No Arbitrage Pricing and
the Term Structure of
Interest Rates

by

Thomas Gustavsson

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Department of Economics
Uppsala University

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Abstract

This dissertation provides an introduction to the concept of no arbitrage pricing and probability measures. In complete markets prices are arbitrage-free if and only if there exists an equivalent probability measure under which all asset prices are martingales. This is only a slight generalization of the classical fair game hypothesis. The most important limitation of this approach is the requirement of free and public information. Also in order to apply the martingale representation theorem we have to limit our attention to stochastic processes that are generated by Wiener or Poisson processes. While this excludes branching it does include diffusion processes with stochastic variances.

The result is a non-linear arbitrage pricing theory for financial assets in general and for bonds in particular. Discounting of future cash flows is performed with zero coupon bonds as well as with short term interest rates (roll-over). In the presence of bonds discounting is an ambiguous operation unless an explicit intertemporal numeraire is defined. However, with the proper definitions we can dispense with the traditional expectations hypothesis about the term structure of interest rates. Arbitrage-free bond prices can be found simply from the fact that these are assets with a finite life and a fixed redemption value.¹

¹Note for the current reader: Unfortunately there are some serious mathematical errors in sections 5.2 and 6.2 of this work. In particular the single forward-neutral measure described here is confused with the family of forward-neutral measures introduced by El-Karoui and Geman (1991). This is not a simple matter since it involves some rather delicate problems with the economic behavior of market participants and their attitudes towards risk over time and intertemporal pricing of bonds. In more recent work I show that the single forward-neutral measure described here can, in fact, be identified with the original probability measure, denoted by $Q$ in this text.
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1 Introduction

Similarities between gambling and the trading of financial assets are sometimes considered to discredit the respectability of financial markets. Quite to the contrary I would say. This dissertation shows in detail how the gambling aspect of the behavior of traders can enable them to reach a consensus on the current value of any number of uncertain future prospects. The evaluation procedure is independent of the preferences of traders with respect to risk and investment horizons. To simplify matters we restrict ourselves to the case when all relevant information is public or symmetric among traders. If asset prices fully reflect all relevant information no trader should be able to earn excess returns from trading rules based on historical information - whether public or private. This is known as the efficient market hypothesis. Early examples of this type of approach can be found in Cootner (1964), for an interesting survey see Fama (1970). The basic idea was exploited in a number of empirically oriented papers during the sixties and the seventies. Typically it was claimed that, in "efficient" markets, prices or rates of return should be serially uncorrelated or follow random walks. Unfortunately each researcher only tested his own particular version of how to "beat the market". Comparisons were rare, and most of the suggested empirical rules of thumb were unfounded. The remaining result of these efforts seem to be mainly methodological. They started a strong empirical tradition of research on asset pricing. Several connections were made to traditional statistical methods, and, in particular, to the theory of "fair" games. In a fair game the expected value of winnings and losses should be zero. Alternatively the expected value of the gambler’s fortune should always equal its current value, i.e. the evolution of his fortune over time should follow a martingale. This provided a key link to the development of more general pricing principles for "efficient markets". One of the most popular is known as no arbitrage pricing (or as arbitrage-free pricing).

No arbitrage pricing is an invariance principle for markets with public information. No arbitrage means that all opportunities to make a riskfree profit have been exhausted by traders. This should certainly be a basic requirement for an "efficient" market. As a result of the arbitrage activities relative prices will be constrained. In the case of complete markets the basic theorem of no arbitrage tells us exactly how. Intuitively the theorem claims that any asset price must equal the expected value of its discounted future cash payoffs to preclude arbitrage. This is a surprisingly strong result. It should, however, be noted that for this to hold we cannot calculate the expected value with respect to any probability measure. We have to construct a very special probability measure for this to be true. This is the fundamental difference between no arbitrage pricing and the concept of a fair game. No arbitrage prices can be calculated under fairly general conditions. All involved stochastic processes should have finite variance and expectation. Basically what follows here is an elaboration of this result. Clearly the no arbitrage pricing principle is a statement about the development of asset prices in relation to each other over time. Neither forward
nor spot prices need follow martingales, see Lucas (1978). Instead the focus is on relative prices. This is where discounting enters. The role of discounting is to cancel out any common time trends in absolute prices. Disregarding growth trends in this way no arbitrage means that trading is in some sense a fair game. Although the actual odds need not be fair it should be possible in principle to tilt the odds slightly and get an equivalent game that is fair. As noted before the gambler’s fortune in such an equivalent game will follow a martingale. Therefore the constructed probability measure is known as an equivalent martingale measure. The general theory of no arbitrage pricing and its relation to the famous mathematical theorem of separating hyperplanes (Hahn-Banach theorem) was first developed by Ross (1976 and 1978). He did not make the connection to fair games and equivalent martingale measures. This was done by Harrison and Kreps (1978), and Harrison and Pliska (1981 and 1983). Duffie and Huang (1985) showed the power of the martingale toolbox to replace dynamic programming. The equivalent martingale approach generalizes traditional capital asset pricing models. Optimal portfolio rules can be found in Cox and Huang (1990). In the special case of constant interest rates the no arbitrage principle is also called the risk-neutral evaluation principle. This principle was made famous by option pricing.

This dissertation provides a systematic introduction to no arbitrage pricing of financial assets in general and to that of bonds in particular. The pricing of (zero coupon) bonds is often referred to as the term structure of interest rates, the TSIR for short. For a long time bonds have been treated as an isolated topic dwelling in a maze of technical detail. Here the purpose is to show how bonds fit into the general framework of no arbitrage pricing. What makes bonds special? How can bonds be used for the discounting of future cash flows? In what way do stochastic interest rates influence the pricing of bonds and other assets? What role is played by the term structure of interest rates? Do we need the traditional expectations hypotheses about the term structure? In particular, how do prices of long term bonds relate to short term interest rates? What is the role of local risk-neutrality? These are the main questions that will be discussed here.

Ho and Lee (1986) were the first to use the current TSIR for no arbitrage pricing of bonds. They used an event-tree approach (a binomial model) with both discrete time and discrete state space. Unfortunately their model suffered from some inconsistencies. For example, they did not rule out the possibility of negative interest rates. The first consistent treatment of bond pricing and the TSIR was done by Heath, Jarrow, and Morton (1987, 1989, published in 1992), and (independently) by Artzner and Delbaen (1989). Unfortunately Heath, Jarrow, and Morton did not relate their model to the basic theorem of no arbitrage. Instead they chose to start from scratch using a framework completely unique to bonds. They mapped bond prices onto implied forward spot rates from the current TSIR and derived a new form of the basic no arbitrage theorem within their particular framework. This resulted in a drift condition for arbitrage-free bond pricing which refers to roll-over cash as the numeraire. As will be shown
here the essence of their approach comes out more naturally when using bonds
as the numeraire (no local risk-neutrality). This approach also brings out the
fundamental role of implied forward rates in the pricing of other assets. Geman
(1989) pioneered the use of bonds as discount factors in a general no arbitrage
pricing framework. She showed how discounting future cash flows with bonds
from the current TSIR corresponds to a particular choice of numeraire in an
intertemporal model of asset prices. To properly understand the role of the
TSIR we have to go beyond the cash price convention prevailing in finance and
explicitly identify the micro-economic concept of an intertemporal numeraire.
This provides an interesting analytical alternative to rolling-over of money at
short term rates of interest, the traditional choice of numeraire. As we shall see
both alternatives are equally valid ways to discount future money prices. So why
prefer short term roll-over to pure discount bonds? Indeed, it does not seem to
be widely recognized that the simultaneous existence of these two alternatives
makes discounting ambiguous in terms of money!

In a way the approach of this dissertation follows that of Artzner and Delbaen
(1989). They started with the well-established general theorem of no arbitrage
and subsequently derived the pricing of bonds as a special case. But they too
used roll-over money as numeraire (local risk-neutrality), and formally they only
price one bond in relation to short term interest rates (roll-over money). This
dissertation attempts to close the gap by showing that the general no arbitrage
pricing approach results in the same prices and the same drift conditions as
can be found in Heath, Jarrow, and Morton (1992). Also several of the results
derived for bonds by Artzner and Delbaen (1989) are shown here to hold for
any type of asset. In addition, their results are extended to the case of no local
risk-neutrality using discount bonds as numeraire. Furthermore, I elaborate on
the economic interpretation of the results, hopefully making them accessible to
a wider audience.

The basic method used here is stochastic calculus. It must be remembered
that the theoretical calculations ignore important empirical aspects of asset
pricing. In particular, transaction costs, bid-ask spreads and differences between
lending and borrowing rates of interest are not considered. There is no rationing
of credits and all assets are assumed to be infinitely divisible. This is a suitable
framework only for those who trade regularity in markets with high turnover.
Another limitation is that all relevant information is assumed to be public. This
means that all traders have free access to the same information. Obviously this
ignores the possibility of gaining more information by trading more, paying
extra for forecasting services, or paying for access to privileged information.

Section 2 provides an introduction to the concept of no arbitrage pricing
within a single period framework. In particular, its relationship to the existence
of implicit state prices and probability measures is explained in detail. Section
3 takes on the same topics in a multi-period framework. The key concept here
is that of a self-financing trading strategy. This is followed by a statement and
a proof of the basic theorem of no arbitrage pricing. I relate the concept of no arbitrage to fair games and martingales. Several definitions are provided in order to increase the readability of the proof. Special care is taken to isolate the economic arguments from the mathematical foundations which are delegated to an Appendix. After these preliminary efforts the different methods of discounting and their relation to particular choices of an intertemporal numeraire are described in section 4. Here arbitrage-free prices are derived in their most general form. In continuous time these results can be strengthened, if we’re willing to make specific assumptions about the nature of the flow of information over time. This is done in section 5. With continuous Wiener processes generating the flow of information more specific pricing results are obtained using the martingale representation theorem. In particular, this assumption completely determines arbitrage-free bond prices. To prove this calculations are made with local risk-neutrality and without it. For the second ”locally stochastic” case this has not been done before as far as I know. The resulting bond prices in both cases are shown to satisfy the drift condition in Heath, Jarrow, Morton (1992 p 94). Thus their model is derived here as a special case of the general pricing of assets and bonds in particular. Furthermore, in section 6, the general form of the drift term for arbitrage-free prices is shown to be a non-linear Asset Pricing Theory, cp the linear APT of Ross (1976). This completely determines the market price of risk. Finally, using this result, the validity of the traditional hypotheses about the term structure of interest rates is examined. In contrast to Cox, Ingersoll, Ross (1981) I find that both the local riskneutral hypothesis and the unbiased forward-neutral hypothesis are compatible with general equilibrium.
2 Single period market

A financial market consists of a fixed number $N$ of assets with random future payoffs $A_n, n = 1, 2, \ldots, N$, and their current prices, a column vector $S = (S_1, \ldots, S_N)$. To begin with let there be a finite number of future states

$$
\Omega = (\omega_1, \ldots, \omega_M)
$$

(2.1)

In this case the future payoffs at the end of the period are often written as a matrix

$$
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1N} \\
a_{21} & a_{22} & \cdots & a_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
a_{M1} & \cdots & a_{MN}
\end{pmatrix}
$$

(2.2)

Here each column is a vector showing the nominal (cash) payoffs of an asset in all of the $M$ different states. This is sometimes called the state-space tableau. Typically each asset is a stock and in view of limited liability all its future payoffs are non-negative. The market is called complete if it is possible to obtain any future payoff profile by trading and combining the available assets in different proportions. For this to be possible there must, of course, be at least as many assets as states. However, just counting the number of states and assets is not enough, as in the case of finding solutions to linear equation systems. A necessary and sufficient condition for the market to be complete is that the rank of the matrix $A$ is equal to $M$, the number of possible states.

In this context no arbitrage has a very clear meaning. The prices $S$ preclude arbitrage with respect to the market $A$ if you can’t get something for nothing (or less than nothing). In other words there does not exist any portfolio $\theta = (\theta_1, \ldots, \theta_N)$ with non-negative payoffs in all states and a negative market value. No arbitrage means that any such portfolio should have a positive cost, i.e.

$$
\sum_{n=1}^{N} \theta_n a_{mn} \geq 0 \text{ for } m = 1, \ldots, M \Rightarrow \sum_{n=1}^{N} \theta_n S_n > 0
$$

(2.3)

In order to ensure positivity at least one of the leftmost inequalities must be strict. Clearly a portfolio that does not pay anything should have zero cost.

This simple concept of no arbitrage pricing also has a clear geometric meaning. If the portfolio payoffs are non-negative in each state $m$ and positive in at least one, the vector $\theta A$ of portfolio payoffs lies in the positive convex cone spanned by the columns of $A$. Then according to implication (2.3), the price vector $S$ must form an acute angle to the portfolio vector $\theta$. (The rightmost sum in (2.3) is just the scalar product $\theta S$.) We say that $S$ lies outside the orthogonal complement to the cone spanned by $A$. But this is just the negative cone spanned by the transpose of $A$. So to preclude arbitrage there must exist an implicit state price system $\lambda = (\lambda_1, \ldots, \lambda_M) > 0$ such that

$$
S_n = \sum_{m=1}^{M} \lambda_m a_{mn} \text{ for } n = 1, \ldots, N
$$

(2.4)
The implicit prices give the current value of a future dollar in each of the states. If the market $A$ is complete they will all be positive. In this case no arbitrage is equivalent to condition (2.4). The strict form of this statement (not given here) is known as Farka’s lemma, see e.g. Gale (1964).

