ is the correlation parameter to capture the dependence between $\widetilde{p}(t,T)$ and $\widetilde{N}(t)$.

The dynamics of function α is uniquely determined by

$$\alpha(t,T) = \sum_{i=1}^{p} \sigma_i(t,T)\int_t^T \sigma_i(t,u)du.$$

Theorem 6. *Suppose that* $\sum_{i=1}^{p}[s_i(t,T) - \upsilon_i(t)]$ *is deterministic for all* $t \in [0,T]$ *and* $T \in [0,T^*]$, *then the European call option* C_t *at the time* $t \in [0,T^*]$ *and time to maturity* $T \in [t,T^*]$ *with strike price* K *in €, written on power forward contract with time to maturity* $U \in [T,T^*]$ *is given by the following price formula:*

$$C_t = P(t,U)\Phi(d_+) - \exp(-r(T-t))K\Phi(d_-), \tag{6.9}$$

where $\Phi(d)$ *is the normal distribution function and* $P(t,U) = \frac{\widetilde{p}(t,U)}{\exp(-rt)\widetilde{N}(t)} = \frac{p(t,U)}{\exp(-rt)N(t)}$ *is the € price at time* t *for the underlying forward contract and*

$$d_{\pm} = \frac{\ln(\frac{P(t,U)}{K}) + r(T-t) \pm \frac{1}{2}\Sigma^2}{\Sigma}$$

$$\Sigma^2 = \int_t^T \sum_{i=1}^{p} \left\| s_i(u,U) - \upsilon_i(u) \right\|^2 du \tag{6.10}$$

Where $s_i(t,T) = -\int_t^T \sigma_i(t,u)du$ [4].

6.2 HISTORICAL CALIBRATION OF THE MODEL

Let us now explain how we will extract the exact model ingredients $f(0,\cdot)$ and volatility σ from market data. The historical forward contract prices will be specified in $EURO$ and will satisfy

$$P(t,T) = \frac{\widetilde{p}(t,T)}{\exp(-rt)\widetilde{N}(t)} = \frac{p(t,T)}{\exp(-rt)N(t)},$$

where $\widetilde{N}(t)$ is defined in equation (6.8).

The forward rate curve $f(0,\cdot)$ will be derived from the initial observed EURO prices:

$$f(0,t) = -\frac{\partial}{\partial t}\ln p(0,t) = -\frac{\partial}{\partial t}\ln P(0,t).$$

Now we can discretize the equation (6.1) as follows. Let us fix a time scale $0 = t_0 < t_1 < t_2 ... < t_m$ where $t_{j+1} - t_j = \Delta t$, Δt is fixed time step and $t_0 = 0$ is the starting date for the forward rates in our sample. We will also look at contracts starting in our data at time points

$$t \in \{t_j = t_0 + j\Delta t : j = 1, ...m\} = \{t_1, ..., t_m\}.$$

The maturity dates are defined as

$$T \in \{t_j + k\Delta T : k = 1, ...p,\ j = 1, ..., m\ \}, \quad where\ \Delta T = T_{k+1} - T_k.$$

We define forward differences for f as $\Delta f(t, T_k) = f(t + \Delta T, T_k) - f(t, T_k)$, so we can write equation (6.1) as :

$$\Delta f(t, t + k\Delta T) = \alpha(t, t + k\Delta T)\Delta t + \sum_{i=1}^{p} \sigma_i(t, t + k\Delta T)\Delta W_i(t). \tag{6.11}$$

We will be assuming that functions α and σ are deterministic and depend only on time to maturity $\tau = T - t$, so

$$\alpha(t, T) = \alpha(T - t)\,, \quad \sigma_i(t, T) = \sigma_i(T - t).$$

The discrete description of the HJM stochastic differential equation becomes then

$$\Delta f(t, T) = \alpha(T - t)\Delta t + \sum_{i=1}^{p} \sigma_i(T - t)\Delta W_i(t),$$

where t and T belongs to the above finite sets of dates. From now on, for the sake of simplicity, we will assume that $\Delta t = \Delta T$.

