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Variance Dependent Pricing Kernels in GARCH Models

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A large, faint watermark of the Uppsala University seal is visible in the bottom right corner of the page. The seal features a sun with rays and the Latin motto 'ALERE FLAMMAM VERITATIS' around the perimeter.

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Abstract

In this thesis we study three of the striking contributions of Steve Heston to the context of option valuation. In the continuous time stochastic volatility model of Heston (1993) and discrete time GARCH model of Heston and Nandi (2000), we investigate the whole procedure towards obtaining the closed form formula for the value of the European call option. We then shape our discussion in the work of Christoffersen, Heston and Jacobs (2011) on the role of pricing kernel in option valuation. They have developed an option pricing model that incorporates the aversion to variance risk in the pricing kernel. Among reasons that feature their work in the context of option pricing is a new version of Heston and Nandi's GARCH model which not only overcomes the difficulties in the estimation of original stochastic volatility model of Heston, but also is able to produce a U-shaped pricing kernel observed in empirical studies.

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Chapter 1

Introduction

1.1 Overview

The literature on the option pricing models has witnessed extensive progress since the invention of Black and Scholes model (1973). A considerable amount of this effort has been channeled in providing more realistic assumptions for the underlying price process (see Bakshi, Cao and Chen (1997) for a collection of models). One of the most successful price processes is the bivariate diffusion model of Heston (1993).

Despite being successful, the Heston's stochastic volatility model is difficult to estimate because the volatility is not observable. This problem was successfully addressed by a GARCH framework in Heston and Nandi (2000). Yet another problem to be tackled in option valuation models was due to the pricing kernel which was a function of the index return only. Recently, by incorporating variance risk premium in the pricing kernel, Christofesson et al (2011) have solved this problem. This project is aimed to highlight the crucial role of pricing kernel in the context of option valuation. To do so, we investigate the option valuation framework in the SV model of Heston (1993), GARCH model of Heston and Nandi (2000) and finally the recent work of

Christofesson et al (2011). We focus on the derivation of the closed-form option price in the Heston (1993) work in chapter 3, whereas, in chapter 4 we mostly discuss on the physical and risk neutral processes in the Heston and Nandi's work (the valuation techniques are almost the same). Finally, in chapter 5 we talk less about the physical and risk neutral processes as well as the valuation techniques, but, put more emphasis on the construction of the pricing kernel in Christofesson et al (2011) which distinguishes the valuation model among other models.

In the following sections, we briefly explain techniques and concepts needed for the subsequent chapters.

1.2 Stylized Facts of Financial Returns

Properties of the financial data used for valuation of options show the way one has to take in order to construct or improve the underlying models. Therefore, knowing these properties helps us to better understand why different stochastic volatility models (e.g. Heston or GARCH diffusion) are employed in option valuation as we will see in the following chapters.

Stylized facts refer to a set of common features of financial data. In fact, they are some statistical properties common across various instruments and markets which have been revealed by empirical studies since almost a century ago. Here we only list some of these properties which are more relevant to this study, and refer interested reader to Cont (2001) for an inclusive survey.

1. Absence of serial correlation: except at high frequency, there is no linear autocorrelation in returns.
2. Fat-tailness: The unconditional distribution of returns has fatter tails than that expected from a normal distribution, meaning that, using the normal distribution for the purpose of

modeling financial returns, we will underestimate the number and magnitude of crashes and booms.

3. Volatility clustering: there exists significant serial correlation in volatility of returns. This means that a large (positive or negative) return tends to be followed by another large (positive or negative) return.
4. Leverage effect: the stock returns are typically negatively correlated with the volatility. This suggests that the debt-equity ratio or leverage of the firm increases when the stock price declines.
5. Asymmetry: there exists negative skewness in the unconditional distribution of returns, meaning that, extreme negative returns are more likely to happen than extreme positive returns.

Properties two and five exist, at lower intensity, even after correcting for volatility clustering.

1.3 Volatility Smile

Before going through option pricing with stochastic volatility in the next chapters, we essentially need to know why stochastic volatility models are so important in the context of option valuation.

Time varying volatility has been one of the long lasting issues in financial economics. Early comments on this include Mandelbrot (1963). Therefore, the assumption of constant volatility in the geometric Brownian motion governing the security price process in the Black-Scholes world was suspected soon after the breakthrough of Black and Scholes (1973). Later on, in particular after the 1987 market crash, the options' data observed in the market found not to be in harmony with the Black-Scholes prices. The value of volatility that equates these two prices, called

implied volatility, revealed the answer to the disharmony, the smile. Volatility smile is simply the relation between the implied volatility and the strike price (or some function of the strike price) of the option. More precisely, the term volatility smile is used for FX markets or equity index options as the graph turns up at either ends, whereas, for options such as equity options the graph is downward sloping and therefore the term volatility skew is often used. From the perspective of distribution of returns, volatility smile could be think of as the fat tail of option-implied return distribution that reconciles the empirical distributions of spot returns with the risk neutral distribution underlying option prices.