Is this evaluation procedure unique for any fixed price vector $S$ in the given market $A$? In other words, could there exist another implicit state price system? Yes, in general it could. For example, if the market is not complete the rank of the matrix $A$ will be less than $M$ and no unique shadow price solution need exist. Also restrictions on short sales (some negative values for the components of $\theta$ not permitted) may preclude uniqueness. Finally it is interesting to note that transaction costs too may spoil the uniqueness by driving a wedge between the prices on individual assets and that of replicating portfolios.

2.1 Probabilistic interpretation

There is an important relation between no arbitrage and probability measures $Q$ on the sample space $\Omega$. Each row $m$ in the matrix $A$ represents one possible outcome $\omega_m$ of a random vector $A = (A_1, A_2, \ldots, A_N)$, where $a_{mn} = A_n(\omega_m)$. Thus each asset in the market can be identified with a random variable on the probability space $(\Omega, Q)$. The market $A$ becomes a vector of random variables. Pursuing this interpretation we find that each weighted average of column elements in (2.4) is the expected value of the random variable $A_n$ except for a scale factor. Although state prices are always positive (and less than one) their sum need not equal one. So in general they are not probabilities, i.e., positive numbers between 0 and 1 defined for all events in $\Omega$ and summing to one. But this is easily taken care of. Define a probability measure $Q^*$ for all elementary events in $\Omega$ in the following way

$$Q^*(\omega_m) = \frac{\lambda_m}{\|\lambda\|_1} \text{ where } \|\lambda\|_1 = \sum_{m=1}^{M} \lambda_m$$

(2.5)

The denominator ensures that these new numbers sum into one. Being defined for all subsets of $\Omega$ and additive they are obviously probabilities. (For any event, i.e., for any subset of $\Omega$, simply sum the elementary events involved.) Formally, a numerically valued set function $Q$ is a probability measure provided

$$i) \quad Q(B_k) \geq 0 \quad \text{for all subsets } B_k \in \Omega$$

$$ii) \quad Q(\bigcup B_k) = \sum Q(B_k) \quad \text{for disjoint subsets } B_k$$

$$iii) \quad Q(\Omega) = 1$$

(2.6)

In general the subsets $B_k$ can be any combination of the elementary events $\omega_m$. Clearly $Q^*$ as defined in (2.5) fulfills these requirements.

Next, we define mathematical expectation with respect to the new probability measure $Q^*$. The expected value of the (discrete) random variable $A_n$ with
respect to the probability measure $Q^*$ is

$$E^*[A_n] = \sum_{m=1}^{M} A_n(\omega_m)Q^*(\omega_m) = \sum_{m=1}^{M} a_{mn}Q^*(\omega_m)$$

(2.7)

Using this and definition (2.5) of $Q^*$ the no arbitrage condition (2.4) can be written as

$$S_n = E^*[\beta \cdot A_n] \text{ where } \beta = \| \lambda \|_1$$

(2.8)

The scale factor $\beta$ has a very interesting interpretation. It is the current price of a portfolio that pays $1$ for sure in each of the future states. In other words it is simply a discount factor. To see this set $A_n \equiv 1$ for all $n$. If markets are complete such a portfolio can always be constructed by combining the given assets $A_n, n = 1, 2, \ldots, N$. Otherwise the vector $(1, 1, \ldots, 1)$, with $M$ components, need not lie in the cone spanned by the columns of $A$. Typically the current value of a sure future $1$ will be less than one. With this in mind the right hand side of equation (2.8) is the expected value of future discounted asset payoffs. Thus, a necessary and sufficient condition for no arbitrage is that all current prices can be represented as their expected future discounted payoffs with respect to such a probability measure $Q^*$. This is a very important result. Most of the remainder of this text is concerned with re formulations and generalizations of this basic statement.

### 2.2 Continuous payoffs

The probabilistic characterization of no arbitrage is useful for generalizations to assets with infinitely many future payoff values. Each asset now becomes a continuous random variable on some probability space $(\Omega, Q)$. Here $Q$ is a given probability measure on the sample space $\Omega$. We restrict our attention to random variables with finite mean and variance. A contingent claim with an uncertain future payoff can be identified with some non-negative random variable $X$. By convention we choose to define contingent claims as having non-negative payoffs. As in the discrete case we start with a finite set of assets $A = (A_1, A_2, \ldots, A_N)$ and their respective initial prices $(S_1, S_2, \ldots, S_N)$. All possible portfolio combinations of the given $N$ assets form a linear subspace of continuous random variables. These contingent claims are called attainable. The set of all attainable claims can be written as

$$L[A] = \{X : X = \sum_{n=1}^{N} \theta_n A_n \text{ and } Q(X \geq 0) = 1\}$$

(2.9)

where the portfolio vectors $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ may have positive, zero, as well as negative components (as long as the sum $\theta A$ is non-negative). $L[A]$ is a positive convex cone spanned by the given assets (former column vectors).

In general, there is an infinite number of contingent claims in $L[A]$. Despite this we cannot be sure that any contingent claim can be written as such a finite
combination. Whether this is possible or not depends on the payoff characteristics of the "basis" assets \( A = (A_1, A_2, \ldots, A_N) \). As before the market \( A \) is called \textit{complete} if all contingent claims are attainable, i.e. if any (non-negative) payoff profile can be obtained by forming portfolios of the \( N \) basis assets. Clearly this formulation abstracts from indivisibilities, bid-ask spreads, or other transactions costs. So by their very construction these concepts are only relevant to standardized markets with high turnover.

Attainable claims can be evaluated using an implicit price system \( \pi \). This can no longer be a finite dimensional \( N \)-vector \( \lambda \) as in the discrete case. Instead we identify the implicit price system \( \pi \) with a non-negative linear functional

\[
\pi : L[A] \to R_+ \\
\pi(X) = \pi(\sum_{n=1}^N \theta_n A_n) = \sum_{n=1}^N \theta_n \pi(A_n)
\] (2.10)

This is a compact notation for the whole system of implicit prices. The word functional means that \( \pi \) is a function of a function. (So called random "variables" \( X \) are actually functions \( X : B \to R \) defined on subsets \( B \) of the state space \( \Omega \)). The definition assigns a unique price, the non-negative number \( \pi(X) \), to each attainable payoff profile \( X \). Buying a multiple \( c \) of the contingent claim \( X \) has the same payoff as buying the contingent claim \( cX \), which is also attainable for \( c \geq 0 \). As both alternatives have the same payoff \( cX \) their prices will be assigned the same number according to the definition of \( \pi \). So \( c\pi(X) \) must be equal to \( \pi(cX) \). Also buying \( X \) and \( Y \) you get the same payoff as from buying the contingent claim \( X + Y \). So these alternatives must have the same price, i.e. \( \pi(X) + \pi(Y) = \pi(X + Y) \). Thus, the implicit price system is linear. Furthermore, the system should be defined in such a way as to be consistent with the given market prices \( S_n \) of the assets forming the market, i.e.

\[
\pi(A_n) = S_n \text{ for } n = 1, \ldots, N
\] (2.11)

The consistent implicit price system \( \pi \) can be used to evaluate any attainable claim with future payoff \( X \). The value is simply the current value of the duplicating portfolio, i.e.

\[
\pi(X) = \pi(\sum \theta_n A_n) = \sum \theta_n \pi(A_n) = \sum \theta_n S_n
\] (2.12)

This is a very useful evaluation procedure. In general there could be many different implicit price systems \( \pi \) that are consistent with the initial market prices \( S \) and the given future payoffs \( A \). But we cannot be sure that all of them will result in the same value of the duplicating portfolio in (2.12). If they do not then we can find two different price systems and two different duplicating portfolios. This is an arbitrage opportunity. Riskfree profits can be made from buying the cheapest portfolio while selling the more expensive one. We want to find prices that preclude arbitrage. What distinguishes arbitrage-free prices? To sort this out we must first reformulate the definition of no arbitrage (2.3) in probabilistic terms: Any attainable contingent claim \( X \) with
a positive probability for positive payoffs and a zero probability for negative payoffs must cost something, i.e.

\[ Q(X \geq O) = 1 \text{ and } Q(X > O) > 0 \Rightarrow \pi(X) > 0 \quad (2.13) \]

For payoffs that are non-negative almost everywhere \( Q(X \geq O) = 1 \), so we have

\[ E(X) = 0 \Leftrightarrow Q(X = 0) = 1 \text{ and } E(X) > 0 \Leftrightarrow Q(X > O) = 1 \quad (2.14) \]

where the expectation is taken with respect to the probability measure \( Q \). Using this (2.13) can be written more compactly: the price system \( \pi \) is \textit{arbitrage-free} if you cannot expect to get something for nothing, i.e.

\[ E[X] > 0 \Rightarrow \pi(X) > 0 \quad (2.15) \]

or, equivalently, if any future zero net payoff has zero cost, i.e.

\[ \pi(X) = 0 \Rightarrow E[X] = 0 \quad (2.16) \]

Note that in combination with (2.14) we even have

\[ \pi(X) = 0 \Leftrightarrow E(X) = 0 \quad (2.17) \]

According to this relation it is impossible to find two duplicating portfolios for \( X \) that does not have the same cost. Thus for arbitrage-free prices the evaluation procedure (2.12) results in a unique current value of any contingent claim \( X \).

To be able to perform the calculations we must devise a procedure for finding at least one arbitrage-free price system \( \pi \). How can this be done? Following Harrison and Kreps (1979) and Harrison and Pliska (1981) we show how to identify the implicit price system \( \pi \) with a particular probability measure \( Q^* \). This uses standard mathematical results from Riesz representation theorem, see Harrison and Kreps (1979), and Harrison and Pliska (1981, Proposition 2.6). In the discrete case a probability measure was defined by (2.6) for all subsets \( B \) of the sample space \( \Omega \). For continuous random variables we have to restrict our attention to those subsets that are included in the \( \sigma \)-algebra used to define \( Q \), the given probability measure. Call this \( \sigma \)-algebra \( F \) (The word \( \sigma \)-algebra is used in measure theory for classes of subsets that are closed with respect to set complementation and infinite unions.) Use the characteristic set function \( 1_B \) to define \( Q^* \) for any \( \beta > 0 \) as

\[ Q^*(B) = \pi(1_B/\beta) \text{ all } B \in F \quad (2.18) \]

As \( \pi \) is linear the rest of definition (2.6) is trivially satisfied. The additional positive scale factor \( \beta \) is needed because the functional \( \pi \) was restricted by the consistency requirement (2.11). Otherwise this will conflict with the basic requirement of a probability measure \( Q^*(\Omega) = 1 \). Defining the expected value with respect to the constructed measure \( Q^* \) we find

\[ \pi(X) = E^*[\beta \cdot X] \quad (2.19) \]
As $\pi(1) = E^*[\beta \cdot 1]$ we can interpret $E^*[\beta]$ as a discount factor just like in the discrete case. Note that to define a discount factor we do not have to assume a constant rate of interest, see Harrison and Pliska (1981 p 225, second paragraph). $\beta$ may be any random variable as long as it is strictly positive. In particular, we can use anyone of the model assets to define the discounting provided its payoffs are strictly positive. For example, if asset number one has strictly positive payoffs $A_1$, then we may define a discount factor from $\beta = S_1/A_1$. Notice that here there is a loss of one degree of freedom involved.

When we identify the linear functional $\pi$ with a probability measure $Q^*$ there must be a normalization procedure to ensure that the "probabilities sum to one", i.e. $Q^*(\Omega) = 1$.

Combining (2.19) with (2.17) we find that the constructed probability measure $Q^*$ and the original measure $Q$ have the same null sets, in other words $Q^*(B) = 0$ if and only if $Q(B) = 0$, for all $B \in F$. When this is the case $Q$ and $Q^*$ are called equivalent measures. Inserting (2.19) in (2.11) we find that under the equivalent measure $Q^*$ the price of any asset equals the expected discounted value of its future payoffs, i.e.

$$E^*[\beta \cdot A_n] = S_n$$

(2.20)

In particular for the asset used to define the discount factor $\beta$ we get

$$E^*[\frac{S_1}{A_1} \cdot A_1] = S_1$$

(2.21)

As a consequence any attainable contingent claim $X$ can be evaluated as the expected discounted value of a portfolio that duplicates its future payoffs. As $X$ is attainable there exists a portfolio $\theta X = \theta A$. Combining (2.19) and (2.12) we find the arbitrage-free price of $X$ as the current value of the duplicating portfolio, i.e.

$$E^*[\beta X] = E^*[\beta \theta A] = \theta E^*[\beta A] = \theta S$$

(2.22)

Furthermore all equivalent measures result in the same value because of the consistency requirement (2.20). Thus we have shown that no arbitrage implies the existence of at least one probability measure $Q^*$ satisfying (2.20). On the other hand, if such an equivalent probability measure exists it is easy to see that the no arbitrage conditions (2.15) or (2.16) are satisfied. In conclusion, no arbitrage is equivalent to the existence of such a probability measure $Q^*$. Note that no arbitrage is defined with respect to undiscounted payoffs and the original (given) probability measure $Q$ while the no arbitrage condition (2.20) refers to discounted payoffs and the constructed measure $Q^*$.

Unfortunately this simple pricing principle only applies to attainable claims. We would like to extend it so that all contingent claims can be priced in this way, thereby precluding arbitrage. Clearly this can be done if markets are complete. Then any contingent claim $X$ is attainable, i.e. its payoff can be duplicated with a portfolio of the given assets $A_1, A_2, \ldots, A_N$ as in (2.9). But, in general, we
don’t know when a random variable can be written as a finite linear combination of some given subset of other random variables. Put differently, we do not know what classes of stochastic processes that are closed with respect to addition and multiplication with scalars. This is still an open mathematical problem. An introduction to this interesting topic is presented in the Appendix. In the text we limit ourselves to complete markets for which known results are available.