Assume that we have a total m observations of d different variables listed in row vectors $x_1, x_2, ..., x_d$ and all of them have the dimension $(1 \times m)$.

$$\begin{aligned} X &= \begin{bmatrix} x_{1,1} & x_{2,1} & \cdots & x_{m,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,d} & x_{2,d} & \cdots & x_{m,d} \end{bmatrix} \\ &= \begin{bmatrix} \Delta f(t_1, t_1 + \Delta T) & \Delta f(t_2, t_2 + \Delta T) & \cdots & \Delta f(t_m, t_m + \Delta T) \\ \Delta f(t_1, t_1 + 2\Delta T) & \Delta f(t_2, t_2 + 2\Delta T) & \cdots & \Delta f(t_m, t_m + 2\Delta T) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta f(t_1, t_1 + d\Delta T) & \Delta f(t_2, t_2 + d\Delta T) & \cdots & \Delta f(t_m, t_m + d\Delta T) \end{bmatrix} \in \mathbb{R}^{d \times m} \end{aligned}$$

According to the assumptions that X is Gaussian with mean

$$E(X) = \Delta t \begin{bmatrix} \alpha(\Delta T) \\ \alpha(2\Delta T) \\ \vdots \\ \alpha(d\Delta T) \end{bmatrix} \in \mathbb{R}^d$$

Hence the natural estimator for the $\alpha(T-t)$ would be

$$\begin{bmatrix} \widehat{\alpha}(\Delta T) \\ \widehat{\alpha}(2\Delta T) \\ \vdots \\ \widehat{\alpha}(d\Delta T) \end{bmatrix} = \frac{1}{m\Delta T} \sum_{j=1}^{m} X_j$$

The principal component analysis is performed on the observation data matrix X and the corresponding covariance matrix of order d is denoted by Γ. The orthogonal decomposition of the covariance matrix is

$$\Gamma = BCB' \tag{6.12}$$

Where

$$B = [u_1, u_2, ..., u_d] = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1d} \\ u_{21} & u_{22} & \cdots & u_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ u_{d1} & u_{d2} & \cdots & u_{dd} \end{bmatrix}$$

and

$$C = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix}$$

C is a diagonal matrix, whose diagonal elements are eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d$ and B is a orthogonal matrix of order d whose rth column, u_r is the eigenvector corresponding to λ_r. $B^{'}$ is a transpose of B. Furthermore, we know that $Cov[X] = \Delta T \sigma \sigma^*$ where

$$\sigma = \begin{bmatrix} \sigma_1(\Delta T) & \sigma_2(\Delta T) & \cdots & \sigma_d(\Delta T) \\ \sigma_1(2\Delta T) & \sigma_2(2\Delta T) & \cdots & \sigma_d(2\Delta T) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1(d\Delta T) & \sigma_2(d\Delta T) & \cdots & \sigma_d(d\Delta T) \end{bmatrix} \in \mathbb{R}^{d \times d}$$

Moreover $\Gamma = (B\sqrt{C})(B\sqrt{C})^{'}$, where $\sqrt{C} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_d} \end{bmatrix}$

As a consequence, we get an natural estimator of $\sigma_i(T-t)$ directly from the eigenvalue decomposition as:

$$\widehat{\sigma} = \frac{B\sqrt{C}}{\sqrt{\Delta T}}. \tag{6.13}$$

In what follows, we will assume for simplicity that $p = d$.