1.4 GARCH Models of Volatility

In this section we lay the foundation for the Heston and Nandi (2000) option pricing model by describing the well-known discrete time volatility model, GARCH. Engle (1982) invented Autoregressive Conditional Heteroskedasticity (ARCH) model to treat the serial correlation in square returns explained in the previous sections. GARCH model as an extension of this model introduced by Bollerslev (1986), successfully addressed the fat-tail feature present in the distribution of returns. We base our introduction to these models on Jondeau et al (2007). Generally, a volatility model can be structured as follow

$$\begin{aligned}x_t &= \mu_t(\theta) + \varepsilon_t, \\ \varepsilon_t &= \sigma_t(\theta)z_t,\end{aligned}\tag{1.1}$$

where

$$\begin{aligned}\mu_t(\theta) &= E[x_t | F_{t-1}], \\ \sigma_t^2(\theta) &= E[(x_t - \mu_t(\theta))^2 | F_{t-1}].\end{aligned}\tag{1.2}$$

The dynamic of conditional mean $\mu_t(\theta)$ in (1.2) is usually assumed to be an ARMA(p,q) process, $\sigma_t^2(\theta)$ is the model for conditional variance, θ is the vector of parameters and F_{t-1} is the information set available at time t . According to (1.1), ε_t has a time varying volatility conditioned on the information available at time $t-1$. Finally, z_t has a distribution with mean zero and variance one, such as standard normal or student-t. In case of GARCH model z_t is a strong white noise process. In a volatility model, volatility can be described in either of these two ways: as an exact function of a set of variables (e.g. GARCH models) or as a stochastic function (e.g. stochastic volatility models).

In the case of GARCH(p,q) model the dynamic of volatility is as follow

$$\sigma_t^2 = \omega + \sum_{i=1}^p \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2 \quad (1.3)$$

For this model to be well defined and the conditional variance to be positive, the parameters must satisfy $\omega > 0, \alpha_i \geq 0, i = 1, \dots, p, \beta_j \geq 0, j = 1, \dots, q$. In addition to these constraints, when $\sum_{i=1}^p \alpha_i + \sum_{j=1}^q \beta_j < 1$ which ensures the covariance stationary of ε_t , one can obtain the unconditional variance as

$$\sigma^2 = \omega / \left(1 - \sum_{i=1}^p \alpha_i - \sum_{j=1}^q \beta_j \right)$$

In the GARCH models, positive and negative past values have a symmetric effect on the conditional variance, which is against the empirical results suggesting that bad news tend to be followed by larger increases in volatility than equally large positive returns. Several

parameterizations (e.g. EGARCH, TGARCH and GJR) addressed this asymmetry in the response of volatility to shocks. We do not go further on the details of these models. Instead, we briefly explain two other models proposed by Engle and Ng (1993), Non-linear GARCH and Vector GARCH, as they are similar to the Heston and Nandi GARCH model of chapter 4. The conditional variance in the NGARCH(p,q) has the following form

$$\sigma_t^2 = \omega + \sum_{i=1}^p \alpha_i (\varepsilon_{t-i} + \gamma \sigma_{t-i})^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2 \quad (1.4)$$

whereas the VGARCH has the following conditional variance

$$\sigma_t^2 = \omega + \sum_{i=1}^p \alpha_i (\varepsilon_{t-i} / \sigma_{t-i} + \gamma)^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2 \quad (1.5)$$

1.5 No Arbitrage Argument and the Black-Scholes Pricing Equation

In this section we study some basic concepts in option pricing needed for the advanced arguments stated in chapter 3 onward. More precisely, this section reviews the no arbitrage argument of Black and Scholes (1973) and concludes with their well-known pricing equation. In the on arbitrage approach we assume a geometric Brownian motion for the dynamic of the underlying asset and then derive the dynamic of the derivative asset. A risk-free portfolio using the derivative and the underlying will be composed and in the next step in order to avoid the arbitrage opportunity, the instantaneous return of the portfolio must be equal to the risk-free rate of interest. We then arrive at a partial differential equation which has the price of the derivative asset as its solution.

Black and Scholes assumed that the stock price follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t \quad (1.6)$$

Where dS_t is the instantaneous price change, μ is the constant expected return, σ is the constant volatility of the stock return process and B_t is the Brownian motion or Weiner process such that $dB_t \sim N(0, dt)$. We denote the price of a European call option at time t with exercise price K and time of maturity T on the underlying asset S by $C(S, t)$. Applying Ito's lemma to this function we obtain the following dynamics

$$dC_t = \left[\frac{\partial C_t}{\partial t} + \mu S_t \frac{\partial C_t}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} \right] dt + \sigma S_t \frac{\partial C_t}{\partial S_t} dB_t \quad (1.7)$$

Now the value of the portfolio consisting of one unit of the call option and a short position of $\frac{\partial C_t}{\partial S_t}$ units in the stock would be $\Pi_t = C_t - \frac{\partial C_t}{\partial S_t} S_t$ with the dynamics

$$d\Pi_t = dC_t - \frac{\partial C_t}{\partial S_t} dS_t \quad (1.8)$$

Substituting dC_t from (1.7) and dS_t from (1.6)

$$d\Pi_t = \left[\frac{\partial C_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} \right] dt + 0 dB_t \quad (1.9)$$

To avoid arbitrage opportunity the return of this portfolio must be the same as the risk-free rate of interest

$$\begin{aligned}
d\Pi_t &= r\Pi_t dt = r \left(C_t - \frac{\partial C_t}{\partial S_t} S_t \right) dt \\
&= \left(\frac{\partial C_t}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} \right) dt
\end{aligned} \tag{1.10}$$

Finally

$$\frac{\partial C_t}{\partial t} + r S_t \frac{\partial C_t}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} - r C_t = 0 \tag{1.11}$$

Solving this equation along with its boundary condition which is the final payoff of the call option gives us the price of the European call option at time t . We later use these concepts and notations when we deal with option pricing in the presence of stochastic volatility.