2.3 Viability

In the previous section there were no references to preferences. Does this mean that no arbitrage pricing is preference-free? Yes and no. Implicit in the no arbitrage approach is the assumption that there exists traders who pursue asset trading for some purpose. Obviously they must be able to rank opportunities over all states, always preferring more to less, exploiting any arbitrage opportunity. But for this to be a viable model of economic equilibrium we need to make more assumptions about the preferences of traders and the initial distribution of wealth. The traditional micro-economic approach is to consider the purpose of asset trading to be the transfer of wealth or consumption between the beginning and the end of the period. Traders, or as they usually are called here, consumers, are able to rank bundles of current and future uncertain consumption. A representative consumer is assumed to maximize expected utility subject to some budget condition. The expected utility is defined as the weighted average of exogenously given (subjective) probabilities $Q$ and the corresponding utility of the payoff for each state. Acting as a price-taker the consumer adjusts his net trades in different assets until an optimum is found. This gives equilibrium market prices $S$ as a function of the probabilities $Q$, the preferences and the initial distribution of wealth. Thus, by its very construction such an economic equilibrium is arbitrage-free. The implicit state price system $\pi$ simply consists of the marginal rates of substitution between consumption in different states weighted by their respective probabilities, for details see Ross (1989, example 1.1).

Starting from exogeneously given preferences and a given probability measure $Q$ to derive “equilibrium” market prices $S$ may look like the exact opposite of the no arbitrage approach. And it is sometimes argued that such a “general equilibrium” approach is more fundamental than the no arbitrage approach. For some reason the theorist wants to “explain” why prices change by relating them to more ”fundamental” economic factors. But the relationship between the two modelling approaches is really more subtle. In complete markets no arbitrage implies the existence of a unique probability measure $Q^\ast$, which is derived from the exogenously given prices $S$ (and the market payoff structure $A$). Clearly these probabilities can be used to construct the preferences of a representative consumer. All that is needed is to specify some utility function that generates the same marginal rates of substitution as those implied by the given asset prices. Subject to some integrability conditions such a ”revealed preference” approach should always be possible. Thus the question really is: When
does this procedure fail? Clearly, when markets are not arbitrage-free we will not be able to find any unique set of implied state prices or any probabilities. So neither could there exist any unique representative consumer! Turning the procedure around, starting with a given set of preferences and probabilities, it would still be possible to define a representative consumer and derive equilibrium prices. But these derived prices can only be arbitrage-free if markets are not complete. Is such an equilibrium a pathological special case or an interesting object for economic analysis? It is too early to provide any definite answer to this. Current work on the reconstruction of preferences from no arbitrage under different assumptions about the nature of the stochastic price processes include among others Duffie and Huang (1985), Duffie (1986), Mas-Colell (1986), Huang (1987), Cox and Huang (1989), and Mas-Colell and Richard (1991).
3 Multiperiod markets

A multi-period market model with a continuous state space $\Omega$ is a more complicated concept than the single-period model previously described. Here the probabilistic version of no arbitrage is taken even further using axiomatic probability theory. The basic element of this approach is an abstract specification of the flow of information over time by means of an increasing family of $\sigma$-algebras

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_t \subseteq F_{t+1} \subseteq \cdots \subseteq F_T \quad (3.1)$$

where each $F_t$ is a set of subsets of the state space $\Omega$ indexed by time $t$. This is also known as a filtration of the probability space. Each set $F_t$ may simply be perceived of as all available information at the respective date $t$. In particular this includes knowledge on past and present prices. As long as nothing is forgotten and there is never any false information it seems rather plausible that the information sets should grow over time. Karlin and Taylor (1975) provides a good introduction to these concepts, in particular see the example on p 300f. The choice of equidistant integers to mark time here is just made to simplify the notation. Clearly any partition of the time interval $[0, T]$ will do (or a continuous time index).

The economic intuition for this setup is that trading starts at date 0 and proceeds at a series of subsequent dates until some future date $T$ when all activity ceases. Note that the terminal date $T$ is fixed in advance. In the beginning traders only know what possibly could happen and what cannot, i.e. they only know the state space $\Omega$ equipped with $F_0$. In addition, they are able to calculate the probabilities for different events. (In other words, there is an exogenously given probability measure $Q$ on $\Omega$ which is known to all traders). As time goes by events (i.e. subsets of $\Omega$) accumulate and traders learn more and more about the economy. This will lead them to revise their probability assessments for different events as they redo their calculations conditional upon the received information $F_t$. It should be realized that the information is assumed to flow freely, being instantly available to everybody, i.e. all relevant information is public. This is an important and restrictive assumption. For example, it excludes trading in order to exploit insider information as well as trading to learn more about the true state of the market. In these markets there are never any false signals and “bluffing” never pays.

After this description of how information changes over time we now turn to prices and payoffs. As before we assume that there exists a finite number $N$ of traded assets. However, the payoff structure now involves time. For each date $t$ there is a new payoff structure. In the single period case the letter $S$ was used for the current price and another letter $A$ for the future payoff. Here it is more convenient to use the same letter $S$ for all dates. The future payoff of an asset is simply its future price plus any intermediate payments such as dividends. We may identify the beginning of the single period with $t = 0$ and its end with $t = T$. Formally the single period case can be written as

$$S_n(0) = S_n \text{ and } S_n(T) = A_n \text{ for } n = 1, \ldots, N \quad (3.2)$$
Opening up the possibilities of asset trading at intermediate dates will change asset prices over the period \([0, T]\) as a function of the information flow \(F_t\). In general, asset prices are defined as a time dependent stochastic vector \(S(t) = (S_1(t), S_2(t), ..., S_N(t))\). We identify each component of the vector with a separate asset \(n\). This extends the previous concept of an asset as having uncertain payoffs on only one future date \(T\). Here each asset is a stochastic process. The stochastic vector price process will be written as \(\{S(t) : 0 \leq t \leq T\}\), without the asset subscript, or simply as \(S(t)\). For this framework a portfolio of assets is no longer a question of choice at one moment of time alone. Any trader may revise, his holdings at any moment of time. To describe such dynamic portfolio strategies and tie them up with the basic definitions of attainability, completeness, no arbitrage and consistency is a rather delicate procedure. This is the topic of the next section.

### 3.1 Trading strategies

A dynamic portfolio strategy is traditionally called a *trading strategy*. This is a vector process \(\{\theta(t), t = 0, \ldots, T\}\) which is adapted to the given filtration, i.e. \(\theta(t)\) is \(F_t\)-measurable for all \(t\). The trading strategy is a vector with components \(\{\theta_n(t), n = 0, 1, 2, \ldots, N\}\), one for each asset. The trader starts each period with a known value \(\theta(t-1)S(t-1)\) of his old portfolio, and on the basis of the available information \(F_{t-1}\), he revises his holdings into \(\theta(t)S(t-1)\). The outcome of his choice \(\theta(t)S(t)\) is not known until the start of the next period when he learns about the new price vector \(S(t)\) as part of \(F_t\).

A trading strategy is called *self-financing* if no funds are added to or withdrawn from the portfolio after the initial investment \(\theta(0)S(0)\). This means

\[
\theta(t)S(t) = \theta(t-1)S(t) \tag{3.3}
\]

All funds need not necessarily be invested in risky assets. Excess funds may be stored/lent. We allow the trader to transfer purchasing power (borrow or lend) as long as his overall net position is not one of debt, i.e. we require \(\theta(t)S(t) > 0\), all \(t\). There are no other credit limits, and bid-ask spreads are neglected. For a self-financing strategy we must have

\[
\theta(T)S(T) = \theta(0)S(0) + \sum_{t=1}^{T} \theta(t-1)(S(t) - S(t-1)) \tag{3.4}
\]

Basically it means that all changes in the value of the portfolio are due to capital gains (possibly including dividends). Clearly other income or windfall should be treated separately. In continuous time the sum is replaced by a stochastic integral, see Duffie (1988, section 17). Note that the "integrand" in (3.4) is predictable, i.e. the factor \(\theta(t-1)\) in front of the increment \(S(t) - S(t-1)\) is \(F_{t-1}\) measurable. This technical jargon simply means that a trader must choose his desired portfolio holdings \(\theta(t)\) before he knows the price vector \(S(t)\), i.e. \(\theta(t)\) is predictable.
A contingent claim is a random payoff profile $X(T)$, now defined at time $T$ only. In general, the payoffs could occur at any intermediate date(s). The choice of a common evaluation date is not any restriction on the shape and form of payoff profiles. But before they are evaluated they must be transformed into a terminal $T$ payoff for which date the implicit price system $\pi$ and the probability measures $Q$ and $Q^*$ later will be defined.

A contingent claim is called attainable if there exists some self-financing trading strategy $\theta(t)$ with the same payoff, i.e. if

$$X(T) = \theta(T)S(T)$$

(3.5)

The set of attainable contingent claims now becomes a set of random variables defined at time $T$ such that

$$L[S(T)] = \{X : X = \theta(T)S(T) \text{ and } Q(X \geq 0) = 1\}$$

(3.6)

As before the market is called complete if all contingent claims are attainable.

The implicit price system $\pi$ for attainable contingent claims is defined as before. The number $\pi(X(T))$ is the current ($t = 0$) value of the payoff $X(T)$. Note that the price system only applies to payoffs occurring at the terminal date $T$. The implicit price system $\pi$ is said to be consistent with the initial market prices $S(0)$ if for all self-financing strategies we have

$$\pi(\theta(T)S(T)) = \theta(0)S(0)$$

(3.7)

Thus for consistency the terminal payoff of any self-financing strategy should equal its current ($t = 0$) value (the initial investment). And as there exists some self-financing strategy for any attainable claim the same goes for them. Whether we define no arbitrage as a property of attainable claims or as a property of self-financing trading strategies is just a matter of taste.

An arbitrage opportunity means there is a chance to get something (during the time the trading goes on) without having to risk anything and without having to pay any. Transferring any intermediate date gains/losses to the terminal date $T$, before evaluation with $\pi$, we see that the situation is basically the same as in the one-period case (2.15) and (2.16). For attainable contingent claims we can link no arbitrage to self-financing trading strategies. No arbitrage means

$$E[\theta(T)S(T)|F_0] > 0 \Rightarrow \theta(0)S(0) > 0$$

(3.8)

or equivalently

$$\theta(0)S(0) = 0 \Rightarrow E[\theta(T)S(T)|F_0] = 0$$

(3.9)

Here the expected value is taken with respect to the given initial measure $Q$. Note that the expected value is conditional on the information at the starting date $t = 0$. Although this was the case in the single period case too it was
not important to write it down explicitly. With only two dates the risk for confusion was minimal. However, in the multi-period framework we need to be clear about the role of dates. For prices to be consistent and arbitrage-free over time it seems reasonable to require that the relation should hold for all intermediate dates \( t : 0 \leq t \leq T \) too, i.e.

\[
E[\theta(T)S(T) | \mathcal{F}_t] > 0 \Rightarrow \theta(t)S(t) > 0
\] (3.10)

Indeed the relation should apply not only to terminal payoffs but to any future payoff date \( u \leq T \), i.e.

\[
E[\theta(u)S(u) | \mathcal{F}_t] > 0 \Rightarrow \theta(t)S(t) > 0
\] (3.11)

This takes us very close to the concept of a fair game. In a fair game the expected (discounted) future payoff should always equal its current value, i.e.

\[
E[\theta(u)S(u) | \mathcal{F}_t] = \theta(t)S(t) \quad \forall t : 0 \leq t \leq u
\] (3.12)

In other words, in a fair game the expected (discounted) net gain/loss is always zero. Clearly this implies the no arbitrage relation (3.11). But no arbitrage is weaker. In (3.11) the expected (discounted) future payoff need not equal its current value. In general, the no arbitrage relation only implies that there exists an equivalent probability measure \( Q^* \) for which the equality will hold. As we shall see in the next section with a slight modification this is the basic content of no arbitrage pricing.

### 3.2 No arbitrage and martingales

The basic no arbitrage pricing result was first proved by Ross (1978). Later Harrison and Kreps (1979) and Harrison and Pliska (1981) extended his result by relating it to the concepts of fair games and martingales. However, their models did not (explicitly) involve any relations between stochastic interest rates and bonds. This special topic was not addressed until Artzner and Delbaen (1989) used stochastic interest rates to find arbitrage-free bond prices. An alternative modelling approach can be found in Heath, Jarrow, Morton (1992). However, the role of stochastic interest rates and discounting is not very clear in these derivations. When pricing bonds we must be very clear about these concepts. This merits a separate statement of the basic no arbitrage pricing result with proof here. As it turns out the only modification of the concept of a fair game (or the equivalent game) that is needed is a clear understanding of how the relative price enters into the self-financing trading strategy.

**Assumption A1:** The set of attainable claims \( L[S(T)] \) is a non-empty convex cone in \( L^2(\Omega, \mathcal{F}_T, Q) \), the set of square-integrable random variables on the probability space \( (\Omega, \mathcal{F}_T, Q) \). It contains at least two primitive assets one of which has strictly positive payoffs at all times \( t \). Call this asset number 1.

**Definition:** The relative price with respect to asset 1 is \( Z^*_n(t) = S_n(t)/S_1(t) \).
Theorem 1: A complete market is arbitrage-free if and only if there exists a unique equivalent probability measure $Q^*$ under which the relative price process $Z^*(t)$ becomes a martingale with respect to $\{Q^*, F_t\}$, i.e. for each fixed $0 \leq t \leq u$ we have

$$E^*[Z^*(u)|F_t] = Z^*(t) \quad \text{all } t: 0 \leq t \leq u$$

Corollary: In a complete market the arbitrage-free value of any contingent claim $X(u)$, $0 \leq t \leq u \leq T$ is equal to the investment cost of a self-financing trading strategy that duplicates its payoff, i.e.

$$E^*[\beta(u)X(u)|F_t] = E^*[\beta(u)\theta(u)S(u)|F_t]$$

and asset number 1 is any asset with strictly positive payoffs at all times.