Now our second step is to estimate the volatility functions $\upsilon_i(t)$ from historical data. To obtain a estimator of functions $\upsilon_i(t)$, let us suppose that we have historical data of forward contracts in the form

$$[P(t_j, T_1), P(t_j, T_2), ..., P(t_j, T_n)], \text{ where } j = 1, ..., m$$

We start with the discretization of equation (6.8), as

$$d\widetilde{N}(t) = \widetilde{N}(t) \sum_{i=1}^{p} \upsilon_i(t) dV_i(t).$$

Let us take the left side first of above equation,

$$\frac{d\widetilde{N}(t)}{\widetilde{N}(t)} = \sum_{i=1}^{p} \upsilon_i(t) dV_i(t),$$

we can write,

$$\frac{\Delta\widetilde{N}(t)}{\widetilde{N}(t)} = \frac{\Delta\widetilde{N}(t_j + \Delta T) - \widetilde{N}(t_j)}{\widetilde{N}(t_j)} = \frac{\Delta\widetilde{N}(t_j + \Delta T)}{\widetilde{N}(t_j)} - 1. \tag{6.14}$$

Since

$$P(t,T) = \frac{\widetilde{p}(t,T)}{\exp(-rt)\widetilde{N}(t)} = \widetilde{N}(t) = \frac{\widetilde{p}(t,T)}{\exp(-rt)P(t,T)}.$$

So, by discretization,

$$\widetilde{N}(t_j) = \frac{\widetilde{p}(t_j,T_k)}{\exp(-rt_j)P(t_j,T_k)} \text{ and } \widetilde{N}(t_j+\Delta T) = \frac{\widetilde{p}(t_j+\Delta T,T_k)}{\exp(-r(t_j+\Delta T))P(t_j+\Delta T,T_k)},$$

putting these values in equation (6.14)

$$\frac{\Delta\widetilde{N}(t)}{\widetilde{N}(t)} + 1 = \frac{\widetilde{p}(t_j+\Delta T,T_k)\exp(-rt_j)P(t_j,T_k)}{\exp(-r(t_j+\Delta T))P(t_j+\Delta T,T_k)\widetilde{p}(t_j,T_k)},$$

$$= \frac{\widetilde{p}(t_{j+1},T_k)P(t_j,T_k)}{\exp(-r\Delta T)P(t_{j+1},T_k)\widetilde{p}(t_j,T_k)}.$$

The solution of equation (6.7) is

$$\widetilde{p}(t,T) = \widetilde{p}(0,T)\exp\left(-\frac{1}{2}\sum_{i=1}^{p}\int_0^t s_i^2(u,T)du + \sum_{i=1}^{p}\int_0^t s_i(u,T)dW_{i,1}(t)\right).$$

It can be written as

$$\ln\frac{\widetilde{p}(t_{j+1},T_k)}{\widetilde{p}(t_j,T_k)} = -\frac{1}{2}\sum_{i=1}^{p}\int_{t_j}^{t_{j+1}} s_i^2(u,T_k)du + \sum_{i=1}^{p}\int_{t_j}^{t_{j+1}} s_i(u,T_k)dW_{i,1}(t),$$

using the Euler scheme for approximating the integrals,

$$\ln\frac{\widetilde{p}(t_{j+1},T_k)}{\widetilde{p}(t_j,T_k)} \approx -\frac{1}{2}\sum_{i=1}^{p} s_i^2(t_j,T_k)\Delta T + \sum_{i=1}^{p} s_i(t_j,T_k)\Delta W_{i,1}(t).$$

Now putting the value of $s_i(t,T) = -\int_t^T \sigma_i(t,u)du$ in above equation,

$$\approx \frac{1}{2}\sum_{i=1}^{p}[\int_{t_j}^{T_k}\sigma_i(t_j,u)du]^2\Delta T - \sum_{i=1}^{p}\int_{t_j}^{T_k}\sigma_i(t_j,u)\Delta W_{i,1}(t)du.$$

Combine equation (6.11) with above equation,

$$\approx \frac{1}{2}\sum_{i=1}^{p}[\int_{t_j}^{T_k}\sigma_i(t_j,u)du]^2\Delta T - \int_{t_j}^{T_k}(\Delta f(t_j,u) - \alpha(t_j,u)\Delta T)du$$

