

Measures of Freedom of Choice

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Abstract

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This thesis studies the problem of measuring freedom of choice. It analyzes the concept of freedom of choice, discusses conditions that a measure should satisfy, and introduces a new class of measures that uniquely satisfy ten proposed conditions. The study uses a decision-theoretical model to represent situations of choice and a metric space model to represent differences between options.

The first part of the thesis analyzes the concept of freedom of choice. Different conceptions of freedom of choice are categorized into evaluative and non-evaluative, as well as preference-dependent and preference-independent kinds. The main focus is on the three conceptions of freedom of choice as cardinality of choice sets, representativeness of the universal set, and diversity of options, as well as the three conceptions of freedom of rational choice, freedom of eligible choice, and freedom of evaluated choice.

The second part discusses the conceptions, together with conditions for a measure and a variety of measures proposed in the literature. The discussion mostly focuses on preference-independent conceptions of freedom of choice, in particular the diversity conception. Different conceptions of diversity are discussed, as well as properties that could affect diversity, such as the cardinality of options, the differences between the options, and the distribution of differences between the options. As a result, the diversity conception is accepted as the proper explication of the concept of freedom of choice. In addition, eight conditions for a measure are accepted. The conditions concern domain-insensitivity, strict monotonicity, no-choice situations, dominance of differences, evenness, symmetry, spread of options, and limited function growth. None of the previously proposed measures satisfy all of these conditions.

The third part concerns the construction of a ratio-scale measure that satisfies the accepted conditions. Two conditions are added regarding scale-independence and function growth proportional to cardinality. Lastly, it is shown that only one class of measures satisfy all ten conditions, given an additional assumption that the measures should be analytic functions with non-zero partial derivatives with respect to some function of the differences. These measures are introduced as the *Ratio root measures*.

Keywords: Freedom of Choice, Measure, Choice Set, Diversity, Cardinality, Evenness, Decision Theory, Metric Space

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would spend the next week or so trying to solve it. Usually, the solution would involve functional analysis.

There are several of my father's ideas included in this thesis. Most importantly, the *Ratio root measures*, which are our joint construction; my father proposing the denominator and me proposing the numerator. The first and last proofs of the appendix are also due to my father. The first proof is an answer to my question whether the elements of a set always are positioned on a straight line when the diameter of the set is close to the supremum (they are). The last proof is the uniqueness proof for the *Ratio root measures*. To construct the proof, my father suggested the *Proportional growth condition*. All of these contributions have been very important.

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Part 1: Introducing Freedom of Choice

Prologue

At the Philosophy Department at Uppsala University there is a meeting concerning the student curriculum. It is late in the day and the meeting is almost over. A list of courses has been considered by the Board. There are courses in epistemology, metaphysics, ethics, logic, the philosophy of language and the philosophy of science. There are also courses on the history of philosophy. All philosophers agree that the list contains the most important courses. No one has any complaints regarding the course requirements or the literature. The Board seems prepared to accept the list as the next term's curriculum. But then one philosopher speaks up.

The suggested curriculum is unacceptable, he says. Not because it lacks any important courses, nor because there is something wrong with the requirements or the literature. But all the courses are obligatory for the student. The curriculum does not offer any freedom of choice.

Freedom of choice was a matter the Board had not considered. It immediately raises a concerned discussion. Should the Department support freedom of choice for its students? Is there any value in freedom of choice that justifies such support? One philosopher argues that freedom of choice should be supported since freedom of choice is good in itself. Another philosopher agrees that freedom of choice should be supported, but only because it is instrumentally good. A third philosopher disagrees with both of his colleagues, claiming that freedom of choice should not be supported since it is intrinsically worthless and mostly instrumentally neutral or bad. The philosophers try to convince their colleagues with various arguments, but no one is convinced before the discussion is paused. This time it is interrupted by the Head of the Department.

Freedom of choice may very well be good, he says, but the Department cannot afford it. No matter how valuable freedom may be, it cannot be as valuable as good finances. This should put an end to the discussion.

It does not. The discussion continues with an argument concerning lexically superior values, followed by an argument concerning incommensurable values. Before any conclusion is reached, there is yet another interruption. A fifth philosopher declares that she has an unusual solution. Freedom of choice may be increased without extra expenditures, she says. What are needed are courses that no student would choose. The students get their choice while the Department keeps its money. This should be the perfect compromise.

All the philosophers (except the third one) are happy with the solution and start to think about courses that no student would choose. “The Phenomenology of Unconsciousness”, says one. “Indecision Theory”, says another. “A course on my research”, says a third. But suddenly the first philosopher expresses some doubt about the endeavor. Is it really possible to increase freedom of choice by offering options that no one would choose? Does not freedom require choice-worthy options?

This question leads into a difficult discussion. One philosopher argues that it is possible to increase freedom of choice with options that no one would choose. The only thing that matters is that the options are available. Another philosopher argues for the opposite view. It is not sufficient that the options are available. It is also necessary that the options are good. A third philosopher says that the matter is trivial and can easily be solved by definition. The important thing is not whether additional courses increase freedom of choice, but whether additional courses increase the value of the curriculum. If freedom of choice is valuable in itself and more freedom of choice is always better than less, then the value of the curriculum would increase with additional courses. But if freedom of choice only has instrumental value, it is harder to assess the situation. The students might get annoyed by the additional courses, and this would be instrumentally bad. Or the students might be amused by the additional courses, and this would be instrumentally good. The possible consequences are so many that the actual consequences are hard to assess. No general conclusion can therefore be drawn.

After this contribution to the discussion, there is a pause. Then the fifth philosopher remarks that she does not know exactly what ‘freedom of choice’ means. The other philosophers admit that they do not know this either. All of the philosophers agree that the matter is in need of further investigation. This thesis is that further investigation.

Chapter 1: Introduction

This study concerns the nature of freedom of choice. It attempts to answer three questions: What are the conditions for a person having freedom of choice? When does a person have more freedom of choice than another person? How should freedom of choice be measured?

The study mainly concerns the third question. It attempts to determine what conditions a ratio scale measure should satisfy and what kind of measure satisfies the conditions.

As a result of the study, I shall propose that a person has freedom of choice if and only if he has at least two options, and that a person has more freedom of choice than another person if and only if the options of the first person are more diverse. I shall further propose that a ratio scale measure of freedom of choice should satisfy ten conditions. There is only one class of measures that satisfy all of them, assuming that the measures are analytic functions with non-zero partial derivatives with respect to some function of the differences between the options. They are measures of the following form:

Ratio root measures:

i) For $n < 2$, $F(A, d) = 0$.

ii) For $n \geq 2$, $F(A, d) = \frac{1}{n-1} \sum_{i=1, i \neq j}^n \sum_{j=1}^n (d(x_i, x_j))^r$, with $\frac{1}{2} \leq r < 1$.

(Here, F is a function from a finite metric space to \mathbf{R} , A is a choice set, d is a distance function representing differences among options, n is the cardinality of A , and x_i, x_j are the options in A .) Thus, I shall propose that the *Ratio root measures* should be used to measure freedom of choice.

More precisely, the study will proceed as follows: In the first part of the thesis I introduce the topic of freedom of choice. Chapter 1 introduces the problem, gives some background and presents the decision-theoretical model that shall be used for the discussion. Chapter 2 discusses the concept of freedom of choice. A distinction is made between the concept of freedom of choice and conceptions of freedom of choice, where the conceptions are regarded as explications of the concept. There is a further distinction between evaluative and non-evaluative conceptions, as well as between preference-independent and preference-dependent ones. Several conceptions

of freedom and choice are then introduced and combined to help to specify the particular concept of freedom of choice that shall be discussed in the thesis, referring to a set of options that a person may have. Next, I specify categorical, comparative and complete freedom of choice. The comparative conceptions that are discussed in the thesis are introduced. There are the three preference-independent conceptions of freedom of choice as cardinality of choice sets, representativeness of the universal set and diversity of options, as well as the three preference-dependent conceptions of freedom of rational choice, freedom of eligible choice and freedom of evaluated choice. The last three conceptions may be combined with the first three, resulting in nine preference-dependent conceptions of freedom of choice. Chapter 3 concerns freedom of choice as a measurable property, and presents different methods for constructing a measure. Chapter 4 discusses different models of choice sets that may be used as a basis for a measure of freedom of choice.

In the second part of the thesis I discuss the different conceptions of freedom of choice, together with conditions for a measure, and previously proposed measures. Chapter 5 presents the model that is going to be used in the thesis to represent the relevant properties of choice sets, a combination of a multi-preference decision-theoretical model and a metric space model, which is used to represent differences between the options.

Chapter 6 investigates the cardinality conception of freedom of choice, as well as conditions for a cardinality measure and preference-dependent versions of the cardinality measure. The chapter is, in part, based on the work of Pattanaik and Xu (1990). They axiomatize a cardinality measure by using three conditions for a measure of freedom of choice, an *Indifference between no-choice situations condition*, stating that all singleton sets offer an equal amount of freedom of choice, a *Limited strict monotonicity condition*, stating that any set of two options offers more freedom of choice than a singleton set, and an *Independence condition*, stating that the addition of the same option to two different sets does not change the ranking of the sets. The first two conditions are judged as reasonable, while the third condition is not. The cardinality conception itself is judged as too simplistic as an explication of the concept of freedom of choice. Three preference-dependent cardinality conceptions are then discussed, together with some alternative conditions. The first conception is freedom of rational choice, the idea that freedom of choice depends only on the options that may rationally be chosen. A measure by Pattanaik and Xu (1998) is discussed in some detail. The measure fails to satisfy several reasonable conditions, most importantly a *Weak monotonicity condition*, which states that freedom of choice never decreases when an option is added to a set. The second conception is freedom of eligible choice, the idea that freedom of choice depends only on the options that are sufficiently valuable. Variants of this conception are discussed, with focus on a lexical measure constructed by Romero-Medina (2001). The measures

are all judged as inadequate. The third conception is freedom of evaluated choice, the idea that freedom of choice depends on preferences in such a way that it may be measured partly by the use of utility functions. Two versions of this conception are discussed, and both are abandoned for their counterintuitive implications, such as violating the *Weak monotonicity condition*.

Chapter 7 investigates the representative conception of freedom of choice. It focuses on the first (and only) measure that is proposed to capture the conception, the *Expected compromise measure*, by Gustafsson (2010). It also discusses his *Domain-sensitivity condition*, which states that the ranking of two sets in terms of freedom of choice depends on the identity of the relevant universal set. This condition is judged as unreasonable, together with the measure and the conception itself.

Chapter 8 investigates the diversity conception of freedom of choice. It begins by discussing some diversity conceptions that seem inadequate as conceptions of freedom of choice. The following chapters concern other conceptions of diversity, as well as conditions for a measure, and previous proposals for measures. Chapter 9 is devoted to conditions that concern how cardinality affects diversity, while the following chapters are devoted to conditions regarding the effects of magnitude of differences (Chapter 10), distribution of a sum of differences among individual differences (Chapter 11), distribution of a sum of differences among individual options (Chapter 12), and distribution of individual differences among individual options (same chapter). Chapter 13 concerns the question as to how the different diversity-relevant properties should be weighed against one another. It ends with a list of eight reasonable conditions for a measure of freedom of choice. *Domain-insensitivity* states that the ordering of choice sets in terms of freedom of choice is independent of the identity of a relevant universal set. *Strict monotonicity* states that freedom of choice increases with the addition of an option. *No freedom of choice* states that singleton sets and the empty set offer no freedom of choice. *Option dominance* (roughly) states that a set with more different options offers more freedom of choice than a set with less different options. *Limited evenness* (roughly) states that a set with more equal differences between the options offers more freedom of choice than a set with less equal differences. *Symmetry* states that the measure is insensitive to how the individual differences are distributed among the individual options. *Spread* states that if two sets have the same total sum of differences and are equally even, the set with lesser cardinality offers more freedom of choice. *Limited growth* states that the addition of an option to a set does not increase the value of the measure by more than the value given by the measure for the set of the added option and the most different option included in the set. The six diversity chapters also include a large number of previously proposed measures. None of the measures that are discussed in the six chapters satisfy all of the eight conditions. The

search for an adequate measure of freedom of choice thus continues in the following chapters.

In the third part of the thesis I focus on the construction of an adequate measure of freedom of choice and present my own proposal for a measure. Chapter 14 discusses derived measures of freedom of choice. These are measures that aggregate individual measures of relevant properties. The chapter introduces an additional *Scale-independence condition*, which states that a measure should be scale-independent. The discovery of a *Ratio measure* that satisfies eight of the nine conditions is the result of this chapter. Chapter 15 discusses additively separable strictly concave measures. These are additive functions of some strictly concave function of the distance function. The reason to take an interest in these functions is that they satisfy the *Limited evenness condition*. The discovery of a class of *Root measures* that satisfy seven of the nine conditions is the result of this chapter. Chapter 16 combines the ratio measure and the root measures into a new class of *Ratio root measures*. These measures are shown to satisfy all of the nine conditions for a measure of freedom of choice. To axiomatize the measure, an additional condition is added. It is the *Proportional growth condition*, which states that when the number of options is n , and the pairwise differences between the options are 1, the measure takes on a value of n . The chapter concludes with a number of objections to the measures, which I discuss and try to answer.

Chapter 17 contains a summary and some conclusions that can be drawn from the work in this thesis. The appendix contains proofs, a list of accepted conditions, a summary of conflicting conditions, and a table showing whether or not each one of the conditions are satisfied by the different measures discussed in the thesis.

1.1 History

In the sense that everyone must choose, freedom of choice is a subject of general concern. This is reflected in the academic world, where discussions of freedom of choice occur in many different areas, such as metaphysics, ethics, political philosophy, psychology, economics, decision theory, and marketing science.

There is a variety of conceptions used in these areas, but they have a common core. It is generally agreed that a person has freedom of choice in a situation, only if he has at least two options to choose among. Although the reasons to take an interest in freedom of choice vary, the concerns are interrelated through the different areas. In metaphysics a main concern is whether a person can have freedom of will, which is the ability to choose any of the options that the person in fact never chooses. Here, freedom of choice is discussed as a necessary condition for freedom of will. The answer

to the metaphysical question is also relevant to ethics, where a main concern is whether freedom of will is a necessary condition for moral responsibility. Ethics is also concerned with the question whether, and to what degree, freedom of choice is valuable. The answer to this question is also relevant for political philosophy, which is concerned with the question whether freedom of choice should be supported by society. An answer to the political question may require studies in history, sociology, anthropology, and psychology. In particular in psychology there are many studies concerning the effects of freedom of choice. These studies are of interest to marketing science, which studies the effects of freedom of choice on consumer behavior. These studies are also of interest to economics, which studies the effects of freedom of choice on economic growth. Through the area of microeconomics, economics is connected to decision theory, which is another area that studies freedom of choice. The main interest here is the nature of rational choice to which freedom of choice is a necessary condition. But decision theory also studies freedom of choice in itself. It studies whether freedom of choice could affect the value of options and how choice sets may be ranked in terms of freedom of choice. The discussions regarding freedom of choice in political philosophy and decision theory merge in the area of social choice.

1.1.1 Discussions of the Nature of Freedom of Choice

Even though freedom of choice is widely discussed, it is seldom discussed by itself. There are just a few essays that directly concern the nature of freedom of choice. Among them are essays by Dan-Cohen (1992), Sen (1998), and Carter (2004).

The discussion of measures of freedom of choice is more developed. It may originate in decision theory, although a related discussion occurs in political science, concerning measures of freedom. Part of the discussion dates back to an essay by Koopmans in which he discusses preference rankings of choice sets that depend on preferences for future freedom of choice (1990). Koopmans's essay has inspired other rankings of choice sets influenced by preferences for freedom of choice; for example by Kreps (1979) and Arrow (1995). Suppes (1987) may be the first author in this tradition to rank sets directly in terms of freedom of choice. He is followed by Pattanaik and Xu (1990) who propose another ranking and by Sen (1991) who proposes a third. The last two essays have been especially influential and are cited in almost all subsequent essays concerning measures of freedom of choice.

In this thesis, I shall propose that measures of diversity may be used as measures of freedom of choice. This increases the number of predecessors greatly. Different conceptions of diversity occur in many areas, together with a great variety of measures. In statistics, diversity is captured by the concepts

of range, variance, standard deviation and mean absolute deviation. In applied mathematics, diversity is captured by the concepts of majorization and entropy. The last concept is also used in physics, information theory and biology, where variants of entropy measures go under the name of ‘diversity indices’.

1.1.2 Discussions on the Effects of Freedom of Choice

An important use of a measure of freedom of choice would be to study the effects of freedom of choice. In fact, many such studies have already been done, partly within psychology, and partly within marketing science. In the majority of these studies, freedom of choice is regarded as a function of the cardinality of choice sets, or the variety of choice sets, or both. As the number of articles in this area is vast, I shall only mention a few of them.

First, there are studies that concern how a person is affected by choosing versus not choosing (such as Langer and Rodin (1976), Botti and Iyengar (2004), and Botti and McGill (2006)). Second, there are studies that concern how a chooser is affected by the number of options (such as Kiesler (1966), Fromkin et al. (1975), Clark et al. (1977), Iyengar and Leppar (2000), Hoffrage and White (2009), and Hogarth and Reutskaja (2009)). Third, there are studies that concern how a chooser is affected by the variety of options (such as Faison (1977), Shugan (1980), McAllister and Pessemier (1982), Kahn (1995), Read and Loewenstein (1995), Lehmann, (1998), Hoch et al. (1999), Kahn et al. (1999), Drolet and Kim (2003), and Gourville and Souman (2005)). Fourth, there are studies that concern how a chooser is affected both by the number and the variety of options (such as Malhotra (1982), Lehmann and Kahnemann (1991), and Dhar (1997)). All these studies are relevant for this thesis, but only in the sense that they exemplify the use of different conceptions in order to study the effects of freedom of choice.

1.1.3 Discussions on the Value of Freedom of Choice

The possible value of freedom of choice is more widely discussed than its nature. Judging from the literature, very few philosophers endorse the idea that freedom of choice may be valuable for its own sake (among these we may note Feinberg (1980), Jones and Sugden (1982), Sen (1988) and Pattanaik and Xu (2000b)). There seem to be more philosophers who argue against the idea (most notably Oppenheim (1961), Warnock (1967), Dworkin (1982), Kymlicka (1990) and Dowding (1992)). However, the idea that freedom of choice has instrumental value is commonly held among philosophers. There are a great number of proposals as to why freedom of choice has instrumental value. It has been proposed that freedom of choice may be good because it is instrumental for the experience of the process of

choice, the choice of a favored option, the rejection of a non-favored option, the increased esteem of a chosen option, the making of a significant choice, the exercise of autonomy, the possibility of responsibility, the expression of self and the development of useful personal traits. Similarly, it has been proposed that freedom of choice may be bad because it is instrumental for all of the phenomena mentioned above, as well as the choice of a disfavored option, the rejection of a favored option, the decreased esteem of a chosen option, the uncertainty of making a correct choice, and the risk of behaving like Buridan's ass. (For more detailed discussions see, for example, Mill (1859), Jones and Sugden (1982), Dworkin (1988), Scanlon (1988), Dowding (1992), Mills (1998), Baharad and Nitzan (2000), Sugden (2003), Schwartz (2004), and Botti and Iyengar (2006).)

In addition to the discussion regarding the value of freedom of choice, there is also a discussion regarding how the value of freedom of choice should be measured. Part of the discussion concerns the rankings of choice sets in terms of the value of the most preferred option, which is affected by freedom of choice (see Gravel (1994), and Baharad and Nitzan (2000)). Yet another part concerns the rankings of choice sets when the chooser has preferences for freedom of choice as well as preferences for options (see Bossert, Pattanaik and Xu (1994), Puppe (1995, 1996), Bossert (1997), Gravel (1998), Baharad and Nitzan (2000) and Erlander (2005)). A third part concerns rankings of choice sets in terms of instrumental preferences for freedom of choice, due to a lack of information about future preferences or future choice sets (see Koopmans (1964), Kreps (1979), Lehmann and Kahn, (1991), Arrow (1995), Arlegi and Nieto (2001), and Sugden (2007)).

Rankings of choice sets in terms of value and in terms of freedom of choice coincide when choice sets are ranked in terms of their ability to satisfy preferences for freedom of choice. Puppe has an article where one and the same ranking is described both as a ranking of sets in terms of freedom of choice and a ranking in terms of a preference for freedom of choice (1996). A similar article has been written by Nehring and Puppe (1999).

1.2 The Choice Situation

What shall concern us here is the situation where a person is trying to choose an option from a set of options. It is to this situation that the question how much freedom of choice a person has applies. Since this is the same situation that interests decision theorists, I shall try to answer the question within a decision-theoretical framework. There may be some limitations of the discussion of freedom of choice resulting from the use of the decision-theoretical model. However, all discussions have some limitations, and this problem cannot be avoided.

1.2.1 Choice of an Act

In the traditional decision-theoretical model it is assumed that there is some person P in some situation S who is trying to decide which act to perform. More specifically, the situation S may be described as the state of affairs holding at some specific time t (or time interval t_i to t_n) and place p_j . The person P may be described as choosing among alternative acts x_1 to x_n that P can perform or begin to perform at t_i and p_j , if P so chooses (in this context, being still and silent is also regarded as an act).

Depending on the situation, an act may result in different outcomes. The outcome of an act is also a state of affairs that holds at some time t_k and place p_j . It causally depends both on the performance of some act at some earlier time t_i and the same place p_j and the state of affairs that holds at the same t_i and p_j .

The traditional model is often expanded to include information regarding probabilities. Since the state of affairs that holds at the time of an act is seldom completely known, it is customary to assign probabilities to different possible states. The choice of an act may then be described as the probabilistic choice of different outcomes. In decision-theoretical terms, the choice of an act may be described as a choice of a lottery among different possible outcomes.

The traditional model is always expanded to include information regarding the chooser's preferences for the different outcomes. For every possible outcome, S_i and S_j , it is assumed that P either prefers the holding of S_i rather than S_j , or prefers the holding of S_j rather than S_i , or is indifferent between the two states of affairs. This assumption implies that P 's preferential attitudes order all the possible outcomes.

For example, a student may, on the 22nd of November 2010 at 1 PM, at the Philosophy Department at Uppsala University, choose between the acts of registering for a course in logic and not registering for a course in logic. The choice of registering for a course in logic may lead to the student passing logic or failing logic, depending on the difficulty of the course (for example). The choice between registering for logic and not registering for logic may thus be described as the choice between a lottery between the outcomes of failing logic or passing logic and the certain outcome of not having taken logic at all. The student prefers passing logic over not taking logic and not taking logic over failing logic.

The traditional decision-theoretical question is what the student should choose. Our question is how much freedom of choice the student has.

1.2.2 Choice in Several Stages

So far we have not assumed anything out of the ordinary. However, the traditional decision-theoretical model is not ideal in terms of explaining the

importance of freedom of choice. One reason why the model is inadequate is that the preferences of the person choosing are considered as fixed. But a chooser is usually only interested in freedom of choice before the options have been evaluated, and the chooser does not know which option to choose. After the options are evaluated, the only option that matters to the chooser is the option that the chooser is going to choose. So, unless freedom of choice positively affects the value of the chosen option there seems to be no point in having any.

To be able to explain why freedom of choice is important, it is customary to modify the traditional decision model slightly. This can be done in different ways. Here we shall look at a model that describes choice as a process in several stages, where degrees of freedom of choice may be assessed at each stage. The model is new for this thesis, although similar models have been used before. Realistically, not every process of choice contains all the stages that are described below.

At the first stage of choice, the pre-information stage, the chooser is unaware of the identities, properties and values of the options. Freedom of choice cannot be assessed at this stage since there is no information that such an assessment can be based on. An example of this stage would be a student who is thinking about applying to a university program without having looked at the options.

At the second stage of choice, the pre-evaluative stage, the chooser has information regarding the identities and properties of the options. She is considering the properties of the options that can provide reasons for evaluating the options in different ways but has yet to evaluate them. All preference-independent measures of freedom of choice may be applied at this stage. An example of this stage would be the same student who has read the descriptions of the different programs and is considering their properties in order to evaluate them. She is thinking of the properties of a philosophy program and the properties of an arts program, as well as the differences between them. The aesthetic challenges could be a reason to rank the arts program over the philosophy program, while the intellectual challenges could be a reason to rank the philosophy program over the arts program, for example.

At the third stage of choice, the evaluation stage, the chooser has information regarding the different reasons for evaluating the options differently. She is trying to weigh the reasons against one another in various ways to come up with possible preference orderings. She has yet to decide which preference orderings she should use as a standard for choice. At this stage, all preference-based measures of freedom of choice may be applied that relate degrees of freedom of choice to several possible preference orderings. In the example the student is now aware of the different programs and the different reasons to rank one program over another. She thinks that an expressive person would rank the arts program over the philosophy

program, while an analytic person would rank the philosophy program over the arts program. But the student does not yet know what kind of ranking she should adopt as a standard for choice.

At the fourth and last stage of choice, the decision stage, the chooser has all the relevant information that is necessary to make an informed choice. She knows what the options are, how the options may be evaluated and how she will evaluate the options. All that is left is the rational decision to choose an option that is at least as preferable as any of the others. There are no measures of freedom of choice that are especially designed for this stage, probably because at this stage, freedom of choice no longer seems to matter. At the last stage of choice the student has decided to adopt the preference ordering of an analytic person and thus applies to the philosophy program.

Chapter 2: The Concept of Freedom of Choice

We shall start the investigation by considering various analyses of the concept of freedom of choice. The analyses may be evaluated using different standards. We shall consider such standards first.

2.1 Evaluating an Analysis

An analysis of the concept of freedom of choice could be guided by beliefs regarding how the expression ‘freedom of choice’ is used, or by beliefs regarding how the expression ‘freedom of choice’ should be used. Most often, an analysis is guided by both types of beliefs. It is not always clear whether an analysis aims to capture some language use or aims to influence language use.

Roughly, we may assess an analysis of the concept of freedom of choice from at least three viewpoints. One viewpoint is how well the analysis captures ordinary language use of the expression ‘freedom of choice’. Another viewpoint is how useful the analysis would be for scientific purposes, either for philosophical argumentation or for empirical investigations concerning the nature and effects of freedom of choice. A third viewpoint is how useful the analysis would be for practical decisions regarding what to do.

An analysis that perfectly captures ordinary language use would rarely be useful for scientific purposes since ordinary language is ambiguous and vague, and science requires unambiguity and precision. To investigate the nature and effects of freedom of choice, we must know exactly what phenomenon we are investigating. But an analysis that completely fails to capture ordinary language use would not be useful for scientific purposes either. If we want to investigate the effects of freedom of choice, we want to investigate some phenomenon that an ordinary language user would recognize.

An analysis that is useful for scientific purposes may not necessarily be useful for practical decisions. An analysis of freedom of choice would be most practically useful if it captures a conception that refers to a phenomenon that is generally regarded as good (or bad). But there may be several conceptions of freedom of choice that can be used for scientific

purposes, without referring to some phenomenon that is generally regarded as either good or bad.

An analysis that would be most useful for practical purposes would not capture ordinary language use. A conception that refers to a phenomenon that is generally regarded as either good or bad is not recognizable by an ordinary language user, who knows that freedom of choice is not generally regarded as either good or bad.

The method of analysis used in this thesis is of a rather ordinary philosophical kind. The starting point is not to investigate ordinary language users' conceptions of freedom of choice. The starting point is rather to categorize different conceptions of freedom of choice that occur in the literature. These conceptions are then specified (when needed) and compared to ordinary language users' (mostly philosophers') beliefs regarding freedom of choice. The end point of analysis is to define 'freedom of choice' by stating all the necessary and jointly sufficient conditions for something being an instance of freedom of choice. The definition should be sufficiently precise to be useful for scientific investigations, and sufficiently similar to ordinary language use to be recognizable. As for the practical usefulness of the definition, I shall not attempt to capture a conception that refers to something that is generally regarded as either good or bad.

2.2 Concept and Conceptions of Freedom of Choice

There is no general agreement regarding how the concept of freedom of choice should be analyzed. Perhaps the only agreement regarding the concept is the following necessary condition:

A person *P* has *freedom of choice* in a situation *S* only if *P* has a choice set of at least two options to choose from in *S*.

This belief may be the only one that is shared by all language users. In addition, most people have beliefs regarding typical instances of freedom of choice. A person's set of beliefs is usually vague in the sense that it is unclear whether or not a specific belief belongs to the set. It may also be incoherent by containing a subset of beliefs that cannot jointly be true.

An analysis of the *categorical concept of freedom of choice* is a class of propositions stating the necessary and sufficient conditions for a person having freedom of choice. An analysis of the *comparative concept of freedom of choice* is a class of proposition stating the necessary and sufficient conditions for a person in some situation having more, less, or an equal amount of freedom of choice as some other (or the same) person in some other situation. An analysis of the *complete concept of freedom of choice* is a class of propositions stating the necessary and sufficient

conditions for a person having as much freedom of choice as he could possibly have.

A specific proposal for how the concept of freedom of choice should be understood is a *conception of freedom of choice*. We shall look at several such conceptions next.

2.2.1 Evaluative and Non-Evaluative Conceptions

Conceptions of freedom of choice may be separated into two types, evaluative conceptions and non-evaluative conceptions.

If the concept of freedom of choice is explicated as an *evaluative conception*, it is defined in such a way that freedom of choice is valuable (good, bad or neutral) by definition. For example, ‘*P* has freedom of choice’ =_{Def} ‘*P* has a valuable choice among at least two options’ or, more commonly, ‘*P* has freedom of choice’ =_{Def} ‘*P* has a good choice among at least two options’. If the concept of freedom of choice is explicated as a *non-evaluative conception*, it is defined in such a way that freedom of choice is not valuable (good, bad or neutral) by definition. For example, ‘*P* has freedom of choice’ =_{Def} ‘*P* has a choice among at least two options’.

The main problem with an evaluative explication is that it is at odds with ordinary language use of the expression ‘freedom of choice’. It is perfectly understandable to say, “It is not always valuable to have freedom of choice”. There are some contexts where we may wish to use an evaluative conception of freedom of choice. But then we should also accept the use of a non-evaluative conception in other contexts. We should distinguish between “valuable freedom of choice” and plain “freedom of choice”.

It seems acceptable to distinguish between “valuable freedom of choice” and plain “freedom of choice”. However, as a basic conception, “valuable freedom of choice” seems superfluous. Once we have the concepts of “value” and “freedom of choice” we can easily form the concept of “valuable freedom of choice”. Hence, there seems to be no need to give any special attention to the evaluative conception of freedom of choice.

2.2.2 Preference-Dependent and Preference-Independent Conceptions

Conceptions of freedom of choice may be further separated into *preference-dependent* and *preference-independent* conceptions. If the concept of freedom of choice is explicated as a preference-independent conception, it is defined without reference to preferences in the definiens. For example, ‘*P* has freedom of choice’ =_{Def} ‘*P* has a choice of at least two options’. If the concept of freedom of choice is explicated as a preference-dependent conception, it is defined with reference to preferences in the definiens. For

example, ‘*P* has freedom of choice’ =_{Def} ‘*P* has a choice of at least two preferred options’, or ‘*P* has freedom of choice’ =_{Def} ‘*P* has a preferred choice of at least two options’.

Preference-dependent and preference-independent conceptions of freedom of choice are both common. Conceptions (and measures) of preference-independent freedom of choice are proposed by, for example, Suppes (1987), Pattanaik and Xu (1990, 2000a), Klemisch-Ahlert, (1993), Gravel et al. (1998), Sugden (1998), Rosenbaum (2000), Bavetta and Del Seta (2001), Xu (2003, 2004), Van Hees (2004), Savaglio and Vannucci (2006) and Gustafsson (2010). Conceptions (and measures) of preference-dependent freedom of choice are proposed by, for example, Puppe (1996, 1998), Pattanaik and Xu (1998, 2000b), Sugden (1998), Romero-Medina (2001), Peragine and Romero-Medina (2006), Puppe and Xu (2010) and Gustafsson (2010).

2.3 Conceptions of Freedom and Conceptions of Choice

The expression ‘freedom of choice’ contains the two words ‘freedom’ and ‘choice’. A first step when analyzing the concept of freedom of choice would be to look at the different conceptions of freedom and the different conceptions of choice, to see how these may be combined into different conceptions of freedom of choice. A second step would be to identify the particular conception that shall be discussed in this thesis.

The concept of freedom has many possible explications. In the literature we may, for example, find the following meanings: absence of impediments, ability, availability of options, indeterminacy, self-determination, self-mastery and absence of want. This list of conceptions is not exhaustive, and the conceptions listed are not mutually exclusive. They may all be specified in many different ways. Ofstad has a similar list, mentioning absence of compulsion, indeterminacy, self-expression, rationality and power (1961), and Gray has too, mentioning absence of impediments, availability of choices, effective power, status, self-determination, doing what one wants and self-mastery (1990).

The concept of choice has rather many possible explications as well. According to Dowding, it may be used in three distinct ways, roughly as the set of options, the act of selecting an option from a set and the selected option (1992: 303). A fourth and fifth use, which are not mentioned by Dowding, but are also common, are the following: the opportunity to select an option from a set and the process of selecting an option from a set. In the first sense of the word ‘choice’ we would say that a person has a choice, in the second sense that he makes a choice, in the third sense that some option is his choice, in the fourth sense that he has an opportunity for choice, and in the fifth sense that he is in the process of making a choice.

In combination with the different meanings of ‘choice’ we may distinguish several meanings of ‘freedom of choice’, or ‘free choice’. In this thesis we shall only discuss freedom of choice as a set of available options. This kind of freedom of choice is obviously related to the other kinds. To have a set of available options is a necessary condition for making a choice, no matter how it is made. Having a few, rather than many, available options is a constraint on the opportunity for choice. Dependence relations go in the other direction too; constraints on opportunities for choice may result in a reduction of the options that are available.

In this context, it may be useful to note that the two words ‘freedom’ and ‘choice’ are sometimes used as synonyms to ‘freedom of choice’. The word ‘freedom’ is used as a synonym by Arneson (1985, 1998), Pattanaik and Xu (1998) and at times by Sen (1991). The word ‘choice’ is used as a synonym by Jones and Sugden (1982) and Beavis and Rowley (1983). Other phrases are also used as synonyms, most often ‘opportunity’, which is used by Koopmans (1964), but also ‘freedom of decision’, used by Suppes (1987) and ‘option freedom’, used by Pettit (2003).

In this thesis we shall not discuss measures of ‘freedom of choice’ in any other sense than in relation to choice sets of options, although there are measures of ‘freedom of choice’ in other senses of the phrase. Measures of freedom of choice, regarded as a process of selecting options, are proposed by Gabor and Gabor (1954), Suppes (1996), and Erlander (2005, 2010). Measures of freedom of choice, regarded as a process of selecting preferences, are discussed by Arrow (1995) and Bavetta (2004).

2.4 Conceptions of Categorical Freedom of Choice

The concept of freedom of choice may be specified in a categorical, comparative or complete sense. The only idea that seems to be common to all conceptions of freedom of choice is the following necessary condition:

A person P has *freedom of choice* in a situation S only if P has a choice set of at least two options to choose from in S .

It is more controversial whether having at least two options is also sufficient for having freedom of choice. Most non-evaluative categorical conceptions are consistent with the idea that two options are sufficient for freedom of choice as well as necessary. Thus, most non-evaluative categorical conceptions of freedom of choice may be characterized by the following definition:

A person P has *freedom of choice* in a situation S if and only if P has a choice set A of at least two options to choose from in S .

There are only a few categorical conceptions that cannot be captured through this definition. An evaluative definition of categorical freedom of choice has to include some evaluative term. For example, that P has a ‘valuable’ choice set. There are also some non-evaluative conceptions that require some additions to the definition as well. For the conception of so-called *freedom of rational choice*, the term ‘options’ must be qualified as ‘options that can rationally be chosen’, and for the conception of so-called *freedom of eligible choice*, the term ‘options’ may have to be qualified as ‘options that are sufficiently valuable’ (although that depends on how we understand the conception). We shall discuss these divergences later in the thesis.

2.5 Conceptions of Comparative Freedom of Choice

Freedom of choice is not only a categorical property that a person may either have or lack in some situation. It is also a comparative property. Persons can have *different degrees of freedom of choice* depending on the situation. But it is not clear what is required for having more or less freedom of choice. Since categorical freedom of choice depends on a person having a choice set with options, comparative freedom of choice should depend on some of the quantitative properties of choice sets or options. But there are many quantitative properties of choice sets and options on which quantities of freedom of choice may depend. For instance, there are the number of options, the values of the options, and the degrees of differences between the various options. There are also the value of a choice set and the degree of difference between a choice set and some other set, such as a set with maximally different options, or some relevant universal set. Each of these quantitative properties may be relevant for quantities of freedom of choice. We shall not discuss all possible ways in which comparative freedom of choice may be specified. But we shall discuss some of the ways that occur in the literature.

2.5.1 Preference-Independent Conceptions of Freedom of Choice

For a non-evaluative explication of the concept of freedom of choice there are at least three different analyses that occur in the literature; the *cardinality conception*, the *representative conception* and the *diversity conception*.

According to the first idea, a person’s degree of freedom of choice depends on the number of options in his choice set. The options contribute equally to freedom of choice just in virtue of being different options. This conception may be defined as follows:

The Cardinality conception of freedom of choice: A person P with a choice set A has *at least as much freedom of choice as* a person P^* with a choice set B if and only if A contains *at least as many options as* B .

This idea may be found in essays by Beavis and Rowley (1983), Pattanaik and Xu (1990) and Arneson (1998), although the conception is not endorsed in the last two essays.

According to the second idea, a person's degree of freedom of choice depends on how representative his choice set is of the universal set of possible options. The universal set of possible options is properly understood as the set of all nomologically possible human acts. However, in this context, it is more conveniently understood as some relevant and proper subset of the set of all nomologically possible acts. To distinguish the two sets, I shall call the larger set the *universal set* and the subset the *relevant universal set*. The options contribute to freedom of choice to the degree that they are representative of all options in the relevant universal set.

The Representative conception of freedom of choice: A person P with a choice set A has *at least as much freedom of choice as* a person P^* with a choice set B if and only if A is *at least as representative of the universal set X as* is B .

'Representativeness' may be specified as follows:

The choice set A of options x_i is *at least as representative of the universal set X as* the choice set B of options y_j if and only if the options x_i in A are *at least as similar, overall, to all of the options in the universal set as* the options y_j in B .

The idea that the options in a choice set are at least as similar, overall, to all of the options in the universal set as are the options in another choice set is not a completely precise idea. We shall look at how it may be specified later. The representative conception of freedom of choice only occurs in one essay, by Gustafsson (2010).

According to the third idea, a person's degree of freedom of choice depends on the diversity of his choice set. It may be defined as follows:

The Diversity conception of freedom of choice: A person P with a choice set A has *at least as much freedom of choice as* a person P^* with a choice set B if and only if A is *at least as diverse as* B .

This idea occurs in some essays on measures of diversity, for example, in essays by Arneson (1998), Pattanaik and Xu (1999), Bossert, Pattanaik and

Xu (2003), and Van Hees (2004). It is also suggested in a book by Wertheimer on coercion (1987: 192).

The concept of diversity may be understood in different ways, but it is customary to think of the diversity of a choice set as in some way dependent on the differences among its options. The options contribute to freedom of choice to the degree that they differ from the other options. If we accept this idea, then ‘diversity’ may be specified as follows:

The choice set A of options x_i is at least as diverse as the choice set B of options y_j if and only if the options x_i in A are at least as different from one another, overall, as the options y_j in B .

This definition of diversity is not sufficiently specific to capture a single conception. There are several ways to understand the idea that the options in a choice set are more different from another, overall, than are the options in another choice set. We shall look at more specific definitions later in this thesis.

To illustrate how the conceptions differ we may use an example. Let us consider the choice sets of three teachers at Uppsala University who are choosing among different amounts of teaching hours per week. We may suppose that the relevant universal set of teaching hours per week at Uppsala University is $X = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}, 40 \text{ h}, 50 \text{ h}\}$. Let us compare the three choice sets and call them $A = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}\}$, $B = \{10 \text{ h}, 50 \text{ h}\}$, and $C = \{20 \text{ h}, 30 \text{ h}\}$. According to the cardinality conception of freedom of choice, A offers most freedom of choice by offering more options than either B or C . These latter sets offer less freedom of choice, but an equal amount. According to the representative conception of freedom of choice, at least in Gustafsson’s sense, C offers most freedom of choice by offering options that are more representative of the relevant universal set than either A or B does, B offers less freedom of choice than C , and A offers the least. According to the diversity conception of freedom of choice (or at least one of them), B offers the most freedom of choice by offering more diverse options than either A or C , A offers less freedom of choice than B , and C offers the least. The three conceptions thus result in three different rankings of sets.

2.5.2 Preference-Dependent Conceptions of Freedom of Choice

There are several conceptions of preference-dependent freedom of choice as well. The conceptions differ with regards to the kinds of preferences that are assumed to affect freedom of choice and the way in which these preferences are assumed to do so. As for kinds of preferences, it has been suggested that freedom of choice depends on P ’s logically possible preferences, or some subset of P ’s logically possible preferences, such as P ’s actual or probable

preferences. As for ways in which preferences may affect freedom of choice, there are at least four main proposals.

A first idea is that it is only the options that a person can rationally choose that contribute to his freedom of choice. A person can only rationally choose an option from a choice set when it is at least as preferable as any of the other options in the set, according to at least one relevant preference ordering. This conception may be called *freedom of rational choice*. We may combine this conception of preference-dependent freedom of choice with the three preference-independent conceptions, and thereby define three conceptions of freedom of rational choice at once, as follows:

The Conceptions of freedom of rational choice: A person P with a choice set A has at least as much freedom of choice as a person P^* with a choice set B if and only if A contains at least as many / as diverse / as representative options that P may rationally choose as B contains options that P^* may rationally choose.

Measures of freedom of choice as freedom of rational choice occur in at least two essays, one by Pattanaik and Xu (1998) and one by Bervoets and Gravel (2007) (although none of the authors use this term). The corresponding idea that freedom of choice is unaffected by options that a person cannot rationally choose occurs in several essays, although not accompanied by a corresponding measure. It occurs, for example, in essays by Benn and Weinstein (1971: 197), Fromkin et al. (1975: 434), Jones and Sugden (1982: 56), Wertheimer (1987: 194), Benn (1988: 138) and Erlander (2005: 514, 2010: 89). Pattanaik and Xu use the cardinality conception, while Bervoets and Gravel use the diversity conception (2007: 268). The authors share the idea that all reasonable preference orderings should be used to select the options that can rationally be chosen. It would also be possible to use other sets of preference orderings, such as P 's present preference ordering or P 's probable future preference orderings. If all logically possible preference orderings are used, then any option can rationally be chosen. Such a conception of freedom of rational choice differs from a preference-independent conception only by definition, not by any difference in the ordering of choice sets.

A second idea is that it is only the options that are sufficiently valuable, according to some selection of preference orderings (possibly only one), that contribute to a person's freedom of choice. The sufficiently valuable options are often called 'eligible', so this conception may be called *freedom of eligible choice*. If the three preference-independent conceptions of freedom of choice are used to define the three conceptions of freedom of eligible choice, it may be defined as follows:

The Conceptions of freedom of eligible choice: A person P with a choice set A has at least as much freedom of choice as a person P^* with a choice set B if and only if A contains at least as many / as diverse / as representative sufficiently valuable options for P as B contains sufficiently valuable options for P^* .

Romero-Medina presents the conception of freedom of eligible choice in an essay from 2001 (although he calls it ‘meaningful choice’). He uses the cardinality conception, and a more complicated lexical one, which we shall look at later. Since there may be different ideas regarding which options should be classified as sufficiently valuable, the definition may be varied in different ways. Romero-Medina shares Pattanaik’s and Xu’s view that all reasonable preference orderings should be used to select the options that affect degrees of freedom of choice. It is also possible to use other selections of preference orderings. If all logically possible preference orderings are used, then the conception of freedom of eligible choice gives the same ordering of choice sets as a preference-independent conception.

A third idea is that options contribute to freedom of choice in proportion to their value. Degrees of freedom of choice are a function of the values of the options. This conception may be called *freedom of evaluated choice*. A suggestion for a definition is the following:

The Conception of freedom of evaluated choice: A person P with a choice set A , has at least as much freedom of choice as a person P^* with a choice set B if and only if A contains at least as valuable options for P as B contains valuable options for P^* .

This conception may be interpreted in different ways, as well as combined with other conceptions. We shall discuss these possibilities later. The relevant preference orderings may be all of P ’s logically possible preference orderings or some selection thereof.

The idea that freedom of choice is a function of the values of the options may not occur in the literature, previous to this thesis. But the idea that freedom is a function of the values of the options is rather common. It is held by Berlin (1969: 130), Loevinsohn (1977: 233), Swanton (1979: 346), Elster (1983: 129), Steiner (1983: 81) and Arneson (1998: 182). At least two measures of freedom captures this view: Steiner’s measure (1983: 81) where freedom is regarded as a function of only the values of the options and Crocker’s measure (1980: 53), where freedom is regarded as a function of the values of the options, among other factors.

To illustrate how the conceptions differ we may again use the example from Uppsala University. The relevant universal set of teaching hours at Uppsala University is $X = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}, 40 \text{ h}, 50 \text{ h}\}$. We may further suppose that we shall take all psychologically possible preference

orderings into account, and that there are just two: maximizing money Y_i : $50\text{ h} > 40\text{ h} > 30\text{ h} > 20\text{ h} > 10\text{ h} > 0\text{ h}$; and maximizing leisure: Y_j : $0\text{ h} > 10\text{ h} > 20\text{ h} > 30\text{ h} > 40\text{ h} > 50\text{ h}$. Once again we compare the sets $A = \{0\text{ h}, 10\text{ h}, 20\text{ h}\}$, $B = \{10\text{ h}, 50\text{ h}\}$, and $C = \{20\text{ h}, 30\text{ h}\}$. According to the cardinality conception of freedom of rational choice, all sets offer the same amount of freedom of choice since all sets offer two options that may rationally be chosen (the option 10 h in A may not rationally be chosen). According to the cardinality conception of freedom of eligible choice, the sets A and B offer the most freedom of choice by offering one of the two most valuable options, while C offers the least, by offering none of the most valuable options. Using the conception of freedom of evaluated choice requires some value assignment to the options. Assuming that the utility function is linear, and that $50\text{ h} = 500\text{ }u$ and $0\text{ h} = 0\text{ }u$ for a utility function of money, and that $0\text{ h} = 118\text{ }u$ and $50\text{ h} = 68\text{ }u$ for a utility function of leisure, gives $u(A) = 624$, $u(B) = 776$, $u(C) = 686$ (other aggregations of values are also possible). Thus, according to some conception of freedom of evaluated choice, B offers more freedom of choice than A , which offers more freedom of choice than C .

The simple example of ordering choice sets at Uppsala University in terms of freedom of choice thus resulted in six different orderings through the use of six different conceptions of freedom of choice.

2.6 Complete Freedom of Choice

We have considered a number of conceptions of comparative freedom of choice that may be found in the literature. One may wonder if there really is a common concept that the different conceptions try to capture. On the one hand, there is some conceptual overlap regarding categorical freedom of choice across the different conceptions. On the other hand, there is very little overlap regarding comparative freedom of choice. However, there is a possibility that there is more conceptual overlap regarding what is involved in having *complete freedom of choice*. We shall look at this possibility here.

One might think that there is no need to define complete freedom of choice since what it means to have complete freedom of choice should follow directly from any definition of comparative freedom of choice. Nevertheless, if we can define a conception of complete freedom of choice that can be understood independently of the comparative conceptions, then the complete conception may be used to compare the different comparative conceptions. These can be understood as different ways of trying to specify what it means to be more or less close to having complete freedom of choice.

The most natural definition of complete freedom of choice is the following:

A person P has *complete freedom of choice* if and only if P has the universal set X to choose from.

This definition is only applicable if the universal set is finite. Since I have assumed that the universal set is finite, it is not problematic here.

Obviously there is no person who ever has complete freedom of choice. But by defining a conception of complete freedom of choice we need not suggest that it is attainable. The purpose of defining the conception is only to understand the different comparative conceptions of freedom of choice as different ways of assessing closeness to complete freedom. With this purpose in mind, it is interesting to note that the definition of complete freedom only fits with some of the comparative conceptions of freedom of choice. It fits with the cardinality conception since the universal set has the greatest number of options. It also fits with the diversity conception since the universal set has the most diverse options. It certainly fits with the representative conception since the universal set is most representative of itself. However, the definition does not fit with the different preference-dependent conceptions of freedom of choice. It is inconsistent with the conceptions of freedom of rational choice since the universal set may not contain the greatest number, or the most diverse, or the most representative options that may rationally be chosen. It is too strong for the conceptions of freedom of eligible choice since a subset of the universal set may contain all eligible options. It is at odds with the conception of evaluated choice since the universal set may not be the most valuable set.

One may wonder if there is any other way to define complete freedom of choice that fits better with the evaluative conceptions, without referring to these conceptions themselves. It seems difficult to find a definition that fits with freedom of rational choice, because it is not a type of freedom of choice that is monotonically increasing. However, there is at least one definition that might fit with the other conceptions. To distinguish this definition from the previous one, we may call it a definition of *perfect freedom of choice*:

A person P has *perfect freedom of choice* if and only if P has a choice set A that allows P to choose whichever option P might prefer to choose.

Similar kinds of conceptions have been criticized for not being attainable as goals (for example, by Green ([1881]1991: 21) and Hayek ([1960]1991: 85)). But here we do not claim that it is an attainable (or even valuable) goal.

The definition of perfect freedom fits more or less well with all conceptions of freedom of choice, except for the conception of freedom of rational choice. The preference-independent conceptions of freedom of choice seem to fit with the ‘whichever option’ part of the definition, but not

with the ‘might prefer’ part (unless P might prefer any option). The conception of freedom of evaluated choice seems to fit with the ‘might prefer’ part of the definition, but not with the ‘whichever option’ part. Only the conception of eligible choice seems to fit with both parts of the definition.

Relative to the cardinality conception of freedom of choice, a person is judged to get closer to attaining complete freedom of choice as he gets more options. But it is unlikely that he gets closer to attaining perfect freedom at the same rate as he gets closer to attaining complete freedom. A person may have many options, but if they all are very similar to one another, then they do not seem to give him anything close to the ability to choose whichever option he might prefer to choose.

Relative to the diversity conception of freedom of choice, a person is judged to get closer to attaining complete freedom of choice as his choice set gets more diverse. This may not imply that he also gets closer to attaining perfect freedom. Although a diverse choice set allows for the satisfaction of very different preferences, most persons might not prefer the options that are most extreme. This is also noted by Baujard (2007: 243), and by Gustafsson (2011: 49).

Relative to the representative conception of freedom of choice, a person is judged to get closer to attaining complete freedom of choice as he gets options that are more representative of the universal set. It is uncertain whether he would also get closer to attaining perfect freedom. He would, if the relevant universal set is such that he prefers options that are more normal and prefers similar options to a similar degree. He would not, if he does not prefer options that are more normal and prefers similar options to a similar degree.

Relative to at least one conception of freedom of eligible choice, a person is judged to get closer to attaining complete freedom of choice as he gets more and better options. This seems to imply that he also gets closer to attaining perfect freedom. Relative to the conception of freedom of evaluated choice, a person is judged to get closer to attaining perfect freedom as he gets better options. He might also get closer to attaining perfect freedom, although that might depend on how important having different options would be for having perfect freedom of choice.

Here it may be useful with an example. Let us consider a case where a person wants a maximal amount of freedom of choice but is only allowed to select a choice set of two options from a relevant universal set. We may assume that the person is a student who wants to take a summer course at Uppsala University. He is choosing among different departments that are offering summer programs, from which he will subsequently choose one course. The departments that he is considering are offering programs containing two courses each. Since the student does not know which course he wants to take, nor to which department he wants to apply, the student

turns to an advisor, saying that he wants a maximal amount of freedom of choice. The advisor can use any of the nine preference-dependent conceptions of freedom of choice. Depending on which conception that the advisor uses, he will give the student different kinds of advice.

If the advisor uses any of the preference-independent conceptions, then he would tell the student to pick any department at random, or to select the department that offers the two most different summer courses, or to select the department that offers the set of most representative courses for the curriculum at Uppsala University. The first advice would be due to the cardinality conception, the second to the diversity conception, and the third to the representative conception. If the advisor uses any of the preference-dependent cardinality conceptions, then he would tell the student to pick any department that offers two courses that the student might value equally or to select the department that offers the two best courses, as judged relative to any preference ordering that the student might adopt. The first advice would be due to the conception of freedom of rational choice, while the second would be due to either the conception of freedom of eligible choice or the conception of freedom of evaluated choice.

From a practical perspective, and from the perspective of this example, the conceptions of freedom of eligible or evaluated choice clearly seem preferable. However, the practical perspective is not the perspective of this thesis.

2.7 Related Concepts

There are a number of concepts that are related to the concept of freedom of choice that should be mentioned to avoid confusion. Here I shall briefly mention a few, namely, overall freedom, opportunity, flexibility, significant choice and freedom of will.

The concept of *overall freedom* occurs in political philosophy. It is generally thought that a person's overall freedom depends on his class of specific freedoms. A specific freedom for P is some act x that P is free to do. That P is free to do x may be understood either in a positive or in a negative sense. P is positively free to do x if and only if P is able to do x . P is negatively free to do x if and only if P is not hindered to do x (usually understood in the sense of not being hindered by others to do x).

The word 'liberty' and the expression 'total freedom' are sometimes used instead of 'overall freedom'. Likewise, 'positive liberty' and 'negative liberty' are generally used instead of the more cumbersome expressions 'overall positive freedom' and 'overall negative freedom'.

Freedom of choice is similar to overall freedom in that both depend on the freedom to do something more specific, either choose a specific option (in the case of freedom of choice) or perform a specific act (in the case of

overall freedom). The concept of overall freedom does not, a priori, include the idea of a choice among specific freedoms, so, in that sense, the concept of overall freedom is not a subclass of the concept of freedom of choice. Neither is freedom of choice a subclass of the concept of overall freedom since the concept of an option may include other things than acts.

Political freedom has been discussed for a very long time (for example by Machiavelli (1513), Hobbes (1651), Locke (1690), Montesquieu (1748), Rousseau (1762), Kant (1790), and Mill (1859)). But the discussion of measures of freedom is rather recent. The discussion of freedom of choice may originate in decision theory, with Koopmans or Suppes. But the discussion of measures of freedom rather originates in political philosophy, with Berlin. He does not propose any measure of freedom in his essay ‘Two concepts of freedom’ (1969), but he analyzes the concept of “more or less freedom”. This analysis inspired others to construct measures of freedom (for example Swanton (1979), Crocker (1980), Elster (1983), Steiner (1983), Carter (1992, 1999), Van Hees (1998), Rosenbaum (2000) and Kramer (2003)). It also led to a critique of the idea that freedom can be quantified and measured (by Benn and Peters (1959), O’Neill (1979), Tännsjö (1985), Gray (1990) and Oppenheim (1995, 2004)).

The concept of *opportunity* occurs in political philosophy as well. It is exemplified either by some specific act that a person can choose, or by a class of such acts. Measures of opportunities always apply to classes of acts. The word ‘opportunity’ is used in rather many different senses. Sometimes, it is used interchangeably with the expression ‘freedom of choice’, for example, by Ok and Kranish (1998) and Sugden (1998). At other times, it is used as a synonym to ‘the value of freedom’, for example by Van Hees and Wissenburg (1999) and by Gekker and Van Hees (2006). At still other times, it is used for some preference-dependent conceptions of freedom of choice, such as the conception of freedom of eligible choice or the conception of freedom of evaluated choice. This kind of use is found in Muller (1970) and Sen (2002). Rawls seems to use the word as a synonym to ‘positive liberty’ (1999). I have used the word for the ability to choose whatever one might prefer to choose, in an essay on a measure of opportunity (Enflo 2011).

Another expression for ‘opportunity’, in the sense of preference-dependent freedom of choice, is ‘effective freedom’. This expression is used by, for example, Muller (1970), Foster (1993, 2011), Romero-Medina (2001) and Bavetta (2004). ‘Effective freedom’ is also used as a synonym for ‘positive freedom’ by Rawls (1999).

The concept of *flexibility* occurs in decision theory. The word ‘flexibility’ is used as a synonym for ‘freedom of choice’, although it only occurs in a specific context. A person is said to have flexibility when he has future freedom of choice. His interest in flexibility is merely due to uncertainty regarding his future preferences. The first author who discusses flexibility may be Koopmans (1964). But there are a number of measures of flexibility

and measures of value that depend on a preference for flexibility proposed by other authors, for example, Kreps (1979), Jones and Ostroy (1984), Arrow (1995) and Arlegi and Nieto (2001). These measures might be used as measures of freedom of choice, but may be more suitable as measures of its instrumental value.

The concept of *significant choice* is discussed by Jones and Sugden (1982) and Baujard (2007). According to Jones and Sugden, significant choice depends both on the number of options that could reasonably be chosen, and the significance of the options. An option can only be regarded as significant if it is regarded as more valuable than the other options, but might have been regarded otherwise. As significant choice is described by Jones and Sugden, it is rather different from freedom of choice. However, some instances of freedom of choice may be instances of significant choice.

Last, I shall just mention the concept of *freedom of will*, which occurs in metaphysics. The word 'freedom of choice' is at times used as a synonym to 'freedom of will'. More often, the two words are used to express different concepts. Freedom of choice is usually regarded as a necessary, but not sufficient, condition for freedom of will. A person P has freedom of choice in a situation S when P has at least two alternative acts that he can perform in S . A person P has freedom of will in S when P has freedom of choice as well as control over which of the alternative acts that he actually performs in S . (At least, this is one way to understand the relation between freedom of choice and freedom of will.)