Let $\theta(t)$ be a self-financing strategy that duplicates the payoffs of $X$, i.e. $X(u) = \theta(u)S(u)$. We have

$$E^*[\beta(u)X(u)|F_t] = E^*[\beta(u)\theta(u)S(u)|F_t]$$

and according to Theorem 1 this equals $\beta(t)\theta(t)S(t)$.

Proof of Theorem 1: Let’s start with the ‘only if’ part. Given is a filtered probability space $(\Omega, \mathcal{F}_T, Q)$, and a market, i.e. a finite set of assets $n = 1, 2, \ldots, N$. Using simple self-financing trading strategies the linear subset of attainable contingent claims $L[S(T)]$ is well-defined. From section 2.2 we know how to construct an implicit price system $\pi$ from these claims. For $\beta(T) = S_1(0)/S_1(T)$ the functional $\pi$ uniquely identifies a probability measure $Q^*$ that is equivalent to $Q$. This takes care of the existence part. Next we focus on why the relative prices become martingales with respect to $Q^*$.

Consider a similar self-financing strategy to that of Harrison and Kreps (1979 p 391f): At time $t$ short-sell asset 1 (“borrow funds”) and use the proceeds to buy asset $k$. Sell it later if the event $B$ happens. At this later time, say $u$, also buy back asset 1 (“repay the loan”). Finally, store the remaining value in the form of asset 1 up to the terminal date $T$. Formally this means the following self-financing trading strategy $\theta(s): t \leq s \leq T$:

$$\theta_k(\omega, s) = \begin{cases} 1 & t \leq s < u \ \omega \in B \\ 0 & \text{otherwise} \end{cases}$$

$$\theta_1(\omega, s) = \begin{cases} -S_k(t)/S_1(t) & t \leq s < u \ \omega \in B \\ S_k(u) - S_1(u) \cdot S_k(t)/S_1(t) & u \leq s < T \ \omega \in B \\ 0 & \text{otherwise} \end{cases}$$

$$\theta_n(\omega, s) = 0 \quad n \neq k$$
Storing the net gain/loss in asset 1 from time \( u \) until time \( T \) gives

\[
\frac{S_1(T)}{S_1(u)} \cdot \left( S_k(u) - S_1(u) \cdot \frac{S_k(t)}{S_1(t)} \right) \cdot 1_B
\]  

where \( 1_B \) is the indicator function for the set \( B \). Evaluating the gain/loss at time \( T \) is just a scaling operation. What matters here is the relative change in value between asset \( k \) and asset 1 over the time period from \( t \) until \( u \) (when event \( B \) occurs). As the strategy is self-financing it must have a zero price to preclude arbitrage. Simplifying (3.13) and taking the expected value with respect to \( Q^* \) we must have

\[
E^*[S_1(T) \cdot (S_k(u)/S_1(u) - S_k(t)/S_1(t)) \cdot 1_B] = 0
\]  

As this holds for all sets \( B \) in \( \mathcal{F}_u \) it must hold in particular for the sets defining the conditional expectation \( \mathcal{F}_t \subseteq \mathcal{F}_u \) so

\[
E^*[Z_k^*(u) \cdot |\mathcal{F}_t] = 0
\]

and as \( Z_k^*(t) \) is \( \mathcal{F}_t \)-measurable and \( t \) arbitrary this implies that \( Z_k^*(s) \) is martingale. Repeating the argument for any asset \( k = 2, 3, \ldots, N \) we see that the whole vector \( Z^*(t) \) is a martingale. This ends the proof of the 'only if' part. For a short and elegant proof using the concept of a stopping time, see Harrison and Pliska (1981 p 227).

For the 'if' part let \( \theta(t) \) be any self-financing strategy with expected future payoff \( E[\theta(T)Z^*(T)|\mathcal{F}_0] > 0 \). Because \( Q^* \) is an equivalent measure to \( Q \) the same probability relations hold with respect to \( Q^* \), so \( E^*[\theta(T)Z^*(T)|\mathcal{F}_0] > 0 \). By assumption \( \theta(t) \) is self-financing and predictable and \( Q^* \) turns prices into martingales. So for \( 1 \leq t \leq T \) we have

\[
E^*[\theta(t)Z^*(t)|\mathcal{F}_{t-1}] = E^*[\theta(t-1)Z^*(t)|\mathcal{F}_{t-1}]
\]

\[
= \theta(t-1)E^*[Z^*(t)|\mathcal{F}_{t-1}] = \theta(t-1)Z^*(t-1)
\]

Iteration shows that

\[
E^*[\theta(T)Z^*(T)|\mathcal{F}_0] = \theta(0)Z^*(0)
\]  

Since the terminal payoff \( \theta(T)Z^*(T) \) is positive by assumption its price \( \theta(0)Z^*(0) \) is positive too. Thus, the market is arbitrage-free as defined in (3.8). Q.E.D.

This clearly shows the relation between no arbitrage pricing and martingales. If markets are complete there must exist an equivalent probability measure \( Q^* \) under which relative prices become martingales. Clearly this will restrict cash prices too. To find out exactly how we must be very careful and identify relative prices as a method of discounting. There are at least two obvious ways to
discount intertemporal prices. The first method is based on short term interest rates rolling over money from one date to another. This method was used by Artzner and Delbaen (1989) to prove the theorem for stochastic interest rates. The second method uses bonds from the current term structure of interest rates to discount future cash flows. Its use was pioneered by Geman (1989), although the basic idea can be found in Jamshidian (1987). Choosing one method of discounting prices results in a unique measure by Theorem 1. Choosing the other method produces another unique measure according to the same theorem. In fact, the method used for discounting is exogeneous to the theorem. To understand how this works we will take a closer look at the discounting of intertemporal prices in the next section.
4 Discounting and the choice of a numeraire

In finance prices are typically defined in relation to money. The cash price convention is often used to define discounting as the change in value of money over time. But money is actually not an asset in the intertemporal model. Holding money earns interest in an intertemporal model. One cannot escape this opportunity cost. So the relevant model asset must be either interest-bearing money, i.e. "deposits", or bonds. By convention we identify borrowing and lending over several periods with (zero coupon) bonds. The expression short term rate of interest is reserved for the return on cash over the shortest possible period only. A trader may either roll over money at short term rates of interest or invest it in long term bonds. The corresponding assets are called roll-over money (the "money market account") or bonds. Over the next period you either get the sure payoff from rolling over at the short term rate of interest or you get a stochastic payoff equal to the change in price of some long term bond. The simultaneous existence of these two alternatives makes discounting ambiguous in terms of money (unless interest rates are deterministic). To sort this out we need to elaborate on the discounting procedure and its traditional microeconomic interpretation as a choice of numeraire in an intertemporal framework with stochastic rates of interest.

In general, the prices of \( N \) assets over time are given by a matrix of numbers

\[
\{p_{nt}|1 \leq n \leq N, 0 \leq t \leq T\}
\]

Initially prices are given while all future prices \( \{p_{nt} : 0 < t \leq T\} \) are random variables. Each row shows how the price of an asset changes over time. Any asset with strictly positive payoffs in every state at all times can be used to define a numeraire for the intertemporal prices. Suppose asset number 1 fulfills this requirement. I call this the numeraire asset, cp the definition on p. 15. Traditional micro-economics defines the numeraire as the value of the numeraire asset at a particular moment of time, say the price at time 0, or \( p_{10} \). Using this numeraire discounted prices are defined as the relative prices \( p_{nt}/p_{10} \). Clearly the relative price of the numeraire asset always equals 1. Discounted prices can be written as the product of two terms

\[
\frac{p_{nt}}{p_{10}} = \frac{p_{nt}}{p_{1t}} \cdot \frac{p_{1t}}{p_{10}}
\]

The first term to the right \( p_{nt}/p_{1t} \) is the price of asset \( n \) at time \( t \) in relation to the price at the same time \( t \) of the asset used as numeraire. The second term \( p_{1t}/p_{10} \) is the price at time \( t \) of the numeraire asset in relation to the numeraire, i.e. to its initial price \( p_{10} \). This part is called the discount factor. Using this terminology any change in the discounted price of an asset over time decomposes into a change in its price relative to the numeraire asset and a change in the price of the numeraire asset itself (the discount factor). The relative prices \( p_{nt}/p_{1t} \) without the discount factor are known as undiscounted prices. The discounted
price is always equal to the undiscounted price times the discount factor. In terms of undiscounted prices the value of the asset serving as numeraire equals one for all dates. Table I and Table II illustrate these concepts more clearly.

In microeconomics this decomposition is fairly standard, see Malinvaud (1972 chapter 10) or Bliss (1975 chapter 3). Note that it is not standard in finance where discounting by default is defined in terms of money and consequently ambiguous. The cash price convention has in fact obscured the role of stochastic interest rates in no arbitrage pricing. To properly understand this it is necessary to identify the relative prices in Theorem 1 as undiscounted prices in terms of some numeraire. This will be done in the following sections. First we choose roll-over money as the numeraire asset, and then, in section 4.2, we use bonds from the current term structure. While adhering to the cash price convention I will be precise in using the word ’discounting’ to avoid misunderstanding.

4.1 Roll-over discounting

In terms of short term interest rates discounting is traditionally defined for cash prices as

\[ Z_n^*(t) = S_n(t)/B(t) \text{ where } B(t) = R(1) \cdot R(2) \cdots R(t) \tag{4.1} \]

Here borrowing/lending $1 at time \( t - l \) requires repayment of principal plus interest at time \( t \), the gross amount \( R(t) \). Starting at time 0, repeatedly reinvesting the money up to time \( t \) accumulates into \( B(t) \) dollars (including compound interest). This is called rolling over. The inverse of the accumulation factor, \( 1/B(t) \), is defined as the discount factor (from time 0 to time \( t \)). This procedure is equivalent to choosing roll-over money as the numeraire asset. But in terms of this numeraire the prices in (4.1) must be identified as undiscounted relative prices. The discounted value of the future payoff of the roll-over asset is equal to one at all future times, for details see Table III. It is only in terms of money that ”discounted” prices should be martingales by Theorem 1. Perhaps the word ”discounted” is not the best choice. All that’s needed to apply Theorem 1 is to measure the prices in some common unit (some numeraire). By using an asset outside the model, i.e., money, as a measure of value ”discounting” enters in a deceptive way.

Another interesting observation can be made if we elaborate a bit on the one period interest factors \( R(t) \). In terms of short term rates of interest \( r(t) \) they are defined as

\[ R(t) = 1 + r(t - 1) \quad t = 1, 2, \ldots, T \tag{4.2} \]

(Here the compounding interval equals that of the filtration and short term rates are defined as simple rates). Notice the time lag involved here. This means that \( R(t) \) and the discount factor \( 1/B(t) \) are both known already on date \( t - 1 \), i.e., they are \( F_{t-1} \)-measurable. In other words, rolling over at the short term rates of interest is a ”locally riskfree” strategy. This is actually more than is needed for the no arbitrage pricing Theorem 1. Clearly the same formulas would

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apply even if the discount factors were not known until $t$, i.e. if they were only $\mathcal{F}_t$-measurable. Such a case can be called "locally stochastic".

**Proposition 1:** Using roll-over money as numeraire arbitrage-free cash prices can be written for all $u \leq T$ as

$$S_n(t) = B(t) E^*[ \frac{1}{R(t+1) \cdots R(u)} S_n(u) | \mathcal{F}_t]$$

**Proof:** Apply Theorem 1 to the martingale $Z^*_n(t) = S_n(t)/B(t)$ where $0 \leq t \leq u$.

In Proposition 1 expectation is defined with respect to the probability measure $Q^*$. In view of (4.2) this particular measure is known as the risk-neutral measure.

If we know more about the characteristics of the asset’s payoff the pricing formula can be further specified. Of particular interest is the pricing of zero coupon bonds. Such a bond is a short-lived asset which pays $1$ for sure at some future date of time say $u$, where $0 \leq u \leq T$. As these bonds have no uncertain future payments only discounting matters for their evaluation. This gives bond prices special properties. To indicate this a special notation is used for bond prices. The price at time $t$ of a bond maturing at time $u$ is written as $P(t,u)$, not as $S_u(t)$. For such a bond no arbitrage pricing means

$$P(t,u)/B(t) = E^*[P(u,u)/B(u) | \mathcal{F}_t]$$ (4.3)

As $P(u,u) = 1$ upon maturity the expression simplifies even further into

$$P(t,u) = E^*[B(t)/B(u) | \mathcal{F}_t]$$ (4.4)

This relates bond prices to short term interest rates in a unique fashion. This should not be misunderstood. The conditional expectation is defined with respect to the risk-neutral probability measure $Q^*$, not the actual one (i.e. $Q$) and the flow of information $\mathcal{F}_t$. In particular, it does not mean that long term bonds are priced from short term interest rates. Conditioning with respect to the information sets $\mathcal{F}_t$ is more general than that. Indeed market segmentation, liquidity, and supply and demand factors for each maturity could be included in the information set at each time. So there’s no presumption about causation between long term bond prices and short term interest rates. For more precise statements about this relation we need to make additional assumptions about the nature of the flow of information.

### 4.2 Using the current term structure

There is an obvious alternative to using short term interest rates for discounting: bonds from the current term structure of interest rates. This is the second way
of ‘discounting’ cash prices. These (zero coupon) bonds give the current market price of any future $1. They are by their very definition discount factors. We formalize this as an assumption to indicate its importance.

**Assumption A2:** At each moment of time $s$ there is a whole family of bonds $\{P(s,u); s \leq u \leq T\}$ maturing at future dates $u$ paying $1$ upon maturity.