Now taking the Natural estimators of function $\alpha(T-t)$ and $\sigma_i(T-t)$

$$\approx \frac{1}{2}\sum_{i=1}^{p}[\sum_{l=1}^{k}\widehat{\sigma}_i(l\Delta T)\Delta T]^2\Delta T - \sum_{l=1}^{k}(\Delta f(t_j,T_l) - \widehat{\alpha}(l\Delta T)\Delta T)\Delta T.$$

From equation (6.8), we conclude that $y_k^j = 1 + \sum_{i=1}^{p} \upsilon_i(t)\Delta V_t^i$ and discretized version of equation (6.8) is

$$Y_k = \begin{bmatrix} y_{1,k} \\ y_{2,k} \\ \vdots \\ y_{m,k} \end{bmatrix} = \begin{bmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,n} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{P(t_1,T_1)a_{1,1}}{P(t_2,T_1)\exp(-r\Delta T)} & \frac{P(t_1,T_2)a_{1,2}}{P(t_2,T_2)\exp(-r\Delta T)} & \cdots & \frac{P(t_1,T_n)a_{1,n}}{P(t_2,T_n)\exp(-r\Delta T)} \\ \frac{P(t_2,T_1)a_{2,1}}{P(t_3,T_1)\exp(-r\Delta T)} & \frac{P(t_2,T_2)a_{2,2}}{P(t_3,T_2)\exp(-r\Delta T)} & \cdots & \frac{P(t_1,T_n)a_{2,n}}{P(t_3,T_n)\exp(-r\Delta T)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{P(t_{m-1},T_1)a_{m,1}}{P(t_m,T_1)\exp(-r\Delta T)} & \frac{P(t_{m-1},T_2)a_{m,2}}{P(t_m,T_2)\exp(-r\Delta T)} & \cdots & \frac{P(t_{m-1},T_n)a_{m,n}}{P(t_m,T_n)\exp(-r\Delta T)} \end{bmatrix} \tag{6.15}$$

and

$$a_{j,k} = \exp\left(\frac{1}{2}\sum_{i=1}^{p}[\sum_{l=1}^{k}\widehat{\sigma}_i(l\Delta T)\Delta T]^2\Delta T - \sum_{l=1}^{k}(\Delta f(t_j,T_l) - \widehat{\alpha}(l\Delta T)\Delta T)\Delta T\right), \tag{6.16}$$

where $i, l = 1, ..., p, k = 1, .., n.$

For every k vector, Y_k is normally distributed and by using PCA, we can find the covariance matrix Σ of a matrix Y. The covariance matrix can be decomposed in $\Sigma = W\Lambda W'$, where W is a matrix of eigenvectors and a diagonal of matrix Λ is eigenvalues. By using these eigenvectors, we can calculate volatility functions for n components as:

$$\begin{bmatrix} \widehat{\upsilon}_i(1\Delta T) \\ \widehat{\upsilon}_i(2\Delta T) \\ \vdots \\ \widehat{\upsilon}_i(n\Delta T) \end{bmatrix} = \frac{W\sqrt{\Lambda}}{\sqrt{\Delta T}} \tag{6.17}$$

The moment estimator $\widehat{\rho}$ of the correlation parameter is given by the formula

$$\widehat{\rho} = \frac{1}{mp}\sum_{k=1}^{p}\sum_{j=0}^{m-1}\frac{(y_{j,k}-1)(a_{j,k}-1)}{\sum_{i=1}^{p} s_i(t_j,T_k)\upsilon_i(t_j)\Delta T}, \tag{6.18}$$

where

$$y_{j,k} = \frac{P(t_j,T_k)a_{j,k}}{P(t_{j+1},T_k)\exp(-r\Delta T)},$$

$$a_{j,k} = \exp\left(\frac{1}{2}\sum_{i=1}^{p}[\sum_{l=1}^{k}\widehat{\sigma}_i(l\Delta T)\Delta T]^2\Delta T - \sum_{l=1}^{k}(\Delta f(t_j, T_l) - \widehat{\alpha}(l\Delta T)\Delta T)\Delta T\right),$$

and

$$s_i(t_j, T_k) = -\sum_{l=1}^{k}\widehat{\sigma}_i(l\Delta T)\Delta T.$$

6.3 EMPIRICAL RESULTS

Let us estimate functions $\sigma_i(T-t), \upsilon_i(t)$ and ρ from Nord Pool data. We consider 6 forward contracts with monthly delivery which are traded six months before maturity. Our observation starts from December 1, 2005 to April 1, 2010 and the data consists of 7×52 rows and columns. We assume that $\Delta T = \frac{1}{12}$, which is equal to one month and we take forward prices of 1st day of each month. All historical contract prices $P(t, T)$ are given in EURO.