Freedom of will has been discussed for a very long time, and almost all philosophers seem to have had something to say on the matter. A short list would include Augustine (5th century), Anselm of Canterbury (11th century), Aquinas (1274), Descartes (1641), Hobbes (1651), Spinoza (1677), Leibniz (1684), Locke (1690), Hume (1739-1740), Kant (1785), and Schopenhauer (1841), just to mention some names from earlier centuries. But despite this extensive interest, no philosopher seems to have proposed that freedom of will could be measured. This is notable, since there is nothing in principle that prevents the measurement of freedom of will. A measure of freedom of will could be constructed from measures of control and freedom of choice. The reason why such a measure has never been attempted may be that it is still undetermined whether freedom of will exists.

Chapter 3: Measures

Because freedom of choice is a comparative property, it is also a measurable property. We shall consider what a measure of freedom of choice involves next.

3.1 Freedom of Choice as a Measurable Property

Only properties that can exist in different quantities can be measured. A measure of a quantitative property Q is a method of assigning mathematical objects (usually real numbers) to objects having (or lacking) Q in such a way that quantitative relations between the objects having (or lacking) Q are represented by quantitative relations between the mathematical objects.

The idea is usually specified as follows, let $A = \{x_1, x_2 \dots x_n\}$ be a set of objects and let Q be an n -place relation that holds between the objects in A . Then $S = \langle A, Q \rangle$ is a relational system. Further, let $B = \{y_1, y_2 \dots y_n\}$ be a set of real numbers and let R be a mathematical n -place relation that holds between the numbers in B . Then $M = \langle B, R \rangle$ is another relational system. The system S is represented by the system M if the two systems are *homomorphic*. This means that there is some function $f: A \rightarrow B$ such that for all $x_i \in A$, it is the case that $Q(x_1 \dots x_n)$ if and only if it is the case that $R(f(x_1) \dots f(x_n))$. The function f is a *measure* of Q by R . The triple $\langle S, M, f \rangle$ is a *scale*. For reference see, for example, Suppes and Zinnes (1963: 5, 7, 11).

A measure of freedom of choice is a function that assigns real numbers either to persons having freedom of choice, or to choice sets, offering freedom of choice. In this thesis, real numbers are assigned to choice sets, rather than persons.

There is a universal set of possible options, the set of all nomologically possible human acts, which is denoted by X . I shall assume that this set is finite, although it is unfathomably large. The power set $P(X)$ is the set of all subsets of X . $P(X)$ is thus the set of all sets of nomologically possible acts. Some of these sets are nomologically possible choice sets. These sets may be ranked for offering different amounts of freedom of choice. Q is the relation *offers at least as much freedom of choice as*, \mathbf{R} is the set of real numbers and \geq is the relation *at least as great as*. Here I shall assume that Q yields a complete ranking of sets, although this assumption is controversial.

The relational system $S = \langle P(X), Q \rangle$ is represented by the relational system $M = \langle \mathbf{R}, \geq \rangle$ if and only if there is some function f that assigns real numbers to the choice sets of $P(X)$ so that for all $A, B \in P(X)$, A offers at least as much freedom of choice as B if and only if $f(A) \geq f(B)$. The function f is a measure of freedom of choice.

If the relation *offers at least as much freedom of choice as* can be represented by the relation *is at least as great as*, it is also possible to represent the relations *offers more freedom of choice than* and *offers equal freedom of choice as* by the relations *greater than* and *equal to*. The first relation is defined as follows:

For all choice sets $A, B \in P(X)$, A offers more freedom of choice than B if and only if A offers at least as much freedom of choice as B and it is not the case that B offers at least as much freedom of choice as A .

The second relation is defined thus:

For all choice sets $A, B \in P(X)$, A offers equal freedom of choice as B if and only if A offers at least as much freedom of choice as B and B offers at least as much freedom of choice as A .

Besides the relations *offers more freedom of choice than* and *offers equal freedom of choice as*, we may want to represent other relations between the sets that are compared, such as differences between amounts of freedom of choice (for example, *offers two units more freedom of choice than*) or ratios between amounts of freedom of choice (for example, *offers twice as much freedom of choice as*). Whether this can be done depends on which scale can be used for measuring freedom of choice.

Scales of measurement may be classified in different ways. Here we shall classify scales in accordance with Stevens's classification (1946: 678). This classification may also be found in Roberts (1979: 64). Generally, a scale of measurement is a triplet $\langle S, M, f \rangle$. An ordinal scale is defined as follows:

A property Q is measured on an *ordinal scale* if and only if it is the case that if the relational system $S = \langle K, Q \rangle$ is represented by $M = \langle L, R \rangle$ via the function $f: K \rightarrow L$, then S is also represented by M via any function g such that $f(x_i) \geq f(x_j)$ if and only if $g(x_i) \geq g(x_j)$.

If Q is measured on an ordinal scale, it is only the relations *is at least as Q as*, *is equally Q as* and *is more Q than* that may be represented by the system M via the function f (or any relation implied by these relations).

An interval scale is defined as follows:

A property Q is measured on an *interval scale* if and only if it is the case that if the relational system $S = \langle K, Q \rangle$ is represented by $M = \langle L, R \rangle$ via the function $f: K \rightarrow L$, then S is also represented by M via a function g if and only if $g = \alpha f + \beta$, where $\alpha > 0$.

If Q is measured on an interval scale, it is possible to represent more relations by M_1 than the just ordinal relations. It is possible to represent the 4-place relation *the difference in quantity Q between x_1 and x_2 is at least as great as the difference in quantity Q between x_3 and x_4* , as well as the 8-place relation *the quotient of the difference in quantity Q between x_1 and x_2 and the difference in quantity Q between x_3 and x_4 is at least as great as the quotient of the difference in quantity Q between x_5 and x_6 and the difference in quantity Q between x_7 and x_8* by the system M via the function f . In simpler terms, it is possible to compare differences between quantities and ratios of differences.

A ratio scale is defined as follows:

A property Q is measured on a *ratio scale* if and only if it is the case that if the relational system $S = \langle K, Q \rangle$ is represented by $M = \langle L, R \rangle$ via the function $f: K \rightarrow L$, then S is also represented by M via a function g if and only if $g = \alpha f$, where $\alpha > 0$.

If Q is measured on a ratio scale, it is possible to represent all of the previous relations and also the 4-place relation *the quotient of quantity Q of x_1 and x_2 is at least as great as the quotient of quantity Q of x_3 and x_4* by the system M via the function f .

Freedom of choice may be measurable on an ordinal scale, an interval scale or a ratio scale, depending on the measure. Many measures of freedom of choice work only as ordinal measures, but there are exceptions. When freedom of choice is measured by the cardinality of options or the range of options (measured on a ratio scale), it is measured on a ratio scale. The *Ratio root measures* that I shall introduce later are ratio scale measures.

3.2 Types of Measures of Freedom of Choice

We shall briefly look at the different methods that have been used to construct measures of freedom of choice. For this purpose, let us first assume that a measure of freedom of choice ought to be responsive to property Q_1 , or properties $Q_1 \dots Q_n$, of the possible choice sets and options. Given this, how should we construct a measure of freedom of choice? There are several general techniques that may be used.

If freedom of choice is dependent on just one instantiation of one property Q_1 , which is measured by some function f_1 , then freedom of choice is measurable by the same function f_1 . This technique is used for two types of measures, *Simple cardinality measures* and *Simple difference measures*. A *Simple cardinality measure* measures freedom of choice either by the cardinality of choice sets or by the cardinality of choice contributing options, which is cardinality minus 1. This type of measure is proposed by Beavis and Rowley (1983), Suppes (1987), Pattanaik and Xu (1990) and Bavetta (2004). A *Simple difference measure* measures freedom of choice by a single difference among the options in a choice set, usually the largest difference. This type of measure is proposed by Rosenbaum, for freedom (2000), and by Gravel, for diversity (2009).

Freedom of choice may also be dependent on several instantiations of one property Q_1 , which is measured by some function f_1 . In such a case, freedom of choice must be measured by an *Aggregation measure*. The most common version is when freedom of choice is assumed to depend only on all the differences between the options. The differences may be added or multiplied to measure freedom of choice. Most often all the differences are added into a total difference sum measure. Such measures are proposed for diversity by Barker and Martin (2000), as well as by Wineberg and Oppacher (2003). More complicated aggregation measures are proposed by Crocker (1980), for positive liberty, and by Eiswerth and Haney (1992), Warwick and Clarke (1995), Wen et al. (1998), and Lacevic and Arnaldi (2010), for diversity.

If freedom of choice is dependent on several kinds of properties $Q_1 Q_2 \dots Q_n$, then freedom of choice may be measured by a *Derived measure*. The method requires two steps. The first step is to find fundamental functions $f_1, f_2 \dots f_n$ for measuring the properties $Q_1, Q_2 \dots Q_n$. The next step is to construct a derived function g that is a function of the functions $f_1, f_2 \dots f_n$, for measuring freedom of choice. It is customary to use additively or multiplicatively separable functions, which are functions of the form $g = f_1 + f_2 + \dots + f_n$ or the form $g = f_1 \times f_2 \times \dots \times f_n$. For the method to work, we must know how relatively important the different properties are to freedom of choice. The relative importance of each property should be reflected by adding a weight on each fundamental function until the functions are appropriately correlated. Although this technique is used for many kinds of measures, it is seldom used for freedom of choice. However, there is a previously proposed measure by Crocker for positive liberty (1980).

Even if freedom of choice is dependent on several properties, it may not be necessary to measure freedom of choice by aggregating functions for each one of the relevant properties. For example, a measure may be based on the cardinality of choice sets but be responsive to other properties in an indirect way. We would then have a *Complex cardinality measure*. One version would be a cardinality measure that responds to both cardinality and diversity by counting only options that are different enough. Another version

would be a cardinality measure that is responsive to both cardinality and value by counting only options that are sufficiently valuable. This type of measure is quite common. Modified cardinality measures are used by Klemisch-Ahlert (1993), Pattanaik and Xu (2000a), who take diversity into account, and by Puppe (1996), Pattanaik and Xu (1998, 1999), Bavetta and Del Seta (2001), Romero-Medina (2001) and Peragine and Romero-Medina (2006), who take value into account. Measures of the cardinality of attributes or the cardinality of categories may also be put in this class. Such measures are suggested by Nehring and Puppe (2002), Connell and Orias (1964), and Lloyd and Gheraldi (1964), for diversity. A complex measure may also be based on the differences between the options. We would then have a *Complex difference measure*. An example would be a measure that aggregates only a few of the differences, selected for some desirable property. This type of difference-based measure is proposed by, for example, Weitzman (1992), Faith (1996), Nehring and Puppe (2002), Bossert, Pattanaik and Xu (2003) and Lacevic and Arnaldi (2010), for diversity.

Last, there are *Comparative measures*. The idea behind these measures is that the degree of freedom of choice offered by a set is a function of how similar the set is to a set that offers a maximal amount of freedom of choice. The method requires two steps. The first step is to identify the choice set that offers a maximal amount of freedom of choice. The next step is to define a measure of similarity to the maximal choice set. This method has previously been used by me (Enflo 2005) and by Gustafsson (2010), although we identify the maximal choice set in two different ways; I identify it as a set containing a maximal number of maximally different options, whereas Gustafsson identifies it as the relevant universal set. Besides these two examples, the method does not seem to have been used in the area of freedom of choice. It has been used in other areas, however. Fager uses the comparative method to measure diversity (1972), and Zabrotsky et al. use the method to measure closeness to symmetry (1993).

Chapter 4: Models

Freedom of choice depends on the properties of choice sets and options. Therefore, the measurement of freedom of choice depends on having a model of the relevant properties of choice sets and options. We shall look at several such models next.

4.1 Types of Models of Choice Sets

As ‘freedom of choice’ is defined here, any useful model needs to incorporate the concepts of choice sets and options. In addition to these concepts, the model should also incorporate concepts for the properties of choice sets and options that are relevant for freedom of choice.

Some models should not be used because using them would make the idea of measuring freedom of choice pointless. We should not use a model that implies that no one has any freedom of choice at all. It is important not to restrict the conception of an option so that it applies only to the option that a person actually chooses. Neither should we use a model that implies that a person always has an infinite amount of freedom of choice. It is also important not to expand the conception of an option so that everyone seems to have an infinite number of possible acts to choose among in every situation. Furthermore, we should not use a model that implies that a person always has the same limited degree of freedom of choice. An example of such a model would be one that postulates that freedom of choice can depend on the ability to choose between an act and its omission, that the difference between an act and its omission is maximal, and that freedom of choice is a function only of maximal differences. Neither should we use a model that implies that a person always has an undetermined amount of freedom of choice. We must assume that it is possible to identify and measure the properties that are relevant for freedom of choice.

Obviously, different models are needed for different conceptions of freedom of choice. The use of the cardinality conception of freedom of choice only requires information about the cardinality of choice sets. The use of diversity and representative conceptions also requires information about similarity and differences among options. The use of preference-dependent conceptions requires information about preferences for options. We shall look at different models that fit with different conceptions next.

4.2 Properties and Values

In the freedom of choice literature, it is customary to distinguish between the values of the options and the other properties of the options that may be relevant for assessing degrees of freedom of choice. It is possible that only value properties are relevant, that only non-value properties are relevant or that both value and non-value properties are relevant.

There are several suggestions for why information regarding the values of the options may be relevant for assessing degrees of freedom of choice. One suggestion is that the values of the options are relevant because only the options that may rationally be chosen affect degrees of freedom of choice. 'Freedom of choice' should thus be understood as "freedom of rational choice". Another suggestion is that the values of the options are relevant because only the options that are sufficiently valuable affect degrees of freedom of choice. 'Freedom of choice' should thus be understood as "freedom of eligible choice". A third suggestion is that the values are relevant because degrees of freedom of choice vary with the values of the options. 'Freedom of choice' should thus be understood as "freedom of evaluated choice".

It is more difficult to defend the idea that information regarding the non-value properties of the options may be relevant for assessing degrees of freedom of choice. It may seem that any information regarding the properties of the options is superfluous when there is information about the value of the options. Rational choice concerns choosing some option that is at least as preferable as any other option. When the chooser knows that an option is at least as preferable as any other, information about the other properties of the option is irrelevant.

But there are at least two perspectives from which we may regard information about other properties than values as relevant for a measure of freedom of choice. One perspective is that we wish to measure freedom of choice before evaluation. At that point, information about the properties of the options is important since it is the properties that give us reason to evaluate the options in different ways. Measures of freedom of choice that are responsive to non-evaluative properties should thus be regarded as measures that apply before evaluation. Another perspective is that non-evaluative properties may be important for some instrumental value of choice other than the ability to choose an option that is at least as preferable as any other, for example, the ability to discriminate between different properties (if this is valuable). A person who chooses between purchasing a cat and a dog may learn something about the differences between them, even though the purchases may be equally good.

Since we are interested in measures of freedom of choice that can be applied both before and after evaluation, we may assume that both value

properties and other properties may be relevant for assessing degrees of freedom of choice.

4.3 Preference-Independent Models

The measures of preference-independent freedom of choice that are suggested in the literature are combined with many different models of choice sets. We shall look at some of the models that occur in the literature here.

The *Cardinality model* includes information only regarding the number of options in each set. This model is only suitable when applying the cardinality conception of freedom of choice. The problem with this model is that freedom of choice may depend on differences between the options, and this information is unavailable. If this model is used, it would be difficult to represent the idea that a voter who chooses between a left-wing party and a right wing party may have more freedom of choice than a voter choosing among three left-wing parties. The cardinality model is used for measures by Beavis and Rowley (1983) and Pattanaik and Xu (1990).

The *Attribute model* can be used with the diversity conception of freedom of choice since the model represents similarity relations. It represents these relations in terms of the options sharing the same attributes. The model is used as a basis for measures of diversity and suggested as a basis for measures of freedom of choice. If it is used, the freedom of choice offered by a set may be regarded as a function of the number of attributes that the options exemplify. Nehring and Puppe (2002) propose an attribute-based diversity measure. The problem with an attribute-based measure is that it is not always responsive to an increase in the number of options and is never responsive to degrees of similarity between attributes. A set containing an anti-EU socialist party, a pro-EU socialist party and a pro-EU liberal party might be ranked as equally diverse as a set containing an anti-EU socialist party and a pro-EU liberal party (unless combined attributes are counted, such as anti-EU-socialism). The options in both sets exemplify four attributes. That there is an additional option in the first set is not reflected by the measure. Also, a set containing a pro-EU socialist party and an anti-EU socialist party might be ranked as equally diverse as a set containing a pro-EU socialist party and a neutral-to-EU socialist party (unless more refined political attributes are counted). The options in both sets exemplify three attributes. That there is a larger degree of similarity among the options in the first set is not reflected by the measure.

The *Category model* includes information about the number of options and the categories to which the options belong. A category is a set of elements that are similar in some particular way. Similarity relations are thus represented in this model, although only those that are used to identify categories. The category model is used extensively in biology as a basis for

diversity measures and has also been suggested as a basis for measures of freedom of choice. The most popular diversity measures used in biology, such as Shannon's index and Simpson's index, are used with a model where a relevant universal set of organisms is partitioned into species. The diversity of a set of organisms may then be regarded in two ways, either as a function of only the number of represented species, or as a function of both the number of represented species and how evenly the individuals are distributed among the various species. The first conception is known as *richness*, while the second conception is known as *heterogeneity* (although this word has several meanings). Shannon's index (1949) and Simpson's index (1949) are heterogeneity measures. Neither richness measures nor heterogeneity measures are responsive to the number of elements in each category, nor are they responsive to degrees of similarity between categories.

To use a richness or heterogeneity measure as a measure of freedom of choice requires partitioning the relevant universal set of possible options into categories in a way analogous to how a set of organisms is partitioned into species. Even though this task need not be problematic, there are other problems with using the measures for freedom of choice. Neither of the measures is responsive to the number of options or the degrees of differences between categories. A set containing four socialist parties and four liberal parties is ranked as equally diverse as a set containing two socialist parties and two liberal parties, assuming that both sets exemplify two categories. Also, the second set is ranked as equally diverse as a set containing two socialist parties and two communist parties since both sets exemplify two categories and the degree of differences between categories does not matter. The category model is criticized in Baumgärtner (2007: 7), Nehring and Puppe (2009: 315) and Gravel (2009: 36–37). (For a more thorough presentation of biological diversity measures, see Peet (1974) or Magurran (2004: 61–99).)

The *Similarity relation model* uses a very basic idea to represent similarity relations. It is assumed that an option x is either similar to another option y , or not. If x is similar to y , then the relation *is similar to* holds between x and y . If x is not similar to y , then the relation *is dissimilar to* holds between x and y . Nothing is assumed regarding degrees of similarity between options. This is a disadvantage of this model. It may not be possible to represent the idea that a set containing a left-wing party and a right-wing party offers more freedom of choice than a set containing a left-wing party and a center party. Both sets contain two dissimilar options. The model cannot reflect that the options in the first set are more dissimilar to one another than the options in the second set. The similarity relation model is used by Pattanaik and Xu (2000a) and Peragine and Romero-Medina (2006) as a basis for measures of freedom of choice.

The *Ordinal distance model* represents options as points and differences as ordinal distances. It assumes that there is a distance function that assigns a

distance d to each pair of options (x, y) based on how dissimilar the options are to one another. The distances are only measured on an ordinal scale. The ordinal distance model is a spatial model since a system S consisting of a set of elements X and a distance function d , defined for every pair of elements (x, y) from X , is regarded as a *space*. Because the distances are only ordinal, it is possible to represent that an option x is more similar to an option y than x is to an option z , but it is not possible to specify any exact degree of difference. The model would reflect that a set containing a left-wing party and a central party has more similar options than a set containing a left-wing party and a right-wing party. It would not reflect how large the difference is between the differences of the options in the two sets. The ordinal distance model is used by Bossert, Pattanaik and Xu (2003) for a diversity measure, and again by Pattanaik and Xu (2008).

The *Ratio distance model* represents options as points in space and differences as ratio distances. It is thus also a spatial model. The advantage of the ratio distance model in comparison to the ordinal distance model is that it contains more information. The disadvantage is that it may be difficult to obtain the information that is required to use the model (in fact, there may be no such information at all). For example, differences between political parties may be represented as ordinal distances without controversy, but it is doubtful whether such differences may be represented as ratio distances. Nevertheless, if it is possible to represent differences as ratio distances, there are several advantages. We could compare differences between differences as well as ratios between differences. The model would reflect not only that a set containing a left-wing party and a right-wing party has more different options than a set containing a left-wing party and a central party but also that this difference is smaller than the difference between a set containing a left-wing party and a right-wing party and a set containing two left-wing parties. The ratio distance model is used for measures of diversity by Weitzman (1992), Bossert, Pattanaik and Xu (2003) and Van Hees (2004), and for a measure of freedom of choice by Gustafsson (2010).

The *Tree model* (usually the *Taxonomic tree model* or the *Phylogenetic tree model*) is another biological model for representing similarity relations. It is a combination of the category model and a particular type of spatial model. In the biological tree model, differences between species are represented as distances in a taxonomic tree, while species are represented as nodes of the tree. Diversity measures that are used with this model are constructed by, for example, Vane-Wright et al. (1991), Eiswerth and Haney (1992), Solow et al. (1993), Faith (1996) and Ricotta (2004). These measures are responsive to degrees of differences, but they are not always responsive to an increase in the number of individuals. The last problem can be corrected, but to be able to respond to an increase in the number of individuals, the model has to be based on differences between individuals and not differences between species. In any case, the model is unsuitable for

representing options that cannot be related by a tree structure. It is doubtful whether it could be used as a basis for measures of freedom of choice.

The *Vector model* is used to represent different kinds of similarity relations between options. The model makes it possible to distinguish between similarity relations that are due to different attributes. To be able to use the vector model, each attribute has to be measurable. Each option may then be represented as a vector of numbers, where each number represents to which degree the option has a specific attribute. Differences between options are represented as differences between vectors. The vector model was introduced by Lancaster in 1966. A measure of freedom by Rosenbaum (2000) is designed for this model. There are also a number of measures of diversity that apply to vectors of numbers. The method of *majorization*, used for a partial ranking of sets in terms of diversity, applies to vectors. So does a class of diversity measures that was proposed by Kreutz-Delgado and Rao (1999). The vector model has the disadvantage that only similarity relations that hold between options in virtue of them having the same attributes are represented, not similarity relations that hold between options in virtue of them having different attributes. For example, if political parties are represented by vectors of numbers that represent their views on the importance of equality, freedom of choice, and autonomy, the model does not capture that freedom of choice is more similar to autonomy than freedom of choice is similar to equality.

The *Metric space model* is a type of ratio distance model, with some additional requirements on the distance function used to represent differences. As with any spatial model, the options are either represented as points or as volumes in space. Discrete options are represented as points, while continuous options are represented as volumes. Metric spaces also have dimensions. In a technical sense, the dimensions of a metric space are the independent coordinates that are required to uniquely specify the points in the metric space. In an intuitive sense, the dimensions of a metric space are the quantitative attribute that are required to uniquely identify the options.

There are several advantages of using metric space distances. One advantage is that metric space distances are most commonly used to represent differences in natural sciences, with *Euclidean distances* and *City block distances* being the most popular (for example to represent differences among unitary or separable stimuli). The use of metric space distances as a basis for a measure of freedom of choice thus allows the measure to be used in a scientific context. Another advantage is that metric space distances are convenient in testing our intuitions regarding freedom of choice since metric distances are the most familiar to us, especially Euclidean distances and City block distances. The *Euclidean space model* is used for measures of freedom of choice by Klemisch-Ahlert (1993), Pattanaik and Xu (2000b), Xu (2003, 2004) and Savaglio and Vannucci (2006). It is used for measures of diversity

by Champely and Chessel (2002), Van Hees (2004) and Lacevic and Arnaldi (2010). The idea that choice sets may be represented as volumes of continuous options is used by Pattanaik and Xu (2000b), Xu (2004) and Savaglio and Vannucci (2006) and Lacevic and Arnaldi (2010). I shall also use the *Metric space model* in this thesis.

4.4 Preference-Dependent Models

The models used when measuring preference-dependent conceptions are less varied than the models used for the preference-independent conceptions. It is customary to just use the cardinality model and add information about the values of the options. The values of the options are usually assigned according to the preferences of the choosing person. Preference relations are assumed to be either ordinal scale or ratio scale. Ordinal preference relations are used in essays by Sen (1991), Foster (1993), Puppe (1996, 1998), Pattanaik and Xu (1998), Sugden (1998), Romero-Medina (2001), Peragine and Romero-Medina (2006), and Puppe and Xu (2010). Ratio scale preference orderings are used in an essay by Jones and Sugden (1982), or so it seems.

The most variety among preference-based measures occurs in the type of preferences that are used. We shall look at four suggestions. The first and most general idea is to assume that all logically possible preference orderings are relevant for freedom of choice. If this is assumed, then there is no practical difference between preference-dependent and preference-independent conceptions of freedom of choice, at least not for the conceptions of freedom of rational choice and freedom of eligible choice. Any option could rationally be chosen relative to the set of all logically possible preference orderings since any option is at least as preferable as all the others, according to at least one logically possible preference ordering. Similarly, any option would be sufficiently valuable relative to all logically possible preference orderings since it would be sufficiently valuable according to at least one logically possible preference ordering. As for freedom of evaluated choice, all options would be of equal value, relative to the whole set of all logically possible preference orderings. One reason for assuming that all possible preferences are relevant is that we wish to measure freedom of choice before an evaluation of the options has been made. Another reason is that we may believe that there are normative reasons not to restrict the freedom-relevant preferences in any way. Especially Sugden, in later essays, is critical of the idea of restricting freedom-relevant preferences at all (see, for example, 2003: 797, 2006: 34).

A second, more restricted idea is to assume that only a person's actual preference ordering at the time of choice is relevant for freedom of choice. This idea is not very popular, for various reasons. One reason is that degrees

of freedom of choice are often assessed for future situations of choice, for which the actual preferences of a person are currently unknown (see Sen (1993: 529)). Another reason is that using only the chooser's actual preferences seems to ignore that the chooser should be regarded as an autonomous person who is able to change his preferences (see Sugden (1998a: 323, 2003: 791)).

A third idea is to assume that only reasonable preferences are relevant for freedom of choice. This is what Pattanaik and Xu (1998) propose. Their idea is also taken up by Romero-Medina (2001), Peragine and Romero-Medina (2006), and Van Hees (2010). A major problem for this approach is that it is uncertain whether reasonableness can be applied to preferences at all, in particular to final preferences. (Hume famously denies this ([1739–1740], part 3, section 3).) The idea is also met by suspicion by those who doubt that there exists any objective standard of reasonableness, such as Sugden (1998: 225), or think that it is difficult to determine which preferences are reasonable, such as Puppe (1998: 53). There are also other critiques of the proposal of using reasonable preferences. One critique is that there is no reason to think that all reasonable preferences should be regarded as relevant for all agents. This critique is made by Arneson (1998: 178) and Puppe (1998: 53). Another critique is that reasonable preferences may not be relevant for a person who is unaware of why the preferences are reasonable. This critique is made by Bavetta and Peragine (2006: 35). Yet another critique is that unreasonable people's preferences are excluded from mattering to freedom of choice. This is Van Hees's and Wissenburg's critique (1999: 76).

A fourth idea is to assume that only a person's potential preferences are relevant for freedom of choice. This is what Jones and Sugden (1982), Sugden (1998), Bavetta and Peragine (2006), Peragine and Romero-Medina (2006) and Foster (2011) assume. 'Potential preferences' could be understood as "psychologically possible preferences". This would either be the preference orderings that a person might develop (given that he is unable to choose among preference orderings) or the preference orderings that a person might choose (given that he is able to choose among preference orderings). An advantage of this suggestion is that we may retain the idea that it is the chooser's preferences that are important without assuming that it is determined which preferences they are. It is possible to assess degrees of freedom of choice before the actual situation of choice occurs.

In this thesis, I shall use psychologically possible preferences as a basis for the preference-dependent measures of freedom of choice. Technically, this matters since the psychologically possible preferences may differ between persons. What one person may possibly prefer may not be possible for another. When comparing the choice sets of different persons, from a preference-dependent freedom of choice point of view, different preference orderings may have to be used for each person.

Part 2: Discussing Conceptions, Conditions and Measures

Chapter 5: Model of Thesis

In this part of the thesis I shall discuss the different conceptions of freedom of choice, together with conditions for their measures and proposals for measures. I shall only use one model as a basis for the discussion; a combination of a decision-theoretical model to represent the situation of choice, and a metric space model to represent differences among the options. The use of the Decision-theoretical model hardly requires a defense, but the Metric space model may require one since it needs a lot of information.

To use the metric space model, we must assume that the options can be counted and that their differences can be represented as ratio scale metric distances. This may be difficult in practice. There are many things that we must be able to do. First, we must be able to individuate acts. This involves separating acts from the continuous natural processes in which they occur. They must be separated from their conditions, causes, circumstances and consequences, as well as from other acts. Second, we must be able to measure differences. This involves recognizing all the different ways in which any two acts may differ and to represent all these differences as a single distance. This presupposes that all differences are comparable, despite being differences of different kinds. Third, we must be able to do all of these things, not just for actual acts, but also for possible acts. Such acts are not even empirically available.

Even though the metric space model may be too demanding for practical use, there are advantages of using it for theorizing. One advantage is that the investigation of freedom of choice is not hampered by the assumption that there is insufficient information about the relevant properties. The metric space model should contain all the information that we need. Another advantage is that we need not exclude any measures from our investigation because they are designed for other kinds of models. The metric space model can be used with most kinds of measures, whether they are designed for the metric space model or not. Measures designed for the cardinality model can be used with the metric space model since that model includes information about the cardinality of sets. Measures designed for the similarity relation model can be used if we decide at which distance two options should be regarded as similar. Measures defined for ordinal distances can be used for ratio distances while measures defined for ratio distances can be used for metric distances. As for categories and attributes, it is possible to define both categories and attributes relative to a spatial model. A category may be

defined as a class of elements that are all located within a certain distance to each other. An attribute may be defined as a spatial dimension. Finally, the vector model is easy to transform into a spatial model by regarding each attribute as a spatial dimension.

5.1 Decision-Theoretical Model

The traditional decision-theoretical model was presented briefly in the first chapter. In accordance with that model I shall assume that some person P in some situation S is trying to decide which act to perform. The acts that P may perform are the options of P . I shall not ascribe probabilities to the different outcomes, but rather assume that the outcomes are known. A person's choice of an option may thus be regarded as a choice of an act and its outcome. As an example we may use a student who is choosing to register for either a course in ethics or a course in logic, at exactly 1 PM on the 11th of April, 2011, at the Philosophy Department of Uppsala University.

5.1.1 The Universal Set

More formally, I shall assume that there is a universal set of possible options. The possible options are referred to by small letters $x, y \dots$ and so on, while the universal set is referred to by X . The universal set may either be understood as the set of all nomologically possible human acts, or as a relevant proper subset of that set. I shall call the first set the 'the universal set', and the second set the 'relevant universal set'. While the first set includes all the acts that it is nomologically possible for a human to perform, the second type of set only includes all the acts that are nomologically possible for a human to perform and that are judged to be relevant for some specific purpose. (For example, all the acts of registering for various summer courses.) I shall assume that both these sets are finite.

No matter how X is defined, we may also define a power set $P(X)$, which is the set of all subsets of X . Some subsets of X are possible choice sets, which are denoted by capital letters $A, B \dots$ and so on. They are the sets of actions that some person P can choose to perform in some situation. For example, the set of registering for a course in ethics and registering for a course in logic may be a choice set for a student visiting a philosophy department during daytime on a regular weekday. The set of registering for a course on flying and registering for a course on diving is not a choice set in the same situation.

5.1.2 Preferences

I shall also assume that the person P has different preferential attitudes towards the different acts, or rather towards the different outcomes of the different acts. The outcomes are states of affairs. A person's preferential attitude towards a state of affairs holds at some specific time t_i (or time interval t_i to t_n).

In a comparative sense, a person may have one of three preferential attitudes towards the holdings of two different states of affairs. The person P may *prefer* the holding of S rather than S^* , or *prefer* the holding of S^* rather than S , or *be indifferent* between the holdings of the two states of affairs. The preferential attitudes may also be more or less intensive. Thus, P may prefer S holding over S^* with a greater intensity than P prefers S^* holding over S^{**} .

In a categorical sense, a state of affairs S is *good* when P prefers S holding over S not holding, *bad* when P prefers S not holding over S holding, and *neutral* when P is indifferent between S holding and S not holding (at least relative to P 's current preferences).

For example, at 1 PM, on the 11th of April 2011, a student may prefer registering for ethics over registering for logic. He may further prefer registering for ethics over not registering for ethics with a greater intensity than he prefers registering for ethics over registering for logic. Since he prefers registering for ethics over not registering for ethics and registering for logic over not registering for logic, both states of affairs are good, relative to P 's current preferences.

5.1.3 Preference Orderings

A person P 's preferential attitudes towards the options x in X , at any given time t_i , will order the options in X . The preference ordering Y over X is constructed through the relation *is weakly preferred to*, which is denoted by R . The relation R is assumed to be reflexive, transitive and complete. In a more technical language, the preference ordering Y is a relational system, $Y = \langle X, R \rangle$. There is a set Z that contains all logically possible preference orderings Y over the options in X . There are also subsets of Z that contain preference orderings of a certain kind, such as P 's actual preference orderings (present, past, future) and P 's probable preference orderings (past, present, future). Here I shall use the set K_P , which is the set of preference orderings that is relevant for some person P for judging P 's freedom of choice. For a student who is registering for one of two possible summer courses, the relevant set of preference orderings may be the student's probable preference orderings before the summer, during the summer and after the summer.

I shall also assume that all the preference orderings are completely comparable across times and persons. It is thus meaningful to say that a

person P with a preference ordering Y at a time t_i prefers an option x at least as much as P with another preference ordering Y^* at another time t_j . It is also meaningful to say that a person P with a preference ordering Y prefers an option x at least as much as a different person P^* with a different preference ordering Y^* prefers an option y . The assumption of complete comparability may seem unrealistic since we cannot make precise intertemporal or interpersonal comparisons regarding strength of preferences. However, the assumption is done to simplify the discussion. In the example with the student we shall assume that we can compare the student's present preferences with his preferences at other times and the preferences of other persons. It is thus meaningful to say that the student favors registering for ethics less before the summer than after the summer and that he favors registering for ethics less than some of the other students.

5.1.4 Utility Functions

A person P 's preferential attitudes towards the options x in X may be represented by real numbers. More precisely put, if \mathbf{R} is the set of real numbers, and \geq is the relation *at least as great as*, then any preference ordering $Y = \langle X, R \rangle$ may be represented by a mathematical system, $M = \langle \mathbf{R}, \geq \rangle$, through a function v that is such that $v(x) \geq v(y)$ if and only if xRy . The function v is a *utility function*. I shall assume that P 's preferences may be represented on a bipolar ratio scale.

As I have mentioned, the set Z contains all logically possible preference orderings Y over the options in X . There is also a set W of all possible scales representing all logically possible preference orderings Y by M via all possible utility functions v . I shall use a subset of W here, a set L , which is a set of utility functions, where each utility function v_i represents a relevant preference ordering Y from the set of relevant preference orderings K_P .

When I speak of *the value of some state of affairs* S , I shall mean the value that is assigned to S by some value function that represents the preferences of some person P . The number 0 is always assigned to indifferent states of affairs, while positive numbers are assigned to good states of affairs and negative numbers are assigned to bad states of affairs.

Because I have assumed that all preference orderings are comparable, the selection of one utility function to represent one logically possible preference ordering automatically implies the selection of a class of utility functions to represent all other logically possible preference orderings. The functions assign the same numerical value to equally valuable options with respect to different preference orderings.

If we index each option x of X as $x_1, x_2 \dots x_n$ and index each utility function v_i of L as $v_1, v_2 \dots v_n$ then we can represent a value assignment in the form of a table. An example of a student's value assignment to a choice set of summer courses would be:

	v_1
Ethics	10
Logic	5

This table could be expanded to show the student's value assignments to the same choice set at different times, for example before taking the course (v_1), during the course (v_2) and after the course (v_3). For example:

	v_1	v_2	v_3
Ethics	10	15	20
Logic	5	0	15

Similar tables can be made to compare the preferences of different students.

(For more on the traditional decision-theoretical model see, for example, von Neumann and Morgenstern (1956) or Savage (1972).)

5.2 Metric Space Model

I shall also specify the *Metric space model* that is used for the discussion more precisely. To do this I shall assume that there is a finite universal set of possible options, X , and that there is a function d that assigns a distance to all ordered pairs of possible options (x, y) from X . The distance $d(x, y)$ represents how different x and y are to one another either overall or in all relevant aspects. The system $X_d = \langle X, d \rangle$ that consists of the universal set X and the distance function d is a *space*. The subspaces of X_d shall be denoted A_d, B_d etc. The set $P(X_d)$ is the set of all subspaces of X_d . For all choice sets $A, B \in P(X)$, there are thus corresponding finite spaces $A_d, B_d \in P(X_d)$. Since the terms 'space' and 'subspace' are more technical and less familiar than the terms 'set' and 'subset', I shall often use the latter terms. Sometimes I shall use the expression 'metric set'. Mostly I shall use the symbols A, B (and so on) rather than the symbols A_d, B_d (and so on). But it is important to keep in mind that I always refer to choice sets with distances defined for each pair of options.

I shall also assume that X is a *metric space*, which means that there are additional requirements on the distance function d . For all $x, y, z \in X$, the distance function d satisfies:

- 1) *Non-negativity*: $d(x, y) \geq 0$.
- 2) *Identity of indiscernibles*: $d(x, y) = 0$ if and only if $x = y$.
- 3) *Symmetry*: $d(x, y) = d(y, x)$.
- 4) *Triangle inequality*: $d(x, y) \leq d(x, z) + d(z, y)$.

The metric space $X_d = \langle X, d \rangle$ can be associated with an indexed metric space \mathbf{M}_X , where the distances are indexed with positive integers and arranged in the form of a matrix. To construct this matrix, the options in X are first indexed from 1 to n , as $x_1, x_2 \dots x_{n-1}, x_n$. Next, the distances between the options are indexed from 1 to N , so that $d_1 = d(x_1, x_1)$, $d_2 = d(x_1, x_2) \dots d_{N-1} = d(x_n, x_{n-1})$, $d_N = d(x_n, x_n)$. Last, the matrix is arranged so that each row shows the distances from one option $x \in X$ to all other options $y \in X$. For example, if $A = \{x, y, z\}$ and $d(x, x) = 0$, $d(x, y) = 5$, $d(x, z) = 10$, $d(y, x) = 5$, $d(y, y) = 0$, $d(y, z) = 5$, and so on, the distances are indexed as d_1 to d_n , where $d_1 = d(x, x) = 0$, $d_2 = d(x, y) = 5$, and so on. The distances are then arranged in the form of a matrix \mathbf{M}_A :

0	5	10
5	0	5
10	5	0

It is also possible to include information regarding the different options:

	x	y	z
x	0	5	10
y	5	0	5
z	10	5	0

A distance between two options in A is denoted by d_{Ai} . The maximal distance between the options in A is denoted either by $\max d_{Ai}$ or by $\text{diam}(A)$. It is called the *diameter* of A .

When the number of options in A is n , the number of distances between the options in A is n^2 , with $n^2 - n$ distances being non-zero distances. The symbol N shall be used for n^2 . When discussing a total sum of distances between the options in A , I shall often use the notation:

$$\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j).$$

This is to make it clear that both the distances $d(x_i, x_j)$ and the symmetrical distances $d(x_j, x_i)$ are summed. At other times I shall use the notation $\Sigma(A_d)$ because it is simpler. The notation $\Sigma(x_A)$ is used for the total sum of distances from an option $x \in A$, to all the other options in A . In the example above, $\Sigma(A_d) = 40$, while $\Sigma(x_A) = 15$.

For some purposes, we do not need to know how the distances are distributed between the different options. We only need to know how the

distances between the options in different sets compare in size. For these purposes, we define a function that maps the elements of a matrix to a vector of suitable size. The vector \mathbf{d}_A contains all the distances between the options in some set A arranged in weakly decreasing order. Thus, $\mathbf{d}_A = (d_{A1}, d_{A2} \dots d_{A_{n-1}}, d_{An})$, where $d_{A1} \geq d_{A2} \geq \dots \geq d_{A_{n-1}} \geq d_{An}$. In the example above $\mathbf{d}_A = (10, 10, 5, 5, 5, 5, 0, 0, 0)$. The vector $\mathbf{d}_A(\mathbf{x})$ is the vector that contains all the distances from an element $x \in A$ to all other options in A arranged in weakly decreasing order. In the example above $\mathbf{d}_A(\mathbf{x}) = (10, 5, 0)$.

Some metric spaces $X_d = \langle X, d \rangle$ can also be associated with a coordinate system. This holds for all finite-dimensional linear metric spaces, such as Euclidean spaces and City block spaces. A coordinate system assigns a vector of coordinates to each option (regarded as a point). The option x is assigned the coordinate vector $(x_1, x_2 \dots x_n)$, where x_i identifies the position of the option relative to the i th dimension. Different types of metric space distances may be identified through different relations between points that are described by coordinate vectors. For *Euclidean space distances*, it holds that if $x = (x_1, x_2 \dots x_n)$ and $y = (y_1, y_2 \dots y_n)$, then the distance between x and y is:

$$\sqrt{\sum_{i=1}^n (x_i - y_i)^2}.$$

For *City block distances*, the distance between x and y is:

$$\sum_{i=1}^n |x_i - y_i|.$$

If we add the assumption that the options x, y and z from the example above only vary in one dimension, we may (for example) represent them by the coordinate vectors $x = (0), y = (5)$ and $z = (10)$. The set $A = \{0, 5, 10\}$ may then be represented by the set of coordinate vectors $A^c = \{(0), (5), (10)\}$.

Let us look at two examples here. In the first example we shall model the situation of a teacher who is choosing among different amounts of teaching hours per week for a specific semester. He can choose to work full time, three quarter time or half time, but not anything in between or below that. He thus has the choice set $A = \{\text{accept 20 h, accept 30 h, accept 40 h}\}$, or simply: $A = \{20 \text{ h, } 30 \text{ h, } 40 \text{ h}\}$. There is a metric distance function that assigns $d(20 \text{ h, } 20 \text{ h}) = 0, d(20 \text{ h, } 30 \text{ h}) = 10, d(20 \text{ h, } 40 \text{ h}) = 20$, and so on.

This is the distance matrix \mathbf{M}_B :

	20 h	30 h	40 h
20 h	0	10	20
30 h	10	0	10
40 h	20	10	0

The options may be represented in Euclidean space as points on a straight line. In this context, it is most natural to consider work hours as one

dimension and represent the options through a single coordinate. For example, 20 h may be represented by the vector (20). The set B may then be represented by the coordinate vector $B^c = \{(20), (30), (40)\}$.

In the second example we shall model the choice of a teacher who is choosing among houses to rent for a semester. She has the choice set $C = \{Shed, Villa, Mansion\}$. For all houses, l = length, b = breadth and h = height. The shed has $l = 3$ m, $b = 3$ m and $h = 3$ m. The villa has $l = 12$ m, $b = 9$ m and $h = 3$ m. The mansion has $l = 30$ m, $b = 18$ m and $h = 9$ m. The three houses are represented in Euclidean space with a coordinate system. Length is the first coordinate, breadth is the second and height is the third. Her choice set may thus be represented by the set of coordinate vectors $C^c = \{(3, 3, 3), (12, 9, 3), (30, 18, 9)\}$. It may also be represented by the distance matrix $\mathbf{M}_C =$

	s	v	m
s	0	$\sqrt{117}$	$\sqrt{990}$
v	$\sqrt{117}$	0	21
m	$\sqrt{990}$	21	0

Similar matrices shall be used throughout the thesis.

Chapter 6: Freedom of Choice as Cardinality

We shall first consider the cardinality conception of freedom of choice. For this, we may repeat the definition from a previous chapter:

The Cardinality conception of freedom of choice: A person P with a choice set A has *at least as much freedom of choice as* a person P^* with a choice set B if and only if A contains *at least as many options as* B .

For example, A contains three options of working hours, while B contains two options, $A = \{0 \text{ h}, 45 \text{ h}, 90 \text{ h}\}$, while $B = \{0 \text{ h}, 99 \text{ h}\}$. One reason we may judge A to offer more freedom of choice than B is that A contains more options than B .

6.1 Measure

The cardinality conception of freedom of choice is very simple. The corresponding measure is equally simple:

The Cardinality measure: $F(A) = |A|$.

F is a function from \mathbf{R} to \mathbf{R} .

The *Cardinality measure* is suggested by Beavis and Rowley (1983). It is also suggested in an ordinal form by Suppes (1987), Pattanaik and Xu (1990), and Bavetta (2004). Furthermore, it is used as an opportunity measure by Ok (1997), and again by Ok and Kranish (1998). It is criticized by Jones and Sugden (1982: 55–56), and Sen (1991: 24–26). It is also criticized by Rosenbaum, for freedom (2000: 208–209). The cardinality conception of choice also occurs in many studies in descriptive decision theory regarding the effects of choice on a person. (Some examples of such studies include Kiesler (1966), Fromkin et al. (1975), Clark et al. (1977), Iyengar and Leppar (2000), Ahlert and Crüger (2004), Hoffrage and White (2009) and Hogarth and Reutskaja (2009).)

It is possible to arrive at the conclusion that the *Cardinality measure* should be used by two ways of reasoning; either it seems immediately obvious or the *Cardinality measure* is accepted as a consequence of the

acceptance of some conditions for a measure of freedom of choice. The first way of reasoning may perhaps be attributed to Beavis and Rowley, who at least think that it is possible to measure freedom of choice by the cardinality of choice sets. The second way of reasoning may be attributed to Pattanaik and Xu who has suggested three conditions for an ordering of choice sets in terms of freedom of choice, which together imply that choice sets should be ordered by their cardinality. Interestingly, Pattanaik and Xu do not endorse the ordinal measure that is the result of their conditions. Instead they have criticized the conditions and formulated other types of measures. These may be found in Pattanaik and Xu (1998, 2000, 2008), Bossert, Pattanaik and Xu (2003), and Xu (2003, 2004). Since Pattanaik's and Xu's essay, Foster has suggested an alternative set of axioms, which are more general, but lead to the same cardinality measure (2011: 695).

Before criticizing the *Cardinality measure*, we shall look at Pattanaik's and Xu's three conditions. If the *Cardinality measure* is unacceptable, at least one of Pattanaik's and Xu's conditions must be unacceptable as well.

6.2 Conditions for a Cardinality Measure

As we mentioned, Pattanaik and Xu's conditions are not the only possible conditions for a cardinality measure. But they are the ones that are most discussed and the ones that we shall consider here.

6.2.1 Indifference between No-Choice Situations

The first condition that Pattanaik and Xu propose is this one (1990: 386):

The Indifference between no-choice situations condition: For all options $x, y \in X$, $\{x\}$ offers as much freedom of choice as $\{y\}$.

According to this condition, the set of working hours $\{20 \text{ h}\}$ offers as much freedom of choice as the set of working hours $\{40 \text{ h}\}$. This seems trivially true since a person who only has one option has no choice. (Perhaps one should even say that one option is not an option at all.) So $\{x\}$ offers as much freedom of choice as $\{y\}$ by offering none. Nevertheless, there are philosophers who have criticized the condition.

One way to criticize the *INS-condition* is to say that since a singleton set does not offer any freedom of choice at all, it cannot offer the *same* amount of freedom of choice as any other set. However, this argument is unconvincing. There is no harm in considering no freedom of choice as a particular degree of freedom of choice. Even if there were, it would be simple to exchange 'offers as much freedom as' for 'offers no freedom of

choice, just as'. This would not change Pattanaik and Xu's conclusion regarding the implication of the three conditions.

The *INS-condition* may also be criticized from the perspective of evaluative conceptions of freedom of choice. Supposedly, if the degree of freedom of choice is dependent on the values of the options, the *INS-condition* may not hold. If x is preferred to y , then the set $\{x\}$ may offer more freedom of choice than the set $\{y\}$. For example, if a person prefers working full time to working half time, then the set $\{40 \text{ h}\}$ may offer more freedom of choice than the set $\{20 \text{ h}\}$. Sen says something along these lines when he suggests that we have more freedom when we are offered a single option which we would have chosen anyway than when we are offered a single option which we would not have chosen counterfactually (1991: 25). However, Sen also says that this comment applies only to the freedom to lead the life we would choose to lead and not to freedom of choice. We have no freedom of choice when we have only one option, no matter what preferences we may have. So even if we use a preference-dependent conception of freedom of choice rather than a preference-independent conception, the freedom of choice offered by a singleton set is nil.

There is another way to criticize the principle from a preference-dependent perspective. The argument is more complicated than the previous one since it mixes different kinds of intuitions regarding freedom of choice. The original argument was made by Jones and Sugden and applied to the value of choice (1982: 57). Later, Gustafsson adapted the argument and applied it to freedom of choice (2011: 52–53). We shall look at Gustafsson's argument here in a simplified version. The argument presupposes that freedom of choice increases with the addition of an option that is strictly preferred to all the other options according to at least one relevant preference ordering (Jones, Sugden and Gustafsson call such an option *significant*.) Gustafsson's version of Jones's and Sugden's first condition looks roughly as follows:

The Significant options strict monotonicity condition: For any choice set $A \in P(X)$, and any option $y \in X - A$, if y is added to A to get $A \cup \{y\}$, and there exists a relevant preference ordering $Y \in K_p$, where y is strictly preferred to x for all $x \in A$, then $A \cup \{y\}$ offers strictly more freedom of choice than A .

The argument also presupposes that freedom of choice remains unchanged with the addition of an option that is not strictly preferred to all the other options according to at least one relevant preference ordering (Jones, Sugden and Gustafsson call such an option *insignificant*.) Gustafsson's version of Jones's and Sugden's second condition looks roughly as follows:

The Insignificant options equivalence condition: For any choice set $A \in P(X)$, and any option $y \in X - A$ if y is added to A to get $A \cup \{y\}$, and there does not exist a relevant preference ordering $Y \in K_p$, where y is strictly preferred to x for all $x \in A$, then $A \cup \{y\}$ offers as much freedom of choice as A .

The two conditions would be more convincing if they contained the phrase ‘is at least as preferable as’ rather than the phrase ‘is strictly preferred to’. But let us not dwell on this detail. Let us instead consider how the conditions may work in an argument against the *INS-condition*. We first suppose that z might be strictly preferred to x , but that x might not be strictly preferred to z . If this is the case, then it follows from the conditions that $\{x, z\}$ offers more freedom of choice than $\{x\}$, and that $\{x, z\}$ offers as much freedom of choice as $\{z\}$. By transitivity, $\{z\}$ offers more freedom of choice than $\{x\}$. Hence, the *INS-condition* does not hold.

This argument is unconvincing. The joint acceptance of the two conditions is a major problem. We have to find some explanation why freedom of choice increases when adding a significant option to a set, but remains the same when adding an insignificant option.

As a start, we may suggest that the conditions should be accepted because ‘freedom of choice’ should be interpreted as “freedom of choice over significant options”, or “significant choice”. But this explanation will not do. Since the chooser might not strictly prefer x over z , the set $\{x, z\}$ does not offer any significant choice. It does not matter if z is added to $\{x\}$, or x is added to $\{z\}$; the degree of significant choice remains the same. Since there is just one significant option in each of the three sets, none of the three sets offer any significant choice. The *Significant options strict monotonicity condition* is therefore incorrect, relative to this analysis. Roughly the same thing can be said regarding freedom of rational choice.

We may also suggest that the conditions should be accepted because ‘freedom of choice’ should be interpreted as “freedom of evaluated choice”, a type of freedom that is a function of the values of the options. But this explanation will not do either. Relative to this analysis, neither of the two conditions is acceptable. If y does not add a positive value to A , then the addition of y to A would not increase freedom of choice, even if y is significant. So the *Significant options strict monotonicity condition* would not hold. Furthermore, if y adds a positive or negative value to A , then the addition of y to A would affect freedom of choice, even if y is insignificant. So the *Insignificant options equivalence condition* would not hold either.

Last, we may suggest that the conditions should be accepted because ‘freedom of choice’ should be interpreted as “freedom of evaluated choice over significant options”. But not even this suggestion works. Let us suppose that A offers a large degree of freedom of choice since A contains a great number of valuable significant options. Next, the option y is added. The

option y is also a significant option. In fact, y is so significant that it makes all other options in A insignificant. In this case, $A \cup \{y\}$ would offer no freedom of choice at all. So the *Significant options strict monotonicity condition* would still be incorrect.

The only conception that the two conditions and the resulting argument would work for may be the conception of the possible future utility gained from choosing an option from either A or $A \cup \{y\}$. But possible future utility is not the same thing as freedom of choice.

It is also doubtful whether Jones and Sugden would accept the revised conditions and the argument above. For one thing, they reject their own version of the *Significant options strict monotonicity condition*, at least for significant choice (1982: 55). For another thing, their conditions and argument concerns the value of choice and conditions that work for the value of choice may not necessarily work for freedom of choice.

In any case, I shall accept the *INS-condition*. Let us thus look at the next condition.

6.2.2 Monotonicity

The next condition that Pattanaik and Xu present is the following (1990: 386):

The Limited strict monotonicity condition: For all options $x, y \in X$, $\{x, y\}$ offers strictly more freedom of choice than $\{x\}$.

(Pattanaik and Xu calls the condition ‘Strict monotonicity’.) This condition seems trivially true since one option offers no freedom of choice but two options offer some freedom of choice.

The *Limited strict monotonicity condition* has been disputed, though. The condition has mostly been criticized from the perspective of preference-dependent freedom of choice. The condition may not hold if the values of the options are taken into account. There are at least two ways this criticism may be construed. One way to argue against the *Limited strict monotonicity condition* is to propose that $\{x, y\}$ does not offer more freedom of choice than $\{x\}$ when y cannot be preferred to x and therefore cannot rationally be chosen. This is to use the conception of freedom of rational choice. For example, if a teacher prefers working full time, then he would not have more freedom of choice when offered the choice of working 0 h and 40 h than when he is forced to work 40 h. We shall consider this type of objection when we discuss freedom of rational choice.

Another way to argue against the *Limited strict monotonicity condition* is to propose that $\{x, y\}$ may not offer more freedom of choice than $\{x\}$ since y may be so bad that it cannot contribute to freedom of choice. This is to use the conception of freedom of eligible choice. Not working may be so bad

that the teacher would not have any more freedom of choice when offered the choice between working 0 h and 40 h than when he is forced to work 40 h. We shall consider this type of objection when we discuss freedom of eligible choice.

A third way to argue against the *Limited strict monotonicity condition* is to propose that some kinds of options do not contribute to freedom of choice because they are irrelevant. If we are interested in the freedom of choice regarding which books to read, having a sandwich to eat is irrelevant and does not contribute to the freedom of choice. This is not a good objection either. It is perfectly sensible to say that when we are interested in freedom of choice of a specific type, an irrelevant option should not be considered an option at all. If we are interested in the freedom of choice of which books to read, having a sandwich to eat should not be considered an option. It is thus possible to save the *Limited strict monotonicity condition* from the critique of irrelevance by putting a limit on what should count as options.

A fourth way to argue against the *Limited strict monotonicity condition* is to propose that an option y may not contribute to the freedom of choice offered by $\{x, y\}$ when y is very similar to x . Perhaps, when the two options x and y are so similar that the difference between them is not detectable by the chooser, $\{x, y\}$ does not offer any freedom of choice. Jones and Sugden bring up this objection (1982: 56). There are different ways to formulate the objection. One could say that $\{x, y\}$ does not offer any freedom of choice when the difference between x and y represents a distance of 0. But this critique is a misunderstanding. The distance function has the property of *Identity of indiscernibles*. This means that the distance 0 holds only between identical options. So if the distance between x and y equals 0, then $x = y$. In this case the condition simply does not apply. If the distance function is designed according to the chooser's ability to discriminate and the chooser cannot discriminate between x and y , then x and y should be regarded as the same option, and thus x and y offer no choice. If the distance function is not designed according to the chooser's ability to discriminate, then the chooser would have a choice between x and y , although it may seem strange.

But what if the difference between x and y is detectable by the chooser, but just barely? We could oppose the condition by saying that some options x and y , even though they are different, are too similar to give any freedom of choice. This is done by, for example, Norman (1987: 38) and Pettit (2003: 392). Whether a teacher may work 40 h or 41 h should not count as a choice between two options. The difference is too small.

This is not a good objection either. The idea that two options must be sufficiently different to offer freedom of choice leads to sorites types of problems. Where should we draw the line between two options being sufficiently different and two options not being sufficiently different? For most areas of interest, any suggestion would be question begging. It would also be unnecessary. We do not have to give up the *Limited strict*

monotonicity condition to solve the problem that the addition of very similar options poses. Even though the addition of a very similar option to a choice set does not contribute very much to freedom of choice, it contributes something. It contributes very little. And very little is more than nothing. For this reason I shall accept the *Limited strict monotonicity condition*.

6.2.3 Independence

The third condition that Pattanaik and Xu propose is this one (1990: 386):

The Independence condition: For all choice sets $A, B \in P(X)$, and any option $y \in X - A \cup B$, A offers at least as much freedom of choice as B if and only if $A \cup \{y\}$ offers at least as much freedom of choice as $B \cup \{y\}$.