Using the current term structure of interest rates as numeraire means using these bonds for the ‘discounting’ of cash prices. This also brings up an important notational problem which was neglected in the previous section. At time 0 the relevant discount factors are $\{P(0,u); 0 \leq u \leq T\}$ and discounted cash prices look like

$$P(0,u)S(u) \quad \text{for } 0 \leq u \leq T$$

As time goes by the term structure will change and another set of bond prices will be appropriate for the discounting. For example, at time $s = 1$ we get the following discounted cash prices

$$P(1,u)S(u) \quad \text{for } 1 \leq u \leq T$$

And in general we have

$$P(s,u)S(u) \quad \text{for all } s : 0 \leq s \leq u \leq T \quad (4.5)$$

Here the discount factors cannot be written down without specifying their dependence on the starting date $s$. This complication was obscured by the notation $1/B(t)$ for the roll-over discount factor. Writing discounted prices as $S(t)/B(t)$ is really valid only for the discounting of prices from time 0 to time $t$, i.e., for $s = 0$. In particular, when discounting from time 1 to time $t$ we should use $B(1)/B(t)$. In general, the roll-over method will use $B(s)/B(u)$ as the discount factor from time $s$ to time $u$. Thus the relevant random variable $Z^*(t)$ always involves the starting date $s$. In fact, starting at a later date $s > 0$ the roll-over martingale actually is

$$Z^*(t) = S(t) \cdot B(s)/B(t) \quad \text{where } s \leq t \quad (4.6)$$

Applying *Theorem 1* to this martingale produces

$$E^*[S(u) \cdot B(s)/B(u)|\mathcal{F}_t] = S(t) \cdot B(s)/B(t) \quad (4.7)$$

In this case the starting date factor $B(s)$ is the same on both sides. So it can be canceled out making the formula identical to the one we get when starting at $s = 0$. Neither the factorization nor the cancelation will occur when we use bonds $P(s,u)$, $s \leq u \leq T$ from the current (time $s$) term structure of interest rates as discounting factors. In the case of discounting using the current term structure each bond price $P(s,u)$ is employed in the same way as the discount factor $B(s)/B(u)$. Using *Theorem 1* on $Z^*(u) = P(s,u)S(u)$ for $0 \leq s \leq u \leq T$ we immediately obtain another measure $Q^*$ such that

$$E^*[P(s,u)S(u)|\mathcal{F}_t] = P(s,t)S(t) \quad (4.8)$$
However, when discounting from time \( t \) to \( u \) the bond prices of the starting TSIR \( P(s, u) \) are no longer used. Instead bond prices from the current time \( t \) TSIR is used. Thus formula (4.8) has one degree of freedom too much. Discarding old bonds for \( s < t \), restricting our interest to the current period by setting \( s = t \) we get the adequate formulation

**Proposition 2:** Using bonds from the current term structure as numeraire no arbitrage cash prices can be written for all \( t \leq u \) as

\[
E^n[P(t, u)S(u)|\mathcal{F}_t] = P(t, t)S(t) = S(t) \text{ as } P(t, t) = 1
\]

Clearly the discount factors \( \{P(t, u), t \leq u \leq T\} \) are not locally riskfree. They are locally stochastic. Although \( P(t, u) \) is \( \mathcal{F}_t \)-measurable its rate of return over the next period is not known at time \( t \). Both the probability measure \( Q^n \) and the relevant martingale \( Z^n_{\cdot, u}(u) \) are different from the roll-over case.

Dividing both sides in Proposition 2 with \( P(t, u) \) we find the distinguishing feature of this alternative cash price discounting method: the forward price for any maturity \( u \) is equal to the expected future price with respect to the probability measure \( Q^n \), i.e.

\[
E^n[S(u)|\mathcal{F}_t] = S(t)/P(t, u)
\]

where the right hand side is the forward price at time \( t \) for delivery at time \( u \). This explains why \( Q^n \) is called the forward-neutral probability measure. Alternatively, we have

**Proposition 3:** Forward prices for any date of maturity \( u \leq T \) are martingales with respect to the probability measure \( Q^n \), i.e.

\[
E^n[S(t)/P(t, u)|\mathcal{F}_s] = S(s)/P(s, u) \text{ where } 0 \leq s \leq t \leq u \leq T
\]

**Proof:**

\[
E^n[P(t, u)S(u)|\mathcal{F}_s] = E^n\left[\frac{E^n[P(t, u)S(u)|\mathcal{F}_t]}{P(t, u)}\right]_{\mathcal{F}_s} =
\]

\[
= E^n\left[\frac{P(t, u)E^n[S(u)|\mathcal{F}_t]}{P(t, u)}\right]_{\mathcal{F}_s} = E^n[S(u)|\mathcal{F}_s] = \frac{S(s)}{P(s, u)}
\]

Q.E.D.

**Example:** It should be pointed out that it doesn’t matter whether interest is paid up front as in the case of a bond, or at the end of the term, such as for a deposit. The only thing that matters is the fixing of interest over several periods instead of only the next one. The simplest possible example of this basic principle involves two periods. Consider rolling over $1. This pays principal
plus interest after one period. Call this amount $R(1)$. Over the single period investing $1/R(1)$ will pay $1$. This is the same payoff as that of a zero coupon bond maturing at the end of the period. So the initial market price of this bond, $P(0, 1)$ say, must be equal to $1/R(1)$. Now repeat the comparison over two periods. Consider a 2-period bond priced initially at $P(0, 2)$. The payoff is $1$ on date 2. $1$ buys $1/P(0, 2)$ bonds at time 0 and pays the sure amount $1/P(0, 2)$ upon maturity. Alternatively rolling over $1$ from time 0 to time 2 produces $R(1) \cdot R(2)$. Discounting the future bond payoff with the roll-over factor we get

$$P(0, 2) = E^*[\frac{1}{R(1)R(2)}]|\mathcal{F}_0$$

On the other hand discounting the future payoff of the roll-over strategy using the current term structure of bonds we get

$$1 = E^*[P(0, 2)R(1)R(2)|\mathcal{F}_0] \Leftrightarrow P(0, 2) = \frac{1}{E^*[R(1)R(2)|\mathcal{F}_0]}$$

According to Jensen’s inequality both relations cannot hold for the same measure $Q^*$. So if the latter holds it must hold with respect to some other measure, here called $Q^\ast$.

### 4.3 Forward and futures prices

The relation between the two ways of discounting cash prices (choosing the numeraire) comes out clearly when we compare the formulas for arbitrage-free forward prices. This was done in the discrete case by Satchell, Stapleton and Subrahmanyam (1989). A forward contract entered into at time $t$ for delivery at time $u$ has future payoff $S(u) - G(t, u)$, where $G(t, u)$ is the forward price. As the current value of a forward contract is zero we have for the risk-neutral probability measure $Q^*$

$$E^*\left[\frac{B(t)}{B(u)}(S(u) - G(t, u))|\mathcal{F}_t\right]$$

Solving for the forward price $G(t, u)$ we get

$$G(t, u) = E^*\left[\frac{B(t)}{B(u)P(t, u)}S(u)|\mathcal{F}_t\right]$$

Comparing this to (4.9), the forward price in the forward-neutral probability measure $Q^\ast$ we get

**Proposition 4:** Expectation with respect to the forward-neutral measure $Q^\ast$ and with respect to the risk-neutral measure $Q^*$ are related as

$$E^*[S(u)|\mathcal{F}_t] = E^*\left[\frac{B(t)}{B(u)P(t, u)}S(u)|\mathcal{F}_t\right]$$
Multiplying both sides with $P(t,u)$ produces two expressions for the spot price $S(t)$

$$P(t,u)E^*[S(u)|\mathcal{F}_t] = E^*\left[ \frac{B(t)}{B(u)} S(u)|\mathcal{F}_t \right]$$

(4.12)

Next using the covariance formula $E[\beta S] = E[\beta] \cdot E[S] + cov(\beta, S)$ the right hand side can be written as

$$E^*[B(t)/B(u)|\mathcal{F}_t] \cdot E^*[S(u)|\mathcal{F}_t] + Cov^*[B(t)/B(u), S(u)|\mathcal{F}_t] =$$

$$= P(t,u) \cdot E^*[S(u)|\mathcal{F}_t] + Cov^*[B(t)/B(u), S(u)|\mathcal{F}_t]$$

Combining this with (4.12) shows that the forward-neutral measure $Q^*$ absorbs all covariance risk from time $t$ to time $u$.

Arbitrage-free futures prices can be related to the expected future spot with respect to the risk-neutral measure $Q^*$. Futures prices are marked to market every day. For simplicity assume that this corresponds to the length of the discrete time intervals $t, t+1,$ etc of the filtration. Then the payoffs of a futures contract are

$$0, H(t+l) - H(t), H(t+2) - H(t+1), \ldots, H(u) - H(u-1)$$

(4.13)

The current value of a futures contract too is zero. But its future payoff occurs on the next day. So its expected discounted value with respect to $Q^*$ is

$$E^*[B(t)/B(t+l) \cdot (H(t+l) - H(t))|\mathcal{F}_t] = 0$$

(4.14)

where $B(t)/B(t+1) = 1/R(t+1) = l/(1+r(t))$ which is $\mathcal{F}_t$-measurable by (4.2). So this factor can be multiplied out from under the expectation and we’re left with

$$E^*[H(t+l) - H(t)|\mathcal{F}_t] = 0$$

(4.15)

in other words the futures price $H(t)$ is a martingale with respect to $Q^*$. Iterating we get

$$E^*[H(u)|\mathcal{F}_t] = H(t)$$

(4.16)

and as $H(u) = S(u)$ we find

$$E^*[S(u)|\mathcal{F}_t] = H(t)$$

i.e we have

**Proposition 5:** The futures price equals the expected value of the future spot price with respect to the risk-neutral measure provided, of course, that the lengths of the marked to market time intervals are matched.
5 Martingale representation

Martingales are rather abstract stochastic processes. To find out what they look like we must make some assumption about the generating filtration. The full power of the equivalent martingale measure approach to no arbitrage pricing is unleashed in continuous time. The remainder of this dissertation will deal with continuous time models. Then the strong results on representation of martingales in terms of Brownian motion (and Poisson processes) can be utilized. Harrison and Pliska (1981 and 1983) showed that with proper assumptions about no arbitrage and self-financing strategies the vector price process in a complete market must have a well-known martingale representation property:

**Lemma 1:** Assume that $X(t)$ is a continuous square-integrable positive real valued martingale with respect to some filtration. If the filtration is generated by a single standard Wiener process $W(t)$, then we can find another stochastic process $\sigma(t)$ so that $X(t)$ can be explicitly represented as

\[
X(t) = X(0) + \int_0^t \sigma(v)X(v)dW(v)
\]

This is a simple formulation of the martingale representation theorem, for stricter versions see Karatzas and Shreve (1988 Theorems 3.4.15 and 3.4.2) or Elliott (1982 Theorem 12.33). There are two equivalent ways of writing this result: in difference form or as an exponential. The difference form is

\[
dX(t) = \sigma(t)X(t)dW(t)
\]

There is no $dt$ term as a martingale does not have any drift part. Writing this as an exponential martingale (Ito’s lemma) we get

\[
X(t) = X(0) \exp \left( \int_0^t \sigma(v)dW(v) - \frac{1}{2} \int_0^t \sigma(v)^2 dv \right)
\]

In what follows these representation results for Wiener processes are applied to find more explicit expressions for arbitrage-free prices. Both roll-over money and term structure bonds will be used as numeraire assets. We tackle each case in turn.

5.1 Roll-over pricing

With continuous trading it is natural to use continuously compounded rates of interest too. Now the discount factors assume the form

\[
1/B(t) = \exp \left( -\int_0^t r(v)dv \right) \quad 0 \leq t \leq T
\]

The interest rate process $r(t)$ can be path dependent (an Ito process) as it only enters in integrated form. Thus by their very definition the discount factors have bounded variation. Indeed they are even differentiable and

\[
\frac{dB(t)}{B(t)} = r(t)dt
\]
Using this we can further simplify the general form of the vector price process $S(t)$ in a complete market with no arbitrage. The relevant martingale is

$$Z^*(t) = S(t)/B(t) \text{ or } S(t) = B(t)Z^*(t)$$

Applying Ito’s lemma we get

$$dS(t) = r(t)B(t)Z^*(t)dt + B(t)dZ^*(t) = r(t)S(t)dt + B(t)dZ^*(t)$$

As $B(t)$ is of bounded variation the $dB(t)dZ^*(t)$ term must be zero (Elliott (1982) Correlarium 12.23), cp Vasicek (1977 p.182). Re-arranging we get for each component $n$

**Proposition 6:** Using roll-over money as numeraire the expected rate of return on any asset is locally riskfree and equals the current short term rate of interest

$$\frac{dS_n(t)}{S_n(t)} = r(t)dt + \frac{dZ^*_n(t)}{Z^*_n(t)}$$

This is really all we can say without further assumptions on the nature of flow of information and the involved stochastic processes. Before doing this we could however note that the formula in Proposition 6 admits a symbolic solution for $S_n(t)$ as the right hand side is a semi-martingale, see Elliott (1982 Theorem 13.5). This solution can be written as

$$S_n(t) = S_n(0) \exp \left( \int_0^t r(v)dv + \int_0^t \frac{dZ^*_n(v)}{Z^*_n(v)} - \frac{1}{2} \int_0^t \frac{d(Z^*_n(v))^2}{Z^*_n(v)} \right) \tag{5.1}$$

where the rightmost term $d(Z^*_n(v))^2$ is the quadratic variation of the process. To get any further we have to make some assumptions about the nature of the flow of information, i.e., about the filtration $\{\mathcal{F}_t; 0 \leq t \leq T\}$.