$$[P(t_j, T_0), P(t_j, T_1), P(t_j, T_2), P(t_j, T_3), P(t_j, T_4), P(t_j, T_5), P(t_j, T_6)]$$

Where $t = 0 = t_0 < t_1 < t_2 ... < t_m$, $T = 0 < T_1 < T_2 < T_3 < T_4 < T_5 < T_6$ and $P(t_j, T_0)$ is the price of the previous matured monthly contract at the date of maturity. We need this price $P(t_j, T_0)$ because when we calculate the forward rates from the equation

$$f(t, T) = -\frac{\Delta \ln P(t, T)}{\Delta T} = -\frac{\ln P(t_j, T_i) - \ln P(t_j, T_{i+1})}{\Delta T}. \quad (6.19)$$

It decreases the dimension of the system and we need our system to be 6 dimensional because we have 6 monthly forward contracts. After using formula (6.19) and $\Delta f(t, T_k) = f(t+\Delta T, T_k) - f(t, T_k)$, we get our data matrix X which consists of 6×51rows and columns.

$$X_j = \begin{bmatrix} \Delta f(t_j, T_1) \\ \Delta f(t_j, T_2) \\ \Delta f(t_j, T_3) \\ \Delta f(t_j, T_4) \\ \Delta f(t_j, T_5) \\ \Delta f(t_j, T_6) \end{bmatrix} = \begin{bmatrix} 0.9436 & 0.0860 & \cdots & \cdots & -2.8718 \\ -0.0389 & 0.1469 & \cdots & \cdots & -3.7067 \\ -0.0108 & -0.3160 & \cdots & \cdots & 0.0605 \\ 0.1916 & 0.2226 & \cdots & \cdots & -0.9573 \\ 0.2286 & -1.4711 & \cdots & \cdots & 0.0309 \\ -1.9123 & 0.8836 & \cdots & \cdots & 0.3397 \end{bmatrix}$$

The principal component analysis is performed on the observation data matrix X and corresponding covariance matrix of order 6 is denoted by Γ. By decomposition our covariance matrix Γ, we get eigenvectors and eigenvalues by equation (6.12).

$$\Gamma = \begin{bmatrix} 1.3179 & 0.3008 & 0.1951 & 0.1051 & -0.3166 & -0.0563 \\ 0.3008 & 1.1158 & -0.4243 & 0.1376 & -0.0499 & -0.0542 \\ 0.1951 & -0.4243 & 1.0992 & -0.2314 & 0.2340 & -0.0935 \\ 0.1051 & 0.1376 & -0.2314 & 0.6482 & -0.2819 & 0.1626 \\ -0.3166 & -0.0499 & 0.2340 & -0.2819 & 0.9512 & -0.1926 \\ -0.0563 & -0.0542 & -0.0935 & 0.1626 & -0.1926 & 0.6208 \end{bmatrix}$$

$$B = [u_1, u_2, ..., u_p] = \begin{bmatrix} -0.4722 & 0.7519 & -0.0868 & 0.0520 & 0.0292 & -0.4479 \\ -0.5398 & -0.1280 & -0.6185 & -0.2970 & 0.1415 & 0.4489 \\ 0.4366 & 0.6239 & 0.0367 & -0.3599 & -0.0608 & 0.5344 \\ -0.3078 & -0.1030 & 0.2891 & -0.3894 & -0.8121 & -0.0027 \\ 0.4367 & -0.0680 & -0.5995 & -0.3821 & -0.1854 & -0.5147 \\ -0.0985 & -0.1172 & 0.4068 & -0.6943 & 0.5307 & -0.2178 \end{bmatrix}$$

$$C = \begin{bmatrix} 1.8310 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1.4517 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1.1062 & 0 & \cdots & \vdots \\ \vdots & \vdots & 0 & 0.5386 & 0 & \vdots \\ \vdots & \vdots & \vdots & 0 & 0.4325 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0.3932 \end{bmatrix}$$