This condition also occurs in Sen, under a different name (1991: 23). The conjunction of this condition with the *INS-condition* and the *Limited strict monotonicity condition* implies a cardinality measure of freedom of choice. But the last condition is not very reasonable if the differences of the options matter. The *Independence condition* can be criticized by pointing out that if y is very different from the options in B , while being very similar to the options in A , then $B \cup \{y\}$ might offer at least as much freedom of choice as $A \cup \{y\}$, even though A offers more freedom of choice than B . This is pointed out both by Sugden (1998: 318) and Carter (2004: 78).

This critique is so convincing that it is hard to come up with any defense of the *Independence condition*. So the *Independence condition* should be abandoned.

6.2.4 Upper and Lower Limits

Before abandoning the *Cardinality measure*, along with the *Independence condition*, we shall just consider some additional conditions that the measure satisfies. The *Cardinality measure* trivially satisfies a condition regarding the maximal amount of freedom of choice that is offered by a choice set:

The Maximal freedom of choice condition: For any choice set $A \in P(X)$, if $A \neq X$, then X offers strictly more freedom of choice than A .

This condition is implied by the *Strict monotonicity condition*, which we shall discuss later. I mention this condition here, not because it is in any sense remarkable that the *Cardinality measure* satisfies the condition, but because we shall look at some measures that fail to satisfy the condition later.

The *Cardinality measure* has no counterintuitive implications regarding the upper limit of freedom of choice, but it has a slightly problematic implication regarding the lower limit. Since a singleton set has one more option than the empty set, the *Cardinality measure* implies that singleton sets offer more freedom of choice than the empty set. This is unacceptable since none of these sets offer any freedom of choice. The *Cardinality measure* thus fails to satisfy the following reasonable condition:

The No freedom of choice condition: For any choice set $A \in P(X)$, if $|A| \leq 1$, then A offers no freedom of choice.

A similar condition is suggested by Rosenbaum (2000: 212). The *No freedom of choice condition* implies the *INS-condition*, but not vice versa. This objection is not a serious objection to the *Cardinality measure* since the measure can easily be changed as follows:

Modified cardinality measure:

- i) For $n \leq 1$, $F(A) = 0$.
- ii) For $n > 1$, $F(A) = |A|$.

F is a function from \mathbf{R} to \mathbf{R} .

However, there are other problems with the *Cardinality measure* that cannot be solved as easily, which we shall look at next.

6.3 Problems

We have considered some objections to the conditions whose acceptance results in an acceptance of the *Cardinality measure*. But there are also objections that can be made directly against the *Cardinality measure*. One objection is that the *Cardinality measure* ignores the values of the options. Why this is objectionable may be explained in different ways. Another objection is that the *Cardinality measure* ignores the differences between the options, which are highly relevant for freedom of choice. When we compare $A = \{0 \text{ h}, 1 \text{ h}, 2 \text{ h}, 3 \text{ h}, 4 \text{ h}, 5 \text{ h}\}$ and $B = \{0 \text{ h}, 40 \text{ h}\}$, it seems wrong to judge that A offers more freedom of choice just because A contains more options than B . Clearly it should matter that the options in B are more different to one another than are the options in A . There is thus something missing from the cardinality conception of freedom of choice.

We shall consider different proposals for how the cardinality conception may be adjusted to remedy its flaws. We shall begin by considering how it may be adjusted to take the values of the options into account. First, we shall consider the proposal that only options that may rationally be chosen should count as options. Next, we shall consider the proposal that only options that

are sufficiently valuable should count as options. Then we shall consider the proposal that each option contributes to freedom of choice in accordance with its value. Last, we shall consider a hybrid measure that depends both on the number and values of the options.

Later on we shall consider different proposals for how the conception may be adjusted to take the differences between the options into account.

6.4 Freedom of Rational Choice

One complaint that may be raised against the original cardinality conception is that it takes too many options into account. It counts options that may not rationally be chosen. As we may recall, an option x may rationally be chosen by a person P from a choice set A if and only if x is at least as preferable as any other option $z \in A$, according to at least one preference ordering that is relevant for P . At least after the evaluation stage of choice, there seems to be no point in having options that may not rationally be chosen. Such options, therefore, do not contribute to freedom of choice. For example, compare the choice sets $A = \{\text{one \$100 bill, another \$100 bill}\}$ and $A \cup \{y\} = \{\text{one \$100 bill, another \$100 bill, one \$1 bill}\}$. In A there are two options, and in $A \cup \{y\}$ there are three. According to the original cardinality conception, $A \cup \{y\}$ offers more freedom of choice than A . But there are no more options in $A \cup \{y\}$ that may rationally be chosen than there are in A . Both sets contain two options that may rationally be chosen (at least assuming some normal preference ordering). So, perhaps A offers as much freedom of choice as $A \cup \{y\}$. If we agree with this reasoning, we may adopt the cardinality conception of freedom of rational choice:

The Cardinality conception of freedom of rational choice: A person P with a choice set A , has at least as much freedom of choice as a person P^ with a choice set B if and only if A contains at least as many options that P may rationally choose as B contains options that P^* may rationally choose.*

This conception occurs in an essay by Pattanaik and Xu (1998). A similar diversity conception occurs in Bervoets and Gravel (2007).

One argument for the adoption of “freedom of rational choice” as an appropriate analysis of ‘freedom of choice’ is that it fits with ordinary language use. There are at least some expressions that can be understood only if ‘freedom of choice’ is interpreted as “freedom of rational choice”; for example, expressions like ‘I had no choice’, when an agent explains why a superior act was chosen over its inferior alternatives. This expression can be used to explain why an agent gave money to a robber (that act was superior to act of defying the robber), got engaged to be married (that act was

superior to the act of declining a proposal) or ordered salad at a restaurant (that act was superior to ordering any of the non-vegetarian dishes).

However, against this argument we may note that there are also expressions that are unintelligible if ‘freedom of choice’ is interpreted as “freedom of rational choice”. We cannot intelligibly say that “I choose to get married” or “I choose to eat salad”, given that these acts were preferred. This subject is also discussed by Benn and Weinstein (1971: 197), and more extensively by Wertheimer (1987: 192–201).

There is another reason to take a special interest in freedom of rational choice, however. It is a particularly useful kind of freedom of choice. After all, it is the choice among the options that the chooser is left to consider after evaluation (assuming that the chooser is rational). Having said that, we should note that freedom of rational choice is not a particularly instrumentally valuable kind of freedom of choice, in the sense of contributing to the choice of a better option. It is possible that some person has more freedom of rational choice when choosing among bad options than when choosing among good options.

6.4.1 Conditions and Cardinality Measure

The options that may rationally be chosen may be related to different preference orderings. The common decision-theoretical model uses only the chooser’s present preference ordering at the time of choice. This idea is problematic if we wish to assess an agent’s future freedom of choice at a time when the agent’s future preferences are unknown. Pattanaik and Xu propose that all reasonable preference orderings should be used to select the options that could rationally be chosen. As we have seen, there are many difficulties involved in this proposal. Here we shall rather use the class of all psychologically possible preferences. However, the exact choice of relevant preference orderings does not matter for the general discussion.

The main intuition behind freedom of rational choice may be formulated in the following condition:

The Rational choice condition: For any choice set $A \in P(X)$ such that $|A| \geq 1$, all options $y, z \in X - A$, and any set of preference orderings K_P , if $Mal(A, K_P) \subset Mal(A \cup \{y\}, K_P)$ and $Mal(A, K_P) \not\subset Mal(A \cup \{z\}, K_P)$, then $A \cup \{y\}$ offers strictly more freedom of choice than $A \cup \{z\}$.

The set $Mal(A, K_P)$ is the set of all x included in A such that x is at least as preferable as all other options in A for at least one preference ordering in K_P , which is the set of relevant preference orderings for P .

There are two ways in which $Mal(A, K_P)$ may be a subset of $Mal(A \cup \{y\}, K_P)$. One possibility is that y is indifferent to some option

$x \in Mal(A, K_P)$ for at least one preference ordering in K_P . Another possibility is that y is preferred to some option $x \in Mal(A, K_P)$ for at least one preference ordering in K_P , but that x is preferred to y for some other preference ordering in K_P .

We should also look at how the conditions for the cardinality conception of freedom of rational choice differ from the ordinary conditions. The *INS-condition* continues to hold for the cardinality conception of freedom of rational choice. The *Maximal freedom of choice* condition no longer holds. The universal set of options may not be the set that offers a maximal amount of freedom of choice. The *No freedom of choice condition* continues to hold only if we add the same assumption that was used for the *Modified cardinality measure*: sets of less than two options do not offer any choice. The *Limited strict monotonicity condition* continues to hold only if it is applied to sets of rationally relevant options; that is $\{x, y\}$ offers strictly more freedom of choice than $\{x\}$ provided that x and y are of equal value. If the *Limited strict monotonicity condition* is applied to other sets than sets of rationally relevant options, then it no longer holds. It should thus be replaced, perhaps with this condition:

The Indifference monotonicity condition: For any choice set $A \in P(X)$ such that $|A| \geq 1$, any option $y \in X - A$, and any set of preference orderings K_P , if $Mal(A, K_P) \subset Mal(A \cup \{y\}, K_P)$, then $A \cup \{y\}$ offers strictly more freedom of choice than A .

A similar condition is suggested by Pattanaik and Xu (1998: 184).

Pattanaik and Xu propose an ordinal cardinality measure that may work as a measure of freedom of rational choice. They do not intend it to be a measure of freedom of rational choice, but rather to be a measure of the intrinsic value of freedom of choice. However, this proposal would only work if the intrinsic value of freedom of choice could be measured in the same way as freedom of choice. This is only possible if the intrinsic value of freedom of choice is a linear function of degrees of freedom of choice. If the intrinsic value of freedom of choice is measured according to strength of preferences, this is doubtful. Thus, the measure seems better suited as a measure of freedom of rational choice.

Regardless of how the measure is used, it is axiomatized in Pattanaik's and Xu's essay. They show that four conditions are necessary and sufficient to characterize the measure (1998: 187). The first condition is the familiar *INS-condition*. The second condition is a version of the *Insignificant options equivalence condition*, previously suggested by Jones and Sugden (1982: 57). The third condition is a *Composition condition* and concerns relationships among unions of sets. It is a variant of a condition suggested by Sen (1991: 23). The fourth condition is an *Indifference monotonicity condition*, similar to the one suggested above (1998: 184). If we accept these

four conditions, we should also accept the ordinal measure which is implied by them. Here we shall consider a ratio scale version of the ordinal measure:

Rational choice measure:

$$F(A, K_P) = |Mal(A, K_P)|.$$

F is a function from \mathbf{R} to \mathbf{R} , K_P is a set of relevant preference orderings for P and $Mal(A, K_P)$ is the set of all x included in A such that x is an option that is at least as preferable as all other options in A for at least one preference ordering in K_P (1998: 187).

This measure could be criticized in roughly the same way as the preference-independent cardinality measure, and in some additional ways that have to do with the idea that freedom of choice should be understood as freedom of rational choice.

6.4.2 Problems

One thing worth noting is that the conception has a curious consequence that conflicts with ordinary intuitions. When the cardinality of options increases; freedom of rational choice may decrease. More specifically, when adding a better option to a choice set, freedom of rational choice will decrease to nothing. For example, a philosopher may be undecided regarding the choice between accepting a position as a teacher and accepting a position as a researcher. Then she gets an offer to become a professor. Because being a professor is preferable to being either a teacher or a researcher, according to all relevant value orders, she must rationally accept the position as a professor. When she had to choose between the teaching position and the research position she had two options, and thus both freedom of choice and freedom of rational choice. When the professor position was added she got three options and more freedom of choice, but she had only one rationally relevant option, and thus no freedom of rational choice. The *Rational choice measure* does not satisfy the *Limited strict monotonicity condition*, or the more general *Strict monotonicity condition*. In fact, it does not even satisfy a weak version of the monotonicity condition:

The Weak monotonicity condition: For any choice set $A \in P(X)$ such that $|A| \geq 1$, and any option $y \in X - A$, $A \cup \{y\}$ offers at least as much freedom of choice as A .

The failure to satisfy this condition is unfortunate.

A related counterintuitive consequence is that the universal set of options may not be the set that offers the most freedom of rational choice. If the universal set of options would include a unique universally best option, which is an option that is best according to all relevant preference orderings,

then the universal set would offer no freedom of rational choice. The universal set of options would offer as little freedom of choice as the singleton set with the universally best option as the only option. The *Rational choice measure*, therefore, does not satisfy the *Maximal freedom of choice condition*. The *No freedom of choice condition* may be satisfied by adding an extra condition concerning singleton sets offering no choice.

The conception of freedom of rational choice may seem suitable to apply after the options have been evaluated. But if we use this conception, we have to accept that it has several counterintuitive consequences.

6.5 Freedom of Eligible Choice

Another objection that can be raised against the original cardinality conception is that it includes options that are insufficiently valuable (or even bad). To handle this objection, the original cardinality conception may be changed into a cardinality conception of freedom of eligible choice. Such a conception may be defined as follows:

The Cardinality conception of freedom of eligible choice: A person P with a choice set A has *at least as much freedom of choice as* a person P^* with a choice set B if and only if A contains *at least as many sufficiently valuable options for P* as B contains sufficiently valuable options for P^* .

The idea here is that freedom of choice only depends on sufficiently valuable options. As I mentioned previously, the sufficiently valuable options are often called ‘eligible options’, as by Peragine and Romero-Medina (2006) and Van Hees (2010). At times they are also called ‘essential options’, as by Puppe (1996), Baharad and Nitzan (2003) and Puppe and Xu (2010). The sufficiently valuable options may be judged relative to different preference orderings. We may use only a person’s actual preference ordering at the time of choice, or a larger class of preference orderings. No matter which preference orderings that are used, it should suffice that an option is regarded as sufficiently valuable according to one relevant preference ordering. For example, if there are just three relevant preference orderings regarding work for a person, one favoring ease, one favoring pay and one favoring importance, then it is not the case that a work option need to be sufficiently easy, well paid and important to count as a sufficiently valuable option; it suffices that it is either sufficiently easy, well paid or important.

One argument in favor of the conception of freedom of eligible choice is that it seems to capture some ordinary language intuitions regarding freedom of choice. Often very bad options are not regarded as options. A traveler sitting on a train might decide between reading a newspaper and looking out

the window. But he would not consider throwing himself out the window. This option is so bad that he simply would not regard it as an option, even though he is physically capable of the act.

However, this intuition may just as well be explained with reference to the conception of freedom of rational choice. The reason that the traveler does not consider throwing himself out the window is because this option cannot rationally be chosen when the options of reading a newspaper and looking out the window are available. If the options of reading and looking out the window were to disappear and another appalling option appear; the traveler might rationally consider throwing himself out the window. For example, if the train were to catch fire, throwing oneself out the window would be considered an option. It is quite odd to say that a traveler has no freedom of choice when he can choose among the options of throwing himself out the window or waiting to get burnt by an approaching fire.

In favor of the conception of eligible choice, we could argue that it is a particularly useful conception. For example, if maximizing freedom of choice is accepted as a guide for action, then it would be useful to exclude very bad options from the set of options that influence freedom of choice. This way we eliminate the possibility that someone, following the guide, would end up choosing among many terrible options. However, this practical use of the measure also eliminates some possible good uses for freedom of choice, such as gaining practice in rejecting bad options.

6.5.1 Conditions and Cardinality Measures

To measure of freedom of eligible choice, any preference-independent measure of freedom of choice may be applied to sets of sufficiently valuable options. A sufficiently valuable option may roughly be defined in relation to one preference ordering as follows:

An option x is *sufficiently valuable according to some preference ordering* Y represented by some utility function v if and only if $v(x) \geq s$, where s is some particular number.

A more general definition is the following:

An option x is *sufficiently valuable* if and only if it is sufficiently valuable according to at least one relevant preference ordering.

We may also formulate a condition regarding sufficiently valuable choice:

The Eligible choice condition: For any choice set $A \in P(X)$ such that $|A| \geq 1$, all options $y, z \in X$ such that $y, z \notin A$, and any set of preference orderings K_P , if there exists at least one preference ordering $Y_i \in K_P$ such that y is *sufficiently valuable* by Y_i and there does not exist at least one preference ordering $Y_j \in K_P$ such that z is *sufficiently valuable* by Y_j , then $A \cup \{y\}$ offers strictly more freedom of choice than $A \cup \{z\}$.

This condition is uncontroversial, given the idea that ‘freedom of choice’ should be understood as “freedom of eligible choice”.

A problem for the conception of freedom of eligible choice is the delimitation of the class of sufficiently valuable options. On the one hand, it seems odd to only include the best of all options, because a chooser rarely encounters those. This way of limiting options seems too exclusive. On the other hand, it seems equally odd to include all options that are preferred to a certain standard, because that would include options that cannot rationally be chosen. This way of limiting options seems too inclusive. If freedom of eligible choice were to be assessed relative to only the best stamps, then this conception would only take Three Schilling Yellow into account (which would offer no freedom of choice, being just one stamp). This suggestion seems to include too little. But if freedom of eligible choice were to be assessed in relation to all valuable stamps, such as all stamps worth more than \$10, then the conception would take \$10 stamps into account along with \$100,000 stamps. Since the \$10 stamps are much less valuable than the \$100,000 stamps, this conception seems to include too much. This is a general problem when choosing any value limit below the highest one. To see how the problem may be dealt with, we shall consider some measures that are especially designed for the conception of freedom of eligible choice (or so it seems).

In a 2001 essay, Romero-Medina suggests three ordinal measures of eligible choice (which he calls ‘meaningful choice’ or ‘effective freedom’). We shall look at two of these measures. Romero-Medina first suggests that freedom depends only on the class of most valuable options, according to any reasonable preference ordering. Here we shall use relevant preference orderings rather than reasonable preference orderings, in order to make the discussion more general. A ratio scale version of the ordinal measure may then be defined as follows:

Eligible choice measure:
 $F(A, K_P) = |Max(AK_P)|.$

F is a function from \mathbf{R} to \mathbf{R} , K_P is the set of all relevant preference orderings for some person P , and $Max(A, K_P)$ is the set of all x included in A such that

x is an option that is at least as preferable as all other options in X for at least one preference ordering in K_P .

The *Eligible choice measure* may be interpreted as a measure of a person's ability to choose whichever option he might prefer the most. Thus, it may also be interpreted as a measure of closeness to perfect freedom of choice.

Romero-Medina axiomatizes his ordinal measure by three conditions, which are rather similar to Pattanaik's and Xu's conditions for their measure of freedom of rational choice. One is Sen's *Composition condition* (1991: 23). Another is an *Inclusion monotonicity condition*, relating to sets and their subsets. A third is a version of the *INS-condition*, a *Simple non-dominance condition*, stating that singleton sets offer an equal amount of freedom of choice if and only if they contain an equal number of options that are included in $\text{Max}(A, K_P)$. This condition is more acceptable for a measure of freedom than for a measure of freedom of choice. If we wish to use Romero-Medina's measure for freedom of choice, we should add that singleton sets offer no freedom of choice.

We may compare Romero-Medina's measure to the original cardinality measure in terms of how well it satisfies some of the reasonable conditions. The *INS-condition* only holds if we add the condition that singleton sets offer no freedom of choice (which we should). The *Limited strict monotonicity condition* continues to hold only if it is applied to the addition of options from the class of most valuable options. Otherwise it should be replaced with a different monotonicity condition, such as this:

The Eligible options monotonicity condition: For any choice set $A \in P(X)$ such that $|A| \geq 1$, any option $y \in X - A$, and any set of preference orderings K_P , if y is *sufficiently valuable* by at least one relevant preference ordering $Y \in K_P$, then $A \cup \{y\}$ *offers strictly more freedom of choice than A*.

The *Maximal freedom of choice* condition no longer holds since subsets of the universal set of options may offer a maximal amount of freedom of choice. The *No freedom of choice condition* only holds if we add the usual condition regarding singleton sets offering no choice.

Romero-Medina seems confident that the class of most preferred options will contain a sufficient number of options to "take into account any valuable option in terms of freedom" (2001: 185). This seems extremely doubtful. Fortunately, Romero-Medina realizes that we might want to compare sets that do not include any of the universally best options (which should be most sets, supposedly). For this purpose, he suggests a lexicographic ordering, where sets are first compared by the number of best options according to the preference orderings of K_P and, secondly by the number of 2nd best options according to the preference orderings of K_P etc.,

until it is possible to discriminate between the sets that are compared (2001: 185). To define the measure we shall let the set $Max^n(A, K_P)$ be the set of all the n th best options x in X that are included in A , according to at least one preference ordering in K_P . Thus, the set $Max^1(A, K_P)$ is the set of the best options in X included in A , while $Max^2(A, K_P)$ is the set of the second best options in X included in A , and so on. The lexical measure may then be presented as follows:

The Lexical ordinal eligible choice measure: For all choice sets $A, B \in P(X)$, and any set of preference orderings K_P , if and only if for all n , $|Max^n(A, K_P)| = |Max^n(B, K_P)|$, then A offers equal freedom of choice as B , and if and only if there exists some m such that $|Max^m(A, K_P)| > |Max^m(B, K_P)|$ and for all $i < m$, it is the case that $|Max^i(A, K_P)| = |Max^i(B, K_P)|$, then A offers strictly more freedom of choice than B .

Romero-Medina presents the measure slightly differently (2001: 186). For comparisons between different persons with different sets of relevant preference orderings the measure has to be changed slightly. However, we shall not bother with this additional complication here. The *Lexical ordinal eligible choice measure* may be interpreted as a measure of a person's ability to choose whichever option he might prefer to choose. Thus, it may also be interpreted as a measure of closeness to perfect freedom of choice.

The lexical measure satisfies the *Limited strict monotonicity condition* and the *Maximal freedom of choice condition*. This is because the measure is strictly increasing. The set $A \cup \{y\}$ always offers strictly more freedom of choice than A . When y is preferred to all the options in A there is a level m such that $|Max^m(A \cup \{y\}, K_P)| > |Max^m(A, K_P)|$ since $1 > 0$, and for all $i < m$ $|Max^i(A \cup \{y\}, K_P)| = |Max^i(A, K_P)|$ since all the other options are the same. When y is equally good as n options in A there is also a level m such that $|Max^m(A \cup \{y\}, K_P)| > |Max^m(A, K_P)|$ since $n + 1 > n$, and for all $i < m$ $|Max^i(A \cup \{y\}, K_P)| = |Max^i(A, K_P)|$ since all the other options are the same. When all the options in A are preferred to y there is also a level m such that $|Max^m(A \cup \{y\}, K_P)| > |Max^m(A, K_P)|$ since $1 > 0$, and for all $i < m$ $|Max^i(A \cup \{y\}, K_P)| = |Max^i(A, K_P)|$ since all the other options are the same. The lexical measure does not satisfy the *INS-condition* since it implies that $\{x\}$ offers more freedom of choice than $\{y\}$ whenever x is more valuable than y . This problem is easily solved by stipulation, however. The same goes for the fact that the measure does not satisfy the *No freedom of choice condition*.

To give an example of how the measure works we may consider the sets $A = \{\text{one } \$100 \text{ bill, another } \$100 \text{ bill, one } \$1 \text{ bill}\}$, $B = \{\text{one } \$1000 \text{ bill, another } \$1000 \text{ bill, one } \$1 \text{ bill}\}$ and $C = \{\text{one } \$100 \text{ bill, another } \$100 \text{ bill, one } \$1 \text{ bill, another } \$1 \text{ bill}\}$. Applying the lexical measure, B is judged as

offering more freedom of choice than C , which offers more freedom of choice than A .

Romero-Medina axiomatizes the lexical measure by five conditions. We shall not consider these here, however; instead we shall go on to discuss some problems for the measure.

6.5.2 Problems

There are several objections to the lexical measure. One objection is that it is only an ordinal measure. In this sense, it is not an ideal measure. This objection is not as serious as some of the other ones, however.

Another objection is that if we wish to capture the freedom to choose valuable options, then it seems strange that options that cannot rationally be chosen should contribute to freedom of choice. The measure does not satisfy the *Rational choice condition*. When z is preferred to y and y is indifferent to x , according to all relevant preference orderings, the set $\{x, z\}$ offers more freedom of eligible choice than the set $\{x, y\}$. The measure is also consistent with the idea that A offers more freedom of choice than B when A and B have equally many best options but A has a few more second best options. But these options seem irrelevant. If the chooser only has the possible preference orderings $Y_i: x > y > z$ and $Y_j: z > y > x$, then it is reasonable to think that $\{x, z\}$ should be ranked as high as $\{x, y, z\}$, because no matter which preference ordering the chooser uses as a standard of choice he cannot rationally chose y when the options x and z are available. The fact that the measure counts y as a contributing option is a weakness of the measure. Nevertheless, this problem can easily be solved by restricting the use of the measure to sets of options that can be rationally chosen.

There are other problems to consider, however. Given that we wish to measure the freedom to choose valuable options, there are several implications of the measure that seem strange. First, it seems strange that a set of ten best options, according to one relevant preference ordering, offers as much freedom of choice as a set of ten best options, according to ten relevant preference orderings. Second, it seems strange that a set of two options that offers the best options according to two relevant preference orderings should be judged as offering more freedom of choice than a set of one hundred options that offers the second best option according to a hundred relevant preference orderings.

There is yet another thing that is strange with the whole conception of eligible choice. The reason that we would wish to explicate the concept of freedom of choice as “freedom of eligible choice” seems to be that we are interested in an especially instrumentally valuable kind of freedom of choice, in the sense of helping us to gain better options. However, if this is the case, why not define degrees of freedom of choice directly in terms of degrees of value? This is the idea behind freedom of evaluated choice.

6.6 Freedom of Evaluated Choice

The main idea behind freedom of evaluated choice is that better options contribute more to freedom of choice than worse options. We shall consider two specific interpretations of this idea. These interpretations are far from being the only possible ones. We shall discuss them just because they are very simple.

The first suggestion is that each option contributes to freedom of choice in accordance with its value. For the original cardinality conception each option contributes equally to freedom of choice and each option contributes as one. For the modified version each option contributes as one, multiplied by its value. This implies that each option contributes with its value. The proper way to measure degrees of freedom of choice in accordance with this conception is thus to sum all the values of the options in a set.

The second suggestion is that each option contributes to freedom of choice both by contributing to cardinality and by contributing to value. Cardinality is one relevant factor for freedom of choice, and value is another. The original cardinality measure should thus be expanded by multiplying cardinality by some measure of the value of the options. This type of measures has been studied previously for measures of freedom under the name of *Hybrid measures*.

Both suggestions will lead us rather far from the original cardinality conception, perhaps so far that we should not regard them as variants of the cardinality conception at all. However, this is a good place to discuss them.

As far as arguing for the conception of evaluated choice from the point of view of ordinary language, it seems possible to argue both for and against the conception. If freedom of choice is interpreted as the ability to choose what one prefers, whatever one might prefer, then it is natural to think that degrees of freedom of choice increase with the values of the options. But, apart from this virtue, it seems odd to say that freedom of choice increases with the value of the options. If a person has two options and complains about not having enough freedom of choice, he would not be satisfied by being offered two better options.

The main virtue of the conception of freedom of evaluated choice is perhaps that it captures a kind of choice that is instrumentally useful for the attainment of better options. It is not the best way to capture this kind of choice, however, since the values of options that may not rationally be chosen are included.

6.6.1 Conditions and Total Value Measure

In the first part of the thesis, I tentatively suggested the following definition for freedom of evaluated choice:

The Conception of freedom of evaluated choice: A person P with a choice set A has at least as much freedom of choice as a person P^* with a choice set B if and only if A contains at least as valuable options for P as B contains valuable options for P^* .

This definition makes degrees of freedom of choice equivalent to the values of the options (possibly the sum). The values of the options should be determined by all relevant preference orderings. Since we have assumed that all preference orderings are comparable, this is unproblematic.

The definition may seem strange in the sense that it equates freedom of choice with the values of the options. Is it really possible to equate freedom of choice with values? Does this not confuse degrees of freedom of choice with the value of freedom of choice?

It may seem strange to equate freedom of choice with values of options. After all, we usually conceive of a choice as a choice among options, not as a choice among the values of options. The values of the options are what make us choose some option, rather than another. They are not the options themselves. But if the values of the options are what we foremost consider before choice, it is not so strange that we should at least regard freedom of choice as a function only of the values of the options, rather than of some other properties.

As for the proposal that freedom of choice is confused with the value of freedom of choice, it is simply incorrect. The value of freedom of choice is not the sum of option values. The value of freedom of choice depends on preferential attitudes towards freedom of choice, not preferential attitudes towards options. It may very well happen that a set of very valuable options is not very valuable in terms of freedom of choice.

If the idea that freedom of choice may be measured by the values of the options is at all acceptable, then freedom of choice might perhaps be measured by a function that sums the values of all the options in a set. More precisely, freedom of choice may be measured by a variant of a measure by Bossert (1997). Bossert's original measure is not a proposal for a measure of freedom of evaluated choice, but a proposal for a measure of overall wellbeing, depending on freedom of choice. However, we can try to apply the measure to freedom of choice. The proposed measure is simply to sum the values of all the options in a set (1997: 101–102). Bossert defines the value of an option in relation to one utility function only. Since we have assumed that several preference orderings may be relevant for freedom of choice, we should define the value of an option in relation to several utility functions. We shall use the value $V_m(x)$, which is a sum of values of x according to the m relevant utility functions, rather than the value $v(x)$, which is the value of x according to one utility function. This leads us to the following measure:

Total value measure:

$$F(A, L) = \sum_{i=1}^m \sum_{j=1}^n v_i(x_j) = \sum_{j=1}^n V_m(x_j).$$

F is a function from \mathbf{R}^n to \mathbf{R} , L is a set of utility functions v_i , m is the number of relevant utility functions, and $n = |A|$.

The *Total value measure* satisfies the following condition:

The Value condition: For any choice set $A \in P(X)$ such that $|A| \geq 1$, and all options $y, z \in X$ such that $y, z \notin A$, if $V_m(y) > V_m(z)$, then $A \cup \{y\}$ offers strictly more freedom of choice than $A \cup \{z\}$.

The measure also satisfies the following monotonicity condition:

The Valuable options monotonicity condition: For any choice set $A \in P(X)$, and any option $y \in X - A$, if $V_m(y) > 0$, then $A \cup \{y\}$ offers strictly more freedom of choice than A .

Other conditions that the measure satisfies may be found in Bossert (1997).

As an example we may consider a person who is a philosopher, poet, and parent and is deciding which local library to visit. At times he resides in Stockholm and at other times in Uppsala. The relevant universal set of libraries contains two university libraries, two city libraries, a state library and a children's library. For the chooser there are three relevant preference orderings for books, namely the preference orderings of a philosopher, a poet, and a parent. We shall compare the freedom of evaluated choice of libraries offered by Stockholm (A) and Uppsala (B). The value assignments are as follows:

	<i>Ph</i>	<i>Po</i>	<i>Pa</i>
<i>University Library</i>	9	5	1
<i>City Library</i>	5	9	5
<i>State Library</i>	9	9	9
<i>Children's Library</i>	1	3	9

If $A = (\text{University Library, City Library, State Library, Children's Library})$, $B = (\text{University Library, City Library})$, then *The Total value measure* gives $F(A) = 74$ and $F(B) = 34$. Thus, if the chooser wants to maximize freedom of evaluated choice for libraries, then he ought to be in Stockholm.

One problem with the *Total value measure* is that it does not satisfy the *No freedom of choice condition*. But this problem can easily be solved by the usual stipulation that singleton sets offer no choice.

Another problem arises if we try to measure freedom of evaluated choice by a sum of option values. The problem is that the inclusion of bad options may result in counterintuitive rankings. First, let us assume that we represent the bad options, as we are wont to do, by negative numbers. Then we get the strange consequence that the addition of a bad option may decrease freedom of choice. The *Total value measure* does not satisfy the *Limited strict monotonicity condition*, nor does it satisfy the *Weak monotonicity condition*. If the option of losing \$5000 is added to the set of options of getting \$5000 or nothing, the freedom of evaluated choice decreases from 5000 to 0. We also get the implausible conclusion that a larger set of bad options may offer less freedom of choice than a smaller set of bad options. For example, the set that includes the three options of losing any one of three \$100 bills offers less freedom of choice than the set that includes losing any one of just two \$100 bills. This is counterintuitive. Even worse, as long as there are bad options that are represented by negative numbers, then the universal set of options will not offer a maximal amount of freedom of choice. A set that offers all good options and no bad options would offer more freedom of choice than the universal set. Thus, the measure does not satisfy the *Maximal freedom of choice condition* either.

We may try to solve these problems, either by assigning 0 to all bad options, or by assigning 0 only to the absolutely worst option, and small positive numbers to all the other bad options. The first solution has the implausible consequence that a set that includes the options of losing \$1 or \$2 offers as much freedom of evaluated choice as a set that offers the options of losing \$100,000 or \$200,000. The second solution has an equally implausible consequence. Suddenly a greater number of bad options may offer more freedom of choice than a smaller number of good options. Losing any one of one hundred \$100 bills may offer more freedom of choice than gaining any one of two \$100 bills. From the perspective of trying to capture the freedom to choose valuable options, this is strange.

All these problems suggest that the *Total value measure* cannot work as a measure of freedom of choice. Next we shall consider yet another way to change the cardinality conception to include information about value.

6.6.2 Hybrid Value and Cardinality Measure

Another idea is that the cardinality measure may be combined with some value information to construct a measure of freedom of evaluated choice. Such a measure would be a hybrid measure of a cardinality measure and a value measure. At least two hybrid measures occur in the area of measuring freedom, proposed by Crocker (1980) and Foster (2011). Related ideas may be found in essays by Swanton (1979: 346) and Elster (1983: 130). The use of hybrid measures for measuring freedom is criticized by Carter (1999: 135–140).

Here, we shall look at a very simple example of a hybrid measure. The number of options is multiplied by the total values of the options. As previously, the total value of the best option is defined in relation to several preference orderings:

Hybrid measure:

$$F(A, L) = |A| \times \sum_{i=1}^m \sum_{j=1}^n v_i(x_j) = |A| \times \sum_{j=1}^n V_m(x_j).$$

F is a function from \mathbf{R}^n to \mathbf{R} , L is a set of relevant utility functions v_i , m is the number of relevant utility functions, and $n = |A|$.

The *Hybrid measure* might satisfy the *INS-condition* and the *No freedom of choice condition*, but only if we add the condition that singleton sets offer no choice. Since negative values are not excluded, the measure would not satisfy the *Maximal freedom of choice condition*, the *Limited strict monotonicity condition* or the *Rational choice condition*. This result should be sufficient to discard the measure.

There are obviously other ways to construct hybrid measures. One could multiply the number of options by the total values of the best options according to each relevant preference ordering, or the total greatest values of the best options according to each relevant preference ordering. One could also multiply the number of options that may rationally be chosen by each of the suggestions just given. But all of these suggestions would have the same problems as the *Hybrid measure*, as long as they would aggregate negative values. They would also have another major problem in common; it seems difficult to make any sense of the combinations. Why would freedom of choice depend on both the number of options and their values? This suggestion just interprets the concept of freedom of choice as a muddled and unnecessary concept. We may want to know how many options a person has, especially pre-evaluation. We may also want to know the values of his options. But why would we want to fuse information regarding cardinality and values into a single measure, thus making it difficult to distinguish between a choice set of a few highly valuable options and a choice set of many less valuable options? The hybrid measure seems to be a way to lose information, rather than a way to gain information.

Another obvious problem with hybrid measures is the difficulty in selecting appropriate weights for cardinality and value in terms of their contribution to freedom of choice. Cardinality and value seem to be incomparable factors. This point is also made by Carter (1999: 136–137). Even Foster, who suggests one hybrid measure, admits that there is something ad hoc about their way of evaluating the non-chosen options (2011: 711).

6.7 Partial Conclusion

As a result of this chapter, four conditions were accepted for a measure of freedom of choice, the *Indifference between no-choice situations condition*, the *Limited strict monotonicity condition*, the *No freedom of choice condition* and the *Maximal freedom of choice condition*.

The cardinality conception itself was judged as too simplistic an explication of the concept of freedom of choice. Two explanations were put forward as to why the cardinality conception failed; one idea was that the conception did not take the differences among the options into account. Another idea was that the conception did not take the values of the options into account. We postponed discussing the first idea and instead started to consider modifications of the cardinality conception that would take the values of the options into account.

The cardinality conception of freedom of rational choice had some appeal. It captured some ordinary language uses of the expression ‘freedom of choice’. It also captured the kind of freedom of choice that should concern us the most after the pre-evaluation stage of choice. But the conception was problematic in other ways. It failed to capture several ordinary language uses of ‘freedom of choice’. It had several counterintuitive implications. It violated the conditions of *Weak monotonicity* and *Maximal freedom of choice*.

The cardinality conception of freedom of eligible choice was more problematic. That the conception captured some ordinary language intuitions could be explained by reference to the conception of freedom of rational choice. It failed to capture other intuitions. The conception also suffered from delimitation problems, at least in a preliminary interpretation. A lexical version could solve the delimitation problem, but it had other problems. For example, it seemed strange that it counted options that could not rationally be chosen.

The cardinality conception of freedom of evaluated choice was perhaps the most problematic of the three. We considered two interpretations. The first interpretation had freedom of choice measured by the total value sum of the options. This approach had many counterintuitive implications, among them the implication that freedom of choice might decrease with the addition of an option. The second interpretation had freedom of choice measured by the cardinality of options multiplied by some value. This approach was quickly dismissed as conceptually confused.

In general, there seems to be some use for the conception of freedom of rational choice, as it is the freedom of choice that is left after evaluation. But the conceptions of freedom of eligible choice and freedom of evaluated choice seem mistaken as explications of the concept of freedom of choice. The first conception could perhaps be used as an explication of the concept of opportunity, although there are other explications that are more

reasonable. The second conception could perhaps be used as an explication of the concept of opportunity value.

Further on, save for one exception, I shall only discuss the preference-dependent conception of rational choice. I shall regard the other two conceptions as failed.

Chapter 7: Freedom of Choice as Representativeness

We have now come to the second idea regarding how “freedom of choice” should be understood, not as cardinality of choice sets, but as representativeness of choice sets. This idea occurs in one essay only: ‘Freedom of choice and expected compromise’ by Gustafsson (2010). Because of this, we shall mostly base the discussion on Gustafsson’s measure. As a result of the discussion, I shall suggest that the concept of representative choice may not be adequately represented by the measure. However, I shall not discuss any alternative proposals for measuring representative choice. The main reason for this is that I shall also suggest that the conception of representative choice differs from the concept of freedom of choice in such fundamental ways that representative choice cannot be regarded as a kind of freedom of choice. Nevertheless, it is quite possible that representative choice in general is a more instrumentally valuable phenomenon than freedom of choice. There is thus reason to take an interest in representative choice regardless.

7.1 Measure

Let us first repeat the definitions given for representative choice. The main idea was this one:

The Representative conception of freedom of choice: A person P with a choice set A has at least as much freedom of choice as a person P^ with a choice set B if and only if A is at least as representative of the universal set X as is B .*

This was specified as follows:

The choice set A of options x_i is at least as representative of the universal set X as the choice set B of options y_j if and only if the options x_i in A are at least as similar overall to all of the options in the universal set as are the options y_j in B .

Gustafsson proposes that given the relevant universal set X , the choice set A offers at least as much freedom of choice as the choice set B if and only if the expected degree of dissimilarity between a randomly picked option x from X and the least dissimilar option y in A is at least as low as the expected degree of dissimilarity between a randomly picked option x from X and the least dissimilar option z in B (2010: 69). The probability that an option should be randomly picked from X is $1/|X|$. Next Gustafsson defines the minimal dissimilarity between the option x and the set A as follows:

$$\delta(x, A) = \min (d(x, y) \text{ such that } y \in A).$$

The expected degree of dissimilarity E between the randomly picked option x and the least dissimilar option y in A is then:

$$E(\delta(x, A_d), X) = \frac{\sum_{x \in X} \delta(x, A_d)}{|X|}.$$

Since the denominator $|X|$ is the same for all sets that we wish to compare, it is not needed here. Gustafsson's measure of expected compromise may thus be defined as follows:

Expected compromise measure:

$$EC(\delta(x, A_d)) = \sum_{x \in X} \delta(x, A_d).$$

In this context, EC is a function from a finite metric space to \mathbf{R} .

Expected compromise is related to freedom of choice in such a way that less expected compromise implies more freedom of choice:

Expected compromise ordinal measure: For all choice sets $A, B \in P(X)$, if and only if $EC(A) \leq EC(B)$, then A offers at least as much freedom of choice as B .

Gustafsson calls this measure the *Unweighted expected compromise measure* since he allows for weights on the options, in accordance to their value or relevance (2010: 73). We shall discuss the weighted version later. To get a ratio scale measure that assigns numerical values to choice sets we may use the original expected compromise measure for freedom of choice. To avoid confusion we may use a negative form of the measure. This would be a diversion from Gustafsson's measure, however, since he only intends his measure to be ordinal.

Negative expected compromise measure:

$$F(\delta(x, A_d)) = - \sum_{x \in X} \delta(x, A_d).$$

F is a function from a finite metric space to \mathbf{R} .

It is important to note that Gustafsson understands the universal set as the relevant universal set, and not as the set of all nomologically possible acts. The universal set should contain all possible options in a relevant domain. However, it is unclear exactly what Gustafsson means by ‘possible’ since he believes that the range of possible options can change (2010: 68).

To give an example of how the *Expected compromise measures* work we may again consider choice sets of teaching hours. Let us suppose that the relevant universal choice set of teaching hours at Uppsala University is $X = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$. We shall rank the sets $A = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}\}$ and $B = \{0 \text{ h}, 40 \text{ h}\}$. The distances for the first set are $\delta(0 \text{ h}, A) = 0$, $\delta(10 \text{ h}, A) = 0$, $\delta(20 \text{ h}, A) = 0$, $\delta(30 \text{ h}, A) = 10$ and $\delta(40 \text{ h}, A) = 20$. Thus, $EC(A) = 30$ and $F(A) = -30$. The distances for the second set are $\delta(0 \text{ h}, B) = 0$, $\delta(10 \text{ h}, B) = 10$, $\delta(20 \text{ h}, B) = 20$, $\delta(30 \text{ h}, B) = 10$ and $\delta(40 \text{ h}, B) = 0$. Thus, $EC(B) = 40$ and $F(B) = -40$. So, A offers more freedom of choice than B .

It is obvious that the *Expected compromise measures* satisfy the *Limited strict monotonicity condition*. When an option y is added to the set $\{x\}$ the value of $\delta(y, \{x\})$, which is greater than 0, is replaced by the value of $\delta(y, \{x, y\})$, which equals 0. Thus the value of the *Expected compromise measure* decreases, while the value of *Negative expected compromise measure* increases. For example, relative to $X = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$, the set $\{x\} = \{10 \text{ h}\}$ offers less freedom of choice than $\{x, y\} = \{10 \text{ h}, 20 \text{ h}\}$. We get $EC(\{x\}) = 70$ and $F(\{x\}) = -70$, as well as $EC(\{x, y\}) = 40$ and $F(\{x, y\}) = -40$. The *Expected compromise measures* also satisfy the *Maximal freedom of choice condition* since the universal set is the only set that is given a value of 0.

7.2 Conditions

In his essay, Gustafsson discusses some conditions that the measure satisfies. This is one of them:

The Domain-sensitivity condition: There exist relevant universal sets X and X^* and choice sets A and B such that $A, B \subseteq P(X)$ and $A, B \subseteq P(X^*)$, where A offers at least as much freedom of choice as B , given X , and A offers strictly less freedom of choice than B , given X^* .

(The condition is here changed from ‘universal sets’ to ‘relevant universal sets’.) Gustafsson believes that this is a reasonable condition for freedom of choice. He refers to Van Hees who states that freedom is sensitive to variations in the domain of technologically feasible options (1998). This view is also shared by Rosenbaum (2000: 217).

The condition requires that there may be different relevant universal sets. Initially this idea may seem rather odd. Why should there be different relevant universal sets for different comparisons of the very same sets A and B ? Finding some explanation for this assumption requires some imagination. Perhaps this scenario will do: the sets A and B are the possible future choice sets of some student. Depending on how history evolves, either X or X^* will be relevant for comparison. If the student in the future chooses University A , she will have some courses to choose from, if she chooses University B , she will have some others. How much freedom of choice the two universities offer depend on whether or not it will be possible to study some particular medical technique that might be discovered in the future. If the medical technique is discovered, X is the relevant set; if it is not, X^* is the relevant set. Neither university offers any course on the medical technique. However, if the technique is available, A offers more freedom of choice than B ; if the technique is not available, B offers more freedom of choice than A . This implication may seem implausible since the choice sets themselves are not affected by the change of relevant universal set. Neither A nor B offers any more or any less courses.

But perhaps the reversal of rankings can be explained by the additional assumption that B is a medical school and ought to offer courses on available medical techniques, while A is a law school and ought not. So B offers less freedom of choice than A , relative to X . This explanation does not fit with how the *Expected compromise measure* works, however. If B is a medical school, it would offer more freedom of choice than A when there is an additional medical technique available. The reason for this is that the courses offered by the medical school would be more similar to the course on the new medical technique than the courses offered by the law school. The student who wishes to study the medical technique has to compromise less when choosing B than when choosing A . So the measure would rank B over A . Whether or not this is acceptable is debatable.

There are some reasons why the *Domain-sensitivity condition* may initially seem reasonable. We cannot practically compare sets to a universal set as large as the set of all nomologically possible options. If other sets are chosen as universal sets, then it is at least possible that freedom of choice should be sensitive to the domain. It would be, if freedom of choice should be understood as representative choice (which is unclear), and the relevant universal set might change (which is also unclear). Furthermore, the condition may seem reasonable if the concept of freedom of choice is understood as a relative concept. Obviously, a person with a choice set of

two options has more freedom of choice, in a relative sense, compared to a relevant universal set of three possible options, than he has compared to a relevant universal set of a thousand possible options. The proportion to which the agent has complete freedom of choice is greater in the first case than it is in the second case. If freedom of choice is understood as an absolute concept, however, the condition of *Domain-sensitivity* seems mistaken. After all, two options are just two options, no matter what the relevant universal set looks like. Understood as an absolute concept, freedom of choice should rather satisfy the very opposite condition:

The Domain-insensitivity condition: There do not exist any relevant universal sets X and X^* and choice sets A and B such that $A, B \subseteq P(X)$ and $A, B \subseteq P(X^*)$, where A offers at least as much freedom of choice as B , given X , and A offers strictly less freedom of choice than B , given X^* .

Since I regard freedom of choice as an absolute concept, I shall accept this other condition here. The freedom of choice offered by a choice set does not depend on the identity of the relevant universal set. It depends on the intrinsic properties of the choice set.

7.3 Problems

Most notably, the *Expected compromise measures* fail to satisfy the *INS-condition*. Singleton sets may offer different amounts of freedom of choice depending on how representative the single option is of the relevant universal set. If the relevant universal set is $X = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$, then $A = \{20 \text{ h}\}$ is ranked as offering more freedom of choice than $B = \{0 \text{ h}\}$. This ranking would be reasonable if we were only to judge options for their *representativeness*. But we should also judge options for offering *choice*. Neither set offers any choice; so the ranking is not reasonable. Gustafsson himself argues that *INS* is false. We considered his argument previously and saw that it was flawed. This is hardly surprising, given that it tries to show that a conceptual truth is false.

The *Expected compromise measures* also fail to satisfy the *No freedom of choice condition*. A set with just one option may be considered as offering as much freedom of choice as a set with several options. Relative to the same X as above, $A = \{20 \text{ h}\}$ is ranked as offering as much freedom of choice as $C = \{0 \text{ h}, 10 \text{ h}\}$. These two problems may be solved by stipulation, however. Gustafsson does this himself when he, at the end of his essay, suggests a modified version of the *Expected compromise measure* where he adds the condition that all singleton sets offer the same degree of freedom of choice (2010: 54). The modified version of the *Expected compromise measure*

would come closer to capturing the concept of choice; at the same time, it would get further from capturing the concept of representativeness.

But there are other problems with the *Expected compromise measures*. For example, a set with a small diameter may be considered as offering as much freedom of choice as a set with a large diameter. Given the same X , $G = \{20 \text{ h}, 30 \text{ h}\}$ is considered as offering as much freedom of choice as $H = \{20 \text{ h}, 40 \text{ h}\}$. Furthermore, a set with options distributed at unequal intervals may be considered as offering as much freedom of choice as a set with options distributed at equal intervals. Given the same X , $I = \{0 \text{ h}, 10 \text{ h}, 40 \text{ h}\}$ is considered as offering as much freedom of choice as $J = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. Relative to intuitions regarding the importance of a large diameter and an equal interval distribution to freedom of choice, these implications are very strange. They are not just strange relative to some diversity conception of freedom of choice. They are also strange relative to the idea of representativeness itself. $H = \{20 \text{ h}, 40 \text{ h}\}$ seems more representative of X than $G = \{20 \text{ h}, 30 \text{ h}\}$ does and $J = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$ seems more representative of X than $I = \{0 \text{ h}, 10 \text{ h}, 40 \text{ h}\}$ does. The *Expected compromise measures* are too blunt to register such distinctions. (In fact, the sets $K = (10 \text{ h}, 20 \text{ h}, 30 \text{ h})$ and $L = (0 \text{ h}, 10 \text{ h}, 30 \text{ h})$ would also be ranked as equal to I and J).

It seems difficult to formulate some general rule regarding representativeness since degrees of representativeness are domain-sensitive. We cannot say that a larger diameter or a greater degree of equal interval distribution generally implies greater representativeness. It only does so in some contexts. A set with a larger diameter is not necessarily more representative, if most options in X are clumped close to one another. Neither is a set that has a greater degree of equal interval distribution necessarily more representative, if the options in X are not distributed at equal intervals. More generally, just because two options in A are more similar than two options in B , this does not imply that A is less representative of X than is B .

It may be possible to change the *Expected compromise measures* so that the measures get closer to our intuitions regarding representativeness in the examples above. In personal conversation, Gustafsson suggests that many problems may be solved if the universal set is selected in a different way than it was selected in my examples, and would include all possible options individuated in a fine-grained way. That may be the case for some of the problems that are shown in my examples. If the universal set has a very large range, then a set with two options and a larger diameter may be ranked above another set with two options and a smaller diameter (but not necessarily). If the universal set is very dense, than a set with options distributed at equal intervals may be ranked above a set with options distributed at unequal intervals (but not necessarily). However, this does not solve all the problems for the *Expected compromise measure*. The relevant

universal set, being finite, does not have an unlimited range. Thus there may be counterintuitive rankings of sets with options close to the edges of the universal set (for example, close to 0 for options that do not have properties admitting of negative degrees). Here sets with smaller diameters may be ranked above sets with larger diameters. It is also possible that the relevant universal set contains gaps, in a similar fashion to the way in which the set $X = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$ contains gaps. If this is the case, the measures may still give counterintuitive rankings for sets of options distributed at more or less equal intervals.

7.4 Preference-Dependent Representative Choice

From the above discussion, we may infer that it is doubtful that any measure of representative choice would work as a measure of freedom of choice. From a previous discussion, we may also infer that it is doubtful that any preference-dependent measure would work. Despite these doubts, we shall briefly look at two preference-dependent measures of representative choice. The first measure is a measure of freedom of rational choice. I bring this up since it may be useful to have such a measure after evaluation. The second measure is a *Hybrid measure*. I bring this up since it was suggested by Gustafsson as an alternative version of his measure.

7.4.1 Rational Representative Choice

Let us begin by discussing freedom of rational choice. In accordance with the conception of freedom of rational choice, it is only the options that can rationally be chosen that influence freedom of choice. Although this idea seems perfectly clear when paired with the cardinality conception of freedom of choice, it is less clear when paired with the representative conception. It is unclear which subset of the possible options in the relevant universal set that is relevant for comparisons. We may look at an example for illustration. Let us suppose that X is a bank deposit of dollar bills, and that $A = \{\text{one } \$100 \text{ bill, another } \$100 \text{ bill, one } \$1 \text{ bill}\}$ and that $B = \{\text{one } \$1000 \text{ bill, another } \$1000 \text{ bill, one } \$1 \text{ bill}\}$. The set $Mal(A, K_p) = \{\text{one } \$100 \text{ bill, another } \$100 \text{ bill}\}$ and the set $Mal(B, K_p) = \{\text{one } \$1000 \text{ bill, another } \$1000 \text{ bill}\}$. How should we assess degrees of freedom of rational choice? There are several options.

One option is that we compare the maximal options of each set only to those possible options in the relevant universal set that are of equal value to the maximal options of the set. So, if we are assessing the freedom of choice offered by A and B , we compare A with the other \$100 bills in the deposit, and B with the other \$1000 bills in the deposit. This suggestion is strange. The values that we would get would only show how representative a set of

best options is to the set of equivalued options in the relevant universal set, not to the relevant universal set itself. Another option is that we compare the maximal options of each set to all the possible options in the relevant universal set. But this suggestion is also strange. It implies that we shall compare the two \$100 bills in $Mal(A, K_p)$ to the \$1 bill in A , as if the \$1 bill was not already included in A . A third option is that we compare the maximal options of each set only to those possible options in the relevant universal set that are of equal or greater value than the maximal options of the set. So, if we are assessing the freedom of choice offered by A and B , we compare A with the bills of a value of at least \$100, and B with the bills of a value of at least \$1000. This is the only acceptable suggestion. An interesting implication of this suggestion is that freedom of rational choice would increase if the value of the maximal options increases. This makes this conception of freedom of rational choice close to some conception of freedom of evaluated choice. It also makes it close to the lexical version of freedom of eligible choice. This measure would capture the idea:

Negative rational expected compromise measure:

$$F(\mathfrak{A}x, A_d) = - \sum_{x \in X} \delta(x, A_d) \text{ such that } x \text{ is at least as good as any } y \in Mal(A, K_p) \text{ for at least one preference ordering in } K_p.$$

F is a function from a finite metric space to \mathbf{R} .

This measure would show how representative a person's rationally choosable options are of the at least as preferable possible options in the relevant universal set. Unsurprisingly, the measure does not satisfy the *Limited strict monotonicity condition*. When an option is added to a set A , which is preferred to all the other options in A but less representative of all the possible options in the relevant universal set, freedom of choice would decrease, according to the measure. The measure also does not satisfy the *Maximal freedom of choice condition* since any set including the universally best option would be given the maximal value of 0. Furthermore, it does not satisfy the *No freedom of choice condition* and the *INS-condition* since singleton sets can offer different amounts of freedom of choice. The second failure may be acceptable, and the last two failures may be solved by stipulation. But the failure to satisfy the *Limited strict monotonicity condition* seems sufficiently serious to abandon the measure. Even though it may be acceptable for a measure of rational choice, it does not seem acceptable for a measure of representative choice.

7.4.2 Evaluated Representative Choice

Let us look at another way to incorporate information about preferences into a measure of representative choice. It is another measure by Gustafsson,

which may be regarded as a hybrid measure (2010: 73). It combines the *Expected compromise measure* with a function $w(x)$ that assigns a value (or perhaps a relevance value) to each option. Here we may again use the value $V_m(x)$, which is a sum of values of x according to the m relevant utility functions. We then have:

$$\begin{aligned} & \textit{Weighted expected compromise measure:} \\ \textit{WEC}(\delta x, A_d, K_p) &= \sum_{x \in X} V_m(x) \delta(x, A_d). \end{aligned}$$

In this context, *WEC* is a function from the family of finite metric spaces to \mathbf{R} . Gustafsson then defines an ordinal measure for freedom of choice, just as he did previously. Since we are looking for a ratio scale measure, we may, once again, either use the original *Weighted expected compromise measure* or that measure in a negative form:

$$\begin{aligned} & \textit{Negative weighted expected compromise measure:} \\ F(\delta x, A_d, K_p) &= - \sum_{x \in X} V_m(x) \delta(x, A_d). \end{aligned}$$

F is a function from the family of finite metric spaces to \mathbf{R} .

Like the unweighted version, the weighted measure fails to satisfy the *No freedom of choice condition* and the *INS-condition*. In addition, it also fails to satisfy the *Limited strict monotonicity condition* and the *Maximal freedom of choice condition*. The reason for this is that options given a value of 0 cannot contribute to freedom of choice.

The major problem with the measure is not the failure to satisfy different conditions, however. The major problem is lack of conceptual clarity. The measure fuses two kinds of information together in a muddled way. It does not give information about how representative a set of options is of the relevant universal set, as the unweighted *Expected compromise measure* does; nor does it give information about how valuable the options are, as the *Total value measure* does. It gives some information about how similar or dissimilar the options are to possible options that are more or less valuable. But the information about similarity and value is fused so the measure does not show the difference between an option being similar to a bad possible option or an option being dissimilar to a good possible option. Even if it would, this does not seem to be especially important. What is important is the values of a person's options (at least the ones that are at least as preferable as any other), not the values of possible options that are similar to a person's options.

Here we may give an example of a strange implication of the measure. Let us suppose that a student is choosing between two summer programs in philosophy at two different universities; *University A* and *University B*. Both

universities offer a course in ethics and a course in logic. All the courses are equally good. Both universities offer ethics and logic every other year, and aesthetics and epistemology every other. The only difference is that University *A* has a slightly better course in aesthetics. Let us further suppose that the relevant universal set is the set of all summer courses ever offered by the two philosophy departments, and that two different courses offered at the same department are more similar than two different courses offered at different departments. If we then apply the *Negative weighted expected compromise measure*, we get an implausible result. University *A* is ranked as offering more freedom of choice than University *B*, even though both universities just offer a course in ethics and a course in logic. This seems wrong. Surely the degree of freedom of choice offered by a course in ethics and a course in logic should not depend on the value of a course in aesthetics.