**Assumption A3*: The filtration $\mathcal{F}_t$ is generated by a finite set of independent standard Wiener processes $\{W^*_j(t); j = 1, \ldots, J\}$.**

With this simplification it is easy to see that any component of the vector martingale $Z^*(t)$ can be written by martingale representation in terms of the common set $\{W^*_j(t); j = 1, \ldots, J\}$. We sum this up in a theorem:

**Theorem 2:** Under Assumption A3* there exists stochastic processes $\sigma_{nj}(t)$ (all square-integrable and predictable) so that any martingale $Z_n(t)$ can be represented as

$$Z_n(t) = Z_n(0) \exp \left( \sum_{j=1}^J \int_0^t \sigma_{nj}(v)dW^*_j(v) - \frac{1}{2} \int_0^t |\sigma_{nj}(v)|^2dv \right)$$

where $\{W^*_j(t); j = 1, \ldots, J\}$ is the common set of standard Wiener processes.
Proof: For a full discussion the reader is referred to Elliott (1982 chapter 12, in particular Theorem 12.33).

The difference form of Theorem 2 is

\[ dZ_n(t) = \sum_{j=1}^{J} \sigma_{nj}(t)Z_n(t)dW^*_j(t) \]  

(5.2)

In particular, for the roll-over martingale \( Z^*(t) = S(t)/B(t) \) we find that spot prices in arbitrage-free markets must have rates of returns that look like

\[ \frac{dS_n(t)}{S_n(t)} = r(t)dt + \sum_{j=1}^{J} \sigma_{nj}(t)dW^*_j(t) \]  

(5.3)

This is much more specific than Proposition 6. To simplify the notation in what follows the explicit summation over the \( J \) Wiener processes is left out, just like the scalar product \( \theta(t)Z^*(t) \) was previously simplified in (3.4). Using martingale representation we can specify what price processes must look like to preclude arbitrage. Applying Theorem 2 to \( Z^*(t) \) we get

\[ S_n(t) = B(t)S_n(0) \exp \left( \int_0^t \sigma_n(v)dW^*(v) - \frac{1}{2} \int_0^t |\sigma_n(v)|^2 dv \right) \]  

(5.4)

In particular this relation must hold for any discrete set of bond prices \( P(s,u) \), where \( 0 \leq s \leq u \), \( u \) is discrete, and \( s \) continuous. Choosing \( n = t \) as the index for a bond maturing at time \( t \) we have \( S_n(s) = P(s,n) \). As bond prices must equal one upon maturity, \( P(t,t) = 1 \), and we can solve for \( B(t) \) in (5.4) as did El-Karoui and Geman (1991). Inserting \( P(t,t) \) in (5.4) we get

\[ \frac{P(t,t)}{B(t)} = \frac{1}{B(t)} = P(0,t) \exp \left( \int_0^t \sigma_t(v)dW^*(v) - \frac{1}{2} \int_0^t |\sigma_t(v)|^2 dv \right) \]  

(5.5)

Here the starting date was written as the number 0 although the same relations will hold for any choice of starting date, say \( s > 0 \). Furthermore from the definition of \( B(t) \) on p 27 we already know that

\[ \frac{B(s)}{B(t)} = \exp \left( - \int_0^s r(v)dv + \int_s^t r(v)dv \right) = \exp(- \int_s^t r(v)dv) \]  

(5.6)

But for this to hold for all choices of \( s \) and \( t \) we must have \( \sigma_t(v) = \sigma_s(v) \), for all \( v : 0 \leq s \leq v \leq t \leq T \). Thus, the coefficients \( \sigma_t \) and \( \sigma_s \) representing \( B(t) \) and \( B(s) \) from Theorem 2 cannot be different for \( s \) and \( t \). To show the special role of these coefficients they will be written as \( \gamma(v) \) from now on. From time \( s \) to time \( t \) we have

\[ \frac{B(s)}{B(t)} = \exp \left( \int_s^t \gamma(v)dW^*(v) - \frac{1}{2} \int_s^t |\gamma(v)|^2 dv \right) \]  

(5.7)
Inserting (5.7) with $s = 0$ for $1/B(t)$ into (5.4) produces

**Proposition 7:** Under Assumption A3* using roll-over money as numeraire arbitrage-free cash prices can be written as

$$S_n(t) = \frac{S_n(0)}{P(0,t)} \exp \left( \int_0^t (\sigma_n(v) - \gamma(v))dW^*(v) - \frac{1}{2} \int_0^t (|\sigma_n(v)|^2 - |\gamma(v)|^2)dv \right)$$

In particular for bond prices we have

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp \left( \int_0^t (\sigma_T(v) - \gamma(v))dW^*(v) - \frac{1}{2} \int_0^t (|\sigma_T(v)|^2 - |\gamma(v)|^2)dv \right)$$

(5.8)

It is reassuring to see that for $t = T$ both integrands vanish as $\sigma_t(v) = \gamma(v)$, cp (5.7), to make $P(t,t) = 1$, thereby satisfying the fixed maturity value of any bond.

Note that the $J$ Wiener processes here are the same for all assets. This is the general case with common pricing of all model assets, bonds, stocks etc. If we only want to price bonds it is possible that we can manage with only a subset of the common set 1, 2, ..., $J$ of Wiener processes. Indeed this is what we hope for in empirical applications.

### 5.2 Forward-neutral pricing

Using bonds from the current term structure of interest rates as numeraire produces another set of intertemporal prices. These form a vector martingale $Z^n(t)$ with respect to the forward-neutral measure $Q^n$. As the prices are different so are the martingales and so are the measures $Q^n$ and $Q^*$. For a given starting point $s = 0$ we have

$$Z^n(t) = P(0,t)S(t) \text{ or } S(t) = \frac{1}{P(0,t)} Z^n(t)$$

Differentiating using Ito’s lemma

$$dS_n(t) = - \frac{Z^n_n(t)}{P(0,t)} dP(0,t) - \frac{dP(0,t)}{P(0,t)} Z^n_n(t) + \frac{dZ^n_n(t)}{P(0,t)}$$

The middle term is zero because $P(0,t)$ has bounded variation in $t$. Dividing both sides with $S_n(t)$ and simplifying we get

$$\frac{dS_n(t)}{S_n(t)} = - \frac{dP(0,t)}{P(0,t)} + \frac{dZ^n_n(t)}{Z^n_n(t)}$$

And for an arbitrary starting date $s : 0 \leq s \leq t$ we have

$$\frac{dS_n(t)}{S_n(t)} = - \frac{dP(s,t)}{P(s,t)} + \frac{dZ^n_n(t)}{Z^n_n(t)}$$

(5.9)
This form is similar to that for the roll-over measure. Here the locally stochastic return of the bond \( P(s,t) \) replaces the locally risk-free short term rate of interest \( r(t) \). This does not matter as both are \( \mathcal{F}_t \)-predictable, and that’s all we need for Theorem 1.

To better describe the rate of return on the bond \( P(s,t) \) Heath, Jarrow, Morton (1987) introduced a transformation between bond prices and the instantaneously compounded implied forward rate \( f(s,t) \). The idea is that although \( P(s,t) \) is a stochastic process in \( s \) it is always a smooth curve in the second parameter \( t \), the maturity date. This family \( \{P(s,t), s \leq t \leq T \} \) is the market discount function underlying the current term structure of interest rates. It is no big deal even to use a spline as a discount function. Here we only need to assume that the family is differentiable in \( t \). Then we can introduce the transformation

\[
\frac{-\partial \ln P(s,t)}{\partial t} = f(s,t) \quad \Leftrightarrow \quad \ln P(s,t) = -\int_s^t f(s,v)dv
\]

or simply

\[
P(s,t) = \exp \left( -\int_s^t f(s,v)dv \right)
\]

Using the new variable \( f(s,t) \) the bond price return can simply be written as

\[
\frac{dP(s,t)}{P(s,t)} = -d(\ln P(s,t)) = d \left( \int_s^t f(s,v)dv \right) = f(s,t)dt
\]

For a discrete time version of this transformation see Hicks (1946 p 145). Inserting this expression into (5.9) we get

**Proposition 8:** Using bonds from the current term structure as numeraire the expected rate of return on any asset equals the current implied forward rate of interest \( f(s,t) \). It is not locally riskfree. Formally this means

\[
\frac{dS_n(t)}{S_n(t)} = f(s,t)dt + \frac{dZ^n_n(t)}{Z^n_n(t)}
\]

To complete the analogy with the roll-over case we next assume that the martingales are generated by a common finite set of Wiener processes:

**Assumption A3**: The filtration \( \mathcal{F}_t \) is generated by a finite set of independent standard Wiener processes \( \{W^n_j(t); j = 1, \ldots, J \} \).

This replaces Assumption A3* in the roll-over case. Under Assumption A3* martingale representation implies that there exists stochastic processes \( \eta_{n,j}(t) \) (all square integrable and predictable) so that

\[
Z^n_{n}(t) = Z^n_{n}(0) \exp \left( \sum_{j=1}^{J} \int_0^t \eta_{n,j}(v)dW^n_j(v) - \frac{1}{2} \int_0^t |\eta_{n,j}(v)|^2dv \right)
\]

(5.11)
where \( \{W^n_j(t); j = 1, \ldots, J\} \) is the common set of standard Wiener processes.

This is the exact analogue of Theorem 2 for an arbitrary starting \( s : 0 \leq s \leq t \).

Rewriting (5.11) in difference form we get

\[
dZ^n_n(t) = \sum_{j=1}^J \eta_{nj}(t)Z^n_n(t)dW^n_j(t) \tag{5.12}
\]

Inserting this into the spot price formula in Proposition 8 simplifies it into

\[
dS_n(t) = f(s,t)dt + \sum_{j=1}^J \eta_{nj}(t)dW^n_j(t) \tag{5.13}
\]

Leaving out the explicit summation and writing this on exponential form we find

**Proposition 9:** Under Assumption A3" using bonds as numeraire arbitrage-free cash prices can be written as

\[
S_n(t) = \frac{S_n(s)}{P(s,t)} \exp \left( \int_s^t (\sigma_n(v) - \gamma(v))dW^*(v) - \frac{1}{2} \int_s^t (|\sigma_n(v)|^2 - |\gamma(v)|^2)dv \right) \tag{5.14}
\]

This is an extremely convenient form. The first term \( S_n(s)/P(s,t) \) is the forward price for asset \( n \) at time \( s \) for delivery at time \( t \). The other term is a martingale with respect to \( Q^* \). So the future spot price equals the current forward price times a martingale.

Finally, we show how the \( \eta \)'s are related to the \( \sigma \)'s. In order to do this we derive another expression like Proposition 9 involving the \( \sigma \)'s and \( W^* \). Start with the, roll-over Proposition 7 re-written here for an arbitrary starting date \( s \leq t \) as

\[
S_n(t) = \frac{S_n(s)}{P(s,t)} \exp \left( \int_s^t (\sigma_n(v) - \gamma(v))dW^*(v) - \frac{1}{2} \int_s^t (|\sigma_n(v)|^2 - |\gamma(v)|^2)dv \right) \tag{5.14}
\]

Completing the square \( |\sigma_n(v)|^2 - |\gamma(v)|^2 \) and collecting terms we get

\[
S_n(t) = \frac{S_n(t)}{P(s,t)} \exp \left( \int_s^t (\sigma_n(v) - \gamma(v))dW^*(v) - \frac{1}{2} \int_s^t (|\sigma_n(v)|^2 - |\gamma(v)|^2)dv \right) \tag{5.15}
\]

These algebraic manipulations correspond to a change of measure from \( W^* \) to \( W^* \) according to Girsanov’s theorem (see section 6.1 below) with

\[
\rho(t) = \frac{B(s)}{P(s,t)B(t)} = \exp\tag{5.16}
\]

Thus we may identify the relation between \( W^*(v) \) and \( W^n(v) \) for \( s \leq v \leq t \) as

\[
dW^n(v) = dW^*(v) - \gamma(v)dv \text{ or } W^n(v) = W^*(v) - \int_s^v \gamma(x)dx \tag{5.17}
\]
Thus (5.15) and Proposition 9 provide two representations of the same stochastic process under $Q’$ and we may identify $\eta(v) = \sigma_n(v) - \gamma(v)$. Using this arbitrage-free prices under $Q’$ can be written in terms of the $\sigma$’s as

$$S_n(t) = \frac{S_n(s)}{P(s,t)} \exp \left( \int_s^t (\sigma_n(v) - \gamma(v)) dW^n(v) - \frac{1}{2} \int_s^t (|\sigma_n(v) - \gamma(v)|^2) dv \right)$$

(5.18)

In particular for bonds we get

$$P(t,T) = \frac{P(s,T)}{P(s,t)} \exp \left( \int_s^t (\sigma_T(v) - \gamma(v)) dW^n(v) - \frac{1}{2} \int_s^t (|\sigma_T(v) - \gamma(v)|^2) dv \right)$$

(5.19)

### 5.3 The drift condition

Heath, Jarrow, and Morton (1987) were the first to derive a formula for arbitrage-free bond prices. This was also done independently by Artzner and Delbaen (1989) who used a similar approach to this text. Unfortunately, it is not easy to see that both approaches are really equivalent. The calculations in this section attempts to provide the missing link. We show that the drift conditions in Heath, Jarrow, Morton (1992 Proposition 4 or Lemmas 1 and 2) are satisfied for the model of this text too. Thus the general approach here includes theirs as a special case. This identification is based on an idea of Ingemar Kaj.