1.6 Characteristic Function and Fourier Inversion Theorem

What made Heston's option pricing model superior to its preceding works was the closed-form formula he obtained via the characteristic functions. In this section, we review the basic concept of characteristic function and refer to them in the proceeding chapters.

Definition. Let X be a random variable with probability distribution F_X . The characteristic function of X or F_X is the function ϕ defined for real u by

$$\phi_X(u) = E[e^{iuX}] = \int_{-\infty}^{\infty} e^{iuX} dF_X(x) \tag{1.12}$$

or for distribution F_X with a probability density function f_X

$$\phi_X(u) = \int_{-\infty}^{\infty} e^{iuX} f_X(x) dx \tag{1.13}$$

We will be assuming that X is such that F_X is continuous and $\int_{-\infty}^{\infty} |\phi_X(u)| du < \infty$.

For instance, characteristic function of $X \sim N(\mu, \sigma^2)$ is $\phi_X(u) = \exp(\mu i u - \frac{1}{2} \sigma^2 u^2)$.

However, the right hand side of (1.13), with u replaced by $-u$, is known as the Fourier transform of f where f is any integrable function. Note also that there are different definitions of Fourier transform such as

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}} e^{-2\pi i \xi X} f(x) dx \\ \hat{f}(v) &= \int_{\mathbb{R}} e^{-ivX} f(x) dx \\ \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega X} f(x) dx\end{aligned}\tag{1.14}$$

for any integrable function f . Using characteristic function it is possible to characterize distributions that cannot be described by a probability distribution function. Using the inversion theorem, which we bring it here without proof, it is possible to recover the distribution function from its characteristic function.

Inversion theorem. Under the earlier assumptions about X

$$F_X(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{\exp(-itx) \phi_X(t)}{it} \right] dt.\tag{1.15}$$

Using this theorem the following equivalent expressions can also be derived

$$\begin{aligned}
F_X(x) &= \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(-itx)\phi_X(t)}{it} dt \\
&= F_X(0) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(-itx)-1}{it} \phi_X(t) dt \\
f_X(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_X(t) dt
\end{aligned} \tag{1.16}$$

Similar to the last line in (1.16), for the Fourier transforms in (1.14) we have

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbb{R}} e^{-2\pi i \xi X} f(x) dx & \Rightarrow f(x) &= \int_{\mathbb{R}} e^{2\pi i \xi X} \hat{f}(\xi) d\xi \\
\hat{f}(v) &= \int_{\mathbb{R}} e^{-ivx} f(x) dx & \Rightarrow f(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ivx} \hat{f}(v) dv \\
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega x} f(x) dx & \Rightarrow f(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\omega x} \hat{f}(\omega) d\omega
\end{aligned}$$

We are therefore equipped with flexible formulas which enable us to derive the density function from the characteristic function and vice versa. Using (1.16) for the example of normal random variable $X \sim N(\mu, \sigma^2)$ we can get the density as

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} e^{\mu iu - 0.5\sigma^2 u^2} du$$

We will come back to this later in chapter 3 and 4.

In our context, the Fourier transform is called characteristic function when the function used in the integration is a density. There are techniques (Fast Fourier Transform) in the context of Fourier analysis that make the computations of the integrals more efficient.

1.7 Pricing Kernel

Discussing on the pricing kernel inevitably takes us to the utility theory. In theory, a utility function U is an increasing (positive first derivative) and concave (negative second derivative) function that takes some observable variable such as consumption (of stock for instance) and gives the utility of this consumption that is not observable. Nevertheless, individuals have different utility functions. Three types of utility functions concave, convex and linear are assigned to risk averse, risk seeker and risk neutral individuals, respectively. These three functions however share a common feature, called non-satiation property, that is, they increase with the increase in the wealth, or more wealth is preferred to less wealth. This means that the first derivative of utility function is always positive. What differentiate the three forms of utility function is the risk preference of individuals that is reflected in the second derivative of U . The risk averse (lover) investor has a concave (convex) utility function which makes the second derivative negative (positive), and the risk neutral investor has a linear utility function and therefore second derivative of zero for its utility function.

It is therefore possible to analyze risk attitudes of investors by looking over their utility functions. One of the concepts that link the utility function of an investor to what is happening in the real world is the pricing kernel. Pricing kernel, as we will discuss it below, is the ratio between the risk neutral and the historical densities.

From asset pricing, we have the price of a security at time zero in an equilibrium model as

$$P_0 = E^P [\psi(S_T) M_T] \quad (1.17)$$

where $\psi(S_T)$ is the payoff function of security S_t at maturity T , which could be the stock price plus its dividend for a stock, or, the payoff of the call option at maturity for an option. E^P is the expectation with respect to the physical or historical measure P conditional on the information set available at time zero, and M_T is stochastic discount factor which we explain it further below. For the derivation of equation (1.17), which is the first-order condition for an optimal consumption and portfolio choice, we refer to Cochrane (2001).