The measure also seems strange because it seems to combine two different ways of looking at choice; pre-evaluation and post-evaluation. Pre-evaluation, the properties and differences between the options seem important since they provide us with reasons for evaluating the options in different ways. Post-evaluation, the properties of the options seem less important than their values since it is the values that give us reasons to choose one option over another. The *Weighted expected compromise measure* must be applied after evaluation, when it does not seem especially relevant to know the differences between the options, much less the values of the other options.

7.5 Partial Conclusion

Some progress was made in this chapter. One condition was accepted for a measure of freedom of choice, the *Domain-insensitivity condition*. The conception of representative choice was judged as an inappropriate explication of the concept of freedom of choice. At least, the *Expected compromise measures* were judged to be unsuitable as measures of freedom of choice.

Chapter 8: Freedom of Choice as Diversity

At the end of their 1990 essay, Pattanaik and Xu suggest that information about the similarities of the options is also relevant for assessing degrees of freedom of choice. In other words, a measure of freedom of choice should incorporate information about the diversity of choice sets. (This formulation was used by Pattanaik and Xu in 2000a.) Following Pattanaik's and Xu's essay, there has been a discussion regarding how the diversity of options affects degrees of freedom of choice. Carter has argued that diversity is not relevant for freedom of choice (1999), while authors such as Dowding and Van Hees have argued for the opposite view (2009).

A few authors equate freedom of choice with diversity of choice. This is done by Bervoets and Gravel, who propose that their measure of diversity can be used as a measure of freedom of choice (2007: 268). It is also done by Bavetta and Del Seta, who propose several measures of freedom of choice, where one is a measure of variety of choice (2001: 222–224). Other authors who (at times) seem to equate freedom of choice with diversity of choice include Pattanaik and Xu (2000a), Xu (2004), and Savaglio and Vannucci (2009). Other authors think that diversity is one of several factors that contribute to freedom of choice. For example, an anonymous author of Penguin's *English Dictionary*, defines 'choice' as "a sufficient number and variety to choose among" (2004: 240). Klemisch-Ahlert proposes that freedom of choice depends both on the number of options and their range (1991: 191). Wertheimer proposes that choice depends both on the distances between the options and their value (1987: 192). In related areas, Arneson proposes that freedom depends on both the number of options and their degree of significant differences (1998: 172). Pettit proposes that (what he calls) 'option freedom' depends on the number, diversity and values of the options (2003: 389–393). Gabor and Gabor propose that free acts of choice depend on both the diversity and independence of choice (1954: 332). Finally, Crocker proposes that positive liberty depends on diversity, together with values and probabilities (1980: 54–57).

There are several measures of freedom of choice that incorporate information about diversity, for example, in Pattanaik and Xu (2000a, 2008), Rosenbaum (2000), Bavetta and Del-Seta (2001), and Van Hees (2004).

8.1 Conceptions of Diversity

In the following chapters I shall investigate the conception of freedom of choice as diversity. This means that I shall assume that freedom of choice and diversity of choice sets are the same thing. I shall investigate intuitions regarding diversity with the assumption that they are intuitions also regarding freedom of choice.

Earlier in this thesis, I defined ‘freedom of choice as diversity’ as follows:

The Diversity conception of freedom of choice: A person P with a choice set A has at least as much freedom of choice as a person P^ with a choice set B if and only if A is at least as diverse as B .*

I also specified the concept of diversity as follows:

The choice set A of options x_i is at least as diverse as the choice set B of options y_j if and only if the options x_i in A are at least as different from one another overall as are the options y_j in B .

This definition is not sufficiently precise to function as a standard for rankings of choice sets. There are many ways to understand the idea of options being more different from one another overall than other options are. I shall immediately reject three interpretations.

First, when I speak of the options in A being more different from one another overall than the options in B , I shall not mean that “the diameter of A is larger than the diameter of B ”. The diameter of A only involves the difference between the two farthest options in A . The conception of diversity that I am interested in here involves the differences between all the options in A .

Second, when I speak of the options in A being more different from one another overall than the options in B , I shall not mean that the options in A on average are more different from one another than the options in B . This idea captures the conception of diversity as *heterogeneity*, which is a conception common in biology. There is a reason why the conception of diversity as heterogeneity is unsuitable as a conception of freedom of choice as diversity. The reason is that the heterogeneity of a choice set might decrease when an option is added to the set. The heterogeneity of $A \cup \{y\}$ may be less than the heterogeneity of A since the average difference between the options in $A \cup \{y\}$ may be less than the average difference between the options in A . But $A \cup \{y\}$ cannot offer less freedom of choice than A .

Third, when I speak of the options in A being more different from one another overall than the options in B , I shall not mean that “the total sum of differences between the options in A is greater than the total sum of

differences between the options in B'' . We shall discuss why this interpretation is unsuitable later.

The three specifications of the conception of diversity that I have rejected all have the virtue of being exact. I do not have an equally exact definition of diversity to offer yet. What I shall do is to analyze intuitions regarding diversity for different cases. These intuitions shall be used to formulate conditions that a measure of freedom of choice as diversity should satisfy. If the intuitions and conditions are consistent, it is possible to find a measure, or a class of measures, that satisfies all the conditions. If there is just one measure that satisfies the conditions, then the conception of freedom of choice as diversity has been specified in a way that is equally precise as the three rejected specifications. This is a goal of this thesis.

8.2 Relations to Universal Set

We have proposed that the diversity of a set is a function of how dissimilar the elements of the set are to one another. But not everyone agrees on this understanding of the concept of diversity. We shall begin by looking at some alternative conceptions of diversity here and see why they are problematic as explications of the concept of freedom of choice.

In an essay on diversity, Gravel proposes that the diversity of a set may be measured by aggregating a diversity value for each element of the set. As a general technique, he proposes a two-step procedure in which the first step involves measuring the contribution to diversity of every element in the relevant universal set by a function f , and the second step involves comparing the sets on the basis of the sum of the contributions of their elements as measured by the function f (2009: 26). This proposal is not sufficiently specific for any detailed discussion. However, there is a general problem with this type of technique that suggests that it cannot be used to construct a measure of diversity. The problem is to determine the numerical contribution of an element to diversity.

There seem to be two ideas that are compatible with the description by Gravel. The first idea is that the diversity contribution of an element depends on only the intrinsic properties of the element. This idea is unreasonable since diversity contribution partly is an extrinsic property. How much an element contributes to the diversity of a set depends on the other elements of the set. For example, the intrinsic properties of an environmental party do not reveal how much it contributes to the diversity of a set of political parties. The second idea is that the diversity contribution of a set depends on its relations to the other elements of the relevant universal set. This idea is unreasonable as well. For example, let us suppose that an environmental party contributes very much to the diversity of the relevant universal set of political parties, while a central party contributes very little. From this it

does not follow that a set of two environmental parties is more diverse than a set of an environmental party and a central party.

Gravel's general measuring technique is thus unacceptable for measuring diversity. We shall look at a similar and equally unacceptable measuring technique next. It may be viewed as an interpretation of Gravel's idea, and is suggested by Eiswerth and Haney (1992). Eiswerth and Haney propose that the diversity of a set should be measured by aggregating for each element x its average distance to the other elements of the relevant universal set, excluding x itself (1992: 241). The average distance is called the *distinctiveness* C of an element x . It is defined as follows:

Distinctiveness:

$$C(x) = \frac{\sum_{i=1}^n d(x, y_i)}{n-1} \text{ such that } x, y_i \in X, x \neq y_i, \text{ and } n = |X|.$$

The measure is defined as follows (the name is my own):

Distinctiveness measure:

$$F(A, d) = \sum_{i=1}^n C(x_i) \text{ such that } x_i \in A \text{ and } n = |A|.$$

F is a function from a finite metric space to \mathbf{R} .

The main problem with judging the diversity of a set as a function of its relations to the relevant universal set is that sets are judged differently depending on what the relevant universal set looks like. This is also what makes Gustafsson's measure of freedom of choice problematic. His measure is relevant also in this context since Gustafsson suggests that his measure of freedom of choice may be appropriate as a measure of diversity.

Relative to both measures, $A = \{0 \text{ h}, 20 \text{ h}\}$ may be ranked as equally diverse, less diverse, or more diverse than $B = \{20 \text{ h}, 40 \text{ h}\}$, depending on the nature of the relevant universal set X . For Eiswerth's and Haney's measure, A is ranked as equally diverse as B if the relevant universal set of working hours $X = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$, A is ranked as less diverse than B if $X = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$, and A is ranked as more diverse than B if $X = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}, 60 \text{ h}\}$. For Gustafsson's measure A is ranked as equally diverse as B if the relevant universal set $X = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$, A is ranked as less diverse than B if $X = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}, 60 \text{ h}\}$, and A is ranked as more diverse than B if $X = \{0 \text{ h}, 5 \text{ h}, 10 \text{ h}, 15 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$.

Another problem is that of two sets of two options, the set with the larger diameter may be judged as less diverse. For example, let $G = \{0 \text{ h}, 20 \text{ h}\}$ and $H = \{20 \text{ h}, 30 \text{ h}\}$, where G has a diameter of 20 and H has a diameter of 10.

According to Eiswerth's and Haney's measure, G may be ranked as less diverse than H if the relevant universal set is $X = \{0 \text{ h}, 5 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}\}$. According to Gustafsson's measure, G may be ranked as less diverse than H if the relevant universal set is $X = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}, 40 \text{ h}, 50 \text{ h}\}$. Both these rankings are counterintuitive since sets of two options seem more diverse if they have a larger diameter.

Eiswerth's and Haney's measure has also been criticized by Lacevic et al. (2007: 1864), for rewarding an unequal distribution of points in one-dimensional space. Some further criticism of their measure may be found in Solow et al. (1993: 62).

Eiswerth's and Haney's and Gustafsson's measures seem inadequate to measure diversity. The main reason for this is that the diversity of a set is simply not a function of its relations to the relevant universal set. The diversity of a set is a function of relations among the elements of the set itself.

8.3 Relations to Mean Point

Another alternative to the idea that the diversity of a set is a function of how dissimilar the elements of the set are to one another, is the idea that the diversity of a set is a function of the distances from the options to some mean option. This idea is the basis of several statistical conceptions of diversity, such as *variance*, *standard deviation*, and *mean absolute deviation*. It is also the basis of a measure suggested by Crocker.

In his book *Positive Liberty* from 1980, Crocker proposes a rather complicated measure of positive liberty. According to him, positive liberty is a function of several factors, one of them being diversity. It is Crocker's idea of how to measure diversity that is interesting in this context. He suggests that we should aggregate a value d_i for each option, where d_i represents the degree to which the i th option differs from the other options in the same choice set. This sounds as if we should aggregate average differences. But Crocker then continues to say that if the options may be represented as points in a space with a metric, then we may understand d_i as the distance of the i th point from the center of the populations of n points. We thus have the following measure:

Mean measure:

$$F(A, d) = \sum_{i=1}^n d(y_i, C).$$

F is a function from a finite metric space to \mathbf{R} , and C is the centroid (arithmetic mean) of A_d .

A variant of this measure is presented by Morrison and De Jong, who sum the squared distances from each element to the mean (2002: 35). Another variant would sum the distances from each element to the mean point of the relevant universal space. This type of measure occurs as a factor in a measure by Ursem (2002: 463). However, we may conclude from the previous section that this variant is worse.

We should note that Crocker's diversity measure cannot be used for all metric spaces. Any metric space can be embedded in a Banach space, and Banach spaces have arithmetic means. Nevertheless, it may be the case that a metric space can be embedded in several Banach spaces, with different arithmetic means. So there is not always a definite answer to the question of what the arithmetic mean is for a set of options whose differences are represented as metric space distances. However, we need not worry about this point here. To show how the measure works, it suffices to use Euclidean space examples. Let us compare $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ to $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. We must first calculate the arithmetic means of the two sets. The results are 30 h for A and 20 h for B . Then we must calculate the distances to the mean. For A , d_1 for 0 h is 30, d_2 for 30 h is 0 and d_3 for 60 h is 30. This adds up to 60. For B , d_1 for 0 h is 20, d_2 for 20 h is 0 and d_3 for 40 h is 20. This adds up to 40. Thus, A is ranked as more diverse than B . This seems correct.

But even if this particular ranking is correct, there are problems with the measure. One problem is that the option at the center of the set always gets a value of 0. Thus we cannot distinguish between $C = \{0 \text{ h}, 60 \text{ h}\}$ and $C \cup \{30 \text{ h}\}$ in terms of diversity. Both sets get a diversity value of 60. This seems incorrect. Another problem is that options that are very similar to one another, but dissimilar to the mean, get an exaggerated diversity value. For example, compare $J = \{0 \text{ h}, 40 \text{ h}, 60 \text{ h}\}$ to $K = \{0 \text{ h}, 59 \text{ h}, 60 \text{ h}\}$. The mean of $J \approx 33.3 \text{ h}$, while the mean of $K \approx 39.7 \text{ h}$. The *Mean measure* gives the value $F(J) \approx 66.7$ while $F(K) \approx 79.3$. Thus, Crocker's diversity measure ranks K above J . However since 59 h is very similar to 60 h, much more similar than 40 h is to 60 h, it seems obvious that J is more diverse than K . A third problem is that the measure may be insensitive to *dominance relations* between sets. There may exist two sets, G and H , with distance vectors \mathbf{d}_G and \mathbf{d}_H having distances indexed in decreasing order, such that for each pair of distances (d_{Hj}, d_{Gj}) , it holds that $d_{Hi} \geq d_{Gi}$ and there exists some pair of distances (d_{Hj}, d_{Gj}) such that $d_{Hj} > d_{Gj}$ and yet the *Mean measure* may rank H over G . For example, let $G = \{0 \text{ h}, 29 \text{ h}, 30 \text{ h}, 31 \text{ h}, 60 \text{ h}\}$ while $H = \{0 \text{ h}, 1 \text{ h}, 2 \text{ h}, 31 \text{ h}, 32 \text{ h}\}$. The distance vector $\mathbf{d}_G = (60, 60, 31, 31, 31, 31, 30, 30, 30, 30, 29, 29, 29, 29, 2, 2, 1, 1, 1, 1, 0, 0, 0, 0, 0)$ while distance vector $\mathbf{d}_H = (32, 32, 31, 31, 31, 31, 30, 30, 30, 30, 29, 29, 2, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)$. Thus, G dominates H in terms of distances. But the mean of G is 30 h, while the mean of H is 13.2 h and the *Mean measure* gives the value $F(G) = 62$ while $F(H) = 73.2$. This seems incorrect as well.

All these examples show that a greater value by Crocker's measure is neither a sufficient nor a necessary condition for greater diversity. I believe that the reason why Crocker's idea does not work is that the diversity of a set does not depend on how different the options are to some mean point. The diversity of a set depends on how different the options are to all the other options. For this reason, we also cannot use the statistical measures of variance, standard deviation or mean absolute deviation as sole indicators of diversity.

Chapter 9: Conditions Concerning Cardinality

In the coming chapters we shall assume that freedom of choice may be equated with the diversity of choice sets, and consider the reasonableness of the ideas that the diversity of a choice set depends on the cardinality of the set, the magnitudes of the differences between the options in the set, the distribution of the total sum of differences among individual differences and the distribution of the total sum of differences among the individual options. These four properties hardly influence diversity independently of one another. However, we shall begin by discussing each property somewhat independently of the other properties, before discussing all properties at once. We start with cardinality.

9.1 Cardinality

One factor that may matter for freedom of choice and the diversity of a choice set is the cardinality of the set. Cardinality obviously does not matter above all other considerations. A greater number of options is not a sufficient condition for greater diversity. $A = \{0 \text{ h}, 1 \text{ h}, 2 \text{ h}\}$ is not more diverse than $B = \{0 \text{ h}, 10 \text{ h}\}$ just because A contains more options than B . Neither is a greater cardinality a necessary condition for more diversity. B is more diverse than A , even though A contains more options. However, the diversity of a choice set may increase if cardinality is increased. We shall discuss this possibility later.

Cardinality and diversity seem to be two quite different conceptions of freedom of choice. But there are at least two authors who propose that diversity may be measured by the cardinality of options: Van Hees (2004) and Gravel (2009). Van Hees notes that the cardinality of options may be used as a measure of diversity, given the assumption that all the (non-zero) distances between all the options are equal (2004: 260). Since all the distances between the options are seldom equal, the cardinality measure can seldom be used. Gravel says that the cardinality of options may be used as a measure of diversity if cardinality is the only available information (2009: 25). This is not very relevant in this context since we have assumed that more information is available.

9.2 Monotonicity

There is a specific case where the cardinality of a choice set has some influence on diversity. If an option y is added to a set A , then the resulting set $A \cup \{y\}$ is more diverse than A . The only exception is the addition of an option to the empty set since neither the empty set nor singleton sets are diverse. This idea is captured in the following strict monotonicity condition, which is applied to freedom of choice:

The Strict monotonicity condition: For any non-empty choice set $A \in P(X)$, and any option $y \in X - A$, $A \cup \{y\}$ offers strictly more freedom of choice than A .

A similar condition is suggested as a condition for a measure of freedom of choice by Gravel et al. (1998: 82) and Puppe (1996: 178, 1998: 51), and for a measure of diversity by Van Hees (2004: 258), Xu (2003: 112, 2004: 283), Pattanaik and Xu (2008: 264) and Puppe and Xu (2010: 167). It has also been discussed by Gravel (2009: 20).

The condition is reasonable since $A \cup \{y\}$ includes an option that is different from the options in A , whereas A does not include any option that is different from the options in $A \cup \{y\}$. Moreover, all the differences that hold between the options in A also hold between the options in $A \cup \{y\}$, but there are some differences that hold between the options in $A \cup \{y\}$ that do not hold between the options in A . Therefore, $A \cup \{y\}$ is more diverse than A . For example, if we compare $A \cup \{y\} = \{15 \text{ h}, 17 \text{ h}, 20 \text{ h}\}$ to $A = \{15 \text{ h}, 20 \text{ h}\}$, we would judge that $A \cup \{y\}$ is more diverse than A because it contains an additional option. The additional option adds to diversity since it is different from the other options already included in the set. Thus, it also adds to freedom of choice.

We have already considered some criticisms of the *Strict monotonicity condition* when we discussed the *Limited strict monotonicity condition*. However, there is an additional objection to the *Strict monotonicity condition* that could not be made against the limited condition. It is possible to oppose the *Strict monotonicity condition* by assuming that there is some maximal level of freedom of choice beyond which an extra option cannot contribute. If A offers the maximal degree of freedom of choice, then $A \cup \{y\}$ cannot offer more. One way of arguing for this idea is to say that a person's freedom of choice cannot increase beyond the number of options that he is able to consider before making his choice. An option that the chooser does not have time to consider cannot increase his freedom of choice. But this argument may be opposed by making some distinctions. If we speak of freedom of choice regarding options that some person is able to consider, there is a limit to the number of options that can be included in a choice set. This means that if the maximal number of options that can be considered is

reached, there are no options to add. Thus it still holds that $A \cup \{y\}$ offers more freedom of choice than A , because this notation implies that y is an option. If we speak of freedom of choice in a more general sense, there seems to be no reason to assume that there is some upper limit to the number of options that can contribute to freedom of choice. So $A \cup \{y\}$ offers more freedom of choice than A in this case as well.

9.3 Upper and Lower Limit

The cardinality of a choice set may also be used to define a lower and upper limit to the diversity of options offered by a choice set. We discussed such limits in the chapter on cardinality measures. The *Maximal freedom of choice condition* held for freedom of choice as cardinality. It should hold for freedom of choice as diversity as well. This is not to say that there is an upper limit to diversity, per se. There is just an upper limit to the number of options that may be diverse. Also the *No freedom of choice condition* held for freedom of choice as cardinality and should hold for freedom of choice as diversity as well. Obviously there is a difference between a singleton set and the empty set; there is one element in the first set and no elements in the second set. But this difference is not a difference in diversity. There is no diversity in either set.

Perhaps one could press the point that there is a difference between the reasons for there being no diversity in the empty set and in a singleton set. There is no diversity in the empty set because there are no options, and thus no differences between the options. There is no diversity in the singleton set because there is no difference between an option and itself. Mathematically, one could express the difference by saying that the diversity of the empty set is undefined, while the diversity of a singleton set is 0. But this does not seem to reflect any substantial metaphysical difference.

9.4 Cardinality of Attributes

The cardinality of options is not an appropriate measure of the diversity of a choice set, but another kind of cardinality has been suggested for a measure of diversity, the cardinality of the attributes of the options.

It may be difficult to decide how attributes should be identified, which would make the counting of exemplified attributes difficult. Yet, the problem of identifying attributes is hardly any more difficult than the problem of identifying options.

There are different ways of representing attributes in relation to a metric space model. We shall consider several ideas. The first idea would identify attributes independently of the metric space model, while the other ideas

would identify attributes as properties that can be referred to in metric spatial terms.

Nehring and Puppe suggest that an attribute α may be represented by the set of all options that share the attribute in question. This cannot be regarded as a definition since such a definition would be circular. But let us ignore this complication here. A set A realizes the attribute α if and only if $A \cap \alpha \neq \emptyset$. Nehring and Puppe suggest that each attribute should be given a weight in accordance with its importance. Diversity should then be measured by aggregating the weights of the exemplified attributes (2002: 1161). Ignoring the idea of weights, a simplified version of Nehring's and Puppe's measure is the following:

Attribute measure:

$F(A) =$ The number of α such that $A \cap \alpha \neq \emptyset$.

F is a function from \mathbf{R} to \mathbf{R} .

If this idea is presented in a metric space language we may regard an attribute as a subspace of a metric space. We usually do not regard any subspace as an attribute. How attributes should be distinguished from other subsets is a problem that must be solved before the measure is applied. However, we shall not discuss this problem yet. We may just assume that we can identify the subsets that exemplify attributes in some way that is consistent with ordinary language use.

Since we have assumed that the universal set is finite, and that attributes are identified by subsets of the universal set, there is no risk that there is an infinite number of attributes exemplified by each option (which might have been a risk if attributes had been identified in some other way).

An advantage of counting attributes instead of options is that very different options may count for more, and that very similar options may count for less. If attributes are counted, we can draw the proper conclusion that $A = \{\text{pro-EU socialist party, anti-EU liberal party}\}$ is more diverse than $B = \{\text{pro-EU socialist party, anti-EU socialist party}\}$. A exemplifies four attributes, while B exemplifies three attributes. Since both sets contain two options, we cannot draw this conclusion by counting options.

In the last case, the property that similar options count for less when counting attributes is to the advantage of the *Attribute measure*. In other cases the property is a disadvantage. One problem is that options that are similar to others in different respects may not count at all. If we compare $A = \{\text{pro-EU socialist party, anti-EU liberal party}\}$ to $B = \{\text{pro-EU socialist party, anti-EU socialist party, pro-EU liberal party, anti-EU liberal party}\}$, the attribute counting method makes no difference between these sets at all. Both sets exemplify four attributes. Yet B seems to be more diverse than A since A is a subset of B . The measure thus fails to satisfy the *Strict monotonicity condition*. The reason for this is that the measure implies that

the addition of an option to a set only increases diversity if the additional option exemplifies at least one attribute that is not already exemplified by the other options in the set.

This problem can perhaps be solved by counting conjunctions of attributes as well. However, there are other problems with the *Attribute measure* that cannot be solved this way. One such problem is that differences within attributes cannot be represented. $G = \{0 \text{ h}, 1 \text{ h}\}$ and $H = \{0 \text{ h}, 40 \text{ h}\}$ exemplify only one attribute, time. This problem may partly be solved by dividing each attribute into several attributes according to the degree that the attribute is represented. We could then say that 0 h and 1 h both exemplify “a rather short time”, while 40 h exemplify “a rather long time”. However, this solution leads to sorites problems. Another problem with the measure is that differences between attributes cannot be represented. $A = \{\text{pro-EU socialist party, anti-EU liberal party}\}$ and $C = \{\text{pro-EU socialist party, anti-EU communist party}\}$ both exemplify four attributes. It should matter that the options are more different in C than in A . But attribute counting does not reflect this fact. Yet another problem with the measure is that it does not satisfy the *INS-condition*. Singleton sets may offer different degrees of diversity depending on how many attributes the single options exemplify. Singleton sets may even offer more diversity than sets of several options (as noted by Gravel, (2009: 51). However, this problem may be solved by stipulation.

If attributes are identified with sets, there are not so many advantages of counting exemplified attributes rather than options. However, there are other ways to represent attributes in a metric space model, and these might be more advantageous. Nehring and Puppe have several suggestions for how attributes may be represented in a model where diversity is regarded as a function of the pairwise dissimilarities between options. One idea is that when the options differ in several dimensions, they may be represented as vectors of coordinates. Each coordinate represents an attribute. When an option exemplifies a certain attribute its coordinate is 1, and when it does not exemplify an attribute its coordinate is 0 (2002: 1174). However, this idea seems too crude. As we mentioned, $G = \{0 \text{ h}, 1 \text{ h}\}$ and $H = \{0 \text{ h}, 40 \text{ h}\}$ exemplify only one and the same attribute, time. Yet the two sets do not seem to offer the same amount of diversity of choice.

Another idea is to define an attribute more directly in relationship to the metric space model. My suggestion is to define an attribute as ‘the state of being at a particular distance t of a particular option x ’. From the point of ordinary language use this definition seems rather odd. However, it has some technical advantages. If we use this definition we may define another attribute measure:

Extrinsic attribute measure:

$F(A)$ = The number of T such that for some $x \in A$, x has T , where T is the attribute of being at a particular distance t to a particular option $y \in X$.

F is a function from \mathbf{R} to \mathbf{R} .

The measure is called the *Extrinsic attribute measure* since distances to possible options outside a set A is included in defining the attributes exemplified by the options in A .

When the number of options is n , each option has n extrinsic attributes. However, a set containing m options is not necessarily going to contain $m \times n$ attributes since some options may have the same attributes as some others. For example, let us suppose that the relevant universal set is $X = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$. We compare the sets $A = \{0 \text{ h}, 40 \text{ h}\}$, $B = \{0 \text{ h}, 20 \text{ h}\}$ and $C = \{0 \text{ h}, 10 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$. The relevant universal distance matrix looks as follows:

	0 h	10 h	20 h	30 h	40 h
0 h	0	10	20	30	40
10 h	10	0	10	20	30
20 h	20	10	0	10	20
30 h	30	20	10	0	10
40 h	40	30	20	10	0

Consider A ; each option in A exemplifies 5 attributes. But since the two options in A share the attribute of being at a distance of 20 from the option 20 h, the options in A only exemplify 9 attributes. The options in B also exemplify 9 attributes. The options in C exemplify 18 attributes. So C is ranked as offering more diversity of choice than A and B , which offer an equal amount.

The *Extrinsic attribute measure* is in some ways better than the *Attribute measure*. It satisfies the *INS-condition*. This is because each single option has as many attributes as there are possible options in the universal set. Thus, singleton sets exemplify the same number of attributes. The measure also satisfies the *Strict monotonicity condition*. This is because each option x has an attribute that is unique for that option, namely the attribute of being at a distance of 0 to x . So for each option that is added one more attribute is exemplified. The measure satisfies the *Maximal freedom of choice condition* but not the *No freedom of choice condition*. The last problem may be solved by stipulation.

There are problems with the measure, however. One problem is that the number of exemplified attributes differs depending on the nature of the relevant universal set. The measure does not satisfy the *Domain-insensitivity condition*. To solve this problem we may define a new measure that is based

on the intrinsic distances between the options in a set. We then have the following measure:

Intrinsic attribute measure:

$F(A)$ = The number of T such that for some $x \in A$, x has T , where T is the attribute of being at a particular distance t to a particular option $y \in A$.

F is a function from \mathbf{R} to \mathbf{R} .

The *Intrinsic attribute measure* satisfies the *Domain-insensitivity condition*. Apart from this fact, there is not much that speaks in favor of this measure, in comparison to the previous measure. The *Intrinsic attribute measure* also satisfies the *INS-condition* since each single option only exemplifies one intrinsic attribute, the attribute of being at a distance of 0 to itself. It also satisfies the *Strict monotonicity condition* since each option has an attribute that is unique for that option, namely the attribute of being at a distance of 0 to itself. Thus, for each option that is added, one more attribute is exemplified. The measure satisfies the *Maximal freedom of choice condition*, but not the *No freedom of choice condition*. But, as with the last measure, the *No freedom of choice condition* can be satisfied by stipulation.

There are many problems that the two measures share. One problem is that the measures punish an even distribution of options since reoccurring attributes only count once. Another problem is that the measures are insensitive to the degree of the differences between options. Because of this, the measures imply that the sets $A = \{0 \text{ h}, 40 \text{ h}\}$ and $B = \{0 \text{ h}, 20 \text{ h}\}$ offer an equal amount of diversity of choice, even though A seems to offer more.

One could try to remedy the last problem in different ways. One method would be to rank sets in accordance with a lexical ordering of attributes, where attributes defined by larger distances would be given lexical priority to attributes defined by smaller distances. A problem with this solution is that sets with larger diameters would always be ranked above sets with smaller diameters. This would be counterintuitive. Another method would be to give attributes defined by larger distances greater weights. However, if we rely on this solution, there seems to be no point in counting attributes at all since we might as well aggregate distances directly. In any case, none of the proposed modifications would solve the first problem since reoccurring attributes would count only once, no matter their weight.

None of the attribute measures seem adequate to measure the diversity of choice sets. Let us thus consider another attempt to measure diversity using a kind of cardinality, the cardinality of exemplified categories.

9.5 Cardinality of Categories

If identifying attributes and options is difficult, it is not any easier to identify categories. Categories may be regarded as subsets, or subspaces, of the relevant universal set and space. In this sense, they are similar to attributes. A category differs from an attribute by being a set of options that may be similar with regard to several attributes. Usually, but not necessarily, categories also differ from attributes in the respect that categories cannot overlap, which attributes can. Categories are generally created by partitioning the relevant universal set of options. The categories that are thus created may in turn be partitioned into subcategories, which may in turn be partitioned into sub-subcategories, and so on. For each level of partitioning, each option belongs to one category and no more than one category.

Let us first assume that the categories are identified in accordance with some ordinary language use. We shall also assume that categories cannot overlap. A category measure may then be defined as follows:

Category measure:

$F(A)$ = The number of C such that there is at least one option $x \in A$ such that $x \in C$.

This type of measure is common in biology, where it is customary to count the number of species to measure the diversity of a population. Two early mentions of this type of measure may be found in Conell and Orias (1964) and Lloyd and Ghelardi (1964) under the name of *richness*. In practice, this type of measure has been used long before it was named.

The *Category measure* has the virtue of satisfying the *INS-condition* since each option exemplifies the same number of categories. The exact number depends on how many times the relevant universal set has been partitioned. $A = \{\text{pro-EU socialist party}\}$ offers as much diversity of choice as $B = \{\text{anti-EU liberal party}\}$ since they exemplify the same number of categories. If the relevant universal set is partitioned once, for example, into socialist parties and liberal parties, or into pro-EU and anti-EU parties, A and B each exemplify one category. If the relevant universal set is partitioned twice, for example, first into socialist parties and liberal parties, and then into pro-EU socialist, anti-EU socialist, pro-EU liberal and anti-EU liberal parties, then A and B each exemplify two categories.

The measure has some obvious problems, though. It is not sensitive to the number of options, nor is it sensitive to differences between categories. It does not satisfy the *Strict monotonicity condition*. Neither does it satisfy the *Maximal freedom of choice condition* or the *No freedom of choice condition*. But the last problem may be solved by stipulation.

We may try to define categories in a more technical way. A category may be thought of as a set of options that are located within a certain distance d to

each of the other options in the same set. The options in A exemplify a category C if and only if $A \cap C \neq \emptyset$. To simplify matters we may skip the assumption that categories cannot overlap.

A *category individuated at a distance t* is a set C_t such that $C_t \in P(X)$ and for all pairs of options (x, y) such that $x, y \in C_t$, it holds that $d(x, y) \leq t$ and for any option z such that $d(z, x) \leq t$ and $d(z, y) \leq t$, it holds that $z \in C_t$.

Since there are an infinite number of possible distances, there are an infinite number of ways to define a category. If this is allowed, each option would exemplify an infinite number of categories and each set would be judged as equally infinitely diverse. To be able to rank choice sets by the cardinality of categories, we must find a method of selecting which categories should count. It is a bad idea to use only the intrinsic distances that occur between the options in a set to define the categories that should count. This implies that sets of two options always offer the same amount of diversity of choice, even when they have different diameters. Instead, we may use all the distances that hold between the possible options in the relevant universal set. We may call these distances the *relevant distances*. The measure can then be defined as follows:

Distance category measure:

$F(A)$ = The number of C_{t_i} , defined for all relevant distances t_i , such that there is at least one option $x \in A$ such that $x \in C_{t_i}$.

F is a function from \mathbf{R} to \mathbf{R} .

Let us suppose that the relevant universal set is the set of possible working hours $X = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$. We shall compare $A = \{0 \text{ h}, 40 \text{ h}\}$ and $B = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}\}$. The category counting proceeds as follows: first we list the distances between the elements of X , thus $d(0 \text{ h}, 0 \text{ h}) = 0$, $d(0 \text{ h}, 10 \text{ h}) = 10$, $d(0 \text{ h}, 20 \text{ h}) = 20$, etc. Then we make a list of the distances in decreasing order: 40, 30, 20, 10, 0; these distances shall be used to identify the categories. Next we shall see how many categories of working hours can be identified at each distance. We may start with the largest distance $d = 40$ and ask which categories can be identified at this distance. The answer to this question is one category: $C_1 = \{0 \text{ h}, 10 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$. We then continue with the second largest distance $d = 30$, and ask the same question. This time the answer is two: $C_2 = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}\}$ and $C_3 = \{10 \text{ h}, 20 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$. For the third largest distance $d = 20$, the answer is three: $C_4 = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}\}$, $C_5 = \{10 \text{ h}, 20 \text{ h}, 30 \text{ h}\}$, and $C_6 = \{20 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$. For the second smallest distance $d = 10$, the answer is four: $C_7 = \{0 \text{ h}, 10 \text{ h}\}$, $C_8 = \{10 \text{ h}, 20 \text{ h}\}$, $C_9 = \{20 \text{ h}, 30 \text{ h}\}$, and $C_{10} = \{30 \text{ h}, 40 \text{ h}\}$. For the smallest distance $d = 0$, the answer is five: $C_{11} = \{0 \text{ h}\}$, $C_{12} = \{10 \text{ h}\}$, $C_{13} =$

{20 h}, $C_{14} = \{30 \text{ h}\}$, and $C_{15} = \{40 \text{ h}\}$. The options in A exemplify the categories $C_1, C_2, C_3, C_4, C_6, C_7, C_{10}, C_{11}$, and C_{15} . The options in B exemplify the categories $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{11}, C_{12}$, and C_{13} . Thus, A exemplifies $1 + 2 + 2 + 2 + 2 = 9$ categories, while B exemplifies $1 + 2 + 3 + 3 + 3 = 12$ categories. Therefore, B offers more diversity of choice than A .

The *Distance category measure* satisfies the *Strict monotonicity condition* since an extra option always exemplifies one extra category at the distance of 0. For the same reason, it satisfies the *Maximal freedom of choice condition*. The measure does not satisfy the *INS-condition* since the single options of two different singleton sets may exemplify different number of categories. It also does not satisfy the *No freedom of choice condition*, but this problem may be solved by stipulation.

A possible objection to the *Distance category measure* is that it does not satisfy the *Domain-insensitivity condition*. The rankings are sensitive to the identity of the relevant universal set. A ranking done in relation to a relevant universal set that includes possible options at many different small distances will not be the same as a ranking done in relation to a relevant universal set that includes possible options at many different large distances. The measure thus does not satisfy the *Domain-insensitivity condition*.

Another objection to the *Distance category measure* is that it may rank sets with a larger diameter above sets with a smaller diameter that are otherwise equal. If the universal set X is very small, $X = \{0 \text{ h}, 12 \text{ h}, 24 \text{ h}\}$, then $A = \{0 \text{ h}, 12 \text{ h}\}$ and $B = \{0 \text{ h}, 24 \text{ h}\}$ are ranked as equally diverse since the options of each set exemplify six categories. A related problem is that the measure does not give categories defined by larger distances a greater weight even though they are more important for diversity. This problem may be solved in at least two ways. One solution is to order the categories lexically, so that a set that exemplifies a greater number of categories defined by a larger distance would be judged as offering more diversity than a set exemplifying fewer categories defined by the same and larger distances. If we use this method, however, then the set that has the largest diameter may be ordered over any other set, no matter how the sets compare in other aspects. This seems counterintuitive. Another solution is to give the categories numerical weights, depending on the magnitude of the distance that identifies the category. However, if we use this method, it seems better to measure diversity using the magnitude of the distances directly. We shall therefore consider such proposals next.

Chapter 10: Conditions Concerning Magnitude of Differences

The magnitudes of the differences between the options in a choice set should matter for the diversity of a choice set. This seems trivial. But they could matter in different ways and this is not trivial. All differences between the options may matter, or only some. All differences may contribute equally or some may contribute more than others. The contribution of a difference may be proportional to the magnitude of the difference or it may not. The contribution of a difference may be independent of the other differences or it may not. There are a many possibilities.

If all the differences matter to diversity, they can do so in different ways. The differences may matter either collectively or individually. If all the differences matter collectively, they could matter either in the form of a *total sum* of differences or in the form of an *average difference*. The total sum of differences and the average difference capture information about all differences in the form of a single statistic. If all the differences matter individually, they may be used for difference-by-difference comparisons between choice sets and ranked by the principle of *dominance*. The principle of dominance takes all differences into account, but it is not always applicable.

If only some differences matter, these differences may also matter collectively, in the form of a sum of differences, or individually, through dominance comparisons between individual differences. An individual difference that may be especially important is the diameter. Sets of differences that may be especially important include the set of minimal differences from each option to the others, the set of maximal differences from each option to the others, and the set of minimal path distances connecting all the options to one another.

We shall not discuss all possible suggestions here, but we shall discuss some of them. Many of the suggestions will be discussed together with a corresponding measure.

10.1 Total Sum of Differences

A good start for a discussion is to assume that all differences matter to diversity. One idea is that the *total sum of all differences* matters for the diversity of a set. The total difference sum is defined as follows:

The *total difference sum* of a metric choice set A_d equals

$$\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j) \text{ for all } x_i, x_j \in A, \text{ where } n = |A|.$$

The total difference sum shall also be written as $\Sigma(A_d)$. For example, let $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ with the distance matrix \mathbf{M}_A :

	0 h	30 h	60 h
0 h	0	30	60
30 h	30	0	30
60 h	60	30	0

Let $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$ with the distance matrix \mathbf{M}_B :

	0 h	20 h	40 h
0 h	0	20	40
20 h	20	0	20
40 h	40	20	0

A may be more diverse than B for the reason that the total sum of differences is greater in A_d than in B_d (240 vs. 160). If a greater total difference sum would be both a sufficient and necessary condition for greater diversity, we should accept the following measure:

Total difference sum measure:

$$F(A, d) = \sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j).$$

F is a function from a finite metric space to \mathbf{R} .

This measure is suggested by Barker and Martin (2000: 1005), although they use only half the distances (not the symmetrical ones). It is also suggested by Wineberg and Oppacher for a special type of metric distances, the Hamming distance (2003: 1494). It is criticized by Rosenbaum (2000: 223), Van Hees (2004: 263) and Lacevic et al. (2007: 1864). The first two authors criticize the measure for the excessive growth of the function when an element is added to a set. The last authors criticize the measure for awarding clusters of elements, rather than elements that are more diverse.

Initially, the *Total difference sum measure* may seem to capture the concept of diversity perfectly. For example, $C = \{0 \text{ h}, 15 \text{ h}, 30 \text{ h}\}$ is considered more diverse than $G = \{0 \text{ h}, 30 \text{ h}\}$ since $F(C) = 120$ and $F(G) = 60$. This is intuitively correct. A set is more diverse than any of its proper subsets. Also, $G = \{0 \text{ h}, 30 \text{ h}\}$ is considered more diverse than $H = \{0 \text{ h}, 15 \text{ h}\}$ since $F(H) = 30$. This is also intuitively correct. A set with options at larger distances is more diverse.

Nevertheless, there are reasons to think that a greater total difference sum is neither sufficient nor necessary for diversity. For example, the sum of differences for $J = \{0 \text{ h}, 1 \text{ h}, 29 \text{ h}, 30 \text{ h}\}$, which is equal to 236, is greater than the sum of differences for $C = \{0 \text{ h}, 15 \text{ h}, 30 \text{ h}\}$, which is equal to 120. But the ranking of J over C is not a reasonable ranking; C is more diverse than J . If a greater total difference sum matters to diversity, it does not do so by being sufficient for greater diversity. Neither does it do so by being necessary for greater diversity, as is evident from the same example.

The total sum of differences may matter as one factor that affects the diversity of choice sets. If several factors are relevant for diversity, then the diversity of A may be greater than the diversity of B , if the other factors are equal but A has a greater total difference sum (if this is at all possible).

10.2 Limited Growth

An important argument against the idea that the total difference sum matters for diversity is that the total difference sum grows at a greater rate than diversity. It grows excessively fast with the addition of options. This is pointed out by Rosenbaum (2000: 223) and Van Hees (2004: 263).

If we only wish to make ordinal comparisons between sets that are subsets of one another or between sets of equal cardinality, excessive growth of the diversity function is not a problem. Problems arise for other cases. Excessive growth is a problem if we want a diversity measure that is ratio scale since it may result in counterintuitive ratio relations between the diversity values of sets. Excessive growth is also a problem for some ordinal diversity measures since it may result in counterintuitive rankings of sets. It is, for example, a problem for comparisons of sets of different cardinality that are not subsets of one another. Let us look at an example. We compare the sets $A = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$ and $B = \{0 \text{ h}, 29 \text{ h}, 30 \text{ h}\}$. The *Total difference sum measure* gives A and B the values $F(A) = 160$ and $F(B) = 120$, respectively, and orders A above B . Next we add the option 1 h to B and get $C = B \cup \{1 \text{ h}\} = \{0 \text{ h}, 1 \text{ h}, 29 \text{ h}, 30 \text{ h}\}$. The *Total difference sum measure* gives C the value $F(C) = 236$ and orders C above A . This seems intuitively wrong. A seems more diverse than C , despite C having an extra element. This is a problem for the measure.

The general characterization of this problem is that the *Total difference sum measures* increase too fast with the addition of options. When there are n options in a set A , there are n^2 distances in the distance vector \mathbf{d}_A . All but the n zero distances contribute to the diversity value of the *Total difference sum measure*. For each addition of an option to the set A there is an addition of n non-zero distances to the distance vector \mathbf{d}_A . If there are 100 options in A , there are 9900 non-zero distances in \mathbf{d}_A . If an option y is added to A , there are 101 options in $A \cup \{y\}$ and 10100 non-zero distances in $\mathbf{d}_{A \cup \{y\}}$. This is a growth of 200 distances contributing to the total difference sum. It is generally absurd to think that the addition of an option to a set increases the diversity to a degree that is proportional to the increase of the total difference sum. The contribution of an option to the diversity of a set can hardly depend that much on the size of the set. If there is a choice between adding an option to a larger set or a smaller set in order to maximize diversity, we do not think that the option should be added to the larger set. A political party added to a larger set of political parties would not add significantly more to the political diversity than the same political party added to a smaller set of political parties. In more general terms, the contribution to diversity by an option added to a larger set should not be significantly greater than the contribution to diversity by the same option added to a smaller set. However, the contribution to diversity by an option added to a smaller set should also not be significantly greater than the contribution to diversity by the same option added to a larger set. The contribution by an option added to a smaller set is obviously *proportionally* greater than the contribution by the same option added to a larger set. But there are no reasons to think that the contribution by an option added to a smaller set is *absolutely* greater than the contribution by the same option added to a larger set.

All these considerations suggest that a measure of diversity should not grow proportionally to the number of distances. A measure of diversity should at most grow proportionally to the number of options. This assures that the contribution to the diversity of a set by an option is independent of the size of the set.

In his essay on diversity, Weitzman proposes a condition regarding the maximum diversity that can be added by a species (1992: 392). If the condition is rewritten slightly to fit the notation of this thesis, and applied to freedom of choice, we get the following condition:

The Maximal additional freedom of choice condition: For a measure of freedom of choice F , any metric choice set $A_d \in P(X_d)$, and any option $y \in X - A$, it holds that $F((A \cup \{y\})) \leq F(A) + \max d(x_i, y)$ such that $x_i \in A$.

As the condition is stated, it is rather reasonable, but one detail should be changed to make it more general. We should not require that $F((A \cup \{y\})) \leq$

$F(A) + \max d(x_i, y)$ since this would exclude measures that aggregate factors other than distances, as well as measures that aggregate both the distance $\max d(x_i, y)$ and its symmetric counterpart, the distance $\max d(y, x_i)$. The last class of measures might satisfy $F((A \cup \{y\})) \leq F(A) + 2 \max d(x_i, y)$, but not Weitzman's condition. We should instead require that $F((A \cup \{y\})) \leq F(A) + F(\{x_i, y\})$, where $d(x_i, y)$ is the maximal distance between y and any $x_i \in A$. That is to say, whatever value the diversity function gives to the set of y and the furthest option x_i in A should be the maximal value that the addition of y can add to the diversity value of A . With this slight change of Weitzman's condition we get the following condition:

The Limited growth condition: For a measure of freedom of choice F , any metric choice set $A_d \in P(X_d)$ and any option $y \in X - A$, it holds that $F((A \cup \{y\})) \leq F(A) + F(\{x_i, y\})$, where $d(x_i, y) = \max_k d(x_k, y)$ and $x_i, x_k \in A$.

The *Limited growth condition* only applies to measures that assign a numerical value to the compared sets. It is thus not generally applicable.

Even though the *Limited growth condition* is not generally applicable, it is satisfied by many diversity measures. Some measures satisfy the condition because they do not aggregate all differences. Other measures satisfy the condition because they do not merely aggregate all differences but also divide the aggregated sum by some factor that prevents excessive growth.

10.3 Diameter and Lexical Difference Orderings

Instead of assuming that all differences matter for diversity, we may assume that only some matter. The most extreme idea is that only one difference per set matters for diversity, the diameter. When ranking the sets $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ and $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$, we may rank A above B , just because a diameter of 60 is larger than a diameter of 40. This idea occurs in Gravel for a measure of diversity (2009: 43). It also occurs in Rosenbaum for a measure of freedom, although his measure also incorporates some information regarding probabilities (2000: 216).

Ranking choice sets in accordance with their diameter may seem reasonable when there are just two options in a set. A offers more diversity than B when A has two options which are more different to one another than are the options in B . We may thus accept the following condition for a ranking of sets in terms of freedom of choice as diversity:

The Limited diameter condition: For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B| = 2$, and all options $x, y \in X$ such that $x \in A$ and $y \in B$, if $d(x_i, x_j) > d(y_i, y_j)$, then A offers strictly more freedom of choice than B .

This condition occurs in an essay by Pattanaik and Xu (2008: 264). Similar conditions also occur in essays by Rosenbaum (2000: 212) and Bossert et al. (2003: 418). According to the *Limited diameter condition*, $C = \{0 \text{ h}, 41 \text{ h}\}$ offers more freedom of choice than $G = \{0 \text{ h}, 15 \text{ h}\}$, due to C having a larger diameter than G .

The *Limited diameter condition* seems to describe something that is trivially true about diversity. But its use is very limited. One may wonder if the diameter can be used to rank sets of a greater cardinality than two. We may think that the ranking of sets in accordance with their diameter is reasonable since the diameter is the largest difference between the options in a choice set and therefore supposedly the most important one. If a larger diameter would be both a sufficient and necessary condition for greater diversity, then the following measure of diversity may be used:

Diameter measure:

$$F(A, d) = \max (d(x, y) \text{ such that } x, y \in A).$$

F is a function from a finite metric space to \mathbf{R} .

This measure is basically the same as Rosenbaum's measure. It is also discussed by Gravel, who calls it the *Maxi-max ranking* (2009: 43).

However, the idea that a larger diameter is either a sufficient or necessary condition for greater diversity is unreasonable. As for the sufficiency condition, we cannot reasonably judge that $C = \{0 \text{ h}, 41 \text{ h}\}$ offers more diversity than $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. As for the necessity condition, we cannot reasonably judge that B cannot offer more diversity than C . (A similar critique is made by Nehring and Puppe (2002: 1173), Dowding and Van Hees (2009: 379), and by Rosenbaum himself (2000: 222).) An additional problem is that the measure does not satisfy the *Strict monotonicity condition*. For example, $C = \{0 \text{ h}, 41 \text{ h}\}$ is judged as equally diverse as $C \cup \{y\} = \{0 \text{ h}, 20 \text{ h}, 41 \text{ h}\}$. The measure does satisfy the *Limited growth condition*, however.

The idea of ranking choice sets according to their diameter is too crude. The smaller distances should also matter to diversity. This is the idea behind a measure by Bervoets and Gravel (2007: 264). They propose a lexical diversity ordinal measure, where sets are first compared by their largest distance, and if equal, are compared by their second largest distance and so on, until one distance in one of the sets is found to be larger than the compared distance in the other sets. This set is then judged as more diverse

than the others (2007: 265). Even though the measure is designed for ordinal distances, it should work for ratio scale distances as well. We have:

The Lexi-max ordinal measure:

For all metric choice sets $A_d, B_d \in P(X_d)$ with distance vectors \mathbf{d}_A and \mathbf{d}_B , if and only if for all i , $d_{Ai} = d_{Bi}$, then A offers equal freedom of choice as B , and if and only if there exists some m such that $d_{Am} > d_{Bm}$ and for all $i > m$, $d_{Ai} = d_{Bi}$, then A offers strictly more freedom of choice than B .

This measure is slightly more reasonable than the *Diameter measure*, but not much. The advantage lies in the fact that the *Lexi-max ordinal measure* takes more distances into account. Consequentially, a set such as $C \cup \{y\} = \{0 \text{ h}, 20 \text{ h}, 41 \text{ h}\}$ is ranked as more diverse than a set such as $C = \{0 \text{ h}, 41 \text{ h}\}$. But most counterexamples that were used against the *Diameter measure* can be used against the *Lexi-max ordinal measure* as well. Both measures give too much weight to the largest distance.

The *Lexi-max ordinal measure* satisfies the *No freedom of choice condition* since all singleton options only have a distance of 0, between the singleton option and itself. It does not satisfy the *Limited growth condition*. The reason for this is that the *Lexi-max ordinal measure* is only an ordinal measure. The *Lexi-max ordinal measure* does not satisfy the *Strict monotonicity condition* either since the new option may not imply the addition of any new distances. For the same reason, it also fails to satisfy the *Maximal freedom of choice condition*. (Further criticisms may be found in Wineberg and Oppacher (2003: 1495).)

10.4 The Exclusive Average Difference

If only one difference per choice set matters for diversity, the most reasonable candidate is some average difference. An average difference has the advantage of representing all the differences among the options in a set without excessive growth. It may be computed in two different ways: either by excluding zero differences, or by including them. The first type of average may be called the *exclusive average*, while the second may be called the *inclusive average*. Let us first consider the exclusive average difference. It is defined as follows:

The *Exclusive average difference* of a metric choice set A_d equals

$$\frac{\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j)}{n^2 - n} \text{ for all } x_i, x_j \in A, \text{ where } n = |A| \text{ and } n \geq 2.$$

A measure based only on the exclusive average is the following:

Exclusive average distance measure:

$$F(A, d) = \frac{\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j)}{n^2 - n} \text{ for all } x_i, x_j \in A.$$

F is a function from a finite metric space to \mathbf{R} .

The *Exclusive average distance measure* is suggested as a measure of diversity by Wen et al. (1998: 773). It is also suggested by Warwick and Clarke, although they apply the measure to species and not to individuals (1995: 302). Similar measures also occur in the literature. The square root of the exclusive average distance measure is suggested as a measure of diversity by Lacevic and Arnaldi, (2010: 3). The square root of the total squared differences divided by $n^2 - n$ is suggested as a measure of spread by Heffernan (1988: 100).

If the *Exclusive average distance measure* is used to rank the sets $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ and $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$, then A is ranked above B just because the exclusive average difference of 40 (240/6) in A is greater than the exclusive average difference of roughly 26.7 (160/6) in B . This seems correct. Using the exclusive average difference as the only condition for diversity is not generally a good idea, however. The *Exclusive average distance measure* may come in conflict with the *Strict monotonicity condition*. When an element is added to a set the exclusive average difference may decrease. This happens, for example, for $C = \{0 \text{ h}, 40 \text{ h}\}$ and $C \cup \{20 \text{ h}\} = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. The exclusive average difference for C is 40, while the exclusive average difference for $C \cup \{20 \text{ h}\}$ is roughly 26.7. This example shows that a greater exclusive average difference is neither sufficient nor necessary for greater diversity.

10.5 The Inclusive Average Difference

Most of the comments that apply to the exclusive average difference also apply to the inclusive average difference. One advantage of the inclusive average difference is perhaps that it is more directly related to the total

difference sum. The inclusive average difference equals the total difference sum divided by the cardinality of differences. It is defined as follows:

The *Inclusive average difference* of a metric choice set A_d equals

$$\frac{\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j)}{n^2} \text{ for all } x_i, x_j \in A, \text{ where } n = |A| \text{ and } n \geq 2.$$

A measure based only on the inclusive average is the following:

Inclusive average distance measure:

$$F(A, d) = \frac{\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j)}{n^2} \text{ for all } x_i, x_j \in A.$$

F is a function from a finite metric space to \mathbf{R} .

A similar measure is suggested by Heffernan who proposes the square root of the total squared differences, divided by n^2 as a measure of spread (1988: 101). If the *Inclusive average distance measure* is used to rank the sets $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ and $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$, then A is ranked above B because the inclusive average difference of roughly 26.7 (240/9) in A is greater than an average difference of roughly 17.8 (160/9) in B .

The *Inclusive average distance measure* has the same major disadvantage as the *Exclusive average distance measure*; it may come in conflict with the *Strict monotonicity condition*. In the previous example, comparing $C = \{0 \text{ h}, 40 \text{ h}\}$ with $C \cup \{20 \text{ h}\} = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$, the set C gets an inclusive average difference of 20 and the set $C \cup \{20 \text{ h}\}$ gets an inclusive average difference of roughly 17.8. The inclusive average is thus neither a necessary nor sufficient condition for greater diversity.

10.6 Exclusive Average Differences

We considered the idea that only one difference per choice set matters for the diversity of the set. It is more common to assume that only one difference per option matters. For each choice set, one difference per option is aggregated in the form of a sum that measures the degree of diversity. The selected difference δ is known as *the distance between an element and a set*. The distance δ between an element and a set may be defined in different ways. Van Hees (2004) discusses several versions, some of which we shall consider shortly.

All measures that aggregate only one distance per option satisfy the *Limited growth condition*. This is because the aggregated distance is never larger than the diameter of the set. The satisfaction of the *Limited growth condition* is an advantage of these measures.

The first measure of this kind that we shall study is very similar to the last two proposals. Instead of comparing the total average difference for each set, we compare the aggregated average differences for each option. More precisely, we first define:

Exclusive average element to set distance:

$$\delta_{\text{exave}}(A_d, y) = \frac{\sum_{i=1}^n d(x_i, y)}{n} \quad \text{such that } x_i \in A \text{ and } y \notin A, \text{ where } n = |A|$$

and $n \geq 2$.

(See Van Hees (2004: 263).) The *Exclusive average element to set distance* ignores the zero distances. If $A = \{0 \text{ h}\}$ and $y = 30 \text{ h}$, then the average element to set distance between A and y is $60/2 = 30$.

We then define an aggregation measure to calculate the diversity of a set A from the element to set distances δ_i :

General representative distances measure:

$$F(A, d) = \sum_{i=1}^n \delta(A - \{x_i\}, x_i) \quad \text{such that } x_i \in A.$$

F is a function from a finite metric space to \mathbf{R} .

We may use the *Exclusive average element to set distance* for the *General representative distances measure* to construct a *Representative exclusive average distances measure*. This measure is equivalent to a measure that consists in a ratio between the *Total difference sum measure* and the cardinality of a set minus one (when $n \geq 2$):

Exclusive ratio measure:

$$F(A, d) = \frac{1}{n-1} \sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j) \quad \text{such that } x_i, x_j \in A, \text{ where } n = |A| \text{ and } n \geq 2.$$

F is a function from a finite metric space to \mathbf{R} .

To illustrate how this measure works we may again compare $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ and $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. As we may recall, this is the distance matrix \mathbf{M}_A :

	0 h	30 h	60 h
0 h	0	30	60
30 h	30	0	30
60 h	60	30	0

This is the distance matrix \mathbf{M}_B :

	0 h	20 h	40 h
0 h	0	20	40
20 h	20	0	20
40 h	40	20	0

For A , we aggregate $90/2 + 60/2 + 90/2 = 120$. For B , we aggregate $60/2 + 40/2 + 60/2 = 80$. This is equivalent to $240/2 = 120$ and $160/2 = 80$, respectively.