We start like Heath, Jarrow, Morton (1992 p 80) by specifying an arbitrary stochastic process for the implied forward rate $f(t,u)$. Assume for simplicity here that only one standard Brownian motion $W(t)$ generates the flow of $F_t$ information over time. The plan is to show that the drift condition in Heath, Jarrow, Morton (1992 Proposition 4) is satisfied for the model of the previous section. Then the conclusion is that the model of the text includes their model as a special case. Write the implied forward rate process as

$$f(t,u) = f(0,u) + \int_0^t \alpha(v,u) dv + \int_0^t \zeta(v,u) dW(v) t \leq u \leq T$$

(5.20)

where $\alpha(v,u)$ and $\zeta(v,u)$ are arbitrary Ito processes in $v$ (possibly path-dependent). Inserting this functional form for $f(t,u)$ into the bond price transformation formula (5.10) we get the following expression for $P(t, T)$

$$\exp \left( -\int_t^T f(0,u) du - \int_t^T \left( \int_0^t \alpha(v,u) dv \right) du - \int_t^T \left( \int_0^t \zeta(v,u) dW(v) \right) du \right)$$

(5.21)

Evaluating the leftmost integral separately we get the forward price

$$\exp \left( -\int_t^T f(0,u) du \right) = \exp \left( -\int_0^T f(0,u) du + \int_0^t f(0,u) du \right) = \frac{P(0, T)}{P(0, t)}$$

(5.22)
Interchanging the order of integration in the other two integrals in (5.21) gives

\[
P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left( -\int_0^t \left( \int_t^T \alpha(v, u) du \right) dv - \int_0^t \left( \int_t^T \zeta(v, u) du \right) dW(v) \right)
\]

(5.23)

On the other hand, in the previous section, formula (5.19), we found what arbitrage-free bond prices must look like under the forward-neutral measure. Identifying terms in (5.23) and (5.19) for \( s = 0 \) as if both referred to the same measure we get

\[
\int_t^T \zeta(v, u) du = \sigma_T(v) - \gamma(v) \quad \text{and} \quad \int_t^T \alpha(v, u) du = \frac{1}{2} |\sigma_T(v) - \gamma(v)|^2
\]

(5.24)

This means that the coefficient processes \( \alpha(v, u) \) and \( \zeta(v, u) \) must be related as in Heath, Jarrow, Morton (1992, Proposition 4, Lemmas 1 and 2), i.e., as

\[
\int_t^T \alpha(v, u) du = \frac{1}{2} \left( \int_t^T \zeta(v, u) du \right)^2
\]

(5.25)

While this takes care of the main identification argument we also need to consider the possible change of measure in between (5.19) and (5.23). The formulas in Heath, Jarrow, Morton include a non-zero market price of risk \( \phi \) because they allow for such a possible change of measure. This topic will be dealt with next.
6 General pricing formulas

The theory of martingale pricing developed so far in this dissertation has been concerned with no arbitrage prices under two equivalent measures: the roll-over measure and the forward-neutral measure. As relative prices are martingales with respect to these measures the formulas did not include any drift terms. Also under Assumption A3* the generating Wiener processes were assumed to be independent. Thus there were no cross-covariation terms at all. However, in applications we start with prices that are observed under some other given measure. What can we say about the nature of no arbitrage prices under another measure? Fortunately, the necessary modifications are straightforward. According to Theorem 1 any given measure \( Q \) must be equivalent to the roll-over martingale measure \( Q^\ast \). This restricts the class of observed measures that are compatible with the martingale approach. Equivalent measures are related to each other, in particular to the martingale measure \( Q^\ast \), via simple drift transformations. The relation between the forward-neutral measure and the roll-over measure at the end of section 5.2. was an example of this, cp formula (5.22). The exact form of the relation between equivalent measures for the case of Wiener processes is known as Girsanov’s theorem. Using this result we examine in this section what prices can look like under any equivalent measure. In general, both drift and cross-covariation will complicate the previously derived arbitrage-free pricing formulas.

6.1 The market prices of risk in a non-linear APT

According to Theorem 1 no arbitrage in complete markets implies the existence of a probability measure under which relative prices become martingales. Each choice of numeraire results in a unique measure. And for each measure we can derive formulas for arbitrage-free pricing. But what about the development of prices under some other given measure? Observed prices, for example, need not follow martingales. It is only under the constructed martingale measure that they do. Actual prices develop over time in some other measure and they will in general have different variance and drift. To properly compare the derived martingale pricing formulas to observed prices we must take this change of measure into account. From Theorem 1 we know that the martingale measure must be equivalent to the given measure. In this case the link between any equivalent measure \( Q \) and the roll-over martingale measure \( Q^\ast \) can be described as a strictly positive continuous linear functional \( \rho(u) \):

\[
E^*[\beta(u)S(u)|\mathcal{F}_t] = E[\rho(u)\beta(u)S(u)|\mathcal{F}_t] \quad (6.1)
\]

This is known as the Radon-Nikodym theorem (for details see Appendix). We have already encountered an example of such a functional. In section 4.3 we found that the forward-neutral measure was linked to the roll-over measure via a factor \( B(t)/B(u)/P(t,u) \), cp Proposition 4. This factor is also known as a Radon-Nikodym derivative, often symbolically written as \( dQ/dQ^\ast \). In general
this likelihood ratio will depend upon the terminal date \( u \leq T \). In fact, it is a martingale too as
\[
E[\rho(u) | \mathcal{F}_u] = \rho(t) \tag{6.2}
\]
Under Assumption A3* the martingale representation of this measure change can be written as
\[
\rho(t) = \exp \left( \int_0^t q(v) dW^*(v) - \frac{1}{2} \int_0^t |q(v)|^2 dv \right) \tag{6.3}
\]
as \( \rho(0) = 1 \). The change of measure relates Wiener processes \( W^*(t) \) with respect to the roll-over measure to Wiener processes \( W(t) \) under any given measure via simple drift transformations
\[
W(t) = W^*(t) - \int_0^t q(v) dv \tag{6.4}
\]
This is known as Girsanov’s Theorem, see Elliott (1982 Corollary 13.25) or Karatzas and Shreve (1988, Theorem 3.5.1). Inserting for the martingale \( Z^*(t) \) in (4.1) we can rewrite the no arbitrage martingale pricing formula in (5.4) for any equivalent measure \( Q \) as
\[
S_n(t) = 1 \rho(t) B(t) S_n(0) \exp \left( \int_0^t \sigma_n(v) dW^*(v) - \frac{1}{2} \int_0^t |\sigma_n(v)|^2 dv \right) =
\[
= B(t) S_n(0) \exp \left( \int_0^t (\sigma_n(v) - q(v)) dW^*(v) - \frac{1}{2} \int_0^t (|\sigma_n(v)|^2 - |q(v)|^2) dv \right) \tag{6.5}
\]
Completing the square in the rightmost integral, and inserting the short term rate of interest for \( B(t) \) we get a formula with another set of local variance processes \( g_n(v) = \sigma_n(v) - q(v) \)
\[
S_n(t) = S_n(0) \exp \left( \int_0^t g_n(v) dW^*(v) - \frac{1}{2} \int_0^t |g_n(v)|^2 dv + \int_0^t g_n(v) dv \right) \tag{6.6}
\]
This shows how the local variance coefficients \( g_n(t) \) of the Wiener processes \( W(t) \) under the new measure \( Q \) are related to the coefficients \( \sigma_n(t) \) of the Wiener processes \( W^*(t) \) under the risk-neutral measure \( Q^* \). Note that the \( \sigma_n \) coefficients may be different for each asset \( n \) while the measure change coefficient \( q(t) \) is the same for all assets. Collecting drift terms this can be written in difference form as

**Proposition 10:** Under assumption A3* arbitrage-free cash prices under any measure \( Q \) can be written as
\[
\frac{dS_n(t)}{S_n(t)} = \{r(t) + q(t)g_n(t)\} dt + g_n(t) dW(t)
\]
This shows what the drift term must look like under any measure $Q$ that is equivalent to the martingale measure $Q^*$. In general, the local variance will not be the same as under the martingale measure, i.e., $g_n(t)$ is not equal to $\sigma_n(t)$. In addition, Assumption $A3^*$ and Girsanov’s theorem restricts the possible form of the drift. The traditional hedging argument, see Vasicek (1977), results in exactly the same restriction of the drift term. Note, however, that the coefficients of the measure change (6.3) replaces the market price of risk. This pinpoints the problem with the traditional argument. It is flawed because it leaves the market price of risk indeterminate. Using the equivalent martingale approach we can identify it as the change in probability measure that is necessary to turn the drift under the roll-over measure into the (locally) riskfree rate $r(t)$. Thus starting with a particular form of drift $\mu_n(t)$ (as the traditional approach does) instead of $r(t) + q(t)g_n(t)$ the market price of risk $q(t)$ is uniquely determined as that change of measure which transformed the exogeneously specified drift $\mu_n(t)$ under $Q$ into the riskfree rate $r(t)$ for the equivalent measure $Q^*$. To solve explicitly for the market price of risk we need at least as many asset drifts as Wiener processes in the model, under Assumption $A3^*$ in this case $J$. It is interesting to note that in the special case of just one Wiener process (as in many bond pricing models) we simply get

$$q(t) = \frac{\mu_n(t) - r(t)}{g_n(t)}$$

(6.7)

In other words the market price of risk must equal the excess rate of return on the bond divided by the duration of the bond (its local price volatility). In the case of diffusion processes we need to make some additional technical assumptions about possible parameter choices. These are called Novikov conditions. Chen (1990) shows that these conditions are in fact violated by previously used processes for bond pricing like the Brownian bridge process and the Ornstein-Uhlenbeck process.

This transformation of martingale pricing formulas into formulas relevant for any measure cannot be omitted in applications of martingale pricing. It is in fact indispensable for estimations of the variance process used in the martingale price formulas. Essentially the martingale approach calls for the application of a non-linear filter to get rid of trends in observed returns. The resulting returns model allows the identification of the relevant variance process. Indeed, the procedure amounts to a non-linear APT (Arbitrage Pricing Theory). This generalizes the linear filter model pioneered by Ross (1976). Clearly each asset $n$ may involve different proportions of the "factors" $W_j$ so the "factor loadings" $g_{nj}(t)$ should be different too. This is indicated by the additional index $n$ in $g_{nj}$. Like APT each asset may have its own unique composition $g_{nj}$ of the common "risk factors" $q_j$. However, here the factor loadings $g_{nj}(t)$ are stochastic processes and may vary over time. This is what makes the model non-linear. Under Assumption $A3^*$ the factors are independent, a so called orthogonal factor model. If observed prices cannot be modeled in this way they must be inconsistent with the martingale model. That means either that prices
are not arbitrage-free or that the particular martingale approach is invalid. Note that the latter case is not at all inconceivable. In order to get specific pricing formulas several simplifying assumptions were made, in particular, that the filtration was generated by a finite set of independent Wiener processes (*Assumption A3*). Whether this is relevant or not for the observed set of prices can only be decided on the basis of empirical data. Clearly, we should try the martingale approach under alternative assumptions about the generating processes. An obvious alternative would be to use correlated Wiener processes, Poisson processes or a mixture of both, see Shirakawa (1991).

6.2 Traditional expectation hypotheses

Traditional expectations hypotheses about the term structure of interest rates relate long term rates (bond yields) to expected future short term (money market) rates. Such hypotheses have a long history in economics. As noted by Morton (1988 p.37) the word ‘expected’ here is typically used tentatively, as a synonym for ‘anticipated’. One does not necessarily do these hypotheses full justice by formulating them rigorously in terms of mathematical expectation. However, in order to check their consistency with no arbitrage pricing this cannot be avoided. Cox, Ingersoll, Ross (1981) distinguish four different hypotheses: the local expectations hypothesis, the return to maturity hypothesis, the yield to maturity hypothesis, and the unbiased forward rate hypothesis. Now we already know what arbitrage-free bond prices must look like. Formulas were derived for two equivalent probability measures: the rollover measure and the forward-neutral measure. A valid hypothesis must fit either of them or some other equivalent measure.

The local expectations hypothesis claims that the expected rate of return on any bond over the next period should be equal to the short term rate of interest for the same period. In other words, the drift of the bond should be equal to the locally riskfree rate.

\[ E\left[ \frac{dP(t,T)}{P(t,T)} \right]_{\mathcal{F}_t} = r(t)dt \] (6.8)

This holds for any asset under the roll-over measure \( Q^* \) according to Proposition 5. So the local expectations hypothesis is consistent with arbitrage-free pricing. In this case all market prices of risk must be zero because there is no other equivalent probability measure with this property holding for bonds. According to the second hypothesis the return-to-maturity on any bond should always equal the expected return from rolling-over at short term rates of interest until maturity.

\[ \frac{1}{P(t,T)} = E[\exp \left( \int_t^T r(u)du \right) | \mathcal{F}_t] \] (6.9)

Cox, Ingersoll, Ross, (1981) claims that this can only hold for all bonds at all times if bond prices are deterministic. This is not completely true. It is
enough for bond prices to be globally deterministic. Still prices may be locally stochastic. This is the case when bonds from the current term structure are used as numeraire, cp formula (4.5). Thus the return-to-maturity hypothesis can be identified with the forward-neutral probability measure.