Cochrane (2001) also explains that M_T is stochastic because otherwise (i.e. in case of no uncertainty) we could discount the payoff with the gross risk free rate R_f as $P_0 = \frac{1}{R_f} \psi(S_T)$ and for a certain riskier asset as $P_0 = \frac{1}{\tilde{R}} E[\psi(\tilde{S}_T)]$ where $\frac{1}{\tilde{R}}$ is an asset specific risk-adjusted discount factor. So, M_T is considered as a general discount factor that incorporates all risk adjustments, and is the same for each asset. Stochastic discount factor could also be shown as

$$M_T = \beta \frac{U'(S_T)}{U'(S_0)} \quad (1.18)$$

where β is a fixed discount factor and U' is the first derivative of the utility function.

We now try to obtain the form of pricing kernel in case of Black and Scholes model, and in chapter 5 we will see how this form will be generalize for appropriately explaining options data.

Recalling the price of an option under risk neutral measure Q , and following Detlefsen et al (2010) we can write

$$\begin{aligned}
P_0 &= \mathbb{E}^Q \left[e^{-rT} \psi(S_T) \right] \\
&= \mathbb{E}^P \left[e^{-rT} \frac{q(S_T)}{p(S_T)} \psi(S_T) \right]
\end{aligned} \tag{1.19}$$

Hence, from (1.17)-(1.19) we have

$$\beta \frac{U'(s)}{U'(S_0)} = e^{-rT} \frac{q(s)}{p(s)} \tag{1.20}$$

where r is the risk free rate and $q(s)/p(s)$ is the pricing kernel. Since both physical and risk neutral distributions in the Black and Scholes option pricing model have log-normal distribution, with r replaced by μ in geometric Brownian motion (1.6) for the risk neutral distribution, we have

$$p(s) = \frac{1}{s} \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \exp\left\{-\frac{1}{2}\left(\frac{\log s - \tilde{\mu}}{\tilde{\sigma}}\right)^2\right\}, \quad \begin{cases} \tilde{\mu} = (\mu - \sigma^2 / 2)t + \log S_0 \\ \tilde{\sigma} = \sigma\sqrt{t} \end{cases}$$

Therefore, one can derive the pricing kernel as

$$\tilde{M}(s) = \left(\frac{s}{S_0}\right)^{\frac{\mu-r}{\sigma^2}} \exp\left\{(\mu-r)(\mu+r-\sigma^2)T / (2\sigma^2)\right\} \tag{1.21}$$

which, as we expect from the Black and Scholes model, has form of the derivative of the power utility function. However, the fact that we cannot observe this monotonically increasing and concave utility function from the market data, has given rise to what is called the pricing kernel puzzle.

Pricing kernel puzzle first arose by Jackwerth (2000). To explain this puzzle, we could say that, theoretically, the marginal utility of investors (pricing kernel) should be decreasing when the

aggregate wealth of an economy rises. This means that behavior of investor is similar to the risk averse investor with a concave and increasing utility function. Empirical studies however proved the converse true. This study does not concern about various parametric and non-parametric methods that allows one to derive the pricing kernel, instead, we would like to mention that a successful option pricing model should be able to explain the pricing kernel puzzle. We leave the detailed discussion to be continued in chapter 5, but, briefly, Christoffersen et al (2011) generalized the pricing kernel and showed that the natural logarithm of such kernel is not decreasing but has a U-shape, and then developed an option pricing model which is able to accounts for such U-shaped pricing kernel.

Chapter 2

Stochastic Volatility Models

2.1 Overview

Of the highly controversial failures of the Black and Scholes model (1973), which has aged as long as the model itself, is the volatility smile problem discussed in the previous chapter. It is out of the scope of this thesis and rather difficult to give a full account of attempts done to address this problem. Various authors enhanced the literature by providing a richer structure for the price process compare to the geometric Brownian motion in the Black-Scholes model. Here we describe and group some major effort and then narrow down the discussion to stochastic volatility models and GARCH diffusion models which will be the focal points of this work in the next chapters.

Much research has been done in providing a more realistic description for asset price dynamics in the Black-Scholes model. The simple geometric Brownian motion has been evolved in two aspects. Firstly, in Merton (1976) a Poisson process accompanied the diffusion process of the price dynamics. Merton's work, which cares about the jumps in the price process, was highly acknowledged and followed by various works such as Cox and Ross (1976), Jones (1984), Ball