We have estimated values of multifactor parameter $\sigma_i(T-t)$ by using equation (6.13). The estimated value of function v is calculated by equation (6.17) and we observed historical data of forward contracts in the form $P(t_j, T_1)$, $P(t_j, T_2)$, ..., $P(t_j, T_6)$ where $j = 1, ..., m$ and we took forward prices of $1st$ day of each month. The estimated value of correlation parameter is $\rho = -0.3979$ and it is calculated by equation (6.18). The results are give in (Table 1) and (Table 2).

$$\begin{bmatrix} & i=1 & i=2 & i=3 & i=4 & i=5 & i=6 \\ \sigma_i(1\Delta T) & -2.2132 & 3.1382 & -0.3164 & 0.1321 & 0.0665 & -0.9729 \\ \sigma_i(2\Delta T) & -2.5301 & -0.5340 & -2.2536 & -0.7550 & 0.3223 & 0.9750 \\ \sigma_i(3\Delta T) & 2.0464 & 2.6041 & 0.1336 & -0.9149 & -0.1386 & 1.1607 \\ \sigma_i(4\Delta T) & -1.4426 & -0.4301 & 1.0534 & -0.9898 & -1.8501 & -0.0059 \\ \sigma_i(5\Delta T) & 2.0471 & -0.2837 & -2.1843 & -0.9712 & -0.4224 & -1.1179 \\ \sigma_i(6\Delta T) & -0.4616 & -0.4892 & 1.4823 & -1.7650 & 1.2090 & -0.4731 \end{bmatrix} \quad \text{(Table 1)}$$

$$\begin{bmatrix} & i=1 & i=2 & i=3 & i=4 & i=5 & i=6 \\ v_i(1\Delta T) & -0.7225 & 1.655e^{-8} & -9.873e^{-10} & 8.499e^{-10} & -2.485e^{-9} & -4.008e^{-10} \\ v_i(2\Delta T) & -0.7229 & -4.883e^{-9} & -3.711e^{-9} & 7.856e^{-9} & 8.757e^{-10} & 7.575e^{-10} \\ v_i(3\Delta T) & -0.7244 & 2.692e^{-9} & -2.185e^{-9} & -3.071e^{-9} & 5.734e^{-9} & -7.700e^{-10} \\ v_i(4\Delta T) & -0.7266 & -5.970e^{-10} & 8.183e^{-9} & -1.199e^{-9} & 3.208e^{-10} & 2.427e^{-9} \\ v_i(5\Delta T) & -0.7296 & -6.987e^{-9} & 5.074e^{-9} & 4.665e^{-10} & -1.516e^{-9} & -2.790e^{-9} \\ v_i(6\Delta T) & -0.7332 & -7.088e^{-9} & -6.368e^{-9} & -4.824e^{-9} & -2.889e^{-9} & 7.810e^{-10} \end{bmatrix}$$

(Table 2)

Now we calculate the option valuation. Since the volatilities functions are deterministic, EURO price of a European call is calculate from equation (6.9) and Σ^2 from equation (6.10) with strike price $K = 42$, Risk-free interest rate $r = 0.025$ and $P(t,T) = 45$.

Σ^2	*Time to maturity*	*Call option price*
0.4473	1 month=1/12	6.3306
0.3478	2 month=2/12	4.6889
0.3036	3 month=3/12	3.9825
0.2118	4 month=4/12	2.4887
0.1899	5 month=5/12	2.1594
0.1810	6 month=6/12	2.0464

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