The *Exclusive ratio measure* has the advantage that it does not ignore any information. But as a measure it hardly fares any better than the *Exclusive average distance measure*. The reason is that the *Exclusive ratio measure* may come into conflict with the *Strict monotonicity condition*. If an option is added to a set, then the sum of exclusive average element to set differences may not increase. We may compare $C = \{0 \text{ h}, 60 \text{ h}\}$ and $C \cup \{30 \text{ h}\}$, where the *Exclusive ratio measure* gives C a value that equals $60 + 60 = 120$ and gives $C \cup \{30 \text{ h}\}$ a value that equals $45 + 30 + 45 = 120$. Thus, a greater sum of exclusive average element to set differences is not necessary for greater diversity. It is not sufficient either since $G = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$ is given a value of 80 and $H = \{0 \text{ h}, 1 \text{ h}, 39 \text{ h}, 40 \text{ h}\}$ is given a value of 158, although G is more diverse than H .

Further on we shall see that the sum of exclusive average element to set differences can be included as a part of a more complex measure of diversity. But initially, there seem to be a better contender for such a part, namely, the sum of inclusive average differences.

10.7 Inclusive Average Differences

Instead of using an exclusive average element to set distance, we may use an inclusive average element to set distance. The *Inclusive average element to set distance* does not ignore any zero distance between an element and itself. It may be defined as follows:

Inclusive average element to set distance:

$$\delta_{\text{inave}}(A_d, y) = \frac{\sum_{i=1}^n d(x_i, y)}{n+1} \text{ such that } x_i \in A \text{ and } y \notin A, \text{ where } n = |A|.$$

If $A = \{0 \text{ h}\}$ and $y = 30 \text{ h}$, then the inclusive average element to set distance between A and y is $60/4 = 15$. In the same way that we constructed an *Exclusive average representative distances measure* we may construct a *Representative inclusive average distances measure*. This measure is equivalent to a measure that consists in a ratio between the *Total difference sum measure* and the cardinality of a set:

Inclusive ratio measure:

$$F(A, d) = \frac{1}{n} \sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j) \text{ such that } x_i, x_j \in A, \text{ where } n = |A| \text{ and } n \geq 1.$$

F is a function from a finite metric space to \mathbf{R} .

We may note that it is only the denominator that distinguishes this proposal from the *Exclusive ratio measure*. There is a very important difference between the implications of the two measures, however. When differences are represented as metric space distances, the *Inclusive ratio measure* always increases when an element is added to a set. So the *Inclusive ratio measure* satisfies the *Strict monotonicity condition*. Since this fact is not intuitively obvious, a proof is included in the appendix.

As an example, we may again consider $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ and $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. For A , the measure aggregates $90/3 + 60/3 + 90/3 = 80$. For B , the measure aggregates $60/3 + 40/3 + 60/3 \approx 53.3$. This is equivalent to $240/3 = 80$ and $160/3 \approx 53.3$, respectively.

The *Inclusive ratio measure* does not work as a measure of diversity on its own, however. It is not necessary to compare sums of inclusive average element to set differences. $C = \{0 \text{ h}, 1 \text{ h}, 40 \text{ h}\}$ and $G = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$ is given the same value of roughly 53.3, but G is more diverse than C . It is not sufficient to compare average element to set differences either since the set $H = \{0 \text{ h}, 1 \text{ h}, 39 \text{ h}, 40 \text{ h}\}$ is given a value of 79 without being more diverse than G . When we discuss derived measures of diversity we shall discuss whether the sum of inclusive average element to set difference can work as a part of a derived measure of diversity.

10.8 Maximal Differences

Let us next consider the idea that a diversity measure should aggregate only maximal differences. We may define:

Maximal element to set distance: $\delta_{\max}(A_d, y) = \max d(x_i, y)$ such that $x_i \in A$ and $y \notin A$.

(This definition also occurs in Van Hees (2004: 261).) An aggregation measure based on the maximal element to set distances is the following:

Maximal distances measure:

$$F(A, d) = \sum_{i=1}^n \delta_{\max}(A - \{x_i\}, x_i) \text{ such that } x_i \in A.$$

F is a function from a finite metric space to \mathbf{R} .

The appeal of the idea of aggregating only maximal element to set differences is similar to the appeal of the idea that only the diameter matters. The maximal differences, being the largest differences, may intuitively seem to be the most important ones. But the appeal of aggregating maximal element to set differences also seems greater than the appeal of using the diameter. The maximal differences incorporate more information.

Let us again consider the example with $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ and $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. For A , the measure aggregates 60 for the option 0 h, 30 for the option 30 h and 60 for the option 60 h, giving a value of $60 + 30 + 60 = 150$. For B , the measure aggregates 40 for the option 0 h, 20 for the option 20 h and 40 for the option 40 h, giving a value of $40 + 20 + 40 = 100$. If we would judge the relative diversity of A and B by looking only at aggregated maximal differences, we would judge that A is more diverse than B . This is intuitively correct.

The *Maximal distances measure* satisfies the *Strict monotonicity condition*. The reason is that the maximal differences between the options in A cannot change when another option, y , is added to A . So the aggregated differences for A , as a subset of A , and A as subset of $A \cup \{y\}$ are always equal. Since there is one more maximal difference aggregated for $A \cup \{y\}$ as compared to A , namely the maximal difference between y and some option in A , $A \cup \{y\}$ will always be ranked as more diverse than A .

Nevertheless, the intuitive appeal of the idea of aggregating maximal differences quickly disappears when looking at some concrete examples. In fact, the idea of aggregating maximal differences leads to rankings that are absurd. Look at the following example: $C = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}\}$ and $G = \{0 \text{ h}, 1 \text{ h}, 20 \text{ h}\}$. It seems obvious that C offers more diversity than G since C contains two options, 0 h and 1 h, which are very similar to one another. But

the sum of maximal element to set differences for C is 50, while the sum of maximal element to set differences for G is 59. A greater sum of maximal element to set differences is thus neither sufficient nor necessary for greater diversity.

10.9 Minimal Differences

The idea that we should aggregate only maximal element to set differences may have absurd results, because in some cases it is the minimal differences that contain the most relevant information. Let us thus consider the idea that a diversity measure should aggregate the minimal element to set differences between each option and the other options in the set, disregarding the difference of 0 between the option and itself. This idea is presented by Weitzman (1992: 367). It is also the basis for a measure of diversity by Bordewich, Rodrigo and Semple (2008). We may define:

Minimal element to set distance: $\delta_{\min}(A_d, y) = \min d(x_i, y)$ such that $x_i \in A$ and $y \notin A$.

(See Van Hees (2004: 260).) A corresponding aggregation measure is the following:

Minimal distances measure:

$$F(A, d) = \sum_{i=1}^n \delta_{\min}(A - \{x_i\}, x_i) \text{ such that } x_i \in A.$$

F is a function from a finite metric space to \mathbf{R} .

The *Minimal distances measure* may seem intuitively appealing because we often judge the contribution to diversity by an option x to a set A by looking at how different x is to the option in A that is most similar to x . As a consequence, we often think that an option that is very similar to another option in the same choice set contributes very little to the diversity of that set. This judgment is based on the idea that minimal differences are more important for diversity assessments than other differences.

Once again let us consider the example with $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ and $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. For A , the measure aggregates 30 for the option 0 h, 30 for the option 30 h and 30 for the option 60 h, resulting in a value of $30 + 30 + 30 = 90$. For B , the measure aggregates 20 for the option 0 h, 20 for the option 20 h and 20 for the option 40 h, resulting in a value of $20 + 20 + 20 = 60$. If we would use the *Minimal distances measure* to judge the relative diversity of A with B , we would judge that A is more diverse than B . This is intuitively correct.

Even though the idea of aggregating minimal element to set differences may seem intuitively appealing, it does not hold up to scrutiny. The idea may come in conflict with the *Strict monotonicity condition*. If an option is added to a choice set, the sum of minimal element to set differences may decrease. This happens in the following example where $A = \{0 \text{ h}, 20 \text{ h}\}$ and $B = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}\}$. The sum of minimal element to set differences in A is 40. The sum of minimal element to set differences in B is 30. Therefore, a greater sum of minimal element to set differences is neither sufficient nor necessary for greater diversity.

10.10 Minimal Path

As we have seen, one problem with aggregating minimal differences is that the sum of minimal differences may decrease when options are added to a set. This problem may be solved if the diversity measure, rather than aggregating minimal differences, aggregates minimal path distances. In other words, we should compare lengths of minimal paths. This proposal is made by Lacevic and Arnaldi (2010: 5). It is also made by Nehring and Puppe for individuals that differ in one dimension (2002: 1158) and by Faith for a taxonomic tree model (1996: 1286).

The idea of a minimal path is usually presented in the language of graph theory, in terms of sets of vertices and edges. However, the idea is easily translated into metric space terminology. A set of vertices may be represented by a set of points of a metric space (representing the options). A set of edges may be represented by a set of pairwise distances between the points (representing the pairwise differences between the options). Each edge that connects two vertices (points) has a weight corresponding to the distance between the two points (vertices). In metric space terminology, a path may be defined as follows:

If $|A| = n$ and the options in A are indexed as x_1 to x_n , then a *path* P through A is an ordered set of distances $(d(x_1, x_2), d(x_2, x_3) \dots d(x_{n-1}, x_n))$.

A minimal path may be defined as follows:

A *minimal path*, ($\min P_i$) through A is a path P_i through A such that for all paths P_j through A , defined through any indexing of options, it holds that $\sum_{i=1}^{n-1} d_i \leq \sum_{i=1}^{n-1} d_j$, where $d_i \in P_i$ and $d_j \in P_j$.

A corresponding measure:

Minimal path measure:

$$F(A, d) = \sum_{i=1}^{n-1} d_i \text{ such that } d_i \in \min P_i \text{ through } A.$$

F is a function from a finite metric space to \mathbf{R} .

Aggregating only minimal path distances may seem intuitively appealing for the same reason that it seems appealing to aggregate only minimal differences; that is, because minimal distances are especially important when judging the contribution to diversity by an option to a choice set.

As an example we may compare the usual sets $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ and $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. The minimal path for A includes the distances $d(0 \text{ h}, 30 \text{ h})$ and $d(30 \text{ h}, 60 \text{ h})$. The length of the minimal path is $30 + 30 = 60$. The minimal path for B includes the distances $d(0 \text{ h}, 20 \text{ h})$ and $d(20 \text{ h}, 40 \text{ h})$. The length of the minimal path is $20 + 20 = 40$. Once again, A is correctly ranked as more diverse than B . But comparing lengths of minimal paths leads to problems for other comparisons.

One problem with the *Minimal path measure* is that it fails to satisfy *Strict monotonicity*. For example, the sets $C = \{0 \text{ h}, 60 \text{ h}\}$ and $C \cup \{y\} = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ have the same lengths of minimal paths, 60 for C and $30 + 30 = 60$, for $C \cup \{y\}$. This seems unacceptable.

Another problem with the measure is that the following two sets, G and H , are ranked as equally diverse. This is G with the distance matrix \mathbf{M}_G :

	x_1	x_2	x_3	x_4
x_1	0	1	2	3
x_2	1	0	1	2
x_3	2	1	0	1
x_4	3	2	1	0

This is H with the distance matrix \mathbf{M}_H :

	y_1	y_2	y_3	y_4
y_1	0	1	1	1
y_2	1	0	1	1
y_3	1	1	0	1
y_4	1	1	1	0

G has a minimal path consisting of the distances $d(x_1, x_2)$, $d(x_2, x_3)$, $d(x_3, x_4)$, while H has a minimal path consisting of the distances $d(y_1, y_2)$, $d(y_2, y_3)$, $d(y_3, y_4)$. Thus, both sets have a minimal path of 3. But it seems obvious that G is more diverse than H . Thus, a longer minimal path is not a necessary condition for greater diversity. It is not a sufficient condition either, as is evident if we compare the sets $J = \{0 \text{ h}, 41 \text{ h}\}$ and $K = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$. J has a longer minimal path, but K seems more diverse.

10.11 Minimal Circular Path

One way to remedy the problem that the sets G and H are ranked as equally diverse is to compare lengths of minimal circular paths rather than just lengths of minimal paths. That is to say, the diversity measure should not just aggregate all the distances of a minimal path but also aggregate the distance between the first and the last connected option so that it aggregates all the distances of a minimal circular path. The minimal circular path of G is 6, while the minimal circular path of H is 4. Thus we can use the length of minimal circular paths to judge that G is more diverse than H .

The minimal circular path differs from the minimal path in that the minimal circular path includes an additional distance, the distance between the first option and the last option of the minimal path. We may thus define a circular path as follows:

If $|A| = n$ and the options in A are indexed as x_1 to x_n , then a *circular path* through A is a set of distances $(d(x_1, x_2), d(x_2, x_3) \dots d(x_{n-1}, x_n), d(x_n, x_1))$.

A minimal circular path may be defined as follows:

A *minimal circular path*, (*min CP_i*) through A is a circular path CP_i through A such that for all circular paths P_j through A , defined using any index of options, it holds that $\sum_{i=1}^n d_i \leq \sum_{i=1}^n d_j$, where $d_i \in CP_i$ and $d_j \in CP_j$.

A corresponding measure:

Minimal circular path measure:

$$F(A, d) = \sum_{i=1}^n d_i \text{ such that } d_i \in \text{min } CP_i \text{ through } A.$$

F is a function from a finite metric space to \mathbf{R} .

Aggregating the distances of the minimal circular path seems more appealing than aggregating the distances of the minimal path since the first method can award a larger diameter. As an example we may again consider the sets $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ and $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. The minimal circular path for A includes the distances $d(0 \text{ h}, 30 \text{ h})$, $d(30 \text{ h}, 60 \text{ h})$ and $d(60 \text{ h}, 0 \text{ h})$. The length of the minimal circular path for A is thus $30 + 30 + 60 = 120$. The minimal circular path for B includes the distances $d(0 \text{ h}, 20 \text{ h})$, $d(20 \text{ h}, 40 \text{ h})$ and $d(40 \text{ h}, 0 \text{ h})$. The length of the minimal circular path for B is thus $20 +$

$20 + 40 = 80$. Comparing lengths of minimal circular paths, A is judged as more diverse than B . This seems intuitively correct.

The appeal of aggregating distances of minimal circular paths has its limits, though. Comparing the sum of minimal circular path distances is neither necessary nor sufficient to judge one set as more diverse than another. The *Minimal circular path measure* does not satisfy the *Strict monotonicity condition*. For example, the set $C = \{0 \text{ h}, 20 \text{ h}\}$ and the set $C \cup \{10 \text{ h}\} = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}\}$ have the same minimal circular path of a length of 40. Since $C \cup \{10 \text{ h}\}$ is more diverse than C , lengths of minimal circular paths are not necessary to rank sets in terms of diversity. A longer minimal circular path is not sufficient for ranking one set as more diverse than another either. It seems obvious that $C \cup \{10 \text{ h}\}$ is more diverse than C , even though they have the same minimal circular paths.

10.12 Minimal Non-Eliminated Differences

We shall consider yet another method of aggregating distances. It is a measure proposed by Weitzman (1992), and again by Bossert et al. (2003). The measure also occurs in Solow, Polasky and Broadus (1993), who use it, and in Solow and Polasky (1994: 98), Baumgärtner (2007: 8) and Gravel (2009: 39–41), who criticize it, and in Weikard (2002), who defends it. There is no particular name for the measure in any of the essays, but we may call it the *Elimination measure*. Here we shall look at the version of Bossert et al. They present the measure as an ordinal measure, but it may also work as a ratio scale measure. Bossert et al. define their measure in two stages. First, they define a distance between an element x and a set A . Second, they define a function that aggregates the distances between each element of A . For the *Elimination measure* the distance between an element x and a set A is defined as follows:

$$\begin{aligned} \delta(x, A) &= 0 \text{ if } A = \{x\}, \\ \delta(x, A) &= \min \{d(x, y) \text{ such that } y \in A - \{x\}\} \text{ if } A \neq \{x\}. \end{aligned}$$

The *Elimination measure* aggregates the distances between each element of a set and the set, with the help of an iterative procedure (2003: 416).

Let us consider a set A . For all x in A , we record the distances between x and all other elements of A (including x itself) in a vector $\mathbf{d}_A(x)$. After we have recorded these vectors for each element in A , we compare them according to a leximin condition. We begin by comparing the minimal distances of each vector. All of the vectors contain the same minimal distance, 0. We continue to compare the second-to-minimal distance of each vector. At least two vectors should contain the same smallest second-to-minimal distance. (Because $d(x, y) = d(y, x)$, the vectors for x and y must

have one distance in common.) We then compare the vectors that contain the smallest second-to-minimal distance by looking at their third-to-minimal distances. If there is a unique element x that has a vector that contains the smallest second-to-minimal distance, as well as the smallest third-to-minimal distance, then we call this element a_1 . If there are several elements in A that have vectors that contain the smallest second-to-minimal distance, as well as the smallest third-to-minimal distance, then we continue to compare their fourth-to-minimal distances. This procedure continues until it either yields a unique element that has the smallest n th distance in its distance vector, or there are no more distances to compare. If there is a unique element, then we call this element a_1 . If there are several elements with the same distance vectors, then we select any one of these elements and call that element a_1 .

When the selection procedure is done, we record the value of $\delta(a_1, A)$. After recording the value of $\delta(a_1, A)$, we eliminate a_1 and repeat the procedure with the set $A - \{a_1\}$. The process is continued until all the elements of A are eliminated. We have then obtained the values $\delta(a_1, A)$, $\delta(a_2, A - \{a_1\}) \dots \delta(a_{m-1}, A - \{a_1 \dots a_{m-2}\})$, $\delta(a_m, \{a_m\})$, where m is the number of elements in A (2003: 417–418). These are the values that we should aggregate. We thus have the following measure:

Elimination measure:

$$F(A, d) = \delta(a_1, A) + \delta(a_2, A - \{a_1\}) + \dots + \delta(a_{n-1}, A - \{a_1 \dots a_{n-2}\}) + \delta(a_n, \{a_n\}).$$

F is a function from a finite metric space to \mathbf{R} .

The *Elimination measure* may be illustrated by an example. Let $A = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}\}$ and let $B = \{0 \text{ h}, 10 \text{ h}, 29 \text{ h}, 30 \text{ h}\}$. We get the following vectors for A : $\mathbf{d}_A(0 \text{ h}) = (0, 10, 20, 30)$, $\mathbf{d}_A(10 \text{ h}) = (10, 0, 10, 20)$, $\mathbf{d}_A(20 \text{ h}) = (20, 10, 0, 10)$, $\mathbf{d}_A(30 \text{ h}) = (30, 20, 10, 0)$; and the following vectors for B : $\mathbf{d}_B(0 \text{ h}) = (0, 10, 29, 30)$, $\mathbf{d}_B(10 \text{ h}) = (10, 0, 19, 20)$, $\mathbf{d}_B(29 \text{ h}) = (29, 19, 0, 1)$, $\mathbf{d}_B(30 \text{ h}) = (30, 20, 1, 0)$. Using the leximin condition for A , we get $a_1 = 10 \text{ h}$ or 20 h . If we pick 10 h , then 10 h is eliminated with a value of 10 . For $A - \{10 \text{ h}\} = \{0 \text{ h}, 20 \text{ h}, 30 \text{ h}\}$, we get $a_2 = 20 \text{ h}$, which is also eliminated with a value of 10 . For $A - \{10 \text{ h}, 20 \text{ h}\} = \{0 \text{ h}, 30 \text{ h}\}$, we get $a_3 = 0 \text{ h}$ or 30 h . If we pick 30 h , then 30 h is eliminated with a value of 30 . We are then left with $A - \{10 \text{ h}, 20 \text{ h}, 30 \text{ h}\} = \{0 \text{ h}\}$, and get $a_4 = 0 \text{ h}$. The element 0 h is eliminated with a value of 0 . The diversity value of A is thus $10 + 10 + 30 + 0 = 50$. If we apply the same method for B , it gets a diversity value of $1 + 10 + 30 + 0 = 41$. Since $50 > 41$, A is more diverse than B . This seems correct.

The *Elimination measure* satisfies the *Limited diameter condition*. It also satisfies the *Strict monotonicity condition*. The reason for this is that the elements of A are always eliminated in such a way that the diversity value of A is as great as possible. (This fact may be more obvious in Weitzman's presentation since he explicitly defines the measure as the maximal value of

the iterative procedure (1992: 375).) Because the elements of A are eliminated in such a way that the diversity value is as great as possible, and the elements of $A \cup \{y\}$ are eliminated in the same way, the diversity value of $A \cup \{y\}$ cannot be any smaller than the diversity value of A . The only way for the diversity value of $A \cup \{y\}$ to be as small as the diversity value of A would be if y could be eliminated with a second smallest distance of 0. But this could only happen if y could be identical to some other element x and, by the property of the *Identity of indiscernibles*, it cannot.

However, the *Elimination measure* also has some problems. One problem is that it results in some counterintuitive rankings. Let us compare $C = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 30 \text{ h}\}$ and $G = \{0 \text{ h}, 7.5 \text{ h}, 15 \text{ h}, 30 \text{ h}\}$. The *Elimination measure* gives $F(C) = 50$, while $F(G) = 52.5$. So G is ranked as more diverse than C . This is at least a debatable ranking. On the one hand, G might be considered as more diverse than C , at least assuming that the option 15 h contributes more to the diversity of G than the two options 10 h and 20 h contributes to the diversity of C . After all, 15 h is more dissimilar to 0 h and 30 h than either of 10 h and 20 h is dissimilar to 0 h and 30 h. On the other hand, C might be considered as more diverse than G since the options in C cover up the range of C better than they do in G . There is a greater gap between 15 h and 30 h than there is between 20 h and 30 h. This might seem more important than covering the middle position.

A more serious problem with the *Elimination measure* is that it fails to distinguish between two sets that are alike in all respects, except that one set has a larger distance between two of its options than the other set does. We may consider the following Euclidean space example. There is a set J that contains three elements, x_1 , x_2 , and x_3 . There is another set K that contains three elements, y_1 , y_2 , and y_3 . The elements of J may be represented as vertices of a triangle in two dimensions. The elements of K lie in a straight line in one dimension. The vectors of J are equal to $\mathbf{d}_J(x_1) = (0, 10, 20)$, $\mathbf{d}_J(x_2) = (10, 0, 20)$, and $\mathbf{d}_J(x_3) = (20, 20, 0)$. When we eliminate the elements in J , we get the value $10 + 20 + 0 = 30$. The vectors of K are equal to $\mathbf{d}_K(y_1) = (0, 10, 20)$, $\mathbf{d}_K(y_2) = (10, 0, 10)$, and $\mathbf{d}_K(y_3) = (20, 10, 0)$. When we eliminate the elements in K , we also get the value $10 + 20 + 0 = 30$. Thus, J is judged as equally diverse as K . However, J seems more diverse than K . The assignment of a greater number by the *Elimination measure* is, thus, not necessary for greater diversity.

The last example shows that the *Elimination measure* fails to satisfy a condition that we shall discuss next, the *Dominance condition*. It fails to satisfy the condition since it ignores information about the differences between the options. But this is not just a problem for the *Elimination measure*. It is general problem for all measures that do not aggregate all differences. Whenever we disregard information about the differences between the options in a set, we disregard information that may be relevant for assessing the diversity of the set.

10.13 Dominance

Not having found any reasonable selection of differences that are exclusively relevant for diversity, let us keep the idea that all differences matter for diversity. But instead of assuming that the differences matter collectively, in the form of a total sum, we may assume that each difference matters individually. The magnitudes of differences matter for a distance-to-distance comparison between the distance vectors of different sets. At least in some cases, it is then clear which of the compared sets is more diverse. A is more diverse than B , when A *dominates* B in terms of magnitudes of differences. The concept of dominance may be defined as follows:

For all metric choice sets $A_d, B_d \in P(X_d)$ with distance vectors \mathbf{d}_A and \mathbf{d}_B , A *dominates* B if and only if $|A| = |B|$ and for each number i and each pair of distances (d_{Ai}, d_{Bi}) , it holds that $d_{Ai} \geq d_{Bi}$ and there exists some pair of distances (d_{Aj}, d_{Bj}) such that $d_{Aj} > d_{Bj}$.

Remember that the distances of \mathbf{d}_A and \mathbf{d}_B are indexed in decreasing order. Here is one example: we compare the sets $A = \{0 \text{ h}, 30 \text{ h}, 60 \text{ h}\}$ and $B = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. A has the distance vector $\mathbf{d}_A = (60, 60, 30, 30, 30, 30, 0, 0, 0)$, while B has the distance vector $\mathbf{d}_B = (40, 40, 20, 20, 20, 20, 0, 0, 0)$. Because 60 is greater than 40, 30 is greater than 20, and 0 is equal to 0, it holds that for each number i and for each $d_{Ai} \geq d_{Bi}$ and at least one $d_{Ai} > d_{Bi}$, and thus A *dominates* B in terms of differences. When a set dominates another set, it is reasonable to say that it is more diverse. This captures the idea that the options in a set should be as different from one another as possible to maximize diversity. We may thus formulate a condition regarding dominance as follows:

The Dominance condition: For all metric choice sets $A_d, B_d \in P(X_d)$, if A *dominates* B , then A *offers strictly more freedom of choice than* B .

Similar conditions are used for diversity by, for example, Weitzman (1992: 392) and Pattanaik and Xu (2008: 263).

However, the *Dominance condition* may be opposed. Gustafsson objects to the condition (or rather a similar, weak dominance condition) since two very dissimilar options may be less representative of the universal set than two more similar options (2011: 49). However, this objection presupposes that the concept of freedom of choice should be explicated as the representative conception of freedom of choice. As we have seen, this view is problematic.

Another objection to the condition is made by a commentator. The objection is that the number of dimensions in which the options differ may

be more important for diversity than relations of dominance. There are two sets, A and B . A has the distance matrix \mathbf{M}_A :

	x_1	x_2	x_3
x_1	0	10	20
x_2	10	0	10
x_3	20	10	0

B has the distance matrix \mathbf{M}_B :

	y_1	y_2	y_3
y_1	0	10	10
y_2	10	0	10
y_3	10	10	0

The differences between the options in A are such that the options can be represented as lying on a straight line, while the options in B can be represented as lying on the vertices of a triangle (in Euclidean space). An additional assumption is that the options in A vary in one dimension, while the options in B vary in two. The commentator suggests that the options in B are more diverse than the options in A since the options in B vary in two dimensions, while the options in A vary in one. Hence, the *Dominance condition* does not hold.

The problem with this counterexample is that it presupposes that there is something amiss with the distance function. If two options, x_i and x_j , that vary in two dimensions are more different than two other options, y_i and y_j , that vary in one dimension, the diversity function that assigns distances between x_i and x_j and y_i and y_j should already reflect this. If the distance function does not reflect this, it is flawed. But if the distance function is flawed, it cannot be used as a basis for judgments of diversity. It certainly cannot be used as a counterexample to the *Dominance condition*. So this counterargument does not work.

A third objection to the *Dominance condition* is to point out that it ignores how the distances are correlated. As long as there is one bijection between A and B that fulfills dominance, A is regarded as more diverse than B . This ignores the way in which the distances in \mathbf{d}_A and \mathbf{d}_B are distributed among the options in A and in B . It is not obvious that A is more diverse than B , if the distances in \mathbf{d}_A and \mathbf{d}_B are distributed very differently among the options in A and B . Let us look at an example, where A dominates both B and C . This is the matrix for A :

	x_1	x_2	x_3	x_4
x_1	0	20	20	20
x_2	20	0	40	40
x_3	20	40	0	40
x_4	20	40	40	0

This is the matrix for B :

	y_1	y_2	y_3	y_4
y_1	0	30	30	30
y_2	30	0	20	20
y_3	30	20	0	20
y_4	30	20	20	0

This is the matrix for C :

	z_1	z_2	z_3	z_4
z_1	0	20	20	20
z_2	20	0	30	30
z_3	20	30	0	30
z_4	20	30	30	0

Represented in Euclidean space, the options in the three sets can be regarded as vertices of three-dimensional tetrahedrons of various breadth and height. A dominates both B and C . But there is a structural similarity between A and C that is lacking between A and B . If we compare the distances for each row of the matrices of A , B and C , we can see that A dominates C for each row of distances. There is no such relation between A and B . This specific relationship of dominance may be called *option dominance*.

To define the concept of option dominance we must first identify a type of vector that involves only the distances from one option x_i to the other options in A , the vector $\mathbf{d}_A(\mathbf{x})$. The distances of this vector may be called d_{Axi} . The relation of *dominance option by option* can then be defined as follows:

For all metric choice sets $A_d, B_d \in P(X_d)$ with distance vectors \mathbf{d}_A and \mathbf{d}_B , options $x, y \in X$ such that $x \in A$ and $y \in B$, and distance vectors $\mathbf{d}_A(\mathbf{x}_i)$ and $\mathbf{d}_B(\mathbf{y}_i)$, A dominates B option by option if and only if $|A| = |B|$ and there is at least one bijection between the options in A and the options in B such that for each pair of options (x_i, y_i) , $\mathbf{d}_A(\mathbf{x})$ dominates $\mathbf{d}_B(\mathbf{y})$.

A corresponding condition:

The Option dominance condition: For all metric choice sets $A_d, B_d \in P(X_d)$, if A dominates B option by option, then A offers strictly more freedom of choice than B .

If a set dominates another set option by option, it also dominates it. Thus, satisfaction of the *Option dominance condition* implies satisfaction of the *Dominance condition*. The reverse relation does not hold; satisfaction of the *Dominance condition* does not imply satisfaction of the *Option dominance condition*. Since the *Option dominance condition* seems reasonable, I shall accept the condition here.

Chapter 11: Distribution of a Sum of Differences among Differences

Besides cardinality and magnitude of differences, it may also matter how a sum of differences is distributed among individual differences. If different sets have the same cardinality and the same sum of differences, it may be important how the sum is distributed among the individual non-zero differences. There are a few possibilities regarding the importance of the distribution of differences for diversity. First, distribution may matter in that diversity is maximized when the diameter is maximized given some fixed sum of differences. Second, distribution may matter in that diversity is maximized when some fixed sum of differences is equally distributed among the individual non-zero differences. Third, distribution may matter both in terms of equal distribution and diameter, where these factors are weighted against one another in some specific way. Fourth, distribution may not matter to diversity at all.

To give an example, let us suppose that there is a total sum of differences of 240 for some sets of four options. The sets could represent different choices among buildings that Uppsala University could rent, where the buildings differ in length, breadth and height. The first idea suggests that the set A , with the distance matrix below, is the maximally diverse set under the given circumstances:

	x_1	x_2	x_3	x_4
x_1	0	ε	ε	$120 - 5\varepsilon$
x_2	ε	0	ε	ε
x_3	ε	ε	0	ε
x_4	$120 - 5\varepsilon$	ε	ε	0

In this example, the diameter is maximal. (This is obviously not a metric space example since it violates the triangle inequality. It is used here as it better illustrates the point.) Set A cannot be interpreted as a choice among buildings, other than approximately. If so, it could be viewed as a choice among two normal buildings and two abnormal buildings. One normal building would have 10 m in length, breadth, and height and another normal building would have 95 m in length and breadth, and 10 m in height. The two abnormal buildings would have close to 10 m height, but have their length and breadth vary between something close to 10 m, and something

close to 95 m, depending on if they were compared to the smaller or the larger building. Such buildings could hardly exist (or at least they would be uninhabitable). For these examples, the coordinate vectors are $x_1 = (10, 10, 10)$, $x_2 = (\approx 10 \text{ or } \approx 95, \approx 10 \text{ or } \approx 95, \approx 10)$, $x_3 = (\approx 10 \text{ or } \approx 95, \approx 10 \text{ or } \approx 95, \approx 10)$, and $x_4 = (95, 95, 10)$.

The second idea suggests that it is rather the set B with the distance matrix below that is the maximally diverse set under the given circumstances:

	y_1	y_2	y_3	y_4
y_1	0	20	20	20
y_2	20	0	20	20
y_3	20	20	0	20
y_4	20	20	20	0

In this example equal distribution is maximized. This example can be represented in Euclidean space. The options are represented as the vertices of an equilateral simplex. If the set is interpreted as a choice among four buildings, it could represent a choice among one building of 10 m in length, breadth, and height, another building of 30 m in length and 10 m in breadth and height, a third building of 20 m in length, roughly 17 m in breadth, and 10 m in height, and a fourth building of 20 m in length, roughly 16 m in breadth, and roughly 26 m in height. If length is the first coordinate, breadth the second, and height the third, then the coordinate vectors are $y_1 = (10, 10, 10)$, $y_2 = (30, 10, 10)$, $y_3 = (20, \approx 17, 10)$, $y_4 = (20, \approx 6, \approx 16)$.

The third idea suggests that the ideally diverse distribution lies somewhere in between the first two ideals, perhaps at the set C with the distance matrix:

	z_1	z_2	z_3	z_4
z_1	0	12	24	36
z_2	12	0	12	24
z_3	24	12	0	12
z_4	36	24	12	0

This example can also be represented in Euclidean space. The options lie on a straight line. It could be interpreted in several different ways as a choice among buildings. One interpretation is that it represents the choice among four tall buildings of the same length and breadth of 10 m, but with different heights, where the first building has a height of 10 m, the second building has a height of 22 m, the third building has a height of 34 m, and the fourth building has a height of 46 m. Here the coordinate vectors are $z_1 = (10, 10, 10)$, $z_2 = (10, 10, 22)$, $z_3 = (10, 10, 34)$, $z_4 = (10, 10, 46)$. Another interpretation is that it represents the choice among four buildings of various length, breadth and height, the first building of 10 m in length, breadth, and

height, the second building of roughly 17 m in length, breadth, and height, the third building of roughly 24 m in length, breadth, and height, and the fourth building of roughly 31 m in length, breadth, and height. Here the coordinate vectors are $z_1 = (10, 10, 10)$, $z_2 = (\approx 17, \approx 17, \approx 17)$, $z_3 = (\approx 24, \approx 24, \approx 24)$, $z_4 = (\approx 31, \approx 31, \approx 31)$. We have to assume that the two interpretations offer the same amount of diversity, or else there would be something wrong with the underlying distance function. This last compromise ideal requires some principle of weighing between maximal diameter and maximally equal distribution.

The fourth idea suggests that distribution does not matter to diversity at all, so that there are no differences in diversity between A , B and C in the examples given. There is not much that speaks in favor of the fourth idea. Most people have the intuition that distribution matters to diversity. The problem is not that there is a lack of intuitions regarding diversity in cases of different distributions, but rather that there are several intuitions that are mutually exclusive. The two ideas of maximizing diameter and maximizing equal distribution are obviously mutually exclusive. A set cannot at the same time have a maximal diameter and a maximally equal distribution (if there are more than two options in the set). If the diameter would increase in relation to a fixed sum of differences, then equal distribution would decrease. If equal distribution would increase in relation to a fixed sum of differences, then the diameter would decrease. Either only one of these properties matters for diversity, or they have to be weighed against one another in some specific way.

The ideal of a maximized diameter and the ideal of a maximally equal distribution both occur in the literature. But the ideal of equal distribution is more common. The ideal of maximal diameter is used for at least two measures, Rosenbaum, for freedom (2000), and Bervoets and Gravel (2007), for diversity. But the ideal of equidistance is used for many more measures, in many different areas. The idea that a vector of numbers is maximally diverse if all the numbers are equal occurs often in the literature. The idea is the basis for *majorization*, a method used in mathematics for partial rankings of sets in terms of diversity. The idea also figures in essays by Kreutz-Delgado and Rao (1999) and Dowden (2011). The ideal of equal numbers has an analogy in biology as well. An equal distribution of a fixed number of individuals among a fixed number of species is considered to be more diverse than an unequal distribution of the same number of individuals among the same number of species (see for example Pielou (1975: 7)). In biology, this property is called *evenness*. Since the ideal of equal distribution is analogous to the ideal of evenness, I shall use the term ‘evenness’ here as well.

Intuitively, there is no overwhelming reason to favor one diversity ideal over the other. We shall thus continue to look at all three ideals in more detail. First we shall consider the idea that diversity is maximized when the

diameter is maximized, given a fixed sum of differences. I shall call this conception of diversity *diameter diversity*. Then we shall consider the idea that diversity is maximized when the differences are the same, given a fixed sum of differences. I shall call this conception of diversity *evenness diversity*. Last, we shall consider a compromise between the two ideals.

11.1 Diameter

The idea of a *maximal diameter* is easily defined.

A metric choice set $A_d \in P(X_d)$ such that $\Sigma(A_d) = M$, has a *maximal diameter* if and only if $\text{diam}(A_d) = M/2$.

Any set of two options has a maximal diameter. But if a set has more than two options, it cannot have a maximal diameter. The diameter is restricted by the property of *Identity of indiscernibles*, which allows distances of 0 only between identical options. It is also restricted by the property of *Triangle inequality*, which excludes some combinations of distances. Relative to metric spaces, one may suggest that we should rather speak of a *maximized diameter*. Such a diameter may be defined as follows:

A metric choice set $A_d \in P(X_d)$ such that $|A| = n$ and $\Sigma(A_d) = M$, has a *maximized diameter* if and only if $\text{diam}(A_d) = M/(2n - 2)$.

A maximized diameter would be the largest possible diameter of any metric set having a fixed number of options and a fixed total difference sum. But, as it turns out, there is no set with more than three options that can have a maximized diameter. Such a set can only have a diameter approaching the maximized diameter, which is never obtained. So, we should rather call the maximized diameter, the *supremum diameter*. The fact that the supremum diameter is equal to $M/(2n - 2)$ for all n is proven in the appendix. There, it is also shown that for all $n > 3$, the supremum can never be attained. For two options it is always attained, and for three options it is attained whenever the three options are lying on a straight line. Although there are no sets with a cardinality above three that has a maximized diameter, we can still say something about sets that have diameters that are close to the supremum. These sets have options that are, or are very close to being, positioned on a straight line. The proof of this fact is also included in the appendix.

There would be no point in formulating a condition regarding the importance of a maximal or maximized diameter for diversity of choice since such a condition would rarely apply. Instead, we may formulate a condition regarding the importance of a diameter close to the supremum:

The Supremum diameter condition: For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$ and $\Sigma(A_d) = \Sigma(B_d)$, if the diameter of A is close to the supremum and the diameter of B is not, then A offers strictly more freedom of choice than B .

The concept of being close to the supremum is rather vague. But we shall ignore its vagueness here and just discuss the condition as it stands.

There are several problems with the *Supremum diameter condition*. We may illustrate one problem with an example using Euclidean distances. Let us compare sets of a total sum of differences of 240 and a cardinality of 3. A has the distance matrix \mathbf{M}_A :

	x_1	x_2	x_3
x_1	0	$60 - \varepsilon$	60
x_2	$60 - \varepsilon$	0	ε
x_3	60	ε	0

B has the distance matrix \mathbf{M}_B :

	y_1	y_2	y_3
y_1	0	31	59
y_2	31	0	30
y_3	59	30	0

The *Supremum diameter condition* implies that A is more diverse than B . This is very unintuitive. Even though A has a maximized diameter, there are other differences in magnitude among the smaller distances that seem relevant for rankings. In the comparison between A and B the distances of ε and 31 seem as relevant as the distances of 60 and 59. In fact, B seems more diverse than A .

Another problem is that the diameter ideal is insufficiently captured by the *Supremum diameter condition*. It needs to be supplemented with other conditions to be able to rank all possible sets. There is a need for a condition that ranks sets with diameters equally close to the supremum. There is also a need for a condition that ranks sets that do not have diameters that are close to the supremum or even equal. Let us look at one problem at a time.

The first problem is that the ideal needs to be supplemented to be able to rank sets with diameters that are close and equally close to the supremum. Since a diameter close to the supremum cannot be almost equal to the total difference sum there will be many sets that have diameters that are equally close to the supremum. If we compare sets of a total difference sum of 240 and a cardinality of 3, we can, for example, compare A and C . A is the above-mentioned set with the distance matrix \mathbf{M}_A :

	x_1	x_2	x_3
x_1	0	$60 - \varepsilon$	60
x_2	$60 - \varepsilon$	0	ε
x_3	60	ε	0

C is the following set, with the distance matrix \mathbf{M}_C :

	z_1	z_2	z_3
z_1	0	30	60
z_2	30	0	30
z_3	60	30	0

How should A and C be ranked? The *Supremum diameter condition* says nothing about this type of comparison. One way to expand the application of the condition would be to reformulate it as a lexical ranking condition, a *Lexical diameter condition*. If two sets have the same total difference sum and cardinality, then a set with a diameter close to the supremum is more diverse than a set with a smaller diameter. If two sets have the same diameter, equally close to the supremum, then a set with a second largest difference is more diverse than a set with a smaller second largest difference, and so on. But the problem with the *Lexical diameter condition* is obvious; it awards the inclusion of options in a set that are very similar to the most different options in that set. A is judged as offering more diversity than C , although x_2 is very similar to x_3 , while z_2 is not so similar to z_3 . We started with the idea that more dissimilar options are more diverse, interpreted the idea in terms of a larger diameter, and ended up with the idea that more similar options are more diverse. This seems incoherent. No matter what idea one could reasonably have of diversity, surely C is more diverse than A . But if we want to explain why C is more diverse than A , it seems as if we have to appeal to considerations of evenness; the options in C are more evenly distributed than the options in A . But in doing this we have appealed to the very opposite ideal to the ideal of maximizing the diameter.

The second problem is that the ideal also needs to be supplemented to be able to rank sets that do not have a diameter close to the supremum. A more general diameter condition can be defined as follows:

The Diameter condition: For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$ and $\Sigma(A_d) = \Sigma(B_d)$, if the diameter of A is larger than the diameter of B , then A offers strictly more freedom of choice than B .

But this is not an acceptable condition either. It is incorrect to judge that whenever two sets, A and B , have the same cardinality and the same total difference sum and A has a larger diameter than B , A is more diverse than B .

Such a judgment implies that in the example above, A is more diverse than B . Since B is clearly more diverse than A , this is unacceptable.

We shall return to the ideal of a larger diameter later. So far, the ideal has not resulted in any reasonable condition.

11.2 Evenness

Maximal evenness may be defined as follows:

A metric choice set $A_d \in P(X_d)$ such that $|A| = n$ and $\Sigma(A_d) = M$, has a *maximally even* distribution if and only if each non-zero distance $d_{Ai} = M/(n(n - 1))$.

A condition that captures the idea that a maximally even set is more diverse than an uneven set, given the same cardinality and the same total difference sum, is the following:

Maximal evenness condition: For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B| = n$ and $\Sigma(A_d) = \Sigma(B_d) = M$, if each non-zero distance $d_{Ai} = M/(n(n - 1))$ and it is not the case that each non-zero distance $d_{Bi} = M/(n(n - 1))$, then A offers strictly more freedom of choice than B .

A similar condition for a type of biological diversity measure may be found in McIntosh (1966: 393) and Pielou (1975: 7). The *Maximal evenness condition* implies that for a total difference sum of 240 and a cardinality of 3, A with matrix \mathbf{M}_A is maximally diverse:

	x_1	x_2	x_3
x_1	0	40	40
x_2	40	0	40
x_3	40	40	0

All the options are as different from one another as possible, and this makes them maximally diverse. (Here we may note that a maximally even distribution of n options represented in Euclidean space requires $n - 1$ dimensions, so that not all distributions of options can be maximally even.)

If evenness matters to diversity, then it should not just matter in the sense that a maximally even set is more diverse than an uneven set (given the same cardinality and total difference sum). Evenness should also matter in the sense that a set that is closer to being maximally even should be more diverse than a set that is further from being maximally even, everything else being equal. The following condition is thus reasonable as well:

Evenness condition:

For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B| = n$ and $\Sigma(A_d) = \Sigma(B_d)$, if A is more evenly distributed than B , then A offers strictly more freedom of choice than B .

There is a problem with this condition, however. It is not clear when one set is closer to being maximally even than another. Obviously, it is clear when a set has a maximally even distribution of a total sum of differences since it is when all individual differences are equal. But it is less clear when a set has a completely uneven distribution of a total sum of differences (as noted by Gregorius for evenness in biology (1990: 703), as well as by Rousseau and Hecke, (1999: 1)). One could argue that a completely uneven distribution of a total sum of differences is the distribution where all the differences between the options are very close to 0, apart from one difference that is very close to being equal to the total difference sum. But one could also argue that evenness is not minimized in a situation where several options are at the same difference of ε to one another. Rather, evenness is minimized in a situation where all the differences are as different from one another as possible. So, instead of the set B (below) being the most uneven distribution of a total difference sum of 240 among three options, the set C (also below) would be the most uneven set. This is B with matrix \mathbf{M}_B :

	y_1	y_2	y_3
y_1	0	ε	$120 - 2\varepsilon$
y_2	ε	0	ε
y_3	$120 - 2\varepsilon$	ε	0

This is C with matrix \mathbf{M}_C :

	z_1	z_2	z_3
z_1	0	40	$80 - \varepsilon$
z_2	40	0	ε
z_3	$80 - \varepsilon$	ε	0

But if we are looking for a type of evenness that may matter to diversity, surely C is more even than B ? The options in C are more different to one another than the options in B . C seems more similar to the completely even set A than B . Thus, B is the most uneven set.

I shall therefore conclude that it is possible to specify both a maximal and a minimal degree of evenness. Problems only arise for the in-between cases. How should we determine closeness to maximal evenness? The only clear cases seem to be those where a sum of two differences is divided between the two individual differences. Here the distribution of the sum of two differences gets closer to being maximally even as the difference between

the two differences gets smaller. For cases of sums of three differences or more, judgments of closeness to evenness are less obvious. What is needed here is some measure of evenness.

The need for a measure of evenness is not problematic in the sense that there are no measures of evenness, because there are many. Problems arise because different measures of evenness result in different rankings of sets. There seems to be no point in committing to any specific measure of evenness when discussing the importance of evenness to diversity in general. Initially, we may just assume that any ordering of sets in terms of evenness is acceptable if it is the result of a measure of evenness that satisfies some reasonable conditions. What we need are thus some reasonable conditions for a measure of evenness.

Here I shall suggest that a measure of evenness E should satisfy at least two conditions. Both conditions apply to cases where the results of the comparisons are obvious. The first one is just a variant of the *Maximal evenness condition* for freedom of choice as diversity:

The Maximal evenness condition for an evenness measure: For a measure of evenness E , and all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B| = n$ and $\Sigma(A_d) = \Sigma(B_d) = M$, if each non-zero distance $d_{Ai} = M/(n(n - 1))$ and it is not the case that each non-zero distance $d_{Bi} = M/(n(n - 1))$, then A is strictly more even than B , and thus $E(A) > E(B)$.

The second condition applies to the special case where the only difference between two sets is in terms of an equal sum of two distances being distributed differently between the two individual distances:

The Limited evenness condition for an evenness measure: For a measure of evenness E , and all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$ and $\Sigma(A_d) = \Sigma(B_d)$, with distance vectors \mathbf{d}_A and \mathbf{d}_B , if for each i and each pair of distances (d_{Ai}, d_{Bi}) , it holds that $d_{Ai} = d_{Bi}$, except for the distances d_{Aj}, d_{Ak}, d_{Bj} and d_{Bk} (and their symmetrical counterparts), which are such that $d_{Aj} + d_{Ak} = d_{Bj} + d_{Bk}$, then if $|d_{Aj} - d_{Ak}| < |d_{Bj} - d_{Bk}|$, A is strictly more even than B , and thus $E(A) > E(B)$.

The second condition applies to the following comparison. Two sets have a total difference sum of 240 and a cardinality of 3. A has the matrix:

	x_1	x_2	x_3
x_1	0	40	40
x_2	40	0	40
x_3	40	40	0

G has the matrix:

	y_1	y_2	y_3
y_1	0	20	60
y_2	20	0	40
y_3	60	40	0

If the sets are represented in Euclidean space, the options in A lie in the vertices of a triangle, while the options in G lie on a straight line. The *Limited evenness condition for an evenness measure* implies that A is more even than G .

As it turns out, the *Limited evenness condition for an evenness measure* implies the *Maximal evenness condition for an evenness measure* (assuming that the relation *more even than* is transitive). It is thus sufficient to use the *Limited evenness condition for an evenness measure*. The *Limited evenness condition for an evenness measure* implies that if two distances are unequal, then there is a potentially greater value for E when the two distances are equal. So, the maximal value for E must occur when all the distances are equal.

We may assume that any acceptable measure of evenness may be used to specify the *Evenness condition*. This would simplify matters considerably. We can then apply the evenness conditions directly to measures of freedom of choice. If a measure of freedom of choice satisfies the *Limited evenness condition for an evenness measure*, it can be used as a measure of evenness. Consequently, it can also be used to specify the *Evenness condition*, which it then automatically satisfies. So instead of using the *Evenness condition* for a measure of freedom of choice, we may use a variant of the *Limited evenness condition for an evenness measure*. We may use the following version:

The Limited evenness condition: For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$ and $\Sigma(A_d) = \Sigma(B_d)$, with distance vectors \mathbf{d}_A and \mathbf{d}_B , if for each i and each pair of distances (d_{Ai}, d_{Bi}) , it holds that $d_{Ai} = d_{Bi}$, except for the distances d_{Aj}, d_{Ak}, d_{Bj} and d_{Bk} (and their symmetrical counterparts), which are such that $d_{Aj} + d_{Ak} = d_{Bj} + d_{Bk}$, then if $|d_{Aj} - d_{Ak}| < |d_{Bj} - d_{Bk}|$, A offers strictly more freedom of choice than B .

Here I shall clear up a possible misunderstanding; the fact that the *Limited evenness condition* only applies to comparisons between sets that are similar

in all respects except for the distribution of the same sum of distances between two (four) individual distances does not mean that evenness only makes a difference to diversity for these limited comparisons. Evenness matters generally for diversity. It is important to distinguish between the ideal of evenness and a condition for evenness. A condition may apply only to limited cases. But the ideal of evenness applies to every distribution of a fixed sum of distances among several individual distances, when evenness is maximized; diversity is maximized relative to the fixed sum.

11.3 Compromises

Another possibility is to try to compromise between the two ideals of diameter diversity and evenness diversity. There are several ways to make such a compromise.

One compromise is to accept the diameter ideal for smaller sums of differences and the evenness ideal for greater sums of differences. When the sum of differences is small, the diameter ideal assures that at least two options are rather different. When the sum of differences is large, the evenness ideal assures that more than two options are rather different. This mixed ideal would obviously lead to sorites problems. But it might be the most reasonable ideal. An example of an application of this diversity ideal would be to use the diameter ideal for choice sets of similar options, such as sets of pens, and to use the evenness ideal for choice sets of dissimilar options, such as sets of books. Applied to choice sets of pens, the diameter ideal recommends a choice among a fountain pen and a few very similar ballpoint pens, rather than a choice among several more dissimilar ballpoint pens. Applied to choice sets of books, the evenness ideal recommends a choice among several dissimilar books, rather than a choice among a children's book and a few similar calculus books.

Another compromise is to accept the *Supremum diameter condition* for the largest distance and the ideal of evenness for all the smaller distances. This diversity ideal may be illustrated by the set A , which has a total difference sum of 240 and a cardinality of 3, and the distance matrix \mathbf{M}_A :

	x_1	x_2	x_3
x_1	0	30	60
x_2	30	0	30
x_3	60	30	0

The diameter is maximized, and the smaller distances are maximally even. Represented in Euclidean space, the options would be positioned on a straight line. Even though this ideal can be perfectly achieved for some metric spaces, this is not always so. If the diameter is close to the supremum,

then the options would be positioned on a straight line. For sets with a cardinality greater than three, this excludes the possibility that all the smaller distances are maximally even. If they were, the diameter would not be close to the supremum. This means that if the compromise should work as a general ideal, there must be some way to measure closeness to the ideal. Since the ideal is identified by two factors, a diameter close to the supremum and maximized evenness for smaller distances, a measure of closeness to the ideal would generally involve weighing the two factors against one another for importance. Is it more important to have a larger diameter or more evenly distributed smaller distances? If any factor is more important, how much more important is it? Even if we could answer these questions, we might also have to weigh the importance of closeness to the ideal distribution against the importance of cardinality, or the magnitudes of all the distances.

A third, somewhat related, compromise is to declare that the ideal distribution is always an equidistant distribution of options on a straight line. In some cases, this diversity ideal coincides with the former, but not always. For the set A above, the two ideals coincide. However, in the following example they may not. Suppose the sets B and C have a total difference sum of 260 and a cardinality of 4. $B = \{0 \text{ h}, 13 \text{ h}, 26 \text{ h}, 39 \text{ h}\}$, while $C = \{0 \text{ h}, 10 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$. B exemplifies the ideal of equidistant options on a straight line, which C does not. B is more even, while C has a larger diameter. Depending on how a larger diameter is weighted against a greater degree of evenness, C may be ranked above B , below B , or equal to B , according to the former ideal. The third ideal is also problematic in terms of determining how close a non-ideal distribution is to the ideal. But perhaps we could use some shortest move technique to measure closeness, such as the one used by Fager (1972: 299).

It is certainly not an argument against any compromise ideal that it is difficult to solve all its problems. It may very well be possible to specify and measure some hybrid ideal. However, I shall not make any further attempts to specify or measure such an ideal here. I shall just assume that evenness diversity is the most reasonable ideal.

Chapter 12: Distribution of Differences among Options

It seems reasonable to think that it matters to diversity how a sum of differences is distributed among individual non-zero differences. But does it also matter how a sum of non-zero differences is distributed among individual options? In some cases, the ideal of maximizing evenness and the ideal of maximizing diameter also implies how a sum should be distributed among options. Satisfaction of the ideal of maximal evenness requires that the non-zero differences are equal, and thus that the sum of differences from each option to all other options are equal. Satisfaction of the ideal of maximal diameter requires that two options are maximally different, and thus that the sum of differences from each of the two options are maximally large. But neither ideal implies anything regarding the ideal distribution of a sum of differences among options, in cases when the degrees of evenness or the diameter are equal but not maximal or maximized.

There are at least two questions that may be asked in relation to the ideal distribution of a sum of differences among individual options. The first question is this: given a sum of differences, what is the ideal distribution among individual options? The second question is this: given a sum of differences, which is already partitioned into individual differences, what is the ideal distribution among individual options? We may call the first question the *general question* and the second question the *limited question*.

12.1 Distribution of a Sum of Differences among Options

We shall begin by looking at the general question: given a sum of differences, what is the ideal distribution among individual options? There are intuitions regarding the distribution of a sum of differences among individual options that are analogous to the intuitions regarding the distribution of a sum of differences among individual differences.

One idea is that the ideal distribution would be when the sum of differences is distributed so that the sum equals the differences from a particular option to all the other options. This would be a *maximally unequal distribution*. Such a distribution is possible only for sets of two options, due

to the principle of the *Identity of indiscernibles*. Another idea is that the ideal distribution would be when the sum of differences is equally distributed among the individual options. This would be a *maximally equal distribution*. A third idea is that the ideal distribution involves both inequality and equality, where these factors are weighted against one another in some specific way. A fourth idea is that the distribution of a sum of differences among individual options does not matter to diversity at all.

Here is an example to illustrate the problem: there is a total sum of differences of 240 for sets of four options. What is the ideal distribution of the total difference sum among the individual options? Again, we may suppose that the sets represent different choices among buildings that Uppsala University could rent, where the buildings differ in length, breadth and height.

As I mentioned, it is not possible to have a maximally unequal distribution in a metric space of four options due to the property of the *Identity of indiscernibles*. So we have to consider an example that is close. Set A with matrix \mathbf{M}_A is as close to a maximally unequal distribution as possible:

	x_1	x_2	x_3	x_4
x_1	0	$40 - \varepsilon$	$40 - \varepsilon$	$40 - \varepsilon$
x_2	$40 - \varepsilon$	0	ε	ε
x_3	$40 - \varepsilon$	ε	0	ε
x_4	$40 - \varepsilon$	ε	ε	0

A can be represented in Euclidean space, with the options as vertices and the distances as edges. The options are the vertices of a high and thin three-dimensional tetrahedron. If the set is interpreted as a choice among buildings, the choice would concern three very similar buildings, differing in size in negligible ways, and a fourth building, which is quite different from the others. One interpretation is that three of the buildings share roughly the same length, breadth, and height of 10, while the fourth building shares the same length and breadth, but has a height of 40. If length is the first coordinate, breadth the second, and height the third, then the coordinate vectors are $x_1 = (10, 10, 40)$, $x_2 = (10, \approx 10, 10)$, $x_3 = (10, 10, \approx 10)$, and $x_4 = (\approx 10, 10, 10)$.

Set B with matrix \mathbf{M}_B has a maximally equal distribution:

	y_1	y_2	y_3	y_4
y_1	0	15	20	25
y_2	15	0	25	20
y_3	20	25	0	15
y_4	25	20	15	0

B can be represented in Euclidean space, with the options as vertices of a two-dimensional rectangle. If the set is interpreted as a choice among buildings, it could represent a choice among one building of 10 m in length, breadth, and height, another building of 10 m in length, 23 m in breadth, and roughly 17 m in height, a third building of 30 m in length, 23 m in breadth, and roughly 17 m in height, and a fourth building of 30 m in length and 10 m in breadth and height. The coordinate vectors are $y_1 = (10, 10, 10)$, $y_2 = (10, 23, \approx 17)$, $y_3 = (30, 23, \approx 17)$, and $y_4 = (30, 10, 10)$.

Set C with matrix \mathbf{M}_C has a distribution that is somewhere in between maximally unequal and maximally equal:

	z_1	z_2	z_3	z_4
z_1	0	12	24	36
z_2	12	0	12	24
z_3	24	12	0	12
z_4	36	24	12	0

C can be represented in Euclidean space, with the options as points on a straight line. I gave two interpretations for this set in the previous chapter. One interpretation was that the set represented the choice among four buildings of the same length and breadth of 10 m, but with different heights, where the first building had a height of 10 m, the second had a height of 22 m, the third had a height of 34 m, and the fourth had a height of 46 m. Here, $z_1 = (10, 10, 10)$, $z_2 = (10, 10, 22)$, $z_3 = (10, 10, 34)$, and $z_4 = (10, 10, 46)$.