and Torous (1985), Aase (1988) and Kou (2002) among others. Secondly, the constant volatility assumption was relaxed through the general class of discrete and continuous stochastic volatility models such as Hull and White (1987), Scott (1987), Wiggins (1987), Heston (1993), Dupire (1994), Derman and Kani (1994), Duan (1995), Heston and Nandi (2000), Christoffersen, Heston, and Jacobs (2006) and Barone-Adesi, Engle and Mancini (2008). The stochastic volatility jump diffusion model as a combination of these two types of models was first emerged in Bates (1996). The model was supported by various studies thereafter, for instance Bakshi, Cao and Chen (1997) and Andersen, Benzoni and Lund (2002). Duffie, Pan and Singleton (2000) added jump to the dynamics of the volatility in the Bates's model. This model was also supported by a number of studies, for example Eraker, Johannes and Polson (2003). Here we skip discussing jump diffusion models, but, using the terminology given in Rebonato (2004), clarify which stochastic volatility models we aimed to study. In general, there can be two distinct sources of stochastic behavior for volatility. The first stems from the functional dependence of volatility on the underlying stochastic price process. The second is due to the fact that the volatility is allowed to be shocked by a second Brownian motion, only imperfectly correlated, if at all, with the Brownian motion driving the price process. Models displaying stochastic behavior in the volatility originating from both of these sources are called fully stochastic volatility models of which the Heston model is a good example. Models for which volatility is stochastic only due to the first source can be described as restricted stochastic volatility models or the well-known local volatility models. The term restricted might refer to the fact that in this models volatility is fully correlated with the stock price which is not the case in reality. The distinction between these two types is important, because as we will discuss later, a risk-neutral valuation cannot in general be used to obtain a unique option price for fully stochastic volatility models

(while it can be done in the restricted stochastic volatility case). We exclude the restricted models from our discussion from now on. Firstly, we discuss the general construction of the fully stochastic volatility models (hereinafter SV models), and briefly introduce some popular models such as SV model of Heston (1993), and then GARCH diffusion model of Heston and Nandi (2000).

2.2 Fully Stochastic Volatility Models

By the explanations we provided in chapter 1, we can identify two reasons that drew the attention of researchers to tackle the specification of constant volatility in the stock price dynamics of the Black-Scholes model and inventing stochastic dynamics for volatility. Firstly, the random character of volatility which had been well documented by statistical analysis of stock price history. Secondly, the well-known discrepancy between Black-Scholes estimated European option prices and market prices, that is, the smile curve. Here we discuss the general framework of SV models but leave the pricing issues to be addressed in the next chapter.

2.2.1 Construction of the Model

In SV models the asset price satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma_t S_t dB_t^1 \quad (2.1)$$

where $\sigma_t = f(Y_t)$ is the volatility process, obviously positive and not perfectly correlated with the Brownian motion B_t^1 . In fact it has the random component of its own. Therefore, contrary to the diffusion model in the Black-Scholes model, here there is a bivariate diffusion model. This

has implication in pricing as it requires the analysis to be done for an incomplete market. We will come back to this in details in the next chapter. We complete the bivariate construction by

$$dY_t = p(S_t, Y, t)dt + q(S_t, Y, t)dB_t^2 \quad (2.2)$$

where $d\langle B^1, B^2 \rangle_t = \rho dt$ and $\rho \in [-1, 1]$ is the correlation coefficient. It is however convenient to write

$$dB_t^2 = \rho dB_t^1 + \sqrt{1 - \rho^2} dW_t$$

where W_t is the Wiener process and $d\langle B^1, W \rangle_t = 0$. The correlation ρ , which is found to be negative in the empirical studies, is also sometimes called leverage effect.

2.2.2 Mean Reversion Property

In most of the models, volatility tends to get back to the mean level of its long run distribution. More precisely, there is a linear pull-back in the drift of the volatility process. Following closely the notation in Fouque et al (2000), the drift term in these models looks like $a(m - Y_t)$ where $a > 0$ (note that $a < 0$ indicates that the process is mean fleeting) is the speed of mean reversion and m is the long-run mean level of Y . To show this, by assuming the volatility of volatility to be zero (i.e. the case of a deterministic process) we have

$$dY_t = a(m - Y_t)dt \quad \Rightarrow \quad Y_t = m + (Y_0 - m)e^{-at}$$

It is clear that as time gets large, Y moves towards m . We do not go further on this and refer interested readers to Rebonato (2004) chapter 13 or Fouque et al (2000) chapter 2 for stronger results, and also chapter 4 of the latter for statistical techniques to estimate the rate of mean

reversion from historical asset prices. We end this section by mentioning that the Ornstein-Uhlenbeck models, see below, are the classic way to describe this property.

2.2.3 Some Popular Models

The fact that volatility is not observable has made it challenging to choose the right model. It is however common to pick a model which produces positive volatility, has the mean reversion property and provides closed-form formulas for European options. The following three models are common models for process Y_t and table below, taken from Fouque et al (2000), shows that they have been frequently used by researchers for the purpose of modeling volatility in option pricing.

Lognormal (LN):

$$dY_t = aY_t dt + bY_t dB_t^2$$

Ornstein-Uhlenbeck (OU):

$$dY_t = a(m - Y_t)dt + b dB_t^2$$

Feller or Cox-Ingersoll-Ross (CIR):

$$dY_t = \kappa(m' - Y_t)dt + v\sqrt{Y_t} dB_t^2$$

This process first investigated by Feller (1951) but was introduced in finance for modeling interest rate by Cox, Ingersoll and Ross (1985).

| Author | Correlation | $f(y)$ | Y Process |
|-------------|---------------|------------|-------------|
| Hull-White | $\rho = 0$ | \sqrt{y} | LN |
| Scott | $\rho = 0$ | e^y | OU |
| Stein-Stein | $\rho = 0$ | $ y $ | OU |
| Ball-Roma | $\rho = 0$ | \sqrt{y} | CIR |
| Heston | $\rho \neq 0$ | \sqrt{y} | CIR |

It is important to note that both Stein and Stein (1991) and Heston (1993) used the OU model but Heston applied Ito's lemma to the OU process and obtained the square root process of Cox-Ingersoll-Ross (1985).