The yield-to-maturity hypothesis asserts that long term bond yields equal the expected value of average future short term rates. This can be written as

$$y(t, T) \equiv -\frac{1}{T-t} \ln P(t, T) = E\left[ \frac{1}{T-t} \int_t^T r(u) du \mid \mathcal{F}_t \right] \quad (6.10)$$

where the (continuously compounded) yield-to-maturity $y(t, T)$ is defined to the left as the average growth rate until maturity, and $(T-t)$ is the time to maturity. Interchanging the order of integration and expectation results in

$$\ln P(t, T) = -\int_t^T E[r(u) \mid \mathcal{F}_t] du \quad (6.11)$$

Here long term bond yields are given as averages of expected future short term rates. This tiny change in formulation should not be overlooked. It is known as the unbiased forward rate hypothesis. Put in a more familiar form it means that implied forward rates are equal to expected future spot rates

$$f(t, T) \equiv -\partial \ln P(t, T) / \partial T = E[r(T) \mid \mathcal{F}_t] \quad (6.12)$$

So the last two expectations hypotheses are equivalent in continuous time, i.e.

$$\exp \left( -\int_t^T E[r(u) \mid \mathcal{F}_t] du \right) = \exp \left( E[-\int_t^T r(u) du \mid \mathcal{F}_t] \right) = P(t, T) \quad (6.13)$$

**Proposition 11:** Under the forward-neutral measure $Q^*$ current forward rates are equal to expected future spot rates

$$f(t, T) = E^*[r(T) \mid \mathcal{F}_t] \quad 0 \leq t \leq T$$

**Proof:**

$$E^*[r(T) \mid \mathcal{F}_t] = E^*[\frac{r(T)}{P(t, T)} \exp \left( -\int_t^T r(u) du \right) \mid \mathcal{F}_t] =$$

These calculations are based on El-Karoui and Geman (1991 p 14). To evaluate the expected future spot rate with respect to the forward-neutral measure change into the roll-over measure and identify the current forward rate as defined in Proposition 4. The last equality follows by identifying the derivative of $\ln P(t, T)$ as in (5.10). Q.E.D.
Proposition 11 identifies the unbiased forward rate hypothesis and the yield to-maturity hypothesis with the forward-neutral measure $Q$. The return-to-maturity hypothesis was also seen to be satisfied in the forward-neutral case. Combining (6.10) and Proposition 11 we get a conspicuous relation

$$\exp\left(\mathbb{E}\left[-\int_t^T r(u)|\mathcal{F}_u|du\right]\right) = P(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r(u)du\right)|\mathcal{F}_t\right] \quad (6.14)$$

In view of Jensen’s inequality, this can only hold for all $T$ if $P(t,T)$ is non-stochastic. But this is exactly what it is under the forward-neutral measure! In addition, the relation is different for each $t$ as the relation is being conditioned upon the information set $\mathcal{F}_t$. So there is no contradiction. Furthermore, this “globally deterministic” and “locally stochastic” measure was seen to correspond to a valid choice of numeraire in the intertemporal model. So the hypothesis must be consistent with arbitrage-free bond pricing. In fact, forward-neutral discounting is the exact opposite - of roll-over discounting as far as expectations hypotheses are concerned. While the local expectations hypothesis is the only one compatible with roll-over discounting the other three hypotheses are united under forward-neutral discounting. It is only under the risk-neutral probability measure that all four hypotheses are pairwise incompatible as claimed by Cox, Ingersoll, and Ross (1981).

7 Conclusions

We have seen how no arbitrage pricing is related to fair games and martingales. Complete markets are arbitrage-free if and only if there exists an equivalent measure under which relative prices become martingales. The concept of no arbitrage pricing can be seen as an extension of the classical fair game hypothesis. We could alternatively talk about the existence of an equivalent fair game as the condition for arbitrage-free prices. Clearly this refers to relative prices denominated in a common intertemporal numeraire. The discounting of future cash flows is an ambiguous operation in the presence of bonds. It is rather surprising that this has not been recognized before. It seems to me that the most natural choice of numeraire should be to use the prices of zero coupon bonds to discount the future, not the use of short-term interest rates in some roll-over strategy such as a savings account or an accumulation factor $B(t)$.

We have also seen that there is a rather close relation between the no arbitrage pricing model and the conventional micro-economic model with a representative agent maximizing expected utility. In fact, we can always use the equivalent martingale measure to reconstruct the optimizing function of the representative consumer. Thus we could say that no arbitrage pricing is a sort of revealed preference approach to the pricing of financial assets.

In contrast to other presentations of the martingale approach I have chosen to be very explicit about the assumptions underlying the results. It is very im-
portant to recognize that the pricing results are conditional upon the availability of free and public information. Alternatively, we could say that the derived pricing results are the requirements of efficient markets. This interpretation opens up interesting comparisons to the classical forms of efficiency: weak, semi-strong and strong. Furthermore, transaction costs are not included in the theory developed here. In general, such costs, as well as the presence of bid-ask spreads, will introduce an element of indeterminacy in the no arbitrage pricing formulas. Another important assumption concerns the nature of the flow of information over time. We must limit ourselves to particular forms of stochastic processes like those generated by Wiener or Poisson Processes. While these are quite general processes they are obviously not the most general type conceivable. Most papers specify prices for a Brownian filtration without explaining that this is an assumption which restricts the possible form of uncertainty evaluated in the pricing formulas. Even the concept of a filtration is restrictive as it involves a steadily increasing amount of information. This excludes any influence of false information and bluffing on the pricing of assets. As option pricing is just a special case of no arbitrage pricing it seems to me that the degree of generality claimed by advocates of different evaluation formulas should be treated with caution and skepticism. Indeed, the whole paradigm of option pricing is rather limited in scope as many volatility traders should have found out. I'm not so sure what mispricing really means in this context.

Finally, the main effort of this study has been directed to the arbitrage-free pricing of bonds. Particular attention has been given to a proper understanding of what role is played by the market price of risk. It should be noted that we do not need any general equilibrium model to solve for arbitrage-free bond prices. Neither do we need any expectations hypotheses about the term structure of interest rates. The pricing of zero coupon bonds can proceed in the same manner as the arbitrage-free pricing of any other asset. The only special thing about bonds is that their value becomes uniquely determined. The reason is that these bonds have a fixed life-time and a fixed redemption value. These factors alone suffice to determine their arbitrage-free values. Other types of hypotheses about the term structure, like market segmentation or preferred habitats, inflation premium, turnover and liquidity particulars, may still be relevant though. In terms of the modeling framework used in no arbitrage pricing such considerations will enter into the information sets, i.e. directly into the u-algebras of the filtration. Thus, although arbitrage-free bond prices have been found in the model of this text there are many other interesting ways to model bond prices left to explore.
References


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Appendix: Mathematical foundations of no arbitrage pricing

It is important to be able to distinguish the mathematics from the economic content of the no arbitrage pricing theorem. I have tried to do this by focusing on the intuitive notion of fair games and martingales in the text reserving the mathematics for this Appendix. Basically what’s at stake here is the need to prove that by adding more and more assets we are able to duplicate the payoff of any square-integrable random variable in $L^2(\Omega, F, Q) = V$ say. In other words we must show that the set $L[S]$ of attainable claims is in $V$. For a constructive proof of the existence of this as well as the existence of a unique equivalent probability measure we have to rely on three famous mathematical theorems: Hahn-Banach theorem on separating hyperplanes, Riesz representation theorem, and the Radon-Nikodym theorem. These are standard theorems in functional analysis and measure theory. They are reproduced here in a convenient form. For more general formulations and strict details see Rudin (1970 p.40, p.105, and p.122).

**Hahn-Banach Theorem:** For any given strictly positive bounded/continuous linear functional $\pi$ on the subspace $L[S]$ there exists a bounded/continuous linear functional $\psi$ extending $\pi$ to all of the vector space $V$ so that

$$\psi(a) = \pi(a) \quad \text{all } a \in V$$

**Riesz Representation Theorem:** For every positive linear functional $\psi$ on $V$ we can find a unique probability measure $Q^*$ on $\Omega$ including some $\sigma$-algebra $F$ so that

$$\psi(A) = E^*[X] \quad \text{all } X \in V$$

**Radon-Nikodym Theorem:** Two probability measures $Q$ and $Q^*$ are equivalent if and only if there exists a random variable $\rho \in L^1(\Omega, F, Q)$ so that

$$E^*[X] = E[\rho X]$$

for all positive random variables $X \in L^1(\Omega, F, Q)$, (here the expectation $E^*$ refers to $Q^*$ and $E$ to $Q$).

Under Assumption A1 on p 16 integration is well-defined with respect to the $\sigma$-algebras $F_t, 0 \leq t \leq T$. Using this we may define open sets in $L[S]$ as the inverse images of open sets in $\mathbb{R}$. It is a classical mathematical result that these so called Borel sets induce a topological structure on $L[S]$. The only problem with the subspace of attainable claims is that including more and more contingent claims we cannot be sure that their limits too are attainable. Hahn-Banach theorem ensures that the given non-negative implicit state-price system $\pi$ on $L[S]$ can be extended to a unique continuous linear functional $\psi$ that can
be used to evaluate any contingent claim in V. Using Riesz representation, this functional defines a unique probability measure $Q^*$ as follows

$$Q^*(B) = \psi(1_B)t \quad \text{and} \quad E^*[X] = \psi(X)$$

The constructed measure $Q^*$ is equivalent to the given measure $Q$. They have the same null sets:

$$Q^*(B) = 0 \quad \Leftrightarrow \quad Q(B) = 0$$

As the measures are equivalent we can apply the Radon-Nikodym theorem to describe the relation with a unique strictly positive continuous and square-integrable functional $\rho(T)$:

$$E^*[X] = E[\rho(T)X]$$

This functional relates the new measure $Q^*$ to the given $Q$ in a unique way for each fixed point in time $T$. Symbolically the relation between the two measures is often written as

$$dQ^* = \rho(T)dQ$$

A more modern approach to the existence problem and completeness is to rely upon known spanning results for Wiener processes and Poisson processes. This is done in section 5 using the martingale representation theorem. In this way the martingale representation results replace the vector space methods of Hahn-Banach.
**Tables:**

**TABLE I: THE NUMERAIRE**

<table>
<thead>
<tr>
<th>Asset</th>
<th>Absolute prices</th>
<th>Relative prices</th>
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</thead>
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<tr>
<td></td>
<td>Time</td>
<td>Time</td>
</tr>
<tr>
<td></td>
<td>0   1   ...   t</td>
<td>0   1   ...   t</td>
</tr>
<tr>
<td>1</td>
<td>( p_{10} )</td>
<td>( \frac{p_{11}}{p_{10}} )</td>
</tr>
<tr>
<td>2</td>
<td>( p_{20} )</td>
<td>( \frac{p_{21}}{p_{10}} )</td>
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<td>\ldots \ldots</td>
</tr>
<tr>
<td>( n )</td>
<td>( p_{n0} )</td>
<td>( \frac{p_{n1}}{p_{10}} )</td>
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**TABLE II: THE DISCOUNT FACTORS**

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<th>Asset</th>
<th>Undiscounted prices</th>
<th>Discounted prices</th>
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<td>Time</td>
<td>Time</td>
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<td>0   1   ...   t</td>
</tr>
<tr>
<td>1</td>
<td>( 1 )</td>
<td>( 1 \cdot \frac{p_{11}}{p_{10}} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{p_{20}}{p_{10}} ) ( \frac{p_{21}}{p_{11}} ) ( \frac{p_{2t}}{p_{1t}} )</td>
<td>( \frac{p_{20}}{p_{10}} ) ( \frac{p_{21}}{p_{11}} ) ( \frac{p_{2t}}{p_{1t}} )</td>
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<tr>
<td>( n )</td>
<td>( \frac{p_{n0}}{p_{10}} ) ( \frac{p_{n1}}{p_{11}} ) ( \frac{p_{nt}}{p_{1t}} )</td>
<td>( \frac{p_{n0}}{p_{10}} ) ( \frac{p_{n1}}{p_{11}} ) ( \frac{p_{nt}}{p_{1t}} )</td>
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### TABLE III: ROLL-OVER NUMERAIRE

<table>
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<th>Asset</th>
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<th>Discounted prices</th>
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</thead>
<tbody>
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<td></td>
<td>Time 0 1 ... t</td>
<td>Time 0 1 ... t</td>
</tr>
<tr>
<td>(Money)</td>
<td>1 [\frac{1}{B(1)}] ... [\frac{1}{B(t)}]</td>
<td>1 1 ... 1</td>
</tr>
<tr>
<td>RollOver</td>
<td>1 1 ... 1</td>
<td>1 [B(1)] ... [B(t)]</td>
</tr>
<tr>
<td>Bonds</td>
<td>1 [\frac{1}{B(1)P(0,1)}] ... [\frac{1}{B(t)P(0,t)}]</td>
<td>1 [\frac{1}{P(0,1)}] ... [\frac{1}{P(0,t)}]</td>
</tr>
<tr>
<td>Stock</td>
<td>[\frac{S(0)}{B(1)}] ... [\frac{S(t)}{B(t)}]</td>
<td>[S(0)] [S(1)] ... [S(t)]</td>
</tr>
</tbody>
</table>

### TABLE III: BONDS NUMERAIRE

<table>
<thead>
<tr>
<th>Asset</th>
<th>Undiscounted prices</th>
<th>Discounted prices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Time 0 1 ... t</td>
<td>Time 0 1 ... t</td>
</tr>
<tr>
<td>(Money)</td>
<td>1 [P(0,1)] ... [P(0,t)]</td>
<td>1 1 ... 1</td>
</tr>
<tr>
<td>RollOver</td>
<td>1 [P(0,1)B(1)] ... [P(0,t)B(t)]</td>
<td>1 [B(1)] ... [B(t)]</td>
</tr>
<tr>
<td>Bonds</td>
<td>1 1 ... 1</td>
<td>1 [\frac{1}{P(0,1)}] ... [\frac{1}{P(0,t)}]</td>
</tr>
<tr>
<td>Stock</td>
<td>[S(0)] [P(0,1)S(1)] ... [P(0,t)S(t)]</td>
<td>[S(0)] [S(1)] ... [S(t)]</td>
</tr>
</tbody>
</table>