The important question is if there is a difference in diversity provided by the three sets. To answer this question, we shall consider one distribution at a time.

12.2 Inequality

We shall begin by discussing the ideal of inequality. For this we shall use the total sum of distances from an option $x \in A$, to all the other options in A , the sum $\Sigma(x_i)$.

A *maximally unequal distribution* may be defined as follows:

A metric choice set $A_d \in P(X_d)$ such that $|A| = n$ and $\Sigma(A_d) = M$, has a *maximally unequal distribution* if and only if there is some option $x \in A$ such that $\Sigma(x_A) = M$.

As I mentioned, a maximally unequal distribution is possible only for sets of two options. For this reason, a distribution with a maximal diameter is equivalent to a maximally unequal distribution. Because we are interested in sets of more than two options as well, we may also (somewhat imprecisely) define:

A metric choice set $A_d \in P(X_d)$ such that $|A| = n$ and $\Sigma(A_d) = M$, has a *close to maximally unequal realizable distribution* if and only if there is some option $x \in A$ such that $\Sigma(x_A) = M - (n^2 - 3n + 2)\varepsilon$, where ε is a very small number.

When there are n options in a set, there are $n^2 - n$ non-zero distances, and $(n - 1)^2 - (n - 1)$ non-zero distances not involving x . This equals $n^2 - 3n + 2$ non-zero distances not involving x . These distances should be as small as possible, so each of these distances should be ε .

We may note that if a set has a maximal realizable diameter, then it also has a maximally unequal realizable distribution. But a set can have a maximally realizable unequal distribution without having a maximal realizable diameter. One such example is the set A from the previous section. Another example is the following: the set B with four options, a total difference sum of 192, and the matrix:

	x_1	x_2	x_3	x_4
x_1	0	32	32	32
x_2	32	0	ε	ε
x_3	32	ε	0	ε
x_4	32	ε	ε	0

If B is represented in Euclidean space the options are arranged in the vertices of a very thin and high tetrahedron.

The ideal of maximizing inequality may seem reasonable in roughly the same way as the ideal of maximizing diameter may seem reasonable. If inequality is maximized, there is at least one option that is as different from the other options as possible. If this ideal is acceptable, then the following is an acceptable condition:

The Maximal realizable inequality condition: For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$ and $\Sigma(A_d) = \Sigma(B_d)$, if A has a *maximally realizable unequal distribution* and B does not, then A offers strictly more freedom of choice than B .

It is also possible to formulate a more general inequality condition:

The Inequality condition: For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$ and $\Sigma(A_d) = \Sigma(B_d)$, if A is *more unequally distributed* than B , then A offers strictly more freedom of choice than B .

This condition requires some measure of inequality. One idea is that degrees of inequality could be measured, in an absolute sense, by the sum of differences in $\mathbf{d}_A(x_i)$, where x_i is the option with greatest sum of differences. Greater values imply greater inequality. However, it might be more appropriate to measure degrees of inequality in a relative sense, for example, by dividing the total difference sum of $\mathbf{d}_A(x_i)$ by the sum of differences in \mathbf{d}_A , where x_i is the option with greatest sum of differences. Again, greater values imply greater inequality.

The exact measure of inequality does not matter much in this context since neither condition is acceptable. Both conditions are inconsistent with the previously accepted *Limited evenness condition*. The conditions award distributions where all options but one are clumped together, rather than being more equally distributed. They overestimate the importance of the differences of a single option to all other options.

12.3 Equality

The ideal of equality may seem more promising, in the sense that it might seem compatible with the *Limited evenness condition*. Let us again use $\Sigma(x_A)$, which is the total sum of distances from an option $x \in A$, to all the other options in A . Maximal equality may be defined as follows:

A metric choice set $A_d \in P(X_d)$ such that $|A| = n$ and $\Sigma(A_d) = M$, has a *maximally equal* distribution if and only if for each $x \in A$, $\Sigma(x_A) = M/n$.

We may note that a maximally even distribution implies a maximally equal distribution. But the reverse relationship does not hold; a maximally equal distribution does not imply a maximally even distribution. The following two distributions are both maximally equal. Both sets have four options and a total difference sum of 192. A has the matrix:

	y_1	y_2	y_3	y_4
y_1	0	16	16	16
y_2	16	0	16	16
y_3	16	16	0	16
y_4	16	16	16	0

B has the matrix:

	y_1	y_2	y_3	y_4
y_1	0	12	16	20
y_2	12	0	20	16
y_3	16	20	0	12
y_4	20	16	12	0

Only the distribution of A is maximally even.

We may define a condition regarding comparisons between maximally equally distributed sets and less than maximally equally distributed sets as follows:

The Maximal equality condition: For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$ and $\Sigma(A_d) = \Sigma(B_d)$, if A has a *maximally equal distribution* and B does not, then A offers *strictly more freedom of choice than B* .

We may also define a more general condition:

The Equality condition: For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$ and $\Sigma(A_d) = \Sigma(B_d)$, if A is *more equally distributed than B* , then A offers *strictly more freedom of choice than B* .

The second condition is problematic in the same way as the *Evenness condition*. Degrees of equality may be measured in several ways. We may, for example, measure equality by the use of a differences among distances measure applied to the sums of distances $\Sigma(x_A)$ for each option in A . (For comparison, see section 14.4 on evenness measures.) Lower values imply greater equality. We may also measure equality by the use of a separable strictly concave function, applied to the sums of distances $\Sigma(x_A)$ for each option in A . Here, we shall not go into these problems in detail. The reason is that equality might not be a reasonable ideal.

Initially, the equality ideal may seem reasonable. After all, the inequality ideal seems unreasonable, so the equality ideal might be reasonable just by representing the opposite ideal. Furthermore, the equality ideal seems similar to the evenness ideal and the evenness ideal seems reasonable, so the equality ideal should be reasonable just by representing a similar ideal. But this is not the case. Strangely enough, the equality ideal has a similar

problem as the inequality ideal; it awards distributions where options are clumped together. It is, in fact, inconsistent with the *Limited evenness condition*.

Let us look at one example. We shall compare two sets. Both sets have a total sum of differences of 240 and a cardinality of 4. The first is a set familiar from the previous section, the set C with the matrix:

	y_1	y_2	y_3	y_4
y_1	0	15	20	25
y_2	15	0	25	20
y_3	20	25	0	15
y_4	25	20	15	0

C has a maximally equal distribution. The second is the set G , with the matrix:

	y_1	y_2	y_3	y_4
y_1	0	15	20	25
y_2	15	0	20	20
y_3	20	20	0	20
y_4	25	20	20	0

G does not have a maximally equal distribution since the sum of each $\mathbf{d}_G(y_i)$ differs from the others. It does not have a maximally even distribution either, but it is more even than C . This follows from the *Limited evenness condition*. There is a bijection from \mathbf{d}_G onto \mathbf{d}_C such that for each pair of distances (d_{G_i}, d_{C_i}) , it holds that $d_{G_i} = d_{C_i}$, except for the two distances 20, 20 in \mathbf{d}_G and 15, 25 in \mathbf{d}_C (and their symmetrical counterparts), which are such that $20 + 20 = 15 + 25$ and $|20 - 20| < |25 - 15|$. According to the *Limited evenness condition*, G offers more freedom of choice than C . According to the *Maximal equality condition*, C offers more freedom of choice than G . So equality and evenness are in conflict. Since evenness matters to diversity, equality cannot matter to diversity in an unqualified way.

There is a possibility, of course, that equality matters for diversity when degrees of evenness are kept fixed. We shall consider this possibility when we discuss the ideal distribution of individual differences among distances. In these cases, degrees of evenness are fixed, due to a fixed metric space.

12.4 Problem

Since neither the inequality ideal nor the equality ideal gave acceptable rankings by themselves, it is difficult to know what to think about the influence of distributions of sums among options on diversity. Perhaps there is some compromise between inequality and equality that influences

diversity. Perhaps the distribution of a total difference sum among options does not matter to diversity at all.

Before we give up on the idea that distributions of sums of differences among options matter for diversity, we shall consider the question from a slightly different angle. We shall consider the question whether distributions of sums of differences among options matter when we keep degrees of evenness fixed. More precisely, we shall consider the question whether the distribution of individual differences among options matters.

12.5 Distribution of Individual Differences among Options

We considered the question whether the distribution of a sum of differences among individual options can affect diversity of options. The answer to this question was unclear. However, there is a related question that we should consider: can the distribution of individual differences among options affect diversity? Given a total sum of differences, which is already partitioned into individual differences, what is the ideal distribution among individual options?

To avoid misunderstanding, we should define exactly what it means that two distance vectors contain the same individual distances:

The individual distances of two distance vectors \mathbf{d}_A and \mathbf{d}_B are *equal* if and only if for each pair of distances (d_{Ai}, d_{Bi}) , it holds that $d_{Ai} = d_{Bi}$.

We may note that when the individual differences of two distance vectors are equal, cardinality, total difference sum and degrees of evenness are also equal.

Here is an example to illustrate the problem: a set has four options, a total difference sum of 240 and the distance vector (25, 25, 25, 25, 20, 20, 20, 20, 15, 15, 15, 15, 0, 0, 0, 0). What is the ideal distribution of these distances among the options?

The first example shows a maximally unequal set given the distance vector above, the set A below with matrix \mathbf{M}_A :

	x_1	x_2	x_3	x_4
x_1	0	25	25	20
x_2	25	0	20	15
x_3	25	20	0	15
x_4	20	15	15	0

If A is represented in Euclidean space, the options are vertices of a tetrahedron in three dimensions. If the set is interpreted as a choice among

buildings, it can represent a choice among one building of 10 m in length, breadth and height, a second building of 30 m in length and 10 m in breadth and height, a third building of 20 m in length, roughly 33 m in breadth, and 10 m in height, and a fourth building of 20 m in length, roughly 15 m in breadth, and roughly 20 m in height. The coordinate vectors are $x_1 = (10, 10, 10)$, $x_2 = (30, 10, 10)$, $x_3 = (20, \approx 33, 10)$, and $x_4 = (20, \approx 15, \approx 20)$.

The second example shows a maximally equal set given the distance vector above, the set B below with matrix \mathbf{M}_B :

	y_1	y_2	y_3	y_4
y_1	0	15	20	25
y_2	15	0	25	20
y_3	20	25	0	15
y_4	25	20	15	0

If B is represented in Euclidean space, the options are vertices of a rectangle in two dimensions. We have already looked at an interpretation of the set as a choice among buildings. It could, for example, represent a choice among one building of 10 m in length, breadth, and height, another building of 10 m in length, 23 m in breadth, and roughly 17 m in height, a third building of 30 m in length, 23 m in breadth, and roughly 17 m in height, and a fourth building of 30 m in length and 10 m in breadth and height. The coordinate vectors are $y_1 = (10, 10, 10)$, $y_2 = (10, 23, \approx 17)$, $y_3 = (30, 23, \approx 17)$, and $y_4 = (30, 10, 10)$.

The third example shows a set that is neither maximally unequal nor maximally equal, the set C with matrix \mathbf{M}_C :

	z_1	z_2	z_3	z_4
z_1	0	20	20	25
z_2	20	0	25	15
z_3	20	25	0	15
z_4	25	15	15	0

C is not a Euclidean space example.

The important question is whether the differences in distribution of distances among the options affect the diversity of the sets. There may be intuitions regarding the distribution of differences among individual options that are similar to the intuitions about distributions of sums of differences among individual differences. First, distribution may matter in that diversity is maximized when the largest differences involve one and the same option. That is to say, diversity is maximized when inequality is maximized, relative to a given set of distances (as in A). Second, distribution may matter in that diversity is maximized when each option has as close to the same sum of differences from all other options as possible. That is to say, diversity is maximized when equality is maximized, relative to a given set of distances

(as in B). Third, distribution may matter both in terms of inequality and equality, where these factors are weighted against one another in some specific way (perhaps as in C). Fourth, distribution among individual options may not matter to diversity at all. The last property is called *symmetry* and occurs in mathematics. Let us consider one intuition at a time.

12.6 Limited Inequality

Since we have already discussed inequality, we shall discuss limited inequality very briefly. A condition regarding limited inequality would look like this:

Limited inequality condition: For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$, with distance vectors \mathbf{d}_A and \mathbf{d}_B where the individual differences are *equal*, if A is *more unequally distributed than B*, then A offers *strictly more freedom of choice than B*.

The *Limited inequality condition* is obviously consistent with the *Limited evenness condition*. In that sense, it is better than the *Inequality condition*. But it shares a problem with the general condition. Both conditions seem to be supported by intuitions similar to those that support the diameter conditions. The intuition behind the inequality conditions is that the more unequal distribution is more diverse. The most unequal distribution is such that the distances from each option to the others are as large as possible. The intuition behind the diameter conditions is that the more uneven distribution is more diverse. The most uneven distribution is such that the distance between one option and some other is as large as possible. The two intuitions are sufficiently similar for it to be strange to dismiss one while accepting the other. Therefore, it would be unreasonable to accept the *Limited inequality condition* after having dismissed the diameter conditions.

12.7 Limited Equality

We have discussed equality as well. A condition regarding limited equality would look like this:

Limited equality condition: For all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$, with distance vectors \mathbf{d}_A and \mathbf{d}_B where the individual differences are *equal*, if A is *more equally distributed* than B , then A offers *strictly more freedom of choice* than B .

The *Limited equality condition* is obviously consistent with the *Limited evenness condition* as well. In that sense, it is better than the *Equality condition*. Nevertheless, it is difficult to assess its reasonableness. On the one hand, the condition is based on the same intuition as the *Equality condition*, which was deemed to be unreasonable. On the other hand, the *Equality condition* was deemed unreasonable for being inconsistent with the *Limited evenness condition*, which the *Limited equality condition* is not. In fact, the intuition behind both equality conditions seems rather similar to the intuition behind the *Limited evenness condition*. The intuition behind the equality conditions is that the more equal distribution is more diverse. The most equal distribution is such that all sums of distances from each option to all the other options are equal. The intuition behind the evenness conditions is that the more even distribution is more diverse. The most even distribution is such that all distances are equal. These two intuitions are strikingly similar.

There is thus some reason to accept the *Limited equality condition*. However, I shall not do that here. I simply do not think that the condition is sufficiently convincing.

12.8 Symmetry

If distributions of differences among options do not affect diversity at all, then the following is a reasonable condition:

The Symmetry condition: For all metric choice sets $A_d, B_d \in P(X_d)$, with distance vectors \mathbf{d}_A and \mathbf{d}_B , if the individual distances in \mathbf{d}_A and \mathbf{d}_B are *equal*, then A offers *equal freedom of choice* as B .

According to the *Symmetry condition*, there is no difference in diversity between the sets A, B and C in the example from section 12.5.

As we may recall, A has the matrix:

	x_1	x_2	x_3	x_4
x_1	0	25	25	20
x_2	25	0	20	15
x_3	25	20	0	15
x_4	20	15	15	0

B has the matrix:

	y_1	y_2	y_3	y_4
y_1	0	15	20	25
y_2	15	0	25	20
y_3	20	25	0	15
y_4	25	20	15	0

C has the matrix:

	z_1	z_2	z_3	z_4
z_1	0	20	20	25
z_2	20	0	25	15
z_3	20	25	0	15
z_4	25	15	15	0

The *Symmetry condition* may be accepted in the absence of more refined intuitions. As an example of problematic comparative cases, we may consider the three sets above, where each set has four options, a total difference sum of 240 and the distance vector (25, 25, 25, 25, 20, 20, 20, 20, 15, 15, 15, 15, 0, 0, 0, 0). It may seem difficult to see any significant differences between the three sets in terms of diversity.

It has been standard in biology to accept the symmetry ideal, although it is unclear if the ideal is argued for at all. Kempton (1979) and Kreutz-Delgado and Rao (1999), for example, simply state that diversity measures should be symmetric. I shall do the same here. One reason is that there seem to be no strong reasons to accept any other ideal. Another reason is that it is mathematically convenient. The acceptance of the symmetry condition allows us to use a symmetric function on the distances to measure freedom of choice as diversity.

Chapter 13: Conditions Concerning Three Relevant Properties

So far I have concluded that at least three things matter for diversity: the cardinality of options, the magnitudes of the differences between the options and the evenness of the distribution of a sum of differences among individual differences. The magnitudes of the differences between all the options can be represented by using any of the following five measures, the *Total difference sum*, the *Total exclusive average difference*, the *Total inclusive average difference*, the *Sum of exclusive average differences* and the *Sum of inclusive average differences*.

The next thing we should do is to investigate how the three properties contribute to diversity, relative to one another. The first step of such an investigation would be to look at the effects on diversity when varying one property, while keeping the other two properties fixed. We have already started such an investigation. For the *Limited evenness condition*, cardinality and total difference sum were kept fixed, while degrees of evenness varied. Using this method we could conclude that evenness had a positive effect on diversity, other things being equal. This method should be used in a more systematic way. In particular, we should investigate the effects of different measures of magnitude, while keeping cardinality and evenness fixed. For this purpose, we may look at some comparative tables. The tables only show comparisons between sets where the degrees of evenness are either equal and at a maximum, or differ from the maximum. The properties of *A* as compared to *B* are listed below.

The first table shows the effects on diversity of varying cardinality, total difference sum and evenness:

Sets	Cardinality	Total difference sum	Evenness	Evaluation for diversity
A vs. B	<i>A</i> equal to <i>B</i>	<i>A</i> equal to <i>B</i>	<i>A</i> maximum and greater than <i>B</i>	<i>A</i> more diverse than <i>B</i>
A vs. B	<i>A</i> greater than <i>B</i>	<i>A</i> equal to <i>B</i>	<i>A</i> equal to <i>B</i> , both maximum	?
A vs. B	<i>A</i> equal to <i>B</i>	<i>A</i> greater than <i>B</i>	<i>A</i> equal to <i>B</i> , both maximum	<i>A</i> more diverse than <i>B</i>

The first comparison can be evaluated by applying the *Maximal evenness condition*. The last comparison can be evaluated by applying the *Option dominance condition*. The second comparison is new and should be discussed. We shall do this shortly.

The second table shows the effects on diversity of varying cardinality, exclusive average difference and evenness:

Sets	Cardinality	Exclusive average difference	Evenness	Evaluation for diversity
A vs. B	A equal to B	A equal to B	A maximum and greater than B	A more diverse than B
A vs. B	A greater than B	A equal to B	A equal to B , both maximum	A more diverse than B
A vs. B	A equal to B	A greater than B	A equal to B , both maximum	A more diverse than B

The first comparison can be evaluated by applying the *Limited evenness condition*. The second comparison can be evaluated by applying the *Strict monotonicity condition* one or more times. The last comparison can be evaluated by applying the *Option dominance condition*.

The third table shows the effects on diversity of varying cardinality, inclusive average difference and evenness:

Sets	Cardinality	Inclusive average difference	Evenness	Evaluation for diversity
A vs. B	A equal to B	A equal to B	A maximum and greater than B	A more diverse than B
A vs. B	A greater than B	A equal to B	A equal to B , both maximum	?
A vs. B	A equal to B	A greater than B	A equal to B , both maximum	A more diverse than B

The first comparison can be evaluated by applying the *Limited evenness condition*. The last comparison can be evaluated by applying the *Option dominance condition*. The second comparison is difficult, and there is probably no general conclusion that can be drawn. That cardinality is greater in A , but the inclusive average difference and evenness is equal in A and in B , implies that the total difference sum is greater in A , but that the individual differences are larger in B . There seems to be no general conclusion that can be drawn in regard to which consideration is most important, a greater total difference sum, or larger individual differences. The proofs regarding the relationship between the sum of inclusive average differences and the total difference sum and the individual differences are included in the appendix.

The fourth table shows the effects on diversity of varying cardinality, total sum of exclusive average differences and evenness:

Sets	Cardinality	Sum of exclusive average difference	Evenness	Evaluation for diversity
A vs. B	A equal to B	A equal to B	A maximum and greater than B	A more diverse than B
A vs. B	A greater than B	A equal to B	A equal to B , both maximum	A more diverse than B
A vs. B	A equal to B	A greater than B	A equal to B , both maximum	A more diverse than B

The first comparison can be evaluated by applying the *Limited evenness condition*. The second comparison can be evaluated by applying the *Strict monotonicity condition* one or more times. The last comparison can be evaluated by applying the *Option dominance condition*.

The fifth table shows the effects on diversity of varying cardinality, total sum of inclusive average differences and evenness:

Sets	Cardinality	Sum of inclusive average difference	Evenness	Evaluation for diversity
A vs. B	A equal to B	A equal to B	A maximum and greater than B	A more diverse than B
A vs. B	Greater	A equal to B	A equal to B , both maximum	?
A vs. B	A equal to B	A greater than B	A equal to B , both maximum	A more diverse than B

The first comparison can be evaluated by applying the *Limited evenness condition*. The last comparison can be evaluated by applying the *Option dominance condition*. The second comparison is as complicated as the comparison involving the inclusive average difference. If cardinality is greater in A , but the sum of inclusive average differences and evenness is equal in A and in B , then the total difference sum is greater in A , but the individual differences are larger in B . (The proofs of these facts are included in the appendix.) It is difficult to judge which set is most diverse.

13.1 Spread

There is one diversity comparison left to assess; what happens when cardinality differs, but total difference sum and degrees of evenness are kept

fixed? For example, let us suppose that there are only two options in A and that the difference between them is 45. In B , there are 10 options and the difference between each pair of options is 1. The sum of differences is the same in each case: 90. Here, most people would judge A to be the more diverse set. For example, a set of ten left-wing parties would not be judged as more diverse than a set of one left-wing party and one right-wing party.

A reasonable measure of diversity should not increase with cardinality, unless the total sum of differences increases as well. The ordering of sets should thus satisfy the following condition:

The Spread condition: For a measure of freedom of choice F , and all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = m$ and $|B| = n$ and $\Sigma(A_d) = \Sigma(B_d) = M$, where each non-zero distance $d_{A_i} = M/(m(m-1))$ and each non-zero distance $d_{B_i} = M/(n(n-1))$, if $m < n$, then A offers strictly more freedom of choice than B , and thus $F(A) > F(B)$.

The *Spread condition* emphasizes the importance of magnitudes of individual differences over cardinality for maximizing diversity. In cases where the difference in magnitude and cardinality is small, the condition may perhaps be opposed. Looking at the most similar comparisons, someone may think that A , with three options and the distance vector $\mathbf{d}_A = (1, 1, 1, 1, 1, 0, 0, 0)$, is no less diverse than B with two options and the distance vector $\mathbf{d}_B = (3, 3, 0, 0)$. But this would be rather odd. The extra option in A hardly makes up for the loss of diameter. As the difference between the diameters increases between the larger set and the smaller set, it gets increasingly strange to continue to insist that the larger set with very similar options is as diverse as the smaller set with very different options. Because of this, I shall accept the *Spread condition*.

13.2 Priority of Diameter

I accepted the *Spread condition* since it is reasonable to judge that a small number of very different options are more diverse than a great number of very similar options. This intuition may be taken much further. It may be reasonable to judge that two very different options are more diverse than *any* number of very similar options. Let us look at one example: A is a set of very similar left wing parties, while B is a set of one left wing party and one right wing party. Perhaps B is more diverse than A , no matter how many left wing parties A contains. There are no number of left wing parties that can be as diverse as a set with just one left wing party and one right wing party.

As I said, this intuition is similar to the intuition captured by the *Spread condition*, but it is more extreme. One may wonder how far the intuition goes. Does the intuition hold only for very small differences, or does it hold

whenever the difference in diameter between the smaller set and the larger set is sufficiently large? In the first case, it is only any number of distances below some limit, say $d \leq 0.1$, that fails to contribute as much to diversity as one distance above some limit, say $d \geq 10\,000$. In the second case, any number of distances below some limit d_i fails to contribute as much to diversity as one distance d_j , whenever the difference between d_i and d_j is sufficiently large. The first kind of intuition may be captured by the following condition:

The Limited diameter priority condition: For all metric choice sets $A_d, B_d \in P(X_d)$, there exist some distances m and k such that $m > k$, and if there exists at least one distance $d_{A_i} \geq m$ and each distance $d_{B_i} \leq k$, then A offers strictly more freedom of choice than B .

The *Limited diameter priority condition* seems somewhat odd. Why should it just be very small differences that fail to contribute as much to diversity as one larger difference? It seems more likely that it is the relative size of the differences that matter, rather than their absolute size. If this is correct, and the intuition itself is reasonable, then a condition such as the following may be reasonable:

The Diameter priority condition: For all metric choice sets $A_d, B_d \in P(X_d)$ and for any distance k , there exists a distance m such that $m = m(k)$ and $m > k$, and if there exists at least one distance $d_{A_i} \geq m$ and each distance $d_{B_i} \leq k$, then A offers strictly more freedom of choice than B .

The *Diameter priority condition* captures an intuition that we may have about diversity. But the acceptance of the condition involves a great number of problems. One problem is the usual sorites type of problem; what exactly is the relation between m and k ? Any specification of how m is a function of k is likely to be arbitrary. We simply do not have intuitions about the effect of differences on diversity that are sufficiently precise to justify any particular choice of function. It is also doubtful that we have intuitions that are sufficiently precise to justify even the choice of some relevant *class* of functions that can specify the relation between m and k . Another problem is that the *Diameter priority condition* is inconsistent with the *Limited evenness condition*. If we can choose a set with a diameter larger than the critical value m , rather than a set where all options are evenly distributed at k , then the *Diameter priority condition* tells us to choose diameter above evenness in the interest of maximizing diversity.

Since a large diameter is an alternative to the ideal of evenness, it is not strange that the conflict should resurface. The conflict cannot really be solved. The ideal of a large diameter is contrary to the ideal of maximizing

evenness. As we have seen previously, the ideal of a large diameter is not very reasonable on its own. If we believe that a large diameter is an integral part of the concept of diversity we should accept some combined ideal.

One idea, mentioned before, is that we should accept the diameter ideal for smaller sums of distances and the evenness ideal for greater sums of distances. If we do this we may at least accept the *Limited diameter priority condition*. Another idea, also mentioned previously, is that we should maximize the diameter for the largest distance, and then maximize evenness for the other distances. If we accept such an ideal we may also accept the *Diameter priority condition*. Nevertheless, I still think that the ideal of evenness diversity is the most intuitively compelling ideal.

13.3 Partial Conclusion

We have suggested that at least eight conditions should be satisfied for an ordering of sets in terms of freedom of choice as diversity. The conditions yield a partial ranking of sets. To get a complete ranking of sets, a measure is needed. If some of the conditions are changed slightly to apply directly to a measure of freedom of choice, we get the following list:

The Domain-insensitivity condition: For a measure of freedom of choice F , there do not exist any relevant universal sets X and X^* and choice sets A and B such that $A, B \subseteq P(X)$ and $A, B \subseteq P(X^*)$, where A offers at least as much freedom of choice as B , given X , and A offers strictly less freedom of choice than B , given X^* , so that $F(A) \geq F(B)$, given X , and $F(A) < F(B)$, given X^* .

The Strict monotonicity condition: For a measure of freedom of choice F , any non-empty choice set $A \in P(X)$, and any option $y \in X - A$, $A \cup \{y\}$ offers strictly more freedom of choice than A , and thus $F((A \cup \{y\})) > F(A)$.

The No freedom of choice condition: For a measure of freedom of choice F and any choice set $A \in P(X)$, if $|A| \leq 1$, then A offers no freedom of choice, and thus $F(A) = 0$.

The Limited growth condition: For a measure of freedom of choice F , any metric choice set $A_d \in P(X_d)$ and any option $y \in X - A$, it holds that $F((A \cup \{y\})) \leq F(A) + F(\{x_i, y\})$, where $d(x_i, y) = \max_k d(x_k, y)$ and $x_i, x_k \in A$.

The Option dominance condition: For a measure of freedom of choice F and all metric choice sets $A_d, B_d \in P(X_d)$, if A dominates B option by option, then A offers strictly more freedom of choice than B , and thus $F(A) > F(B)$.

The Limited evenness condition: For a measure of freedom of choice F , and all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$ and $\Sigma(A_d) = \Sigma(B_d)$, with distance vectors \mathbf{d}_A and \mathbf{d}_B , if for each i and each pair of distances (d_{Ai}, d_{Bi}) , it holds that $d_{Ai} = d_{Bi}$, except for the distances d_{Aj}, d_{Ak}, d_{Bj} and d_{Bk} (and their symmetrical counterparts), which are such that $d_{Aj} + d_{Ak} = d_{Bj} + d_{Bk}$, then if $|d_{Aj} - d_{Ak}| < |d_{Bj} - d_{Bk}|$, A offers strictly more freedom of choice than B , and thus $F(A) = F(B)$.

The Symmetry condition: For a measure of freedom of choice F , and all metric choice sets $A_d, B_d \in P(X_d)$, with distance vectors \mathbf{d}_A and \mathbf{d}_B , if the individual distances in \mathbf{d}_A and \mathbf{d}_B are equal, then A offers equal freedom of choice as B , and thus $F(A) = F(B)$.

The Spread condition: For a measure of freedom of choice F , and all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = m$ and $|B| = n$ and $\Sigma(A_d) = \Sigma(B_d) = M$, where each non-zero distance $d_{Ai} = M/(m(m - 1))$ and each non-zero distance $d_{Bi} = M/(n(n - 1))$, if $m < n$, then A offers strictly more freedom of choice than B , and thus $F(A) > F(B)$.

The conditions were accepted as a result of an investigation of the importance of cardinality, magnitude of differences, distribution of a sum of differences among individual differences and distribution of a sum of differences among individual options for freedom of choice as diversity. They were also accepted as a result of rejecting the importance of preferences for freedom of choice. Obviously, there may be some other relevant factor that we have not discussed. However, it is difficult to see which factor that would be. Have we missed the importance of probabilities for freedom of choice? No, since we allowed for the possibility that the choice among acts should be regarded as a choice among lotteries for different outcomes. Have we missed the importance of distribution of a sum of differences among different dimensions? No, since an adequate distance function should reflect that options that differ in several dimensions may differ more than options that differ in only one dimension. Perhaps there is some further factor that may be relevant for freedom of choice. As I cannot imagine which factor that would be, I shall continue to discuss the construction of a measure that could satisfy the accepted conditions.

Part 3: Constructing a Measure of Freedom of Choice

Chapter 14: Derived Measures

In this part of the thesis we shall discuss the construction of a measure that could satisfy the eight conditions that were accepted in the previous part. We shall begin by discussing derived measures of freedom of choice as diversity. This type of measure is constructed in accordance with the idea that freedom of choice is a function of some properties $Q_1, Q_2 \dots Q_n$ of choice sets (here regarded as finite metric spaces), and that the measure should reflect this in a direct manner. More precisely, we assume that for each relevant property Q_i there is some measure f_i and that freedom of choice may be measured by a derived measure g that is a function of all the functions $f_1, f_2 \dots f_n$. (For a more thorough presentation, see for example Suppes and Zinnes (1963: 17–19)).

14.1 Scale-independence

An important property of a derived measure is that it should be independent of scale. This is brought up by Xu with regard to freedom of choice (2004: 283–284), and by Solow and Polasky (1994: 97–98), Mason et al. (2003: 573) and Dowden (2011: 4), with regard to diversity.

A measure may be scale-independent in several senses. In a weak sense of *scale-independence* it is only required that the ordering of sets in terms of freedom of choice is preserved when the scale by which the other properties are measured is changed. For example, if freedom of choice depends on the spatial differences between options and A is ranked as offering at least as much freedom of choice as B when the differences are measured in kilometers, then A should still be ranked as offering at least as much freedom of choice as B when the differences are measured in meters. The following condition captures the weak sense of scale-independence:

The Ordinal scale-independence condition: For a measure of freedom of choice F , where F is a function of the functions $f_1, f_2 \dots f_n$, and any metric choice sets $A_d, B_d \in P(X_d)$, if it is the case that $F(A) \geq F(B)$, and each function $f_1, f_2 \dots f_n$ is multiplied by some number $K > 0$, then it is still the case that $F(A) \geq F(B)$.

In a stronger sense of *scale-independence* it is also required that ratio relations between degrees of freedom of choice are preserved through a change of scale. For example, if freedom of choice depends on the spatial differences between options and A is ranked as offering twice as much freedom of choice as B when the differences are measured in kilometers, then A should still be ranked as offering twice as much freedom of choice as B when the differences are measured in meters. The following condition captures the strong sense of scale-independence:

The Ratio scale-independence condition: For a measure of freedom of choice F , where F is a function of the functions $f_1 f_2 \dots f_n$, and any metric choice set $A_d \in P(X_d)$, it holds that for all numbers $K > 0$, there is a number $L = L(K) > 0$, where L is an increasing function of K such that if each function $f_1 f_2 \dots f_n$ is multiplied by K , then $F(A)$ is multiplied by L .

The *Ratio scale-independence condition* implies the *Ordinal scale-independence condition*. Whether we should require the ratio scale condition or only the ordinal scale condition depends on whether we intend to measure freedom of choice on a ratio scale or just an ordinal scale. The strong condition is necessary only for ratio scale measures. Since I have assumed that freedom of choice can be measured on a ratio scale, I shall accept the *Ratio scale-independence condition*.

14.2 Separable Functions

Here we shall discuss a type of function that is scale-independent in both senses, namely separable functions. Two examples of such functions are additively separable functions and multiplicatively separable functions. The first one is defined as follows:

A function $g(f_1, f_2 \dots f_n)$ is *additively separable* if and only if $g = f_1 + f_2 + \dots + f_n$.

The second one is defined similarly:

A function $g(f_1, f_2 \dots f_n)$ is *multiplicatively separable* if and only if $g = f_1 \times f_2 \times \dots \times f_n$.

Derived measures are not commonly used as measures of freedom of choice. In fact, there may not be a single one suggested in the area prior to the proposals in this thesis. There are some occurrences of this type of measure in related areas, however. For example, Crocker (1980) proposes a derived

measure of positive liberty. As I have previously shown, this measure is problematic.

From the previous discussion we may conclude that there are at least three properties that affect the degree of freedom of choice: the cardinality of a choice set, the magnitude of all the differences, and the evenness of the distribution of differences. These three properties are also mentioned by Baumgärtner as important for diversity (2007: 3). Here we shall discuss whether it is possible to construct a derived measure of freedom of choice, using functions that represent the three factors and satisfying the previous conditions. The discussion is not going to concern all possible derived measures of freedom of choice; they are far too many. But by discussing some examples of derived measures of freedom of choice, I hope to illustrate some of the problems involved in this kind of construction. Eventually I shall recommend a derived measure of freedom of choice that does not have any of the problems of some of the other measures.

14.3 Three-Term Measures

Let us first consider additively separable functions, or (in other words) sum functions as measures of freedom of choice. The idea is that a measure of freedom of choice should have the following form:

$$\textit{Three-term measures: } F(A, d) = \alpha C(A_d) + \beta M(A_d) + \gamma E(A_d).$$

F is a function from a finite metric space to \mathbf{R} , C represents the cardinality of A , M is a measure of magnitude, E is a measure of evenness and α , β and γ are weights representing the relative importance of each property for freedom of choice.

There are several problems that need to be solved for a derived measure to work as a measure of freedom of choice. The first problem is to select appropriate functions to represent the properties of cardinality, magnitude and evenness. If these properties should be considered as constantly of the same importance relative to one another, it is important to select functions that grow at (roughly) the same rate. The second problem is to assess the relative importance of each property in comparison to the other two. We have to select appropriate values for the weights α , β and γ in the function above. The third problem is to get the measure to satisfy the eight conditions for a measure of freedom of choice.

Let us first consider the problem of selecting functions to represent the three properties that matter for freedom of choice.

14.3.1 Representing Cardinality

There is no problem in finding an appropriate function to represent cardinality since cardinality is a measure in itself. But we have to decide what type of cardinality should be incorporated into a derived measure of freedom of choice. There are several types of cardinality that may be used. We may use the *cardinality of options*, n , or the *cardinality of choice contributing options*, $n - 1$. For example, in regard to $A = \{0 \text{ h}, 20 \text{ h}, 40 \text{ h}\}$, we may use either the cardinality of three or the cardinality of two (reflecting the idea that a singleton set offers no choice). In a choice between the cardinality of options and the cardinality of choice contributing options, it seems most reasonable to choose the second, $n - 1$. This reflects more accurately that the freedom of choice offered by a singleton set is none.

The inclusion of a cardinality factor into a derived measure of freedom of choice is complicated by the *Spread condition*, stating that cardinality has a negative effect on freedom of choice, given a fixed total difference sum. To satisfy this condition, cardinality must be given a negative weight. This may at first seem counterintuitive since we tend to think that cardinality has a positive effect on freedom of choice. But our intuition may be explained by our tendency to think that a greater cardinality implies a greater total difference sum. We rarely think that the total difference sum is equal between two sets before we assess the effects of cardinality. However, if the total difference sum is equal between two sets, a greater cardinality implies a smaller exclusive average difference. Because a greater exclusive average difference has a positive effect on freedom of choice, a greater cardinality must have a negative effect. Thus, cardinality must be given a negative weight.

14.3.2 Representing Magnitude

We also need to select a function that represents the magnitudes of the differences among the options. I have argued that we should take all differences into account. But there are at least five measures of magnitude that take all differences into account: the *Total difference sum*, the *Total exclusive average difference*, the *Total inclusive average difference*, the *Sum of exclusive average differences* and the *Sum of inclusive average differences*. Each of these five measures may be chosen as a part of a derived measure. One may wonder if there are any particular reasons to choose one measure over the others to represent magnitude.

We may first note that the *Total difference sum* and the *Sum of inclusive average differences* have an advantage over the other measures in that both measures satisfy the *Strict monotonicity condition* (for metric space distances). Using one of these measures as part of a derived measure of

freedom of choice may make it more likely that the complex measure satisfies *Strict monotonicity* as well.

The *Total difference sum* has an advantage over the *Sum of inclusive average differences* by being a simpler measure of magnitude. Since the *Sum of inclusive average differences* equals the *Total difference sum* divided by the cardinality of options, this means that the term of cardinality would show up twice in the same measure. The *Total difference sum* also has an advantage over the *Sum of inclusive average differences* by being better known in terms of its effects on freedom of choice. If the cardinality of A is greater than the cardinality of B , while the total difference sum and the degree of evenness are equal in A and in B , then it is clear that B offers more freedom of choice than A . However, if the cardinality of A is greater than the cardinality of B , while the sum of inclusive average differences and the degree of evenness are equal in A and in B , then it is unclear whether B offers more freedom of choice than A , A offers more freedom of choice than B , or A and B offer equal freedom of choice.

The *Sum of inclusive average differences* has an advantage over the four other measures by growing proportionally to the cardinality of options, n . This property is desirable, if cardinality of options and magnitude should be considered as being equally important for sets of different sizes. The proportional growth is not conclusive in a choice between the *Total difference sum* and the *Sum of inclusive average differences*, however. The *Total difference sum* can easily be adjusted to get a function that grows proportionally with the number of options. We just use the positive square root of the total difference sum. This is what we shall do here. Magnitude is thus represented by the function:

$$\begin{aligned} & \text{The Square root of the total difference sum} = \\ & \sqrt{\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j)}, \text{ where } n = |A| \text{ and } x_1 \dots x_n \text{ are the elements of } A. \end{aligned}$$

Since the square root function is a strictly increasing function, the *Square root of the total difference sum* satisfies the *Strict monotonicity condition* as well.

14.3.3 Representing Evenness

We also have to find a suitable function to represent degrees of evenness. A problem here is not only to find a measure that represents evenness, but also to find a measure that works as part of a derived sum measure.

A major problem is that the derived measure should satisfy the *Strict monotonicity condition*. This implies that the value given to evenness should not decrease more than or equal to the combined increase of the total value

given by cardinality and magnitude with the addition of new options to a set. There are at least three methods to sustain *Strict monotonicity*. The first method is to give evenness a small weight in comparison to cardinality and magnitude. This would often be counterintuitive. The second method is to select a measure of evenness that is monotonically increasing. This would also be counterintuitive since evenness does not seem to be a monotonically increasing property. The third method is to give up the idea that evenness should enter into a derived measure as an independent factor. Instead, evenness may matter for the selection of a measure of magnitude. We may measure magnitude through the use of a strictly concave function over the distances. Strictly concave functions give smaller distances a greater weight, with the result that a greater value is rewarded to a more even distribution of a total sum of differences among individual differences. If the measure of magnitude is strictly increasing, then the *Strict monotonicity condition* is satisfied as well.

In this thesis, I shall argue for the third solution. However, I shall not argue for this solution before attempting to show that the first solution is problematic. First, I shall discuss how evenness may be represented as an independent factor.

Any measure that satisfies the *Limited evenness condition for an evenness measure* is a candidate for a measure of evenness. One candidate is to use the negative of the total sum of differences between the non-zero distances between the options in a set. Another candidate is to apply a strictly concave function on the distances, and then add those. The second measure would not be a measure of evenness alone, but of evenness and magnitude jointly. Both measures shall be discussed in this thesis; the first in this chapter, and the second in the next chapter.

We could also have considered the common statistical measures, such as the *variance*, *standard deviation*, or *mean absolute deviation*. These measures can be used as measures of evenness, if they are used in negative form. However, the standard deviation and the mean absolute deviation are measures of how much the average value (distance) in some set of values (distances) differs from the mean value (distance). Here, we are rather interested in how much each distance in a distance vector differs from each of the other distances in the same vector. All the distances in a vector obviously affect the mean distance of that vector, but all distances and the mean distance are not the same thing. Thus we shall not discuss variance, standard deviation or mean absolute deviation here.

14.4 Measuring Evenness

We shall begin by looking at a very simple measure of evenness, based on a measure of lack of evenness, the total sum of differences between distances.

The difference between the distances d_i and d_j is $|d_i - d_j|$. The total sum of differences may thus be defined as follows:

The *total sum* of differences between distances of a metric set A_d equals $\sum_{d_i, d_j \in dA} |d_i - d_j|$ for all d_i, d_j such that $d_i, d_j \neq 0$.

The total sum of differences between distances may be used as a measure of lack of evenness. Since evenness is the opposite of lack of evenness, evenness may be measured by the following measure:

Evenness measure:

$$E(A, d) = - \sum_{d_i, d_j \in dA} |d_i - d_j| \text{ for all } d_i, d_j \text{ such that } d_i, d_j \neq 0.$$

E is a function from \mathbf{R}^N to \mathbf{R} . The *Evenness measure* satisfies the *Limited evenness condition for an evenness measure*. In accordance with this condition, we compare two sets, A and B , which are alike in all respects except for the four distances d_{Aj} , d_{Ak} from A_d and d_{Bj} and d_{Bk} from B_d (and their symmetrical counterparts). It is the case that $d_{Aj} + d_{Ak} = d_{Bj} + d_{Bk}$. It is also the case that $|d_{Aj} - d_{Ak}| < |d_{Bj} - d_{Bk}|$. An appropriate evenness measure should rank A above B in terms of evenness, so that $E(A_d) > E(B_d)$. If $|d_{Aj} - d_{Ak}| < |d_{Bj} - d_{Bk}|$ then since all other distances are equal, the total sum of differences between distances is less in A than in B . Thus, the *Evenness measure* ranks A over B .

Let us look at an example here. There are two sets, A and B , each with a total difference sum of 60 and three options. A has the matrix \mathbf{M}_A :

	x	y	z
x	0	10	10
y	10	0	10
z	10	10	0

A also has the corresponding differences between distances matrix \mathbf{M}_{dA} :

	$d(x, y)$	$d(x, z)$	$d(y, x)$	$d(y, z)$	$d(z, x)$	$d(z, y)$
$d(x, y)$	0	0	0	0	0	0
$d(x, z)$	0	0	0	0	0	0
$d(y, x)$	0	0	0	0	0	0
$d(y, z)$	0	0	0	0	0	0
$d(z, x)$	0	0	0	0	0	0
$d(z, y)$	0	0	0	0	0	0

B has the matrix \mathbf{M}_B :

	u	v	w
u	0	5	15
v	5	0	10
w	15	10	0

B also has the corresponding difference between distances matrix \mathbf{M}_{dB} :

	$d(u, v)$	$d(u, w)$	$d(v, u)$	$d(v, w)$	$d(w, u)$	$d(w, v)$
$d(u, v)$	0	10	0	5	10	5
$d(u, w)$	10	0	10	5	0	5
$d(v, u)$	0	10	0	5	10	5
$d(v, w)$	5	5	5	0	5	0
$d(w, u)$	10	0	10	5	0	5
$d(w, v)$	5	5	5	0	5	0

The *Evenness measure* gives A and B the values $E(A) = 0$ and $E(B) = -150$, thus ranking A over B .

The *Evenness measure* is a valid measure in the sense that it measures the property that it is supposed to measure. Nevertheless, there are reasons to doubt the usefulness of the *Evenness measure* as a part of a derived measure of freedom of choice.

One problem with incorporating the *Evenness measure* into a derived measure is that it is not monotonically increasing. It is, in fact, weakly monotonically decreasing. Since evenness might be a weakly monotonically decreasing property, this is not a problem for the validity of the *Evenness measure*. But it implies that many derived measures incorporating the *Evenness measure* would violate the *Strict monotonicity condition*.

Another problem with incorporating the *Evenness measure* is that evenness may decrease at an excessive rate with the addition of new options. When there are n options, there are n^2 differences, and n^4 differences between differences. Among these, there are $n - 1$ choice contributing options, $n(n - 1)$ non-zero differences and $n^2(n - 1)^2$ non-zero differences between non-zero differences. If one option is added to a choice set, the non-zero differences increase by $2n$, and the non-zero differences between differences increase by $4n^3$. The *Evenness measure* does not decrease strictly monotonically, and it need not decrease at a rate of $4n^3$ either. But the mere possibility that the *Evenness measure* would decrease at a rate of $4n^3$ is sufficient to suggest a different measure of evenness for a derived sum measure of freedom of choice.

There are different ways to modify the *Evenness measure* to get a function that decreases at a slower rate. One method is to divide the *Difference between distances measure* by n^3 . This method is analogous to the way in which the *Total difference sum* can be modified to get the *Sum of*

inclusive average differences. Another method is to use the positive fourth root of the total sum of differences between differences. This method is analogous to how the *Total difference sum* is modified to get the *Square root of the total difference sum*. Since I decided to use the *Square root of the total difference sum* rather than the *Sum of inclusive average differences*, I shall be consistent and use the fourth root rather than division by n^3 here. Thus we have the following measure of evenness:

The Fourth root evenness measure:

i) For $|A| = 1$, $E(A, d) = 0$.

ii) For $|A| \geq 2$, $E(A, d) = -\sqrt[4]{\sum_{d_i, d_j \in dA} |d_i - d_j|}$ for all d_i, d_j such that $d_i, d_j \neq 0$.

E is a function from \mathbf{R}^N to \mathbf{R} .

In the examples above, where the *Difference between distances measure* gave $E(A) = 0$ and $E(B) = -150$, the *Fourth root evenness measure* gives $E(A) = 0$ and $E(B) \approx -3.5$.

14.5 The CSF Sum Measures

Having selected three functions to represent cardinality, magnitude and evenness, the *Cardinality of choice contributing options*, the *Square root of the total difference sum* and the *Fourth root evenness measure*, we get the following derived measures to discuss:

The CSF sum measures:

$$F(A, d) = -\alpha(n-1) + \beta \sqrt{\sum_{d_i \in dA} d_i} - \gamma_4 \sqrt[4]{\sum_{d_i, d_j \in dA} |d_i - d_j|}.$$

F is a function from a finite metric space to \mathbf{R} , $n = |A|$, and α , β and γ are weights.

We have to investigate how the three properties contribute to freedom of choice relative to one another. Part of the investigation is already made. We have investigated the effect on freedom of choice of one property, while keeping the others fixed. From this investigation we concluded that cardinality must be given a negative weight, if the measures should satisfy the *Spread condition*. The next step consists in investigating how the three properties should be weighted relative to one another. What are the appropriate values for α , β and γ ? Is there any set of weights that ensures that the measure satisfies all conditions for a measure of freedom of choice?

We shall consider the problems involved in selecting weights so that the measure satisfies the *Strict monotonicity condition*. Obviously, the unweighted *CSF measure* does not satisfy the condition since an additional option may decrease the value of the measure. (For example when 1 h is added to the set {0 h, 2 h} and the freedom of choice value decreases from 1 to roughly -1.2.) So we must consider some other version.

Let us first suppose that evenness is maximal so that the *Fourth root evenness measure* gives a value of 0. We shall then compare the cardinality of the choice contributing options and the square root of the total difference sum. To satisfy *Strict monotonicity* the square root of the total difference sum must increase above α when cardinality increases by 1. So, what is the smallest possible increase of the square root of the total difference sum when cardinality increases by 1? The smallest increase occurs when a set of one option $\{x\}$ is expanded to a set of two options $\{x, y\}$. (Actually, the smallest increase occurs when the empty set is expanded to a singleton set since there is no increase of the total difference sum at all. But since none of these sets are diverse, we may ignore this case in this context.)

Let us next suppose that the option y that is added to $\{x\}$ is very similar to x . The distance is only ϵ . The total difference sum of $\{x\}$ is 0, while the total difference sum of $\{x, y\}$ is 2ϵ . The square root of the total difference sum of $\{x\}$ is 0, while the square root of the total difference sum of $\{x, y\}$ is $\sqrt{2\epsilon}$. There is thus an increase of the square root of the total difference sum of $\sqrt{2\epsilon}$. At the same time, the cardinality of choice contributing options increases from 0 to 1. For the *CSF sum measure* to satisfy *Strict monotonicity*, it must be the case that $\beta\sqrt{2\epsilon} > \alpha$. If the square root of the total difference sum is given a weight of $\beta = 1$, then $\sqrt{2\epsilon} > \alpha$. This means that cardinality must be given an extremely low weight in comparison to the square root of the total difference sum. In fact, cardinality can hardly matter at all.

We may consider a similar example of selecting weights for magnitude and evenness so that the chosen *CSF sum measure* satisfies the *Strict monotonicity condition*. Since we have just concluded that cardinality must be given a very small weight in comparison to the total difference sum, we shall ignore the cardinality term here. We shall just consider comparisons involving the smallest increase in the square root of the total difference sum that leads to a decrease in degrees of evenness. This occurs when a set of two options, $C = \{x, y\}$, is expanded to a set of three options, $C \cup \{z\} = \{x, y, z\}$. Let us suppose that this is the matrix for C :

	x	y
x	0	τ
y	τ	0

This is the matrix for $C \cup \{z\}$:

	x	z	y
x	0	ε	τ
z	ε	0	$\tau - \varepsilon$
y	τ	$\tau - \varepsilon$	0

Both τ and ε are very small distances; $\tau < 1$, $\varepsilon < 1$, but $\tau > \varepsilon$. The increase of the square root of the total difference sum is equal to $\sqrt{4\tau} - \sqrt{2\tau}$. The decrease of evenness is equal to $\sqrt[4]{16\tau - 16\varepsilon}$. For the *Strict monotonicity condition* to hold, it must be the case that $\beta(\sqrt{4\tau} - \sqrt{2\tau}) > \gamma\sqrt[4]{16\tau - 16\varepsilon}$. Thus $\beta(\sqrt{\tau} - \sqrt{2\tau}/2) > \gamma\sqrt[4]{\tau - \varepsilon}$. For numbers < 1 , such as τ , it is the case that $\sqrt{\tau} < \sqrt[4]{\tau}$. The smaller the number τ , the bigger the difference between $\sqrt{\tau}$ and $\sqrt[4]{\tau}$. For $\beta(\sqrt{\tau} - \sqrt{2\tau}/2) > \gamma\sqrt[4]{\tau - \varepsilon}$ to hold, the weight of β must be much greater than the weight of γ . So, evenness must be given an extremely low weight in comparison to the square root of the total difference sum. Alternatively, distances below 1 cannot be used. In either case, it turns out to be problematic to combine the *Total difference sum* with another term in a derived sum measure of freedom of choice.

We may try all manners of techniques to construct an adequate derived sum measure of freedom of choice. We may try different measures of evenness. We may try different weights. We may also try different terms. From the above discussion, it should at least be clear that it is very difficult to construct a derived sum measure of freedom of choice. Therefore, we shall continue to examine other ways to measure freedom of choice here. First, we shall consider derived product measures.

14.6 Three-Factor Measures

So far we have considered additively separable measures, or sum measures. We shall go on to consider multiplicatively separable measures, or product measures. Assuming that freedom of choice is a function of the three factors of cardinality, magnitude and evenness, the general form for a product measure is the following:

$$\textit{Three-Factor Measures: } F(A, d) = C(A) \times M(A) \times E(A).$$

F is a function from a finite metric space to \mathbf{R} , C represents the cardinality of A , M is a measure of magnitude and E is a measure of evenness.

There is no point in putting any weights on the factors since $\alpha C(A) \times \beta M(A) \times \gamma E(A) = \alpha C(A) \times \gamma M(A) \times \beta E(A)$ and so on.

The above measures do not capture that cardinality has a negative effect on freedom of choice. The factor of cardinality should occur as the denominator in a ratio measure. Thus, we get the following form:

Three-factor ratio measures:

$$F(A, d) = \frac{\alpha(M \times E)}{\beta C}.$$

F is a function from a finite metric space to \mathbf{R} , and α and β are weights.

We cannot use the same functions as factors here that we used as terms for the *Three-term measures*. One problem is that the *Evenness measure* assigns 0 to maximally even sets. This implies that all maximally even sets are assigned 0 by a *Three-factor ratio measure*. This is unacceptable for many reasons. It violates the *Strict monotonicity condition*, the *Option dominance condition* and the *Spread condition*, for example. We could use the *Lack of evenness measure* instead, occurring as a factor in the denominator. However, this would be problematic as well. The *Lack of evenness measure* assigns 0 to maximally even sets. Since the *Lack of evenness measure* occurs in the denominator and division by 0 is undefined, this implies that all maximally even sets would be assigned an undefined degree of freedom of choice. This is also unacceptable.

There are just two things we can do here, remove the factor of evenness altogether, or represent evenness in some other way. Gustafsson argues that evenness cannot matter to freedom of choice (2011: 49–52). If he is right we can remove the factor of evenness and get:

$$\textit{Two-factor ratio measures: } F(A, d) = \frac{\alpha M}{\beta C}.$$

F is a function from a finite metric space to \mathbf{R} .

The cardinality factor can be represented by the *Cardinality of choice contributing options*. Since the measure of freedom of choice should grow roughly at the same rate as the number of options to satisfy the *Limited growth condition*, the magnitude factor may be represented by the *Total difference sum*. We then have:

$$\textit{The CT ratio measures: } F(A, d) = \frac{\alpha \sum_{i=1}^N d_i}{\beta(n-1)}.$$

F is a function from a finite metric space to \mathbf{R} .

If we put $\alpha = 1$ and $\beta = 1$, we get an unweighted *CT ratio measure*. This is, in fact, a measure that we recognize; it is the *Exclusive ratio measure*. This measure was assessed in a previous chapter, with the conclusion that it does not satisfy the *Strict monotonicity condition*. However, the similar *Inclusive ratio measure* satisfies the *Strict monotonicity condition* for metric space distances. It also satisfies the *Domain-insensitivity condition*, the *No freedom of choice condition*, the *Limited growth condition*, the *Option dominance condition*, the *Symmetry condition* and the *Spread condition* (the first fact is not obvious, so a proof is included in the appendix). But, obviously, it does not satisfy the *Limited evenness condition*.

I have previously argued for the acceptance of the *Limited evenness condition*. But I have not argued for the *Limited evenness condition* against Gustafsson's argument against the evenness ideal. We shall thus consider Gustafsson's argument next.

14.7 Defending Evenness

Gustafsson argues that we should give up the idea that evenness matters for freedom of choice (2011: 49–52). The argument is that evenness cannot matter, as there is a conflict between evenness and monotonicity. When an option is added to a set, the degree of evenness may decrease. Gustafsson gives the following comparisons of sets: $B = \{0 \text{ h}, 15 \text{ h}, 30 \text{ h}\}$ and $B \cup \{1 \text{ h}\} = C = \{0 \text{ h}, 1 \text{ h}, 15 \text{ h}, 30 \text{ h}\}$. (The example is changed slightly.) According to the *Strict monotonicity condition*, C offers more freedom of choice than B . Then he says that, with respect to evenness, C is much worse than B . But if evenness is an important feature for freedom of choice, then the great loss of evenness from B to C should outweigh the very minor respect in which C improves on the freedom of choice offered by B . So if monotonicity holds, evenness cannot contribute very much to freedom of choice (2011: 50–51).

Gustafsson may be correct in his judgment that evenness decreases when adding 1 h to B . (Either it decreases or the two sets are incomparable.) If evenness is measured by the *Evenness measure*, for example, evenness decreases from $E(B) = -240$ to $E(B \cup \{1 \text{ h}\}) = -1520$. But Gustafsson's conclusion is premature. The argument does not show that evenness cannot contribute very much to freedom of choice. The argument rather shows that there is a particular way in which evenness cannot contribute much to freedom of choice. This is the way we have investigated in this chapter. Here, evenness is measured as an independent factor, which is assessed with regard only to the total difference sum, and represented by a single value. The value derives from an evenness measure where differences between distances are aggregated under the assumption that the differences matter collectively to freedom of choice. The single value should be summed or multiplied by other values representing other independent factors in a

derived measure of freedom of choice. The result is a total sum or product of values, which is supposed to represent degrees of freedom of choice.