Following the purpose of this study we discuss the Heston model and GARCH diffusion model in more details. This also would be a warming up for our discussion on pricing options through these models which is the topic of the next two chapters. Other models introduced for modeling the dynamic of volatility are Constant Elasticity of Variance Model, SABR volatility model, Chen model, 3/2 model and Schönbucher's stochastic implied volatility model.

2.2.3.1 Heston (1993) Model

Of the contributions of Heston's SV model compare to previous works is that it allows for shocks to return and volatility to be negatively correlated. This is very important because it makes the return distribution skewed which is, as we discussed in chapter 1, one of the stylized facts of equity index returns. Secondly, it provided the option valuation under stochastic volatility with the first closed-form solution for European options and consequently caught the attention of both academia and industry since then.

Assuming $\sigma_t = \sqrt{Y_t}$ in (2.1), by applying Ito's lemma to the square of the process

$$d\sqrt{Y_t} = -a\sqrt{Y_t}dt + b dB_t^2$$

We get directly the following SDE for Y_t namely

$$dY_t = (b^2 - 2aY_t)dt + 2b\sqrt{Y_t} dB_t^2,$$

which can be written also as the CIR process

$$dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t} dB_t^2.$$

We can therefore write the Heston's bivariate diffusion process as follows

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{Y_t} S_t dB_t^1 \\ dY_t &= \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t} dB_t^2 \end{aligned} \tag{2.3}$$

Where κ is the mean reversion rate, θ is the long run mean and $d\langle B^1, B^2 \rangle_t = \rho dt$. We will come back to this model in chapter 3 where we show how the Heston's option pricing formula can be derived.

2.3 GARCH Diffusion Model

The focus of the discussion in this section is on the Heston and Nandi (2000) GARCH diffusion model. Engle and Mustafa (1992) and Duan (1995), among others, also successfully used GARCH model to price options but none of these works could provide a closed-form solution for the price of option. We, therefore, introduce only the GARCH diffusion model of Heston and Nandi (hereinafter HN) as it is also the main pillar of this study. We will come to the pricing problem in chapter 4 and revisit the model in chapter 5 again in order to see how the new pricing kernel is employed in this model.

The following GARCH(1,1) process governs the dynamics of the spot asset price in the HN's diffusion model over time steps of length Δ

$$\ln S_t = \ln S_{t-\Delta} + r + \lambda y_t + \sqrt{y_t} z_t \quad (2.4)$$

$$y_t = \omega + \beta y_{t-\Delta} + \alpha \left(z_{t-\Delta} - \gamma \sqrt{y_{t-\Delta}} \right)^2 \quad (2.5)$$

where r constant is the continuously compounded interest rate for any time interval of length Δ and z_t is a standard normal disturbance. y_t is the conditional variance of the log return between $t-\Delta$ and t , and is known from the information set at time $t-\Delta$. The structure of the price process in (2.4) has two important implications:

Firstly, the expected spot return exceeds r by an amount proportional to the conditional variance y_t

$$\mathbb{E}[r_t] - r = \lambda y_t$$

where $r_t = \ln \frac{S_t}{S_{t-\Delta}}$.

Secondly, the return premium per unit of risk is proportional to the level of volatility

$$\frac{r_t - r}{\sqrt{y_t}} = \lambda \sqrt{y_t} + z_t.$$

It is also important to look at the following two characteristics of the variance process (2.5):

Firstly, recalling from chapter 1 it is clear that the equation for the variance process is slightly different from the standard GARCH(1,1) process of Bollerslev (1986) but similar to the NGARCH and VGARCH models of Engle and Ng (1993).

Secondly, the parameters α and γ determine the kurtosis and skewness of the distribution of log-returns, respectively. The γ parameter makes it possible for shocks to have asymmetric influence. That is, a large negative shock through z_t raises the variance more than a large positive shock. In general, given positive α , positive value for γ makes the correlation between the variance process and the spot return negative

$$Cov_{t-\Delta}[y_{t+\Delta}, \ln S_t] = -2\alpha\gamma y_t.$$

Following Foster and Nelson (1994), Heston and Nandi (2000) showed that their proposed mean and variance equations (2.4) and (2.5) have a continuous time diffusion limit as Δ tends to zero

$$d \ln S_t = (r + \lambda v_t)dt + \sqrt{v_t} dz_t$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dz_t$$

where $v_t = \frac{y_t}{\Delta}$ is the variance per unit of time and dv_t is exactly the square-root process of Feller (1951), CIR (1985) and Heston (1993).

We end this section by saying that the mean and variance equations (2.4) and (2.5) are the physical processes by which we construct the risk-neutral process of the HN's option pricing model in chapter 4.

Chapter 3

Heston's Option Pricing Model

3.1 Overview

Stylized facts of financial returns explained in chapter 1 are an argument against the assumption of constant variance of asset returns. This problem had been well-documented by researchers, e.g. Mandelbrot (1963), before the breakthrough of Black and Scholes (1973). Soon after that, much evidence emerged showing that volatility smile asserts that market prices of options can be obtained by employing a return distribution with fatter tails than the normal distribution. Therefore, the smile problem, in addition to the existing facts before Black and Scholes, highlight the problem of non-normality and constant variance assumption again and centered the attention of researchers. In chapter 2 we briefly reviewed efforts in the context of option pricing to model the smile and introduced the most successful of them, that is, stochastic volatility models. In this chapter we take one step further and clearly discuss that, having such models in hand, how the standard Black-Scholes model was extended to account for stochastic volatility. We do so by focusing on the brilliant work of Heston (1993) which extended the Black-Scholes model to the Heston option pricing model by providing the first closed-form

solution compare to its preceding works. The (semi) closed-form solution was due to using characteristic function, which is a special case of the Fourier transform.

3.2 Theoretical Consideration

In chapter 1 we showed that how the fundamental partial differential equation in the classical Black-Scholes model can be derived given the geometric Brownian motion. We again follow the no-arbitrage strategy and derive the Heston option pricing formula but we note that here we need to use the bivariate Ito's lemma because there are two sources of randomness in the Heston model. We expect a partial differential equation for the pricing function which has two space dimensions of price and volatility. We also expect the pricing function depend on the value of the volatility which is not observable. This has however implication in the hedging argument as we will see in the following subsection.

3.2.1 No Arbitrage Argument and the Pricing Equation

The task here is to construct a hedged portfolio of assets that can be priced by the no-arbitrage principle. Unlike the Black-Scholes case in chapter 1, two derivatives are required to obtain a risk-neutral portfolio. This is because both of the random sources in our diffusion model have to be balanced. Therefore, we consider short position of the call option C hedged by purchasing α units of the underlying asset and β units of the second option D written on the same underlying. C and D however have different specifications. Recalling from previous chapter Heston's bivariate diffusion is

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dB_t^1 \quad (3.1)$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dB_t^2 \quad (3.2)$$

where $d\langle B^1, B^2 \rangle_t = \rho dt$. To keep the notation simple we rewrite it as

$$dS_t = \mu_s dt + \sigma_s dB_t^1 \quad (3.3)$$

$$dv_t = \mu_v dt + \sigma_v dB_t^2 \quad (3.4)$$

Now, we need to modify the argument presented in chapter 1. Assuming $C(S, v, t)$ and $D(S, v, t)$ as the prices of two call options, the dynamic of C (and similarly D) can be obtained using the bivariate Ito's lemma

$$\begin{aligned} dC_t = & \left[\frac{\partial C_t}{\partial t} + \mu_s \frac{\partial C_t}{\partial S_t} + \mu_v \frac{\partial C_t}{\partial v_t} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 C_t}{\partial S_t^2} + \frac{1}{2} \sigma_v^2 \frac{\partial^2 C_t}{\partial v_t^2} + \rho \sigma_s \sigma_v \frac{\partial^2 C_t}{\partial S_t \partial v_t} \right] dt \\ & + \sigma_s \frac{\partial C_t}{\partial S_t} dB_t^1 + \sigma_v \frac{\partial C_t}{\partial v_t} dB_t^2 \end{aligned} \quad (3.5)$$

Therefore, the value of the resulting portfolio $\Pi_t = C_t - \alpha S_t - \beta D_t$ has the following evolution

$$\begin{aligned} d\Pi_t = & dC_t - \alpha dS_t - \beta dD_t \\ = & \left[\frac{\partial C_t}{\partial t} + \mu_s \frac{\partial C_t}{\partial S_t} + \mu_v \frac{\partial C_t}{\partial v_t} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 C_t}{\partial S_t^2} + \frac{1}{2} \sigma_v^2 \frac{\partial^2 C_t}{\partial v_t^2} + \rho \sigma_s \sigma_v \frac{\partial^2 C_t}{\partial S_t \partial v_t} \right] dt \\ & + \sigma_s \frac{\partial C_t}{\partial S_t} dB_t^1 + \sigma_v \frac{\partial C_t}{\partial v_t} dB_t^2 \\ & - \alpha \{ \mu_s dt + \sigma_s dB_t^1 \} \\ & - \beta \left[\frac{\partial D_t}{\partial t} + \mu_s \frac{\partial D_t}{\partial S_t} + \mu_v \frac{\partial D_t}{\partial v_t} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 D_t}{\partial S_t^2} + \frac{1}{2} \sigma_v^2 \frac{\partial^2 D_t}{\partial v_t^2} + \rho \sigma_s \sigma_v \frac{\partial^2 D_t}{\partial S_t \partial v_t} \right] dt \\ & - \beta \sigma_s \frac{\partial D_t}{\partial S_t} dB_t^1 - \beta \sigma_v \frac{\partial D_t}{\partial v_t} dB_t^2 \end{aligned} \quad (3.6)$$