However, there is another way for evenness to contribute to freedom of choice. First, the differences between distances need not matter collectively in the form of a single value. Instead, they may individually affect freedom of choice. Any pair of distances can contribute more to freedom of choice if their differences are smaller. Second, evenness need not contribute independently to freedom of choice. Instead evenness may contribute together with magnitude. The two factors may be measured jointly by the use of a strictly concave function on the distances. This function gives a positive weight for smaller distances, with the result that evenness is rewarded. This function may also be strictly increasing and satisfy the *Strict monotonicity condition*.

In the example above, the set B may be assigned some particular freedom of choice value, which is dependent both on the magnitudes of the distances, and the fact that the distribution of distances is rather close to being even. Had the distances been closer to being even, the resulting set B^* would have been assigned a greater value. The subset of C , which is B , is assigned the same value as B alone, for the same reasons. In addition, C is assigned an extra value for the additional option of 1 h, which also depends both on the magnitude of distances and their degree of evenness. Had the extra option in C been positioned more ideally, the resulting set C^* would have been assigned a greater value. There is thus no loss of freedom of choice due to the loss of evenness from B to C . There is only a loss in terms of the potential for maximizing freedom of choice. The completely even set C^* would have offered more freedom of choice than C . It is in this sense that evenness may matter for freedom of choice.

Chapter 15: Additively Separable Strictly Concave Measures

In the chapter on magnitude, we considered some measures that were functions of the distance function. Such functions could not represent magnitude and evenness jointly. In this chapter, we shall consider measures that can. They are functions of some function of the distance function. More precisely, they are functions of some strictly concave function of the distance function.

We shall begin by adapting the previously accepted *Ratio scale-independence condition* for measures that are functions of some function of the distance function. What is important for these functions is that they should be independent of the scale that is used to measure distances. We may define:

The Distance ratio scale-independence condition: For a measure of freedom of choice F that is a function of the distance function d , and any metric choice set $A_d \in P(X_d)$, it holds that for all numbers $K > 0$, there is a number $L = L(K) > 0$, where L is an increasing function of K such that if d is multiplied by K , then $F(A)$ is multiplied by L .

This condition is just a special case of the previously accepted *Ratio scale-independence condition*. It assures that when sets are compared in terms of freedom of choice, the same ratio relations hold, no matter which distance function is used to measure the differences between the options.

We shall now look at a class of measures that represent magnitude and evenness jointly. Such measures may be based on the *Total difference sum measure*. What is needed is to add a strictly increasing strictly concave measure on the differences. We then get a class of *additively separable strictly concave functions*. They are functions of the following form:

$$G(\mathbf{x}) = \sum_{i=1}^n f(x_i).$$

G is a function from \mathbf{R}^n to \mathbf{R} , \mathbf{x} is a vector of real numbers x_i , and f is a strictly increasing strictly concave function from \mathbf{R} to \mathbf{R} . Strict concavity is defined as follows:

A function f is *strictly concave* if and only if $f(kx + (1 - k)y) > kf(x) + (1 - k)f(y)$ holds for every k in $(0, 1)$ where $x \neq y$.

If an *additively separable strictly increasing strictly concave function* is applied to differences, we get the following class of measures for freedom of choice:

Separable strictly concave measures:

$$F(A, d) = \sum_{i=1}^n \sum_{j=1}^n f(d(x_i, x_j)).$$

F is a function from a finite metric space to \mathbf{R} and f is a strictly increasing strictly concave function from \mathbf{R} to \mathbf{R} .

The *Separable strictly concave measures* belong to a larger class of traditional diversity measures. For example, Kreutz-Delgado and Rao propose the use of separable concave functions as measures of diversity (1999). The measures are also related to the class of Schur-concave functions that are argued to be the only appropriate measures of diversity by, for example, Kempton (1979) and again by Kreutz-Delgado and Rao (1999). Since Schur-concavity is a property of *majorization* and of some biological measures of diversity, such as Shannon's index, there is a long tradition of using Schur-concave functions as measures of diversity.

The *Separable strictly concave measures* satisfy the *Limited evenness condition*. A proof of this is included in the appendix. Since the *Limited evenness condition* differs from the *Limited evenness condition for an evenness measure* only in terms of application, the measures also satisfy the *Limited evenness condition for an evenness measure*. In this sense they may be used as measures of evenness. However, they are not really suitable as measures of evenness alone. This is because the measures are affected not only by the evenness of the distances, but also by the magnitudes of the distances. Evenness is thus not measured as an independent factor. Magnitude is not either. Rather, evenness and magnitude are awarded for every sum of distances.

As far as capturing the factors of magnitude and evenness jointly the *Separable strictly concave measures* are quite successful. Not only do they satisfy the *Limited evenness condition*, but they also satisfy the conditions of *Domain-insensitivity*, *Strict monotonicity*, *No freedom of choice*, *Option dominance*, *Evenness* and *Symmetry*. The *Domain-insensitivity condition* is satisfied because the measures are only functions of the pairwise distances

between the options in a set. The *Strict monotonicity condition* and the *Option dominance condition* are satisfied because the measures are strictly increasing. The *No freedom of choice condition* is satisfied because the empty set and all singleton sets get a freedom of choice value of 0. The *Symmetry condition* is satisfied because the measures are separable sum functions.

15.1 Log Measure

The *Separable strictly concave measures* may initially seem promising as measures of freedom of choice since they satisfy seven important conditions for a measure of freedom of choice. But we have already accepted nine conditions. A first step towards constructing a more adequate measure of freedom of choice is to eliminate any *Separable strictly concave measures* that fail to satisfy the *Distance ratio scale-independence condition*. It is interesting to note that a type of measure based on the logarithmic function falls in this class, namely the following:

Log measure:

$$F(A, d) = \sum_{i=1}^n \sum_{j=1}^n \log(d(x_i, x_j)).$$

F is a function from a finite metric space to \mathbf{R} .

Measures based on the logarithmic (and at times the natural logarithmic) function are popular in the context of measuring diversity. The *log*- and *ln*-functions occur as a part of several diversity measures, such as Shannon's diversity index (1948), and Rényi's index (1961). The log-measure also occurs in Dalton's measure of inequality (1920). Erlander explicitly proposes that Shannon's diversity index should be used as a measure of freedom of choice (2010). Also, Gabor and Gabor propose a measure involving the logarithmic function as a measure of the diversity factor of freedom (1954).

Let us just look at an example where the *Log measure* fails to satisfy the *Distance ratio scale-independence condition* (and in fact, the *Ordinal distance scale-independence condition* as well). We shall compare two sets, A and B , in terms of their freedom of choice. A contains four options at a distance of 2 from one another. B contains three options at a distance of 5 from one another. According to the *Log measure*, $F(A) = 12 \log 2 \approx 3.6$, while $F(B) = 6 \log 5 \approx 4.2$. Since $12 \log 2 < 6 \log 5$, A is ranked as more diverse than B . Next we change scales. All distances are multiplied by 2. According to the *Log measure*, $F(A) = 12 \log 4 \approx 7.2$, while $F(B) = 6 \log 10 = 6$. Since $12 \log 4 > 6 \log 10$, B is now ranked as more diverse than A . But

there have not been any changes made to the sets between the two comparisons. We only changed the scale of measurement. So the *Log measure* does not work as a measure of freedom of choice.

15.2 Root Measures

However, there is a subclass of the *Separable strictly concave measures* that satisfy the *Distance ratio scale-independence condition*. They are functions of the following form:

Root measures:

$$F(A, d) = \sum_{i=1}^n \sum_{j=1}^n (d(x_i, x_j))^r \quad \text{with } 0 < r < 1 \text{ and } n \geq 2.$$

F is a function from a finite metric space to \mathbf{R} .

Not only do the *Root measures* satisfy the *Distance ratio scale-independence condition*; they also satisfy the six conditions of *Domain-insensitivity*, *Strict monotonicity*, *No freedom of choice*, *Dominance*, *Evenness* and *Symmetry*. However, they do not satisfy the *Spread condition* and the *Limited growth condition*. We shall discuss these problems next.

The first problem for the *Root measures* (and the more general class of *Separable concave measures*) is that the measures may give counterintuitive rankings of sets with equal total difference sum but different cardinality. Any strictly concave function has the property of *strict subadditivity*, a property which is defined as follows:

A function f is *strictly subadditive* if and only if $f(d_1 + d_2) < f(d_1) + f(d_2)$ holds for all $d_i > 0$.

For the purpose of measuring freedom of choice, this means that, for a fixed sum of distances, it is possible to obtain a greater value for F by splitting the sum of distances between more individual distances. The *Strict subadditivity* property is not generally problematic for a measure of freedom of choice. If there is a distance vector, $\mathbf{d}_A = (30, 30, 15, 15, 15, 15, 0, 0, 0)$, then it is not problematic that one distance of 30 contributes less to the freedom of choice offered by A than the two distances of 15. *Strict subadditivity* is only problematic for comparisons of sets of different cardinality. If there is an equal total sum of distances for some sets, then the application of a *Root measure* implies that more freedom of choice is offered by a set with more options and smaller individual distances. In other words, the *Root measures* do not satisfy the *Spread condition*.

Another problem for the *Root measures* is the excessive growth of the measures when an option is added to a set. The *Root measures* grow roughly at the same excessive rate as the *Total difference sum measure*. The measures do not satisfy the *Limited growth condition* either. For these reasons we should find another measure of freedom of choice.

Chapter 16: Ratio Root Measures

Let us return to the derived measures that we considered previously, the following measures:

Three-factor ratio measures:

$$F(A, d) = \frac{\alpha(M \times E)}{\beta C} .$$

F is a function from a finite metric space to \mathbf{R} , C represents the cardinality of A , M is a measure of magnitude, E is a measure of evenness, and α and β are weights.

Since magnitude and evenness may be represented jointly, we may attempt the following derived measure instead:

Two-factor ratio measures:

$$F(A, d) = \frac{\alpha(R)}{\beta C} .$$

F is a function from a finite metric space to \mathbf{R} , R is the *Root measures*, C represents the *cardinality of options*, and α and β are weights.

In other words, we may combine the *Root measures* with the *Ratio measures* so that the combined measures do not have the problematic properties of the original ones. The *Root measures* satisfy the conditions of *Domain-insensitivity*, *Distance ratio scale-independence*, *Strict monotonicity*, *No freedom of choice*, *Option dominance*, *Evenness* and *Symmetry*, but not the conditions of *Spread* and *Limited growth*. The *Ratio measures* satisfy all conditions, apart from *Evenness* (and *Strict monotonicity*, for the exclusive one). A combined measure can satisfy all nine conditions. We just have to make some adjustments.

First, we should choose $n - 1$ in the denominator, rather than n because this version has nicer properties (as explained below). To get the measures to satisfy the *Spread condition*, we must set some limits for the root function. The function must be raised to the r th power, where $\frac{1}{2} \leq r < 1$. If we would choose a greater r , then the function would no longer be strictly concave, and if we would choose a smaller r , then the function would fail to satisfy the *Spread condition*. (If we had chosen n , instead of $n - 1$, in the denominator,

then r would have to have been chosen within a smaller range to satisfy the *Spread condition*.) Last, we should add that sets of less than two options offer no freedom of choice. When all these adjustments are done, we end up with the following class of measures of freedom of choice:

Ratio root measures:

i) For $n \leq 1$, $F(A, d) = 0$.

ii) For $n > 1$, $F(A, d) = \frac{1}{n-1} \sum_{i=1, i \neq j}^n \sum_{j=1}^n (d(x_i, x_j))^r$, with $\frac{1}{2} \leq r < 1$.

F is a function from a finite metric space to \mathbf{R} .

The *Ratio root measures* satisfy the nine conditions of *Distance ratio scale-independence*, *Strict monotonicity*, *Domain-insensitivity*, *No freedom of choice*, *Option dominance*, *Evenness*, *Symmetry*, *Spread* and *Limited growth*. The proofs are included in the appendix.

When we have selected a distance function to represent differences, any *Ratio root measure* would also give us a unit for freedom of choice. For example, for $r = \frac{1}{2}$, the *Ratio square root measure* gives that one unit of freedom of choice equals the freedom of choice offered by a choice set of two options that differ from one another to a degree that corresponds to a distance of 0.25. Any value given to a choice set by a *Ratio square root measure* shows how much freedom of choice that is offered by the set in comparison to a choice set of two options at a distance of 0.25. For example, a choice set given a value of 4 should offer four times as much freedom of choice as two options that differ by 0.25, which are given a value of 1.

To give an example of how the *Ratio root measures* work we may look at some examples similar to those previously considered in the thesis. First example: who has more freedom of choice when it comes to accepting a certain number of teaching hours per week, a PhD-student or a Professor? The PhD-student has the choice set $A = \{20 \text{ h}, 30 \text{ h}, 40 \text{ h}\}$, while the Professor has the choice set $B = \{0 \text{ h}, 40 \text{ h}\}$. The distance vector $\mathbf{d}_A = (20, 20, 10, 10, 10, 10, 0, 0, 0)$ and the distance vector $\mathbf{d}_B = \{40, 40, 0, 0\}$. If we use the *Ratio root measures* and choose $r = \frac{1}{2}$, then $F(A) \approx 10.8$ and $F(B) \approx 12.6$. B is thus ordered above A . Other choices of r result in the same ordering. Thus, the Professor has more freedom of choice regarding paid working hours than the PhD-student. Furthermore, the Professor has more than 12 times as much freedom of choice as he would have had, if he had been offered the choice set $C = \{0 \text{ h}, 15 \text{ min}\}$. This is because $F(C) = 1$. Second example: which university has more freedom of choice when it comes to renting suitable buildings, Stockholm University or Uppsala University? Stockholm University has choice set G , which offers four buildings of the same length and breadth of 10 m, but with different heights, one with a height of 10 m, another with a height of 22 m, a third with a

height of 34 m, and a fourth with a height of 46 m. The distance vector $\mathbf{d}_G = (36, 36, 24, 24, 24, 24, 12, 12, 12, 12, 12, 12, 0, 0, 0, 0)$. Uppsala University has a choice set H , which also offers four buildings, one of 10 m in length, breadth and height, another of 10 m in length, 23 m in breadth, and roughly 17 m in height, a third of 30 m in length, 23 m in breadth, and roughly 17 m in height, and a fourth of 30 m in length and 10 m in breadth and height. The distance vector $\mathbf{d}_H = (25, 25, 25, 25, 20, 20, 20, 20, 15, 15, 15, 15, 0, 0, 0, 0)$. If we use the *Ratio root measures* and choose $r = \frac{1}{2}$, then $F(G) \approx 17.5$ and $F(H) \approx 17.8$. H is thus ordered above G . Other choices of r result in the same ordering. Thus, Uppsala University has more freedom of choice regarding buildings than Stockholm University.

16.1 Proportional Growth

The *Ratio root measures* satisfy nine reasonable conditions for a measure of freedom of choice. We may wonder if these measures are the only ones that satisfy the conditions. It turns out that it cannot be shown that the *Ratio root measures* are the only measures that satisfy the conditions, but it can be shown that they are the only measures that satisfy the conditions given two extra assumptions.

The first assumption is that a measure of freedom of choice should be an analytic function with non-zero partial derivatives with respect to some function of the distances. This assumption is made only for the sake of mathematical simplicity. Without the assumption, other measures might satisfy the conditions. However, these measures would have many more terms. They would also be closely related to the *Ratio root measures* (see last note of the appendix).

The second assumption is that the measures of freedom of choice should satisfy an additional condition that may be called the *Proportional growth condition*. The condition is less independently convincing than the nine previous conditions since its reasonableness depends on the choice of distance function. It is similar to the *Limited growth condition*, but more precise.

The Proportional growth condition: For a measure of freedom of choice F , and any metric choice set $A_d \in P(X_d)$ such that $|A| \geq 2$, if $|A| = n$ and each non-zero distance $d_{A_i} = 1$, then $F(A) = n$.

This condition states that the freedom of choice offered by a set equals the cardinality of the set when all the distances between the options are 1.

For the condition to be at all reasonable we must select the distance function in such a way that it seems correct to say that the cardinality of options equals the degree of freedom of choice when all the distances are 1.

For example, a set of four options at a distance of 1 should offer twice as much freedom of choice as a set of two options at a distance of 1, and so on.

Even after we have selected an appropriate distance function, the condition may still seem odd. After all, it does not matter what exact numerical value is assigned to a choice set when the number of options is n and the distances between all pairs of options are 1, as long as the correct ratios between the degrees of freedom of choice offered by different sets is preserved. But since the exact numerical value does not matter, it also does not matter that a set is assigned the number n when the number of options is n and their distances are 1. So the condition is acceptable.

If the *Proportional growth condition* is acceptable, together with the other nine conditions, then the *Ratio root measures* should be acceptable as well. The proof that the *Ratio root measures* are the only analytical functions with non-zero partial derivatives with respect to some function of the distances that satisfy the ten conditions is included in the appendix.

16.2 Problems

We have found a class of measures that satisfies several reasonable conditions for a measure of freedom of choice. But before recommending the *Ratio root measures* as measures of freedom of choice, there are some objections that should be considered. If the objections against the *Ratio root measures* are serious, this could mean that at least one of the conditions is flawed.

16.2.1 Too Many Measures

The first objection is due to Martin Peterson (in conversation). The objection is that since the *Ratio root measures* are not one measure but a class of measures, different choices of r may result in different orderings of choice sets. This is a problem if we want an exact understanding of the concept of freedom of choice. The ideal solution would be to discover additional reasonable conditions that further restrict the value of r , until there is only one r that is permissible. The worst solution would be to choose some value of r without any particular reason. Here I shall propose a solution that is neither ideal, nor arbitrary. The proposal is that we should choose $r = \frac{1}{2}$. One reason to choose $r = \frac{1}{2}$ is that it gives us the familiar square root function. Another reason is that this is the most concave function that is allowed. The reason that we would wish to choose a more concave function is because such a function gives greater weights to smaller distances. This would get the measure closer (although not all the way) to satisfying an intuition that we should consider next.

16.2.2 Overestimating the Importance of Similar Options

The second objection is due to Johan E. Gustafsson (2011). The objection is that the measure exaggerates the importance of very similar options for freedom of choice. Let us look at Gustafsson's example (slightly changed). We have $A = \{0 \text{ h}, 15 \text{ h}, 30 \text{ h}\}$ and $B = \{0 \text{ h}, 30 \text{ h}\}$. The set $B \cup \{15 \text{ h}\} = A$. All the *Ratio root measures* order A over B . If we choose $r = \frac{1}{2}$, then $F(A) \approx 13.2$ and $F(B) \approx 11$. Next we compare A to a set similar to B , the set $B \cup \{1 \text{ h}, 29 \text{ h}\} = C = \{0 \text{ h}, 1 \text{ h}, 29 \text{ h}, 30 \text{ h}\}$. This time all the *Ratio root measures* order C over A , for example, for $r = \frac{1}{2}$, $F(C) \approx 15.7$. This seems intuitively wrong (2011: 51–52).

It is easy to see what has gone wrong; all the additional differences between the options in C , as compared to B , have obscured the fact that the additional two options included in C are very similar to two of the other options in C . The similarity objection arises because we judge the minimum differences between the options in a set as more important for the assessment of freedom of choice than the other differences. In the case of C , we judge the contribution of the option 1 h to C as small due to the small difference between 1 h and 0 h, ignoring the larger difference between 1 h and 29 h.

In general, it seems that when an option y is added to a set A , and y is very similar to an option in A , this makes us judge the choice contribution of y to $A \cup \{y\}$ as very small, even though y may be very different from the other options in $A \cup \{y\}$. If 1 h is added to the set $\{0 \text{ h}, 30 \text{ h}\}$, this seems like a small contribution to the freedom of choice offered by $\{0 \text{ h}, 1 \text{ h}, 30 \text{ h}\}$, because of the similarity between 0 h and 1 h. The distance of 29 between 1 h and 30 h does not seem to matter. However, if we compare this case with cases where options are added that are further from 0 h and closer to 30 h, the distance between the new option and 30 h would seem increasingly important for judging to what degree the new option contributes to freedom of choice (the contribution seems to get greater up to 15 h, and then smaller).

The *Ratio root measures* roughly capture this intuition. They are strictly concave functions and such functions reward evenness by putting a greater weight on smaller distances. Furthermore, the *Ratio square root measure* captures this intuition as far as possible since it puts the greatest weight on smaller distances among all the *Ratio root measures*. However, none of the *Ratio root measures* reflect the intuition to the extreme that it is only the smaller distances that matter.

Even though it is easy to state the intuition, it is difficult to formulate it in terms of an acceptable general condition. The intuition is that an option that is very similar to another option in the same set should not count as a great contribution to freedom of choice. But what type of condition would assure this result? Certainly we should not accept a condition that states that very similar options should count as one option, because that means that a contribution to freedom of choice (although small) is ignored. Neither should

we accept a condition that states that only minimum distances are relevant for freedom of choice since relevant information might then be ignored. Perhaps we should accept a condition that says that smaller distances should be given a much greater weight than larger distances (much greater than the *Ratio root measures* give). But this condition might lead to the ranking of a dominated set over a dominating set. This runs counter to the *Option dominance condition*. Perhaps we should not regard freedom of choice as a function of the differences between the options in a set at all. But then it is unclear of what freedom of choice is a function.

I have one proposal for a general condition that would capture the similarity objection. It is the following condition:

The Extreme limited growth condition: For a measure of freedom of choice F , any metric choice set $A_d \in P(X_d)$ and any option $y \in X - A$, it holds that $F((A \cup \{y\})) \leq F(A) + F(\{x_i, y\})$, where $d(x_i, y) = \min_k d(x_k, y)$ and $x_i, x_k \in A$.

This condition is similar to the *Limited growth condition*, but restricts the growth of the function further than that condition. It says that whatever value the function gives to the set of y and the closest option in A should be the maximal value that the addition of y can add to the freedom of choice value of A .

It is important to note that one must be careful when applying the condition. The condition must be tested against all options included in a choice set. Otherwise one might get the impression that there are different demands on the growth of the function, depending on the order in which options are added to a set. As an example, we may consider the set $A = \{0 \text{ h}, 10 \text{ h}, 40 \text{ h}\}$. In one case, A is created by adding $\{10 \text{ h}\}$ to $\{0 \text{ h}, 40 \text{ h}\}$. In another case, A is created by adding $\{40 \text{ h}\}$ to $\{0 \text{ h}, 10 \text{ h}\}$. In the first case, the condition implies that $F(A) \leq F(\{0 \text{ h}, 40 \text{ h}\}) + F(\{0 \text{ h}, 10 \text{ h}\})$. In the second case, the condition implies that $F(A) \leq F(\{0 \text{ h}, 10 \text{ h}\}) + F(\{10 \text{ h}, 40 \text{ h}\})$. Since $F(\{0 \text{ h}, 10 \text{ h}\}) = F(\{0 \text{ h}, 10 \text{ h}\})$ and $F(\{0 \text{ h}, 10 \text{ h}\}) < F(\{0 \text{ h}, 40 \text{ h}\})$, the condition states different requirements regarding the freedom of choice value for the same set. This does not mean that the condition is inconsistent. It just means that it is slightly difficult to apply.

As it stands, the condition does not seem unreasonable. It is not in conflict with any one of the ten accepted conditions. It assures that a measure that would satisfy the condition would not overestimate the importance of similar options. However, the *Ratio root measures* do not satisfy the condition. This implies that a measure that would satisfy the *Extreme limited growth condition* would not satisfy some subset of the set of ten conditions that the *Ratio root measures* uniquely satisfy.

But which conditions could we sacrifice to accept the *Extreme limited growth condition*? Some measures that we have discussed satisfy the

Extreme limited growth condition since they fail to satisfy the *Option dominance condition*. For some measures, such as the *Cardinality measure* and the *Attribute measure*, this is due to the measures not aggregating distances at all. This does not seem like an acceptable solution. For other measures, such as the *Minimal distances measure*, the *Minimal path measure* and the *Elimination measure*, this is due to the measures not aggregating all distances. Since these measures also fail to satisfy other reasonable conditions, this may not be an acceptable solution either. If we want to keep the *Option dominance condition*, we could use a measure that aggregates all distances and then divide them by some large number. But such a measure would probably fail to satisfy the *Strict monotonicity condition*.

There may be a measure that satisfies the *Extreme limited growth condition* without failing to satisfy the most important of the ten accepted conditions. However, before such a measure is found, it seems better to accept the *Ratio root measures* rather than to accept the *Extreme limited growth condition*.

16.2.3 Underestimating the Importance of Evenness

The third objection is also due to Gustafsson and based on the same example (2011: 52). We again compare $A = \{0 \text{ h}, 15 \text{ h}, 30 \text{ h}\}$ and $C = \{0 \text{ h}, 1 \text{ h}, 29 \text{ h}, 30 \text{ h}\}$. We choose $r = \frac{1}{2}$. The measure gives A and C the values $F(A) \approx 13.2$ and $F(C) \approx 15.7$, and thus C is ranked over A . Other *Ratio root measures* give the same ranking. Gustafsson's objection to this ranking is that A should be ranked over C since the options in A are more evenly distributed than the options in C (given that we should care about evenness at all, which Gustafsson doubts).

If evenness is measured by the *Evenness measure*, A gets a value of -240 and C gets a value of -1840 . So A is rightly judged as more even than C . A perfectly evenly distributed set with the same total difference sum as A , the set A^* , would get a value of $F(A^*) \approx 13.4$ by the *Ratio square root measure*, while a perfectly evenly distributed set with same total difference sum as C , the set C^* , would get a value of $F(C^*) \approx 17.7$, by the same measure. The loss of potential evenness value for A is roughly 0.2, while the loss of potential evenness value for C is roughly 2. So A is, rightly, punished less for a loss of evenness than is C .

A problem with the *Ratio root measures* is that they seem to overvalue the options 1 h and 29 h. The punishment for the lack of evenness of C is not sufficiently severe to rank A over C . But this is hardly due to the measures underestimating evenness. It is rather due to the measures overestimating the importance of similar options. This leads us back to the previous problem, a problem that I could not solve.

16.2.4 Ignoring the Importance of Equal Intervals

The fourth objection is that the *Ratio root measures* do not capture the importance of options being distributed at equal intervals when they are distributed in one dimension. (Such options are generally called a *string*, for example by Van Hees (2004: 257).)

Let us look at an example. We shall compare sets of working hours. $A = \{0 \text{ h}, 15 \text{ h}, 30 \text{ h}, 45 \text{ h}\}$ and $B = \{0 \text{ h}, 12 \text{ h}, 33 \text{ h}, 45 \text{ h}\}$. A has the matrix:

	0 h	15 h	30 h	45 h
0 h	0	15	30	45
15 h	15	0	15	30
30 h	30	15	0	15
45 h	45	30	15	0

B has the matrix:

	0 h	12 h	33 h	45 h
0 h	0	12	33	45
12 h	12	0	21	33
33 h	33	21	0	12
45 h	45	33	12	0

The two sets do not have the same total difference sum since the total difference sum of A is 300, while the total difference sum of B is 312. The degree of evenness, according to the *Evenness measure*, is -1680 for A , and -1920 for B . The *Ratio square root measure* gives the values $F(A) \approx 19.5$ and $F(B) \approx 19.8$. But we tend to judge A to be more diverse than B because we have the following intuition:

The Equal interval condition: For a measure of freedom of choice F , and for all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B| = n$, $\max d_{Ai} = \max d_{Bi} = M$, and A_d and B_d are one-dimensional spaces, if there is at least one distance between all options, $x \in A$ and at least one other option $y \in A$ such that $d(x, y) = M/(n - 1)$ and it is not the case that there is at least one distance between all options $w \in B$ and at least one other option $z \in B$ such that $d(w, z) = M/(n - 1)$, then A offers strictly more of choice than B , and thus $F(A) > F(B)$.

This condition implies that if some options could be selected from a closed interval to maximize freedom of choice, the options should be selected at intervals of equal length. For example, if four options could be selected from the closed interval of $[0 \text{ h}, 30 \text{ h}]$, they should be selected as 0 h , 10 h , 20 h and 30 h to maximize freedom of choice.

One reason that the *Ratio root measures* do not satisfy the *Equal interval condition* is that the *Ratio root measures* take all distances into account, and the total difference sums differ between the two sets. When more than three options are distributed within the same interval, the total difference sum may be smaller for options that are distributed at equal intervals. For example, compare *A* from above to *C* with matrix \mathbf{M}_C :

	0 h	1 h	44 h	45 h
0 h	0	1	44	45
1 h	1	0	43	44
44 h	44	43	0	1
45 h	45	44	1	0

A has options distributed at equal intervals, while *C* does not. *A* has a total difference sum of 300, while *C* has a total difference sum of 356. It is this property of the total difference sum that affects the *Ratio root measures* and gives the counterintuitive implication. (Although, in this example, the *Ratio square root measure* fares better and correctly ranks *A* over *C*, with values of 19.5 and roughly 19.)

Another reason that the *Ratio root measures* do not satisfy the *Equal interval condition* is that the equal interval case may not be the most evenly distributed case, relative to all distances (unless there are only two options, of course). This statement needs to be qualified, however. It is not the case that the *Equal interval condition* is in conflict with the *Limited evenness condition*. The second condition only applies to one-dimensional spaces with three options, and in these cases it gives the same rankings as the *Equal interval condition*. (For one-dimensional spaces with more than three options it cannot be the case that all distances are equal in two sets except for two distances and their symmetrical counterparts.) However, the *Equal interval condition* may be in conflict with the *Evenness condition* since an acceptable measure of evenness may not satisfy the *Equal interval condition*. We have already seen that the *Ratio root measures* do not satisfy the condition, although they may be regarded as acceptable measures of evenness (in a sense). Now we shall see that the *Evenness measure* does not satisfy the condition either. We shall look at the following example; let us suppose that we shall rank two sets of options, $G = \{0 \text{ h}, 53 \text{ h}, 106 \text{ h}, 159 \text{ h}, 212 \text{ h}, 265 \text{ h}\}$ and $H = \{0 \text{ h}, 70 \text{ h}, 105 \text{ h}, 140 \text{ h}, 210 \text{ h}, 280 \text{ h}\}$. Both sets contain six options and have a total difference sum of 3710. The set *G* has options distributed at an equal interval, which is not the case with *H*. So the *Equal interval condition* ranks *G* over *H*. However, the *Evenness measure* gives *G* an evenness value of -65296 and *H* an evenness value of -64260 . So *H* is more even than *G*. Therefore, one acceptable interpretation of the *Evenness condition* would rank *H* over *G*.

It is not so strange that there should be a conflict between equal interval distribution and evenness since they are not the same thing. In a choice between the *Equal interval condition* and the *Evenness condition*, the *Evenness condition* ought to win. It is, after all, the more general condition.

The *Equal interval condition* only applies to comparisons between sets with options of a specific kind, namely those that can be represented as varying in only one dimension. It is doubtful if the *Equal interval condition* captures an intuition that could be applied more generally to all types of spaces. The natural solution to the problem may be to judge degrees of freedom of choice in one-dimensional spaces differently than other spaces. For comparisons of multi-dimensional choice sets we should base our judgments on all distances and apply the ideal of evenness. For comparisons of one-dimensional choice sets we should base our judgments on minimal path distances and apply the ideal of equal intervals. However, this solution may be problematic when we have to compare sets of options that differ in one dimension to sets of options that differ in several dimensions. In these cases it seems better to apply the *Evenness condition*.

16.2.5 Ignoring Preferences

The last objection is due to several commentators on various talks. It is the objection that the *Ratio root measures* ignore information about preferences for options. In a sense, this objection is not really relevant since the *Ratio root measures* are designed to be applied before evaluation of the options. But one may wonder if the *Ratio root measures* can be adjusted to apply also after the options have been evaluated, and include information about the values of the options.

The *Ratio root measures* can easily be applied as measures of freedom of rational choice. Instead of applying the measures to choice sets, they are applied to the subsets of choice sets that contain rationally choosable options. This type of *Ratio root measures* would not satisfy the *Strict monotonicity condition*. Instead they would satisfy the *Indifference monotonicity condition*. They also would not satisfy the *Maximal freedom of choice condition*.

The *Ratio root measures* can also be applied as measures of freedom of eligible choice. They are then applied to the subsets of choice sets that contain eligible options. If they were applied to sets of eligible options, then the measures would not satisfy the *Strict monotonicity condition*, but the *Eligible options monotonicity condition*. They would not satisfy the *Maximal freedom of choice condition*. However, it is not advisable to apply the measures to freedom of eligible choice since this is an implausible explication of the concept of freedom of choice.

The *Ratio root measures* may be multiplied by values to apply to freedom of evaluated choice. This would not be advisable either. It would result in a

loss of information. If we are interested in the values of the options we should rather focus only on these. Perhaps we could design a measure that would apply to the diversity of preferences. But the *Ratio root measures* are not suitable for this use either. The reason for this is that there is no need to divide such a measure by the number of choice contributing options.

16.3 Partial Conclusion

As a result of the last chapters we have a measure of freedom of choice that satisfies all the conditions that were accepted in the previous part of the thesis, as well as some additional conditions that were accepted in this part. Unfortunately, that does not mean that we have a measure without any counterintuitive implications. In the end of this chapter we are thus left with a choice, either to accept the *Ratio root measures* (at least the *Ratio square root measure*) with its counterintuitive implication, or to reject one of the conditions that the measures satisfy. We shall discuss this problem in the next chapter.

Chapter 17: Summary and Conclusions

We have almost come to the end of this discussion concerning measures of freedom of choice. However, before we end, we shall recapitulate the discussion and see if we can draw any general conclusions. Let us first consider the different choices that had to be made when selecting a measure of freedom of choice.

17.1 Choices

One important choice when selecting a measure of freedom of choice was to decide whether the measure should be dependent or independent of preferences. There was at least one reason to prefer a preference-independent conception of freedom of choice; it could be used at a stage of the choice process when the interest in freedom of choice might be greatest, the pre-evaluative stage. Contrariwise, there was at least one reason to prefer a preference-dependent conception of freedom of choice; it was suitable for all later stages in the choice process. The conception of freedom of rational choice was suitable to assess the freedom of choice that was left to consider after evaluation. The conceptions of freedom of eligible choice and evaluated choice might be suitable to measure closeness to perfect freedom of choice, the freedom to choose whichever option one might prefer to choose. Nevertheless, from the discussions concerning the different kinds of preference-dependent freedom of choice it became clear that the preference-dependent conceptions had many problems that were difficult to solve. Therefore, I chose to focus on the preference-independent conceptions.

I discussed three preference-independent conceptions of freedom of choice: the cardinality conception, the representative conception and the diversity conception. All involved a great number of problems in terms of capturing reasonable intuitions regarding freedom of choice. Was there really any reason to prefer one conception to any of the others?

There was at least one reason to prefer the cardinality conception and the representative conception to the diversity conception; the first two conceptions were simpler. At least the cardinality conception did not admit of several sub-conceptions, as the diversity conception did. There were also no inherent, unsolvable conflicts regarding the explication of the cardinality and representative conceptions, as there were with the diversity conception

(but perhaps this was just due to the representative conception not yet being thoroughly investigated). But there was also at least one reason to prefer the diversity conception to any of the other conceptions. The reason was that both the cardinality conception and the representative conception seemed to be in conflict with a greater number of intuitions regarding freedom of choice. The cardinality conception was a very crude conception, but so was representativeness. For example, both conceptions failed to recognize that a set of two very similar options offered less freedom of choice than a set of two more dissimilar options.

In this thesis I chose to focus on the diversity conception, since I believed that this conception was the most reasonable. But since the conception of diversity was complex, it was not sufficient to just choose to focus on diversity. One also had to learn which kind of diversity affects degrees of freedom of choice. There were many conceptions of diversity, but I focused on the two that I found most reasonable: diameter diversity and evenness diversity. The first conception was the idea that diversity was maximized, relative to a total sum of differences, when the diameter was close to the supremum. The second conception was the idea that diversity was maximized, relative to a total sum of differences, when the individual differences were equal.

There was at least one context when it seemed reasonable to prefer diameter diversity over evenness diversity; it was when the total sum of differences was very small. In this context, maximizing diameter would result in at least two options being rather different, rather than all options being very similar. In all other contexts, it seemed more reasonable to prefer evenness diversity over diameter diversity. In these contexts, maximizing evenness would result in all options being equally different, rather than just two options being very different, while all the other options being very similar. Since the conception of evenness diversity seemed reasonable in more contexts than diameter diversity, I chose to focus on the first conception.

One also had to decide which conditions a measure of freedom of choice should satisfy. The choice of a specific conception of freedom of choice reduced the number of possible conditions. Throughout the thesis, several conflicts between conditions were discussed. There was a conflict between the *Rational choice condition* and the *Strict monotonicity condition*. This conflict could be solved in favor of the second condition, given that the measure should be preference-independent. There was also a conflict between the *Supremum diameter condition* and the *Maximal evenness condition*. This conflict could be solved in favor of the second condition, given that the measure should capture evenness diversity. A conflict between the *Limited equality condition* and the *Symmetry condition* was more difficult to solve. However, acceptance of the second condition made it possible to measure freedom of choice by the use of a symmetric function.

Last, one should select a measure. The acceptance of conditions for the measure greatly restricted the choice. After ten conditions had been selected, only one class of measures remained: the *Ratio root measures*. However, accepting these measures as measures of freedom of choice implied accepting some counterintuitive rankings of choice sets as well.

The final choice is to either accept the *Ratio root measures* with its counterintuitive implications, or to go back and further investigate the ten conditions, hoping to dismiss at least one and then try to find another measure. I shall not make an extensive investigation regarding the ten conditions here since I have already discussed them. However, I shall say something about them next.

17.2 The Ten Conditions

I accepted ten conditions for a measure of freedom of choice: the *Ratio scale-independence condition*, the *Domain-insensitivity condition*, the *Strict monotonicity condition*, the *No freedom of choice condition*, the *Option dominance condition*, the *Limited evenness condition*, the *Symmetry condition*, the *Spread condition*, the *Limited growth condition* and the *Proportional growth condition*. Of these ten conditions, some seem more compelling to me than others.

The *No freedom of choice condition* seems as certain as anything might seem. It is conceptually true that the empty set and singleton sets offer no choice. The *Ratio scale-independence condition* seems almost as certain. Unless there is just one acceptable scale for measuring the properties on which freedom of choice depends, a measure must be scale-independent to be a reliable measure. The *Domain-insensitivity condition* seems slightly less certain. But the idea that freedom of choice could depend on other properties than the intrinsic properties of choice sets leads to counterintuitive rankings.

The *Strict monotonicity condition* seems very plausible. It is intuitively correct that freedom of choice grows with the addition of options, although it may grow very slow (and perhaps at a diminishing marginal rate). The *Limited growth condition* also seems very plausible. Freedom of choice does not grow at an excessive rate.

The *Option dominance condition* is debatable in the sense that freedom of choice may not depend on all the pair-wise differences between the options. It is also debatable in the sense that freedom of choice may not be the same thing as the diversity of options. However, if freedom of choice can be equated with the diversity of options, and depends on all differences between the options, then the condition is uncontroversial. The *Spread condition* is also debatable, even given the two previous assumptions. For most comparisons, many similar options do not seem to offer as much freedom of choice as a few dissimilar options. However, there may be exceptions.

Perhaps a choice among a million pencils offers as much freedom of choice as a choice between a pencil and a bicycle. The idea is at least not preposterous. The *Limited evenness condition* is certainly debatable and has been debated throughout this thesis. As the previous condition, it seems reasonable for most comparisons, but there may be exceptions. One exception, which I have mentioned, is that the condition may work for larger differences, but not for smaller differences. If it is applied to smaller differences, it may recommend a choice among a million pencils over a choice between a pencil and a bike. Even though this is not preposterous, it is very strange.

The *Proportional growth condition* is neither reasonable, nor strange. It gives the measure an exact value when the number of elements is n and their pair-wise differences are 1, namely n . This number is appealing in the sense that it assures that the function grows proportionally to the number of options. But there is really no reason why the number should be n , and not some other positive number.

The *Symmetry condition* seems to be the least reasonable condition. It was chosen above the *Limited equality condition*, not so much because it was the more reasonable condition, but to allow the freedom of choice measure to be a symmetric function on the distances.

I believe that a first step towards finding a reasonable alternative to the *Ratio root measures* would be to abandon the *Symmetry condition* (perhaps in favor of the *Limited equality condition*). However, this implies that we must abandon the idea that freedom of choice can be measured by a symmetric function on the distances. This would at least be inconvenient.

17.3 Summary

In this thesis, I set out to find a measure of freedom of choice, capturing an adequate comparative conception of freedom of choice. The task involved several stages and complexities at every stage.

The first part of the thesis included introductory work, where I gave an overview of the problem and classified different conceptions of freedom of choice. In the first chapter, I briefly presented the philosophical background of the problem of finding a measure of freedom of choice. I also presented the decision-theoretical model that was to be used in the thesis. In the second chapter, I presented some of the conceptions of freedom of choice that occur in the literature. The investigation was limited to conceptions where freedom of choice was regarded as a property of some person who has a choice set of least two options. I discussed freedom of choice as a categorical, comparative and complete property. Furthermore, I distinguished between evaluative and non-evaluative conceptions, the first being defined in such a way that freedom of choice was valuable (good) by definition. I noted that

such a conception seemed superfluous since the two conceptions of the value of freedom of choice and freedom of choice could be used instead. I also argued against an evaluative conception of freedom of choice for being contrary to ordinary language use.

This left me with non-evaluative conceptions of freedom of choice, at which point I distinguished between preference-dependent and preference-independent conceptions. I first defined three conceptions of preference-independent freedom of choice since these conceptions could be regarded as more basic. The first, the cardinality conception, was defined so that degrees of freedom of choice were dependent only on the cardinality of choice sets. The second, the representative conception, was defined so that degrees of freedom of choice were dependent only on the representativeness of choice sets relative to the relevant universal set. Finally, the diversity conception was defined so that degrees of freedom of choice were dependent only on the diversity of choice sets. Next I showed three ways in which the preference-independent conceptions could be changed into preference-dependent conceptions (leading to a total of nine non-evaluative conceptions). The first, freedom of rational choice, limited the freedom-relevant options to those that could rationally be chosen. The second, freedom of eligible choice, limited the freedom-relevant options to those that could be regarded as sufficiently valuable. The third, freedom of evaluated choice, suggested that freedom of choice should be regarded as a function of the values of the options. All conceptions were then related to different ways of understanding the concept of complete and perfect freedom of choice.

The third chapter presented freedom of choice as a measurable property, and also presented different methods of constructing a measure of freedom of choice. The fourth chapter contained preliminary work as well. The object of this chapter was to present different ways to model choice sets and their properties, in order to be able to measure freedom of choice. I discussed virtues and problems with different models for the purpose of being used with an adequate measure. As a model for the thesis, I chose a metric space model, in which differences among options were represented as metric space distances. One reason for selecting this model was that it could be used together with most proposed measures. Another reason was to connect to the scientific tradition of representing differences as metric space distances, particularly Euclidean and City block distances. It was acknowledged that the model required rather a lot of information, but since this was a practical problem, rather than a theoretical one, it was not regarded as a sufficient reason to choose another model.

The second part included a detailed discussion of the different conceptions of freedom of choice, as well as conditions for a measure and previously proposed measures. The fifth chapter presented the decision-theoretical and metric space models used in the thesis. The sixth chapter was concerned with the cardinality conception of freedom of choice and its

corresponding measures. The cardinality conception was judged as being too blunt, perhaps because it did not take the values of the options into account, and certainly because it did not take the differences among the options into account.

After this I considered some preference-dependent variants of the cardinality measure. I first discussed the conception of freedom of rational choice, and the virtues and problems with using this type of conception. One problem was that the universal set of options would not offer maximal freedom of choice, according to this conception. Second, I discussed the conception of freedom of eligible choice. There were different ways to interpret this conception more precisely, but all turned out to have counterintuitive implications. Finally, I discussed the conception of freedom of evaluated choice. Despite several attempts to find an acceptable specification of this conception, no such specification was found.

The seventh chapter brought up a rival conception of freedom of choice, the representative conception. Since this conception, and a corresponding measure, had only occurred in an essay by Gustafsson, the discussion revolved around his measure. The alternative conception turned out to be at least as problematic as the cardinality conception, partly because intuitions regarding the importance of a larger diameter, or a greater degree of equidistance, could not be captured.

The diversity conception of freedom of choice was discussed in the eight chapter. It turned out to involve several conceptions in itself. A few conceptions were presented and dismissed as inadequate already in the beginning of the chapter. These involved regarding diversity as a function of relations to the relevant universal set or the mean. The proposal that the diversity of a choice set should be regarded as dependent on the differences among its options was discussed in more detail in chapters nine to twelve. The idea could be specified in many different ways. Two rival conceptions involved maximizing either the largest distance, or the degree of equidistance, among the options. The equidistance conception was regarded as more reasonable, but the conflict between the two conceptions was not completely resolved. The discussion of diversity also involved investigating the properties that might affect diversity, as well as finding reasonable conditions for a ranking of sets in terms of freedom of choice as diversity. Problems involved with regarding all differences as relevant for diversity, or only a selection of differences, were discussed and not completely resolved. The idea that the diversity of a set should be regarded as a function of all distances was regarded as more reasonable, although problematic. Conditions concerning the cardinality of options, the magnitude of differences, the distribution of a sum of differences among individual differences, and the distribution of a sum of differences among options were discussed. The discussion led to acceptance of eight conditions: *Domain-insensitivity*, *Strict monotonicity*, *No freedom of choice*, *Limited growth*,

Option dominance, Evenness, Symmetry and Spread. The thirteenth chapter ended by reformulating the conditions for the ranking of choice sets to fit as conditions for a measure of freedom of choice.

The third part included a discussion regarding the construction of an adequate measure, as well as a presentation of my own proposal for a measure. The fourteenth chapter concerned a particular type of measure of freedom of choice, a derived measure in which each of the freedom-relevant properties was represented by a separate measure and then aggregated. Two kinds of such measures were discussed, additively separable measures and multiplicatively separable measures. A condition regarding the scale-independence of the measure was added, the *Ratio scale-independence condition*. Several measures were shown to be inadequate because they violated one or more of the proposed conditions. A *Ratio measure* was found to satisfy all conditions but one. The fifteenth chapter discussed pure distance-based measures. The use of separable strictly concave functions turned out to be promising. A subclass of such measures, the *Root measures*, was found to satisfy all conditions but two. By combining the *Root measures* with the *Ratio measure*, a new class of measures was found, the *Ratio root measures*. These measures satisfied all of the nine accepted conditions. They were discussed in more detail in the sixteenth chapter.

Having found a class of measures that satisfied all nine conditions, one important question remained; was it possible to show that these measures were the only ones that satisfied the nine conditions? It was not possible to prove that the *Ratio root measures* uniquely satisfied the nine conditions. However, through adding a tenth condition, the *Proportional growth condition*, it was at least possible to prove that the measures were the only functions among analytic functions with non-zero partial derivatives with respect to some function of the distances that satisfied the ten conditions.

The sixteenth chapter ended with a discussion of some problems for the *Ratio root measures*. Since the critique involved intuitions in conflict with the intuitions behind the nine conditions, a tentative conclusion would be that there can be no adequate measure of freedom of choice, if such a measure must capture all of our intuitions.

17.4 Conclusions

In the beginning of the thesis, I stated that I should try to answer three questions. What are the conditions for a person having freedom of choice? When does a person have more freedom of choice than another person? How should freedom of choice be measured?

As an answer to the first question I simply accepted the almost tautological proposal that a person P has freedom of choice if and only if P has at least two options to choose among. This answer was in conflict with

two preference-dependent conceptions of freedom of choice. According to the conception of freedom of rational choice, the options should also be of equal value. According to the conception of freedom of eligible choice, the options should also be sufficiently valuable. Since I wished to define the concept of freedom of choice so that it could also be applied before evaluation of the options, I did not regard the two objections as relevant.

As an answer to the second question, I proposed that a person has more freedom of choice than another person if and only if the options of the first person are more diverse. Before accepting this answer, I considered several other proposals for comparative freedom of choice. I rejected evaluative conceptions of freedom of choice in general. The main reason was that this type of conception could not be regarded as fundamental. I also rejected several kinds of preference-dependent conceptions of freedom of choice, for various reasons. Three preference-independent conceptions remained: the cardinality conception, the representative conception, and the diversity conception. The first and second conceptions were rejected due to their counterintuitive implications. This left me with the third conception, freedom of choice as diversity. The concept of diversity was found to have many possible explications as well. Some of these were rejected as inappropriate for freedom of choice. The most reasonable conception was judged to be the conception of evenness diversity, which was explored in detail in the thesis.

Towards an answer to the third question, I considered a large number of measures. Among these, the *Ratio root measures* were found to be the most promising measures of evenness diversity. It could be shown that they were the only class of measures that satisfied ten reasonable conditions, at least among measures of a certain mathematical kind. Thus, I propose that the *Ratio root measures* should be used as measures of freedom of choice.

Appendix

A. Lists and Table

I. Accepted Conditions

The following conditions have been accepted as conditions for a measure of freedom of choice:

The Ratio scale-independence condition: For a measure of freedom of choice F , where F is a function of the functions $f_1 f_2 \dots f_n$, and any metric choice set $A_d \in P(X_d)$, it holds that for all numbers $K > 0$, there is a number $L = L(K) > 0$, where L is an increasing function of K such that if each function $f_1 f_2 \dots f_n$ is multiplied by K , then $F(A)$ is multiplied by L .

The Domain-insensitivity condition: For a measure of freedom of choice F , there do not exist any relevant universal sets X and X^* and choice sets A and B such that $A, B \subseteq P(X)$ and $A, B \subseteq P(X^*)$, where A offers at least as much freedom of choice as B , given X , and A offers strictly less freedom of choice than B , given X^* , so that $F(A) \geq F(B)$, given X , and $F(A) < F(B)$, given X^* .

The Strict monotonicity condition: For a measure of freedom of choice F , any non-empty choice set $A \in P(X)$, and any option $y \in X - A$, $A \cup \{y\}$ offers strictly more freedom of choice than A , and thus $F((A \cup \{y\})) > F(A)$.

The No freedom of choice condition: For a measure of freedom of choice F and any choice set $A \in P(X)$, if $|A| \leq 1$, then A offers no freedom of choice, and thus $F(A) = 0$.

The Option dominance condition: For a measure of freedom of choice F and all metric choice sets $A_d, B_d \in P(X_d)$, if A dominates B option by option, then A offers strictly more freedom of choice than B , and thus $F(A) > F(B)$.

The Limited evenness condition: For a measure of freedom of choice F , and all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$ and $\Sigma(A_d) = \Sigma(B_d)$, with distance vectors \mathbf{d}_A and \mathbf{d}_B , if for each i and each pair of distances (d_{Ai}, d_{Bi}) , it holds that $d_{Ai} = d_{Bi}$, except for the distances d_{Aj}, d_{Ak}, d_{Bj} and d_{Bk} (and their symmetrical counterparts), which are such that $d_{Aj} + d_{Ak} = d_{Bj} + d_{Bk}$, then if $|d_{Aj} - d_{Ak}| < |d_{Bj} - d_{Bk}|$, A offers strictly more freedom of choice than B , and thus $F(A) = F(B)$.

The Symmetry condition: For a measure of freedom of choice F , and all metric choice sets $A_d, B_d \in P(X_d)$, with distance vectors \mathbf{d}_A and \mathbf{d}_B , if the individual distances in \mathbf{d}_A and \mathbf{d}_B are equal, then A offers equal freedom of choice as B , and thus $F(A) = F(B)$.

The Spread condition: For a measure of freedom of choice F , and all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = m$ and $|B| = n$ and $\Sigma(A_d) = \Sigma(B_d) = M$, where each non-zero distance $d_{Ai} = M/(m(m-1))$ and each non-zero distance $d_{Bi} = M/(n(n-1))$, if $m < n$, then A offers strictly more freedom of choice than B , and thus $F(A) > F(B)$.

The Limited growth condition: For a measure of freedom of choice F , any metric choice set $A_d \in P(X_d)$ and any option $y \in X - A$, it holds that $F((A \cup \{y\})) \leq F(A) + F(\{x_i, y\})$, where $d(x_i, y) = \max_k d(x_k, y)$ and $x_i, x_k \in A$.

The Proportional growth condition: For a measure of freedom of choice F , and any metric choice set $A_d \in P(X_d)$ such that $|A| \geq 2$, if $|A| = n$ and each non-zero distance $d_{Ai} = 1$, then $F(A) = n$.

II. Accepted Implied Conditions

Some conditions are directly implied by the accepted conditions listed above. The *Indifference between no-choice situations condition* is implied by the *No freedom of choice condition*. The *Limited strict monotonicity condition* is implied by the *Strict monotonicity condition*. The *Maximal freedom of choice condition* is implied by the *Strict monotonicity condition*. The *Weak monotonicity condition* is implied by the *Strict monotonicity condition*. The *Limited diameter condition* and the *Dominance condition* are implied by the *Option dominance condition*. The *Maximal evenness condition* is implied by the *Limited evenness condition*, together with transitivity of the evenness relation. The *Ordinal scale-independence condition* is implied by the *Ratio scale-independence condition*. The *Distance ratio scale-independence condition* is implied by the *Ratio scale-independence condition*, when F is a function only of d .

III. Independence of Conditions

Eight of the ten conditions are independent of one another. One exception is the *Strict monotonicity condition*, which is a consequence of the conjunction of the *Distance ratio scale-independence condition*, the *Option dominance condition*, the *Limited evenness condition*, the *Symmetry condition*, the *Proportional growth condition*, and the *Limited growth condition*. Another exception is the *Domain-insensitivity condition*, which is a consequence of the other conditions, as they imply the domain-insensitive *Ratio root measures*.

The conditions are otherwise independent of one another, mostly because they are only sufficient conditions for rankings that apply to different and mutually exclusive cases. The *Scale-independence condition* applies only to comparisons between cases where there is only a change of scale between comparisons of the same sets. The *Strict monotonicity condition* applies only to comparisons between sets of different cardinality and a different total difference sum. The *No freedom of choice condition* applies only to singleton sets and the empty set. The *Option dominance condition* applies only to comparisons between sets with equal cardinality but different total difference sum. The *Limited evenness condition* applies only to comparisons between sets where both cardinality and total difference sum are equal but distribution of distances differs. The *Symmetry condition* applies only to comparisons between sets where cardinality, total difference sum and distribution of distances are equal but distribution between options differs. The *Spread condition* applies only to comparisons between sets where total difference sum and degree of evenness are equal but cardinality differs. Finally, the *Limited growth condition* states how the addition of an option affects the value of the freedom of choice function. It thus applies only to comparisons between sets of different cardinality and different total difference sum. With some extra assumptions, the *Limited growth condition* would imply the *Strict monotonicity condition*, but it does not do so in its present form since it is not excluded by the condition that $F(\{x, y\})$ may be 0. The *Proportional growth condition* also concerns the value of the freedom of choice function. However, it does not imply the *Limited growth condition* since it only applies to cases where all the distances between the options are 1. For this reason, it also does not imply the *Strict monotonicity condition*.

IV. Conflicting Conditions

I shall also summarize which conditions may be regarded as quite reasonable, but are inconsistent with one another. The following conflicts have been discussed:

- 1) The *Rational choice condition* vs. the *Strict monotonicity condition* (second condition judged as more reasonable).
- 2) The *Domain-sensitivity condition* vs. the *Domain-insensitivity condition* (second condition judged as more reasonable).
- 3) The *Supremum diameter condition* vs. the *Maximal evenness condition* (second condition judged as more reasonable).
- 4) The *Limited inequality condition* vs. the *Limited equality condition* vs. the *Symmetry condition* (third condition judged as most reasonable).
- 5) The *Diameter priority condition* vs. the *Limited evenness condition* (second condition judged as more reasonable).
- 6) The *Extreme limited growth condition* vs. the conjunction of the ten accepted conditions (the conjunction judged as more reasonable).
- 7) The *Equal interval condition* vs. the *Evenness condition* (some versions) (second condition judged as more reasonable).

V. Table

In order to get an overview of the thesis, I shall present a comparative table. The table includes measures and conditions. I have only included the ten accepted conditions.

Abbreviations: NA = Not applicable, RI = Ratio scale-independence, DI = Domain-insensitivity, NF = No freedom of choice, SM = Strict monotonicity, OD = Option dominance, LE = Limited evenness, SY = Symmetry, SP = Spread, LG = Limited growth, PG = Proportional growth.

Measures of freedom of choice

Condition/ Measure	RI	DI	NF	SM	OD	LE	SY	SP	LG	PG
<i>Cardinality</i>	Yes	Yes	No/Yes	Yes	No	No	Yes	No	Yes	Yes
<i>Rational choice</i>	Yes	Yes	No/Yes	No	No	No	No	No	Yes	No
<i>Eligible choice</i>	Yes	Yes	No/Yes	No	No	No	No	No	Yes	No
<i>Lexical eligible choice</i>	NA	Yes	No/Yes	Yes	No	No	No	No	NA	NA
<i>Total value</i>	Yes	Yes	No/Yes	No						
<i>Hybrid</i>	Yes	Yes	No/Yes	No						
<i>Neg. expected compromise</i>	Yes	No	No/Yes	Yes	No	No	No	No	No	No
<i>Neg. rational exp. comp.</i>	Yes	No	No/Yes	No						
<i>Neg. weighted exp. comp.</i>	Yes	No	No/Yes	No						
<i>Distinctiveness</i>	Yes	No	No/Yes	Yes	No	No	No	No	No	No
<i>Mean</i>	Yes	Yes	Yes	No	No	No	No	No	Yes	No
<i>Attribute</i>	Yes	Yes	No/Yes	No	No	No	No	No	Yes	No
<i>Extrinsic attribute</i>	Yes	No	No/Yes	Yes	No	No	No	No	Yes	No
<i>Intrinsic attribute</i>	Yes	Yes	No/Yes	Yes	No	No	No	No	No	No
<i>Category</i>	Yes	Yes	No/Yes	No	No	No	No	No	Yes	No
<i>Distance category</i>	Yes	No	No/Yes	Yes	No	No	No	No	Yes	No
<i>Total difference sum</i>	Yes	Yes	Yes	Yes	Yes	No	Yes	No	No	No
<i>Diameter</i>	Yes	Yes	Yes	No	No	No	Yes	Yes	Yes	No
<i>Lexi-max</i>	NA	Yes	Yes	No	No	No	Yes	Yes	NA	NA
<i>Exclusive average</i>	Yes	Yes	Yes	No	Yes	No	Yes	Yes	Yes	No
<i>Inclusive average</i>	Yes	Yes	Yes	No	Yes	No	Yes	Yes	Yes	No
<i>Exclusive ratio</i>	Yes	Yes	Yes	No	Yes	No	Yes	Yes	Yes	Yes
<i>Inclusive ratio</i>	Yes	Yes	Yes	Yes	Yes	No	Yes	Yes	Yes	Yes
<i>Maximal distances</i>	Yes	Yes	Yes	Yes	No	No	No	Yes	Yes	Yes
<i>Minimal distances</i>	Yes	Yes	Yes	No	No	No	No	Yes	Yes	Yes
<i>Minimal path</i>	Yes	Yes	Yes	No	No	No	No	Yes	Yes	No
<i>Minimal circular path</i>	Yes	Yes	Yes	No	No	No	No	Yes	Yes	Yes
<i>Elimination</i>	Yes	Yes	Yes	Yes	No	No	No	Yes	Yes	No
<i>Unweighted CSF</i>	Yes	Yes	No/Yes	No	No	Yes	Yes	Yes	No	No
<i>Log</i>	No	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No
<i>Root</i>	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No
<i>Ratio root</i>	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes

VI. Comments on Table

1) *Ratio scale-independence condition*: All ratio scale measures trivially satisfy this condition. The condition is not satisfied by ordinal scale measures, such as the various lexical measures that we have discussed. That the *Log measure* fails to satisfy the *Distance ratio scale condition* is shown in the corresponding section. A few measures were originally intended as ordinal scale measures, but since I have redefined them as ratio scale measures they are marked by ‘Yes’ in the table.

2) *Domain-insensitivity condition*: All measures that only depend on the domain-insensitive properties of the options in the measured set trivially satisfy this condition. The domain-insensitive measures include the *Cardinality measure*, which only depends on the cardinality of the options in the measured set, the various preference-dependent cardinality measures, which only depend on the value and cardinality of the options in the measured set, and the original attribute and category measures, which only depend on the attributes and categories exemplified by the options in the measured set. Here it is important that the values, attributes and categories are defined in a domain-insensitive way. The domain-insensitive measures also include measures that only depend on the distances between the options in the measured set, which are the majority of the measures in the table.

Measures that depend on domain-sensitive properties of the options fail to satisfy the condition. These include measures that depend on distances between options that do not belong to the measured set, such as the various *Negative expected compromise measures*, the *Distinctiveness measure*, the *Extrinsic attribute measure* and the *Distance category measure*.

3) *No freedom of choice condition*: All measures that depend only on distances between the options in the measured set trivially satisfy this condition since the distance between an element and itself is always 0. Other measures may satisfy the condition by stipulation. These are marked with ‘No/Yes’ in the table.

4) *Strict monotonicity condition*: The *Cardinality measure* trivially satisfies this condition. All strictly increasing functions on sets of distances trivially satisfy the condition as well. The *Total difference sum measure*, the *Distinctiveness measure*, the *Log measure*, the *Root measures* and the *Ratio root measures* are strictly increasing functions and satisfy the condition. That the *Lexical ordinal eligible choice measure*, the *Negative expected compromise measure*, the *Extrinsic attribute measure*, the *Intrinsic attribute measure*, the *Distance category measure*, the *Elimination measure* and the

Maximal distances measure satisfy the condition is shown in their respective sections. That the *Inclusive average distance measure* satisfies the condition is proved later in the appendix.

That the remaining measures all fail to satisfy the condition is shown in their respective sections.

5) *Option dominance condition*: All strictly increasing functions on sets of distances trivially satisfy this condition. The *Total difference sum measure*, the *Log measure*, the *Root measures* and the *Ratio root measures* satisfy the condition. The *Exclusive average distance measure*, the *Inclusive average distance measure*, the *Exclusive ratio measure* and the *Inclusive ratio measure* also satisfy the condition for the reason that these measures aggregate all distances and, when the cardinality of two sets are equal, award a greater value for the set with larger distances.

All measures that do not depend on distances fail to satisfy the condition. These include the various *Cardinality measures* and the original attribute and category measures. Measures that ignore distances among options also fail to satisfy the condition. These include the *Diameter measure*, the *Lexi-max measure*, *Maximal distances measure*, the *Minimal distances measure*, the *Minimal path measure*, the *Minimal circular path measure* and the *Elimination measure*. That the *Negative expected compromise measures*, the *Distinctiveness measure*, the *Extrinsic attribute measure*, the *Intrinsic attribute measure*, the *Distance category measure* and the *Mean measure* fail to satisfy the condition is shown in their corresponding sections. For the first three measures, this is due to their domain-sensitivity. Because lack of evenness is punished by the unweighted *CSF measure*, it also fails to satisfy the condition.

6) *Limited evenness condition*: All strictly concave functions on sets of distances satisfy this condition. These include the *Log measure*, the *Root measures* and the *Ratio root measures*. For a proof of this fact, see the proof concerning the *Ratio root measures*. The unweighted *CSF measure* trivially satisfies the condition as well.

There are several reasons why a measure may fail to satisfy the condition. It may not depend on distances between options at all, like the *Cardinality measure*. It may not depend exclusively on distances of the measured set, like the *Negative expected compromise measures*. It may not depend on all the distances of the measured set, like the *Elimination measure*. Or it may not be strictly concave and put different weights on distances of different sizes, like the *Total difference sum measure*. Some measures fail to satisfy the condition for additional reasons. The *Mean measure* fails to satisfy the condition when an option in the more evenly distributed set is positioned exactly at the mean, while this is not the case for the less evenly distributed set (for example, when {0h, 5h, 10h} and {0h, 6h, 10h} are compared). The

Intrinsic attribute measure fails to satisfy the condition since, for this measure, equal attributes only count once.

7) *Symmetry condition*: All symmetrical functions on sets or vectors of distances satisfy this condition. Symmetrical functions are such that the order of the distances in the vectors does not matter for the value of the function. The *Total difference sum measure*, the *Diameter measure*, The *Lexi-max measure*, the *Exclusive average distance measure*, the *Inclusive average distance measure*, the *Exclusive ratio measure*, the *Inclusive ratio measure*, the unweighted *CSF measure*, the *Log measure*, the *Root measures* and the *Ratio root measures* are symmetric functions and satisfy the condition. The *Cardinality measure* satisfies the condition only because cardinality is fixed in the condition.

Non-symmetrical measures on vectors of distances fail to satisfy the condition. The *Mean measure*, the *Extrinsic attribute measure*, the *Intrinsic attribute measure*, the *Distance category measure*, the *Maximal distances measure*, the *Minimal distances measure*, the *Minimal path measure*, the *Minimal circular path measure* and the *Elimination measure* are non-symmetrical measures and fail to satisfy the condition. Other measures fail to satisfy the condition in so far as they depend on other things than distances between the options in the measured set. This includes the original attribute and category measures, the domain-sensitive measures and all the preference-dependent cardinality measures.

8) *Spread condition*: The *Diameter measure* and the *Lexi-max measure* satisfy the condition since A has a larger diameter than B . The *Exclusive average distance measure* and the *Inclusive average distance measure* satisfy the condition since A has a greater exclusive and inclusive average than B . The *Exclusive ratio measure* and the *Inclusive ratio measure* satisfy the condition since the total difference sum is divided by a greater value for a greater cardinality. The *Maximal distances measure*, the *Minimal distances measure* and the *Minimal circular path measure* satisfy the condition since they give the value $M/(m - 1)$ to A , which is greater than the value $M/(n - 1)$ that they give to B . The *Elimination measure* and the *Minimal path measure* satisfy the condition since they give the value M/m to A , which is greater than the value M/n that they give to B . The unweighted *CSF measure* satisfies the condition due to cardinality having negative weight.

Some measures fail to satisfy the condition since they depend on things other than the cardinality and the distances between the options in the measured set. Preference-dependent measures fail to satisfy the condition because they depend on values. Domain-sensitive measures fail to satisfy the condition since they depend on the identity of the relevant universal set. For example, the *Negative expected compromise measures* and the *Distinctiveness measure* may both rank a set of many similar options above a

set of a few dissimilar options if the first set is either more representative of the relevant universal set (for the *Negative expected compromise measure*), or its options are more distinct (for the *Distinctiveness measure*). The original attribute and category measures fail to satisfy the condition since a set of many similar options might exemplify more attributes or categories than a set of fewer dissimilar options (although it rarely would). The new variants of the attribute and category measures are no better. The *Intrinsic attribute measure* ranks a set of three options at a distance of 1 above a set of two options at a distance of 3 since the first set exemplifies more attributes (six) than the second (four). The *Extrinsic attribute measure* makes the same ranking, for example, if the same sets constitute the relevant universal set and the options are represented as positioned in the vertices of a triangle on a straight line in Euclidean space. The *Distance category measure* may rank a set of many similar options above a set of a few dissimilar options if the relevant universal set contains a lot of smaller distances.

Other measures fail to satisfy the condition because they do not depend on both the cardinality and the distances between the options in the measured set. The *Cardinality measure* fails to satisfy the condition since a set of greater cardinality is always ranked above a set of smaller cardinality, no matter how similar or dissimilar their options are. The *Total difference sum measure* fails to satisfy the condition since the total difference sums are equal in the sets A and B . Other measures fail to satisfy the condition for other reasons. The *Mean measure* fails to satisfy the condition since a greater cardinality may imply either a smaller or greater value for the measure (as an example of a greater value, compare an evenly distributed set of three options to an evenly distributed set of four options, both having a total difference sum of 6). The *Log measure* and the *Root measures* fail to satisfy the condition because they have the property of *Strict subadditivity*.

9) *Limited growth condition*: Most of the measures satisfy this condition. All measures that aggregate at most one distance per option trivially satisfy the condition. These measures include the *Diameter measure*, the *Maximal distances measure*, the *Minimal distances measure*, the *Minimal path measure*, the *Minimal circular path measure* and the *Elimination measure*. The *Exclusive* and *Inclusive average distance measures* satisfy the condition since the average of all distances cannot be greater than the diameter of the set. The *Exclusive* and *Inclusive ratio measures* satisfy the condition since the average distance from the new option to the previous options in a set cannot be greater than twice the diameter divided either by one or two. The *Mean measure* satisfy the condition since the distance to the mean of a set cannot be greater than the diameter of the set. The *Cardinality measure* satisfies the condition since $|A \cup \{y\}| = |A| + 1 < |A| + 2$. The *Rational choice measure* satisfies the condition since it satisfies the condition for all cases, most importantly for the case when y is indifferent to the best options in A

but x is worse. The *Eligible choice measure* also satisfies the condition for all cases, when y and x both are eligible, when neither is, when only y is eligible and when only x is eligible. The original attribute and category measures satisfy the condition since the value of $F(A \cup \{y\}) = F(A) + F(\{y\})$ for both measures. The *Extrinsic attribute measure* satisfies the condition because a set of two options might exemplify $2n$ attributes, where n is the number of options in the relevant universal set, and the addition of an option never exemplifies more than n extra attributes. The *Distance category measure* satisfies the condition since a set of two options exemplifies one category defined by the largest distance, and may exemplify two categories at each smaller distance, while an additional option only may exemplify one additional category defined for each distance smaller than the largest distance.

Ordinal measures cannot satisfy the condition and neither can measures that grow proportionally to the number of distances. Such measures include the *Total difference sum measure*, the *Log measure* and the *Root measures*. The *Negative expected compromise measure* fails to satisfy the condition since the value of $F(A \cup \{y\}) > F(A) + F(\{y\})$. The *Distinctiveness measure* fails to satisfy the condition since the average distance from an added option to the other options in the relevant universal set may be larger than the diameter of the set to which the option is added. The *Intrinsic attribute measure* fails to satisfy the condition since the measure applied to a set of two options exemplifies four attributes, and because the addition of an option might imply the addition of more than four attributes, when the number of options exceeds four. The unweighted *CSF measure* fails to satisfy the condition since it may grow more than $\sqrt{2maxd} - 2$ when an option is added to a set. The *Total value measure* fails to satisfy the condition since the value of $A \cup \{y\}$ may exceed the sum of the value of A and the value of $\{x, y\}$; for example, when A has a positive value and y has a positive value, but the sum of the values of x and y is negative. The *Hybrid measure* fails to satisfy the condition for the same reason.

10) *Proportional growth condition*: Measures that satisfy this condition include the *Cardinality measure*, the *Exclusive ratio measure*, the *Inclusive ratio measure*, the *Maximal distances measure*, the *Minimal distances measure*, the *Minimal circular path measure* and the *Ratio root measures*, which give the value n to a set of n options at a distance of 1 from one another.

Measures that fail to satisfy the *Limited growth condition* cannot satisfy this condition either. The *Total difference sum measure* and the *Root measures* give the value $n^2 - n$. The *Diameter measure* and the *Exclusive average distance measure* give the value 1. The *Inclusive average distance measure* gives the value $1/n$. The *Minimal path* and *Elimination measure* give the value $n - 1$. The *Intrinsic attribute measure* gives the value $2n$. The

Log measure gives the value 0. The *Total value measure*, the *Hybrid measure*, the *Attribute measure*, the *Category measure*, the *Distance category measure*, the *Negative expected compromise measure*, the *Distinctiveness measure* and the *Extrinsic attribute measure* give values that vary, but which are not necessarily n . The *Mean measure* gives values that vary as well, but which are less than n . The *Rational choice measure* and *Eligible choice measure* gives values that vary, but which are at most n . The unweighted *CSF measure* gives a value of $\sqrt{n} - n$, being the most counterintuitive value of them all.

B. Proofs Concerning the Diameter

In this section we shall present two proofs concerning the diameter. They belong to section 11.1. The first proof concerns the size of the largest possible size of the diameter, given a fixed number of options and a fixed total difference sum. It shows how to calculate the supremum of possible diameters, and why the supremum is seldom obtained. The second proof concerns the fact that the options will be positioned on a straight line, given a fixed number of options, a fixed total difference sum and a diameter that is approaching the supremum. Both proofs are due to Per Enflo.

I. The Supremum of Possible Diameters

Here we will study how large the diameter of a finite metric space can be, under different conditions. In particular, we will study how large the diameter can be, given a fixed total sum of all distances between all the elements in a space. We will see that the supremum of possible diameters is not obtained if $n \geq 4$.

We now prove the following theorem:

Theorem 1: If $diam(A) = 1$, $|A| = n$, where $n \geq 4$, then

$$\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j) > 2n - 2, \text{ where } x_i, x_j \in A, \text{ and } \inf_A \left(\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j) \right) = 2n - 2.$$

Proof: We can assume that $d(x_1, x_2) = d(x_2, x_1) = 1$.

Then for $i > 2$, we have by the triangle equality that $d(x_1, x_i) + d(x_i, x_2) \geq 1$ and so $d(x_1, x_i) + d(x_i, x_1) + d(x_2, x_i) + d(x_i, x_2) \geq 2$. This gives, for $x_i, x_j \in A$,

$$\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j) \geq \sum_{i=1}^n (d(x_1, x_i) + d(x_i, x_1) + d(x_2, x_i) + d(x_i, x_2)) +$$

$$d(x_1, x_2) + d(x_2, x_1) + \sum_{i=3}^n \sum_{j=3}^n d(x_i, x_j) \geq 2(n-2) + 2 + \sum_{i=3}^n \sum_{j=3}^n d(x_i, x_j) > 2n - 4 + 2 = 2n - 2.$$

By working on the real line and letting $x_1 = 0, x_2 = 1$ and letting $x_i, i \geq 3$, be close to some fixed number $\alpha, 0 < \alpha < 1$, we see that

$$\inf_A \left(\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j) \right) = 2n - 2.$$

From *Theorem 1* we get the following corollary:

Corollary 1: If $A, \sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j) = T$, where $x_i, x_j \in A$, then $\text{diam}(A) < T/(2n-2)$. Moreover, $\sup_A \text{diam}(A) = T/(2n-2)$.

Proof of Corollary 1: Multiply all numbers in $\langle A, d \rangle$ by $1/(\text{diam}(A))$, and let $\langle A^*, d^* \rangle$ be the metric space that we get. Then $\text{diam}(A_d^*) = 1$ and so $\sum_{i=1}^n \sum_{j=1}^n d(y_i, y_j) > 2n - 2$, where $y_i, y_j \in A^*$.

Thus, in $A, \sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j) > (2n-2)(\text{diam}(A))$, where $x_i, x_j \in A$.

Thus, $T = \sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j) > (2n-2)(\text{diam}(A))$ and so $\text{diam}(A) < T/(2n-2)$.

By letting $\sum_{i=1}^n \sum_{j=1}^n d(y_i, y_j)$ be arbitrarily close to $2n-2$, we get $\text{diam}(A)$ arbitrarily close to $T/(2n-2)$.

Q.E.D.

II. Relation between the Supremum and the Metric Line

In the proofs of *Theorem 1* and *Corollary 1* we have that with a fixed total sum of distances for any metric set A_d , when the diameter of a set A_d approaches supremum, A_d approaches a subset of the real line. Equivalently, with a fixed diameter of a metric set A_d , when the total sum of distances for A_d approaches infimum, A_d approaches a subset of the real line. This is a special case of a more general result which we shall now prove. To formulate and prove this result we need some definitions:

Isometry: Two metric spaces $A_d = \langle A, d \rangle$ and $B_d = \langle B, d \rangle$, with options $x \in A$ and $y \in B$, and distances $d_A \in A_d$ and $d_B \in B_d$, are *isometric* if and only if there is a one-to-one map f from A_d onto B_d such that $d_A(x_i, x_j) = d_B(f(x_i), f(x_j))$ for all $x \in A$.

Isometric embedding: A metric space $A_d = \langle A, d \rangle$ can be *isometrically embedded* in another metric space $B_d = \langle B, d \rangle$ if and only if A_d is isometric to a subset of B_d .

Metric line: The n points (elements) of a metric space $A_d = \langle A, d \rangle$ are on a *metric line* if and only if the set A_d is isometric to a subset of the real line.

We shall prove the following theorem:

Theorem 2: If $\langle A, d \rangle$ is a metric space, where $A = \{x_1, x_2 \dots x_n\}$, then there exists a metric space $\langle A^*, d^* \rangle$, where $A^* = \{x_1^*, x_2^* \dots x_n^*\}$ such that:

- i) $\max d(x_i, x_j) = \max d^*(x_i^*, x_j^*)$,
- ii) $d(x_r, x_s) \geq d^*(x_r^*, x_s^*)$ for all r and s , and
- iii) $x_1^*, x_2^* \dots x_n^*$ are on a metric line.

To prove the theorem we need two lemmas.

First, let l_n^∞ be the space of n -tuples $\mathbf{x} = (x_1, x_2 \dots x_n)$ of real numbers such that with $\mathbf{x} = (x_1, x_2 \dots x_n)$ and $\mathbf{y} = (y_1, y_2 \dots y_n)$ we have $d(\mathbf{x}, \mathbf{y}) = \max_i |x_i - y_i|$. The following lemma was proved in a more general form by Fréchet (1910: 161). See also Banach (1932: 187).

Lemma 1: Any metric space $A_d = \langle A, d \rangle$ with n points can be *isometrically embedded* in l_n^∞ .

Proof of Lemma 1: Let $A = \{x_1, x_2 \dots x_n\}$. Consider the map $f: x_i \rightarrow (d(x_i, x_1), d(x_i, x_2) \dots d(x_i, x_m))$ from A_d into l_n^∞ . We will see that this is an isometric embedding. Consider the n -tuples $(d(x_i, x_1), d(x_i, x_2) \dots d(x_i, x_m))$ and $(d(x_j, x_1), d(x_j, x_2) \dots d(x_j, x_m))$.

We have, by the triangle inequality that for every r , $1 \leq r \leq n$, $d(x_i, x_r) \leq d(x_i, x_j) + d(x_j, x_r)$. Thus, $d(x_i, x_j) \geq d(x_i, x_r) - d(x_j, x_r)$.

Likewise $d(x_i, x_j) \geq d(x_j, x_r) - d(x_i, x_r)$, so

$$d(x_i, x_j) \geq |d(x_i, x_r) - d(x_j, x_r)|. \dots \text{(I)}$$

For $r = i$, we have

$$|d(x_i, x_i) - d(x_j, x_j)| = |0 - d(x_j, x_i)| = d(x_i, x_j). \dots \text{(II)}$$

The formulas (I) and (II) give $d(x_i, x_j) = \max_r |d(x_i, x_r) - d(x_j, x_r)|$. Thus, (I) and (II) give that f is an isometric embedding.

Further, a map P from a metric space A_d onto a subspace $B_d \subset A_d$ is a *projection* if and only if $P \circ P = P$. If A_d and B_d are linear spaces and P is a linear map with $P \circ P = P$, then P is a *linear projection*. If, for all x and y , it holds that $d(P(x), P(y)) \leq d(x, y)$ then P is a *contractive projection*.

The following lemma is a well-known version of the Hahn-Banach theorem (see Banach (1932: 55)). It was proved both by Banach (1932) and by Hahn (1927).

Lemma 2: If B_d is a one-dimensional subspace of a normed space A_d , then there is a contractive, linear projection from A_d onto B_d .

Proof of Theorem 2: We can, by *Lemma 1*, embed A_d isometrically in l_n^∞ by a function $f: A_d \rightarrow l_n^\infty$. Let the diameter of A_d be attained by x_1, x_2 so that $\text{diam}(A) = d(x_1, x_2)$. Consider the line segment l between $f(x_1)$ and $f(x_2)$ in l_n^∞ . By *Lemma 2*, there is a contractive projection P from l_n^∞ onto the line that contains the segment l . Since $f(x_1)$ and $f(x_2)$ are on this line, we have $P(f(x_1)) = f(x_1)$ and $P(f(x_2)) = f(x_2)$. Since $d(x_r, x_2) \leq d(x_1, x_2)$ for every r , $1 \leq r \leq n$, we get $d(P(f(x_r)), P(f(x_2))) \leq d(f(x_r), f(x_2)) = d(x_r, x_2) \leq d(x_1, x_2) = d(P(f(x_1)), P(f(x_2)))$ and so $P(f(x_r)) \in l$ for every r , $1 \leq r \leq n$.

Now, let $\langle A^*, d^* \rangle$ be the metric space with elements $x_1^* = P(x_1^*)$, $x_2^* = P(x_2^*) \dots x_n^* = P(x_n^*)$. Then $\langle A^*, d^* \rangle \subset l$, so iii) holds.

Since P is a contractive projection, we have $d(x_r^*, x_s^*) = d(P(f(x_r^*)), P(f(x_s^*)))$ for all r and s such that $1 \leq r \leq n$ and $1 \leq s \leq n$. So ii) holds and $\max d(x_i, x_j) \leq \max d(x_i^*, x_j^*)$.

Finally, $d(x_1, x_2) = d(P(f(x_1)), P(f(x_2))) = d(f(x_1), f(x_2)) = d(x_1, x_2) = \max d(x_i, x_j)$. Thus $\max d(x_i, x_j) = \max d(x_i^*, x_j^*)$, so i) holds.

From the theorem we get the following two corollaries. The first corollary:

Corollary 2: Let $f = f(d_1, d_2, \dots, d_M)$ be any continuous function of $M = n(n - 1)$ variables which is increasing in each variable, so that if $d_i' \geq d_i$ for some i , $1 \leq i \leq M$, then $f(d_1, d_2, \dots, d_{i-1}, d_i', d_{i+1}, \dots, d_M) \geq f(d_1, d_2, \dots, d_{i-1}, d_i, d_{i+1}, \dots, d_M)$. Then, if $\langle A, d \rangle$ is a metric space, then there is a metric space $\langle A^*, d^* \rangle$ such that:

- i) $\text{diam} \langle A, d \rangle = \text{diam} \langle A^*, d^* \rangle$,
- ii) $f(d_1, d_2, \dots, d_M) \geq f(d_1^*, d_2^*, \dots, d_M^*)$, and
- iii) the points in $\langle A^*, d^* \rangle$ are on a metric line.

Proof of Corollary 2: This corollary follows immediately from *Theorem 2*.

The second corollary:

Corollary 3: If $\langle A, d \rangle$ and f are given as above, then there is a metric space $\langle A^*, d^* \rangle$ such that i) $\text{diam } \langle A^*, d^* \rangle \geq \text{diam } \langle A, d \rangle$, ii) $f(d_1, d_2, \dots, d_M) = f(d_1^*, d_2^*, \dots, d_M^*)$, and iii) the points in $\langle A^*, d^* \rangle$ are on a metric line.

Proof of Corollary 3: Construct $\langle A^*, d^* \rangle$ as above. We have $f(d_1^*, d_2^* \dots d_M^*) \leq f(d_1, d_2 \dots d_M)$. Multiply all distances in A^* by the same number ≥ 1 to get $d_1^{**}, d_2^{**} \dots d_M^{**}$ such that $f(d_1^{**}, d_2^{**} \dots d_M^{**}) = f(d_1, d_2 \dots d_M)$. The space A^{**} with distances $d_1^{**}, d_2^{**}, \dots, d_M^{**}$ obviously fulfills i), ii) and iii).

Note: If we choose $f(d_1, d_2, \dots, d_M) = \sum_{i=1}^M d_i = \sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j)$, we see that

with a fixed total sum of distances, to maximize the diameter we need only to consider metric spaces where the points are on a metric line. So this part of *Theorem 1* is a special case of *Corollary 3*.

Q.E.D.

C. Proofs Concerning the Inclusive Average

These proofs belong to section 10.5 and chapter 13. For the proofs we define:

The Total difference sum: $T(A) = \sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j)$, where $x_i, x_j \in A$

The Inclusive average:

$$I(A) = \frac{\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j)}{n^2} \text{ for all } x_i, x_j \in A, \text{ where } n = |A|$$

Maximal evenness: A choice set A with a distance vector \mathbf{d}_A , having a total difference sum of M , and N non-zero distances, has a *maximally even* distribution if and only if each non-zero distance $d_{A_i} = M/N$.

I. Conditions for a Strictly Larger Total Difference Sum

Here we shall prove that if $|A| > |B|$, $I(A) = I(B)$ and $E(A) = E(B) = \text{Max } E(A)$, then $T(A) > T(B)$.

Proof: The stipulation is that A has a greater cardinality than B , while the inclusive averages of the sets are equal and all the individual non-zero differences are equal. Let us assume that the cardinality of B is $n \geq 2$, while the cardinality of A is $n + 1$.

$$\text{So, } \frac{T(B)}{n^2} = \frac{T(A)}{(n+1)^2} = \frac{T(A)}{n^2 + 2n + 1}. \text{ Thus, } T(A) = \frac{T(B)(n^2 + 2n + 1)}{n^2} = T(B) \left(1 + \frac{2}{n} + \frac{1}{n^2} \right). \text{ Thus, } T(A) > T(B).$$

Since the same proof can be applied to a comparison between B and a set C , where $|B| > |C|$, $I(B) = I(C)$ and $E(B) = E(C) = \text{Max } E(B)$, and so on, and the relation of having a greater total difference sum is transitive, for any set Z such that $|A| > |Z|$, $I(A) = I(Z)$ and $E(A) = E(Z) = \text{Max } E(A)$, it would hold that $T(A) > T(Z)$.

Q.E.D.

II. Conditions for Strict Dominance of Individual Distances

Here we shall prove that if $|A| > |B|$, $I(A) = I(B)$ and $E(A) = E(B) = \text{Max } E(A)$, then, for all non-zero distances, $d_{Ai} < d_{Bi}$.

Proof: Let us assume that the cardinality of B is $n \geq 2$, while the cardinality of A is $n + k$, and that the non-zero distances $d(y_i, y_j) = w$ for all $y_i, y_j \in B$, and that the non-zero distances $d(x_i, x_j) = z$ for all $x_i, x_j \in A$.

$$\text{We then have that } \frac{w(n^2 - n)}{n^2} = \frac{z((n+k)(n+k) - (n+k))}{(n+k)(n+k)}.$$

$$\text{Thus, } \frac{wn(n-1)}{n^2} = \frac{z(n+k)((n+k)-1)}{(n+k)(n+k)}.$$

$$\text{Thus, } \frac{w(n-1)}{n} = \frac{z(n+k-1)}{n+k}. \text{ Thus, } w > z \text{ and so } d_{Ai} < d_{Bi}.$$

Q.E.D.

D. Proofs Concerning the Total Sum of Inclusive Averages

These proofs belong to section 10.7 and chapter 13. For the proofs we define:

The Total difference sum: $T(A) = \sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j)$.

The Total sum of inclusive averages:

$$S(A) = \frac{1}{n} \sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j) \text{ such that } x_i, x_j \in A.$$

Maximal evenness: A choice set A with a distance vector \mathbf{d}_A , having a total difference sum of M , and N non-zero distances, has a *maximally even* distribution if and only if each non-zero distance $d_{A_i} = M/N$.

I. Conditions for a Strictly Larger Total Difference Sum

Here we shall prove that if $|A| > |B|$, $S(A) = S(B)$ and $E(A) = E(B) = \text{Max } E(A)$, then $T(A) > T(B)$.

Proof: The stipulation is that A has a greater cardinality than B , while the inclusive averages of the sets are equal and all the individual non-zero differences are equal. Let us assume that the cardinality of B is $n \geq 2$, while the cardinality of A is $n + 1$.

$$\text{So, } \frac{T(B)}{n} = \frac{T(A)}{(n+1)}. \text{ Thus, } T(A) = \frac{T(B)(n+1)}{n} = T(B) \left(1 + \frac{1}{n}\right).$$

$T(A) > T(B)$.

Since the same proof can be applied to a comparison between B and a set C , where $|B| > |C|$, $S(B) = S(C)$ and $E(B) = E(C) = \text{Max } E(B)$, and so on, and the relation of having a greater total difference sum is transitive, for any set Z such that $|A| > |Z|$, $S(A) = S(Z)$ and $E(A) = E(Z) = \text{Max } E(A)$, it would hold that $T(A) > T(Z)$.

Q.E.D.

II. Conditions for Strict Dominance of Individual Distances

Here we shall prove that if $|A| > |B|$, $S(A) = S(B)$ and $E(A) = E(B) = \text{Max } E(A)$, then, for all non-zero distances, $d_{A_i} < d_{B_i}$.

Proof: Let us assume that the cardinality of B is $n \geq 2$, while the cardinality of A is $n + k$, and that the non-zero distances $d(y_i, y_j) = w$ for all $y_i, y_j \in B$, and that the non-zero distances $d(x_i, x_j) = z$ for all $x_i, x_j \in A$.

$$\text{We then have that } \frac{w(n^2 - n)}{n} = \frac{z((n+k)(n+k) - (n+k))}{(n+k)}.$$

Thus, $\frac{wn(n-1)}{n} = \frac{z(n+k)((n+k)-1)}{(n+k)}$. Thus, $w(n-1) = z(n-1+k)$.

For $w, z > 0, k \geq 1, n \geq 2$, it is the case that $w > z$. Thus, $d_{Ai} < d_{Bi}$.

Q.E.D.

E. Proof Concerning the Inclusive Ratio Measure

This proof belongs to section 10.7. For the proof we define:

The Inclusive ratio measure:

$$F(A, d) = \frac{1}{n} \sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j) \text{ such that } x_i, x_j \in A.$$

The Strict monotonicity condition: For a measure of freedom of choice F , any non-empty choice set $A \in P(X)$, and any options $y \in X$ such that $y \notin A, A \cup \{y\}$ offers *strictly more freedom of choice than* A , and thus $F((A \cup \{y\})) > F(A)$.

We shall now prove that the *Inclusive Ratio Measure* satisfies *Strict monotonicity*.

Proof: For each i, j , where $i \neq j$, we consider $d(x_i, x_j)$. According to the triangle inequality $d(x_i, x_j) \leq d(y, x_i) + d(y, x_j)$. For each x_i we have $n-1$ options x_j , where $d(x_i, x_j) > 0$. We sum all such inequalities and get the inequality:

$$\sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j) \leq \sum_{i=1, i \neq j}^n \sum_{j=1}^n (d(y, x_i) + d(y, x_j)) \dots \text{(I)}$$

For each distance $d(x_i, x_j)$ occurring on the left side we get two distances occurring on the right side; that is, $d(y, x_i)$ and $d(y, x_j)$. Each x_i appears $2(n-1)$ times on the left side in (I), $n-1$ times as the left component in $d(x_i, x_j)$, and $n-1$ times as the right component in (I). Thus, x_i appears $2(n-1)$ times on the right side in (I), which makes the right side of (I) equal to $2(n-1) \sum_{i=1}^n d((y, x_i))$.

This gives $\frac{\sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j)}{n-1} \leq 2 \sum_{i=1}^n d((y, x_i))$, and then we get

$$\begin{aligned}
\frac{\sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j)}{n} &= \frac{\sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j)}{n+1} + \frac{\sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j)}{(n+1)n} \\
&< \frac{\sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j)}{n+1} + \frac{\sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j)}{(n+1)(n-1)} \\
&\leq \frac{1}{n+1} \left(\sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j) + 2 \sum_{i=1}^n d((y, x_i)) \right)
\end{aligned}$$

The left side is $F(A)$ and the right side is $F(A \cup \{y\})$, so thus $F(A \cup \{y\}) > F(A)$.

Q.E.D.

F. Proofs Concerning the Ratio Root Measures

The proofs of this section belong to chapter 16. Here we shall prove the following theorem:

The Sufficiency theorem: any Ratio root measure satisfies the ten conditions of distance ratio scale-independence, domain-insensitivity, strict monotonicity, no freedom of choice, option dominance, evenness, symmetry, spread, limited growth and proportional growth.

Again we define:

The Ratio root measures:

i) For $n < 2$, $F(A, d) = 0$.

ii) For $n \geq 2$, $F(A, d) = \frac{1}{n-1} \sum_{i=1, i \neq j}^n \sum_{j=1}^n (d(x_i, x_j))^r$, with $\frac{1}{2} \leq r < 1$.

I. Satisfaction of Distance Ratio Scale-Independence Condition

The Distance ratio scale-independence condition: For a measure of freedom of choice F that is a function of the distance function d , and any metric choice set $A_d \in P(X_d)$, it holds that for all numbers $K > 0$, there is a number $L = L(K) > 0$, where L is an increasing function of K such that if d is multiplied by K , then $F(A)$ is multiplied by L .

Proof: The *Ratio root measures* satisfy the *Distance ratio scale-independence condition* because if all the distances of A_d are multiplied by K , then $F(A)$ is multiplied by K^r .

Q.E.D.

II. Satisfaction of Domain-Insensitivity Condition

The Domain-insensitivity condition: For a measure of freedom of choice F , there do not exist any relevant universal sets X and X^* and choice sets A and B such that $A, B \subseteq X$ and $A, B \subseteq X^*$, where A offers at least as much freedom of choice as B , given X , and A offers strictly less freedom of choice than B , given X^* , so that $F(A) \geq F(B)$, given X , and $F(A) < F(B)$, given X^* .

Proof: This is trivial since the *Ratio root measures* are functions only of the cardinality and distances of each set that is assessed.

Q.E.D.

III. Satisfaction of No Freedom of Choice Condition

The No freedom of choice condition: For a measure of freedom of choice F and any choice set $A \in P(X)$, if $|A| \leq 1$, then A offers no freedom of choice, and thus $F(A) = 0$.

Proof: This is trivial since for $n \leq 1$, $F(A) = 0$.

Q.E.D.

IV. Satisfaction of Strict Monotonicity Condition

The Strict monotonicity condition: For a measure of freedom of choice F , and any non-empty choice set $A \in P(X)$ and any option $y \in X - A$, $A \cup \{y\}$ offers strictly more freedom of choice than A , and thus $F((A \cup \{y\})) > F(A)$.

Proof: Let us assume that we have a set A , consisting of the options $x_1, x_2 \dots x_n$. To A we add another option y .

According to the triangle inequality, $d(x_i, x_j) \leq d(y, x_i) + d(y, x_j)$. Any strictly increasing strictly concave function f with $f(0) = 0$ has the property that if $d_i, d_j > 0$ and $d_i + d_j = d_k$, then $f(d_i) + f(d_j) > f(d_k)$. Also, if $d_i + d_j > d_k$, then $f(d_i) + f(d_j) > f(d_k)$.

Since the triangle equality holds, we know that $d(x_i, x_j) \leq d(y, x_i) + d(y, x_j)$.

This means that $f(d(x_i, x_j)) < f(d(y, x_i)) + f(d(y, x_j))$ if x_i, x_j and y are distinct.

We can add these inequalities together and get

$$\sum_{i=1, i \neq j}^n \sum_{j=1}^n f(d(x_i, x_j)) < \sum_{i=1, i \neq j}^n \sum_{j=1}^n f((d(y, x_i) + d(y, x_j))) \dots (I)$$

For each distance $d(x_i, x_j)$ occurring on the left side, we get two distances occurring on the right side; that is, $d(y, x_i)$ and $d(y, x_j)$. Each x_i appears $2(n-1)$ times on the left side in (I), $n-1$ times as the left component in $d(x_i, x_j)$ and $n-1$ times as the right component in (I). Thus, x_i appears $2(n-1)$ times on the right side in (I), which makes the right side of (I) equal to:

$$2(n-1) \sum_{i=1}^n f(d((y, x_i))).$$

This gives $\frac{\sum_{i=1, i \neq j}^n \sum_{j=1}^n f(d(x_i, x_j))}{n-1} < 2 \sum_{i=1}^n f(d((y, x_i)))$, and then we get

$$\frac{\sum_{i=1, i \neq j}^n \sum_{j=1}^n f(d(x_i, x_j))}{n-1} = \frac{\sum_{i=1, i \neq j}^n \sum_{j=1}^n f(d(x_i, x_j))}{n} + \frac{\sum_{i=1, i \neq j}^n \sum_{j=1}^n f(d(x_i, x_j))}{n(n-1)} < \frac{1}{n} \left(\sum_{i=1, i \neq j}^n \sum_{j=1}^n d(x_i, x_j) + 2 \sum_{i=1}^n d((y, x_i)) \right)$$

The left side is $F(A)$ and the right side is $F(A \cup \{y\})$, so thus $F(A \cup \{y\}) > F(A)$.

Q.E.D.

V. Satisfaction of Option Dominance Condition

The Option dominance condition: For a measure of freedom of choice F and all metric choice sets $A_d, B_d \in P(X_d)$, if A dominates B option by option, then A offers strictly more freedom of choice than B , and thus $F(A) > F(B)$.

Proof: This is trivial. Since the function is strictly increasing, it is the case that $(d(x_i, x_j))^r > (d(y_i, y_j))^r$ when $d(x_i, x_j) > d(y_i, y_j)$.

Q.E.D.

VI. Satisfaction of Limited Evenness Condition

The Limited evenness condition: For a measure of freedom of choice F , and all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = |B|$ and $\Sigma(A_d) = \Sigma(B_d)$, with distance vectors \mathbf{d}_A and \mathbf{d}_B , if for each i and each pair of distances (d_{Ai}, d_{Bi}) , it holds that $d_{Ai} = d_{Bi}$, except for the distances d_{Aj}, d_{Ak}, d_{Bj} and d_{Bk} (and their symmetrical counterparts), which are such that $d_{Aj} + d_{Ak} = d_{Bj} + d_{Bk}$, then if $|d_{Aj} - d_{Ak}| < |d_{Bj} - d_{Bk}|$, A offers strictly more freedom of choice than B , and thus $F(A) = F(B)$.

Proof: Since the number of options is fixed, we only need to show that a separable strictly concave measure of the following form satisfies the condition:

$$F(A_d) = \sum_{i=1}^N f(d_i).$$

F is a function from \mathbf{R}^n to \mathbf{R} , and f is a strictly increasing strictly concave function from \mathbf{R} to \mathbf{R} .

By hypothesis, $d_{Aj} + d_{Ak} = d_{Bj} + d_{Bk}$ and $|d_{Aj} - d_{Ak}| < |d_{Bj} - d_{Bk}|$, thus $d_{Aj} - d_{Bj} = d_{Bk} - d_{Ak}$.

Suppose that $d_{Aj} < d_{Ak}$ and that $d_{Bj} < d_{Bk}$. In this case since $d_{Aj} + d_{Ak} = d_{Bj} + d_{Bk}$, we have $d_{Bj} < d_{Aj} < d_{Ak} < d_{Bk}$.

We then apply the *Mean Value Theorem*. Assume that the function f is continuous on the closed interval $[d_{Bj}, d_{Aj}]$ and differentiable on the open interval (d_{Bj}, d_{Aj}) . Then there exists some θ_1 in (d_{Bj}, d_{Aj}) such that

$$f'(\theta_1) = \frac{f(d_{Aj}) - f(d_{Bj})}{d_{Aj} - d_{Bj}}.$$

The function f is also continuous on the closed interval $[d_{Ak}, d_{Bk}]$ and differentiable on the open interval (d_{Ak}, d_{Bk}) . So there also exists some θ_2 in (d_{Ak}, d_{Bk}) such that

$$f'(\theta_2) = \frac{f(d_{Bk}) - f(d_{Ak})}{d_{Bk} - d_{Ak}}.$$

We know that $d_{Aj} - d_{Bj} = d_{Bk} - d_{Ak}$. We also know that $\theta_1 < \theta_2$ since they are located in different intervals. This means that $f'(\theta_1) > f'(\theta_2)$ since the function f is concave and the derivative is monotonically decreasing.

So, $f(d_{Aj}) - f(d_{Bj}) = f'(\theta_1)(d_{Aj} - d_{Bj}) = f'(\theta_1)(d_{Bk} - d_{Ak}) > f'(\theta_2)(d_{Bk} - d_{Ak}) = f(d_{Bk}) - f(d_{Ak})$.

So, $f(d_{Aj}) - f(d_{Bj}) > f(d_{Bk}) - f(d_{Ak})$, and therefore $f(d_{Aj}) + f(d_{Ak}) > f(d_{Bj}) + f(d_{Bk})$.

Q.E.D.

VII. Satisfaction of Symmetry Condition

The Symmetry condition: For a measure of freedom of choice F , and all metric choice sets $A_d, B_d \in P(X_d)$, with distance vectors \mathbf{d}_A and \mathbf{d}_B , if the individual distances in \mathbf{d}_A and \mathbf{d}_B are equal, then A offers equal freedom of choice as B , and thus $F(A) = F(B)$.

Proof: The *Ratio root measures* are symmetric functions of the distances and trivially satisfy the *Symmetry condition*.

Q.E.D.

VIII. Satisfaction of Spread Condition

The Spread condition: For a measure of freedom of choice F , and all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = m$ and $|B| = n$ and $\Sigma(A_d) = \Sigma(B_d) = M$, where each non-zero distance $d_{Ai} = M/(m(m-1))$ and each non-zero distance $d_{Bi} = M/(n(n-1))$, if $m < n$, then A offers strictly more freedom of choice than B , and thus $F(A) > F(B)$.

Proof: A measure satisfies the *Spread condition* if and only if it satisfies the following condition:

The Limited spread condition: For a measure of freedom of choice F , and all metric choice sets $A_d, B_d \in P(X_d)$ such that $|A| = m$ and $|B| = n$ and $\Sigma(A_d) = \Sigma(B_d) = M$, where each non-zero distance $d_{Ai} = M/(m(m-1))$ and each non-zero distance $d_{Bi} = M/(n(n-1))$, if $|A| + 1 = |B|$, then A offers strictly more freedom of choice than B , and thus $F(A) > F(B)$.

It is obvious that the *Spread condition* implies the *Limited spread condition*. The *Limited spread condition* also implies the *Spread condition* because if $|A| < |B|$, then $|A| + s = |B|$, for some $s \geq 1$. Thus, the *Spread condition* is obtained by using the *Limited spread condition* s times (assuming, as we have, that the freedom of choice relation is transitive).

The *Ratio root measures* can be shown to satisfy the *Limited spread condition*. Let us assume that all distances between the options in B are $d(y_i, y_j) = 1$. There are n options in B . We then have that

$$\sum_{i=1}^n \sum_{j=1}^n d(y_i, y_j) = n(n-1).$$

There is one additional option in the set B than in A . The number of options in $A = m = n - 1$. The sum of distances between the options in A must also be equal to $n(n-1)$. Each distance of A must then be equal to

$$d(x_i, x_j) = \frac{n(n-1)}{m(m-1)} \text{ because } \sum_{i=1}^m \sum_{j=1}^m d(x_i, x_j) = m(m-1) \frac{n(n-1)}{m(m-1)}.$$

$$\text{This means that } F(B) = \frac{1}{n-1} \sum_{i=1}^n \sum_{j=1}^n (d(y_i, y_j))^r = \frac{n(n-1)}{n-1} = n.$$

$$F(A) = \frac{1}{m-1} \sum_{i=1}^m \sum_{j=1}^m (d(x_i, x_j))^r = \frac{m(m-1)}{m-1} \left(\frac{n(n-1)}{m(m-1)} \right)^r = m \left(\frac{n(n-1)}{m(m-1)} \right)^r.$$

$$\text{If } m = n - 1, \text{ then we get } (n-1) \left(\frac{n(n-1)}{(n-1)(n-2)} \right)^r = (n-1) \left(\frac{n}{n-2} \right)^r.$$

$$\text{If } r > \frac{1}{2}, \text{ then } F(A) = (n-1) \left(\frac{n}{n-2} \right)^r > n = F(B) \text{ for all } n \geq 3.$$

This can be seen as follows: $\left(\frac{n}{n-2} \right)^r$ is an increasing function of r .

Thus the function attains its minimum value when $r = \frac{1}{2}$. By putting $r = \frac{1}{2}$ and squaring both sides we see that the left side squared is equal to

$$\left((n-1) \left(\frac{n}{n-2} \right)^{\frac{1}{2}} \right)^2 = \frac{(n^2 - 2n + 1)n}{n-2} > n^2 = \text{the right side squared.}$$

So the inequality is proved.

Q.E.D.

IX. Satisfaction of Limited Growth Condition

The Limited growth condition: For a measure of freedom of choice F , any metric choice set $A_d \in P(X_d)$ and any option $y \in X - A$, it holds that $F(A \cup \{y\}) \leq F(A) + F(\{x_i, y\})$, where $d(x_i, y) = \max_k d(x_k, y)$ and $x_i, x_k \in A$.

Proof: For the *Ratio root measures* we have that

$$F(\{x, y\}) = \frac{2f(d(x, y))}{2-1} = 2f(d(x, y)) \text{ where } f(t) = t^r.$$

Assume that A has n options. We then have:

$$F(A \cup \{y\}) = \frac{(n-1)F(A)}{n} + \frac{1}{n} \left[\left(\sum_i^n f(d(x_i, y)) + \sum_j^n f(d(y, x_j)) \right) \right] <$$

$F(A) + 2f(\max_k d(x_k, y)) = F(A) + F(\{x_i, y\})$, where $d(x_i, y) = \max_k d(x_k, y)$ and $x_i, x_k \in A$.

Q.E.D.

X. Satisfaction of Proportional Growth Condition

The Proportional growth condition: For a measure of freedom of choice F , and any metric choice set $A_d \in P(X_d)$ such that $|A| \geq 2$, if $|A| = n$ and each non-zero distance $d_{A_i} = 1$, then $F(A) = n$.

Proof: Let us assume that all distances between the options in A are $d(x_i, x_j) = 1$. There are n options in A . We then have that

$$\sum_{i=1}^n \sum_{j=1}^n d(x_i, x_j) = n(n-1).$$

$$\text{Thus, } F(A) = \frac{1}{n-1} \sum_{i=1}^n \sum_{j=1}^n (d(x_i, x_j))^r = \frac{n(n-1)}{n-1} = n.$$

Q.E.D.

G. Necessity Proof for the Ratio Root Measures

The proof of this section belongs to chapter 16, and is due to Per Enflo. Here we shall prove the following theorem:

The Necessity theorem: If a measure F is an analytic function with non-zero partial derivatives with respect to some function f of the distances, and if F satisfies the ten conditions of *distance ratio scale-independence, domain-insensitivity, strict monotonicity, no freedom of choice, option dominance, evenness, symmetry, spread, limited growth and proportional growth*, then F is a *Ratio root measure*.

Again we define:

The Ratio root measures:

i) For $n < 2$, $F(A, d) = 0$.

ii) For $n \geq 2$, $F(A, d) = \frac{1}{n-1} \sum_{i=1, i \neq j}^n \sum_{j=1}^n (d(x_i, x_j))^r$, with $\frac{1}{2} \leq r < 1$.

In this section, we write the non-zero distances in A , where $|A| = n$, as a vector $\mathbf{d}_A = (d_1, d_2 \dots d_m)$ with $m = n(n-1)$. With this notation, we get for $n \geq 2$ that $\frac{1}{2} \leq r < 1$ and that

$$F(A) = \frac{1}{n-1} \sum_{i=1}^m d_i^r$$

F denotes a *Ratio root measure*.

Since the function x^r , $0 < r < 1$, is not differentiable at $x = 0$, we see that $F(A)$ is not differentiable at $(0, 0 \dots 0)$ as a function of the d_i 's, from \mathbf{R}^m to \mathbf{R} . However, $F(A)$ is a differentiable function also at $(0, 0 \dots 0)$ of the variables d_i^r (it is just $1/(n - 1)$ times the sum of the d_i^r 's). Thus, the partial derivatives of F with respect to the d_i^r 's are equal to $1/(n - 1)$ everywhere.

We may repeat the *Necessity theorem*: let a measure F be an analytic function with non-zero partial derivatives with respect to some function f of the m non-zero distances among the elements of A , where $|A| = n$, $m = n(n - 1)$. That is, let F have the form $F(A) = F(f(d_1), f(d_2) \dots f(d_m))$, with non-zero partial derivatives with respect to the $f(d_i)$'s. If F satisfies the ten conditions of *Strict monotonicity, domain-insensitivity, no freedom of choice, option dominance, evenness, symmetry, scale-independence, spread, limited growth and proportional growth*, then F is a *Ratio root measure*.

Remark: it follows from the proof that f cannot be the function $f(d) = d$, so F cannot be an analytic function around $(0, 0 \dots 0)$ of the variables $d_1, d_2 \dots d_m$.

Proof: The *Domain-insensitivity condition* implies that the function is a function only of the distances between the elements of a set.

We first consider the case $n \geq 3$ and put $F(A) = F(f(d_1), f(d_2) \dots f(d_m))$. The *Option dominance condition* implies that F is an increasing function, regarded as a function of the of the variables $d_1, d_2 \dots d_m$. Moreover since F multiplies by $C = C(k)$, if the d_i 's multiply by k , we get that $C(k)$ is an increasing function of k , and also that $d_1 = d_2 = \dots = d_m = 0$ implies $F(f(0), f(0), \dots f(0)) = 0$. If $f(0) = a \neq 0$, then we put $F_0(b_1, b_2 \dots b_m) = F(a + b_1, a + b_2 \dots a + b_m)$. This gives $F_0(0, 0, \dots 0) = 0$, and so by putting $f_0(b) = f(b) - a$ we get that $f_0(0) = 0$ and $F(f(d_1), f(d_2) \dots f(d_m)) = F_0(f_0(d_1), f_0(d_2) \dots f_0(d_m))$, with $f_0(d) = 0$ if $d = 0$. Thus, without loss of generality, we can assume that $f(0) = 0$ and that $F(f(0), f(0), \dots f(0)) = F(0, 0, \dots 0) = 0$.

Since F is symmetric and differentiable as a function of $f(d_1), f(d_2) \dots f(d_m)$, the expansion of an analytic symmetric function via elementary symmetric functions gives

$$F(A) = C_1(n) \left(\sum_{i=1}^m (f(d_i)) \right) + C_2(n) \left(\sum_{i=1, i \neq j}^m \sum_{j=1}^m (f(d_i) f(d_j)) \right) + C_3(n) \left(\sum_{i=1}^m (f(d_i))^2 \right) + \text{higher order terms in the } f(d_i)\text{'s.} \dots \text{(I)}$$

C is some constant dependent on n (See Glaeser (1963: 205–206)).

Since F has non-zero partial derivatives at 0 as a function of $f(d_1), f(d_2) \dots f(d_m)$, we must have $C_1(n)$ different from 0, and without loss of generality, we can put $C_1(n) > 0$.

Now consider A with $d_1 = d_2 = \dots = d_j = d$, where $j \leq m$. If $j < m$, then put $d_{j+1} = d_{j+2} \dots d_m = 0$.

The equality (I) gives that

$$F(A) = jC_1(n)f(d) + \{C_2(n)j(j-1) + C_3(n)j\}(f(d))^2 + \text{higher order terms in the } f(d)^2 \text{ s. } \dots \text{ (II)}$$

The number of $f(d_i)f(d_j)$:s is $j(j-1)$.

Since we consider the case $n \geq 3$, which gives $n(n-1) = m \geq 6$, we have $m(m-1) > m$. We also observe that for higher order terms in (I) the following is true: the number of $(f(d_i))^{g_1}(f(d_i))^{g_2} \dots (f(d_i))^{g_p}$:s in terms of degree $g_1 + g_2 + \dots + g_p$ in the $f(d_i)$:s is equal to $m(m-1) \dots (m-p+1)$, which is $> m$ if $p > 1$.

Now, if we multiply all the distances in A by the same number k , we will multiply $F(A)$ by the number $C = C(k)$. This has to hold equally for all $j \leq m$. It is easy to see, because of the $j(j-1)$ factor, that this implies $C_2(n) = 0$. Correspondingly, for higher degree terms, we get that the coefficients in front of the $(f(d_i))^{g_1}(f(d_i))^{g_2} \dots (f(d_i))^{g_p}$:s terms, which have $p > 1$, must be 0. Putting this information into (II) we get $d_1 = d_2 = \dots = d_m = d$. Thus,

$$F(A) = m(a_1f(d) + a_2(f(d))^2 + a_3(f(d))^3 + \dots) \dots \text{ (III)}$$

Here $a_1 = C_1(n)$, $a_2 = C_2(n)$ etc. In (III), multiplying d by a number k will multiply $F(A)$ by a number $C = C(k)$. It is a well-known mathematical fact that this implies $a_1f(d) + a_2(f(d))^2 + a_3(f(d))^3 + \dots = \alpha d^r$.

Putting this information back into (I) we get

$$F(A) = \left(a_1 \sum_{i=1}^m f(d_i) + a_2 \sum_{i=1}^m (f(d_i))^2 + a_3 \sum_{i=1}^m (f(d_i))^3 + \dots \right) = \alpha \sum_{i=1}^m d_i^r$$

We recall that the *Option dominance condition* implies that this function is non-negative and increasing so we get that $\alpha > 0$ and $r > 0$.

Now, the *Option dominance condition* implies that f is strictly increasing and the *Limited evenness conditions* imply that f is strictly concave, and so $0 < r < 1$. This means that, in (III), we get

$$F(A) = \alpha \sum_{i=1}^m d_i^r, \text{ where } \alpha \text{ is a constant } > 0.$$

The *Limited growth condition* gives that $\alpha n(n-1)$ cannot grow faster than bn for some constant b . The *Proportional growth condition* gives, more precisely, that $\alpha = 1/(n-1)$. So we see that the *Symmetry condition*, the *Scale-independence condition*, the *Proportional growth condition*, the *Limited growth condition*, the *Option dominance condition* and the *Limited evenness condition* imply that F has to be a *Ratio root measure* with $0 < r < 1$. The *Strict monotonicity condition* then follows, as proved above. It

is only by choosing r such that $\frac{1}{2} \leq r < 1$ that the *Spread condition* is satisfied. If we choose $r < \frac{1}{2}$, the *Spread condition* will fail between $n = 2$ and $n = 3$, which is easy to calculate. It is easy to see that the formula for $F(A)$ has to hold also for the case $n = 2$. Finally, the satisfaction of the *No freedom of choice condition* gives the additional stipulation that for $n < 2$, $F(A) = 0$.

Q.E.D.

Note: If we do not assume that F has non-zero partial derivatives at 0 as a function of $f(d_1), f(d_2) \dots f(d_m)$, then we cannot draw the conclusion that $C_1(n)$ is different from 0. This may give us other measures that satisfy all conditions. However, to compute these measures we would have to add many more terms than we have for the *Ratio root measures*. If $C_2(n)$ is different from 0, then the expression will contain $m(m - 1)$ terms, which is of the order of n^4 terms, whereas a *Ratio root measure* contains terms of the order of n^2 terms. Also, except for the terms $(f(d_i))^2$, the expression:

$\sum_{i=1, i \neq j}^m \sum_{j=1}^M f(d_i)f(d_j)$ is equal to the expression $\left(\sum_{i=1}^m f(d_i) \right)^2$, so these measures will be closely related to the *Ratio root measures*.

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