This can be rearranged as

$$\begin{aligned}
d\Pi_t = & \left[\frac{\partial C_t}{\partial t} + \mu_s \frac{\partial C_t}{\partial S_t} + \mu_v \frac{\partial C_t}{\partial v_t} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 C_t}{\partial S_t^2} + \frac{1}{2} \sigma_v^2 \frac{\partial^2 C_t}{\partial v_t^2} + \rho \sigma_s \sigma_v \frac{\partial^2 C_t}{\partial S_t \partial v_t} - \alpha \mu_s \right] dt \\
& - \beta \left[\frac{\partial D_t}{\partial t} + \mu_s \frac{\partial D_t}{\partial S_t} + \mu_v \frac{\partial D_t}{\partial v_t} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 D_t}{\partial S_t^2} + \frac{1}{2} \sigma_v^2 \frac{\partial^2 D_t}{\partial v_t^2} + \rho \sigma_s \sigma_v \frac{\partial^2 D_t}{\partial S_t \partial v_t} \right] dt \\
& + \left[\sigma_s \frac{\partial C_t}{\partial S_t} - \beta \sigma_s \frac{\partial D_t}{\partial S_t} - \alpha \sigma_s \right] dB_t^1 + \left[\sigma_v \frac{\partial C_t}{\partial v_t} - \beta \sigma_v \frac{\partial D_t}{\partial v_t} \right] dB_t^2
\end{aligned} \tag{3.7}$$

In order to neutralize the risk due to dB_t^1 and dB_t^2 the expressions in the two last brackets of (3.7) must be zero. Therefore, we have two hedge ratios

$$\beta = \frac{\frac{\partial C_t}{\partial v_t}}{\frac{\partial D_t}{\partial v_t}} \tag{3.8}$$

$$\alpha = \frac{\frac{\partial C_t}{\partial S_t}}{\frac{\partial D_t}{\partial S_t}} - \frac{\frac{\partial C_t}{\partial v_t}}{\frac{\partial D_t}{\partial v_t}} \tag{3.9}$$

Now, in order to avoid arbitrage opportunity, return of the hedged portfolio must be equal to the return on a risk-free investment. That is

$$\begin{aligned}
d\Pi_t &= r\Pi_t dt \\
&= r(C_t - \alpha S_t - \beta D_t) dt
\end{aligned} \tag{3.10}$$

Therefore, if we compare the dt terms of (3.7) and (3.10) and also substitute the values of α and β from (3.8) and (3.9), we get

$$\left[\frac{\partial C_t}{\partial t} + \mu_v \frac{\partial C_t}{\partial v_t} + rS_t \frac{\partial C_t}{\partial S_t} - rC_t + \frac{1}{2} \sigma_s^2 \frac{\partial^2 C_t}{\partial S_t^2} + \frac{1}{2} \sigma_v^2 \frac{\partial^2 C_t}{\partial v_t^2} + \rho \sigma_s \sigma_v \frac{\partial^2 C_t}{\partial S_t \partial v_t} \right] \bigg/ \frac{\partial C_t}{\partial v_t} = \left[\frac{\partial D_t}{\partial t} + \mu_v \frac{\partial D_t}{\partial v_t} + rS_t \frac{\partial D_t}{\partial S_t} - rD_t + \frac{1}{2} \sigma_s^2 \frac{\partial^2 D_t}{\partial S_t^2} + \frac{1}{2} \sigma_v^2 \frac{\partial^2 D_t}{\partial v_t^2} + \rho \sigma_s \sigma_v \frac{\partial^2 D_t}{\partial S_t \partial v_t} \right] \bigg/ \frac{\partial D_t}{\partial v_t} \quad (3.11)$$

As it can be seen in (3.11) there is a symmetry in the the above equation and the two sides differ only in the type of the option. This means that there is no cross dependencies on either sides of the equation. In fact, we need to have equation (3.11) valid for call option of any maturity and strike price and therefore each side of the equality must be independent from the type of option one considers. We can therefore require each side to be equal to some function, say $\lambda(S_t, v_t, t)$, which does not depend on the maturity or strike price of option

$$\begin{aligned} & \frac{\partial C_t}{\partial t} - rC_t + rS_t \frac{\partial C_t}{\partial S_t} + \mu_v \frac{\partial C_t}{\partial v_t} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 C_t}{\partial S_t^2} \\ & + \frac{1}{2} \sigma_v^2 \frac{\partial^2 C_t}{\partial v_t^2} + \rho \sigma_s \sigma_v \frac{\partial^2 C_t}{\partial S_t \partial v_t} = \frac{\partial C_t}{\partial v_t} \lambda(S_t, v_t, t) \end{aligned} \quad (3.12)$$

Replacing the parameters from (3.1) and (3.2), the fundamental partial differential equation is

$$\begin{aligned} & \frac{\partial C_t}{\partial t} - rC_t + rS_t \frac{\partial C_t}{\partial S_t} + [\kappa(\theta - v_t) - \lambda(S_t, v_t, t)] \frac{\partial C_t}{\partial v_t} \\ & + \frac{1}{2} S_t^2 v_t \frac{\partial^2 C_t}{\partial S_t^2} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 C_t}{\partial v_t^2} + \rho \sigma S_t v_t \frac{\partial^2 C_t}{\partial S_t \partial v_t} = 0 \end{aligned} \quad (3.13)$$

Now by changing the variable $x_t = \ln(S_t)$ and considering the call option, $C(e^{x_t}, t)$, we get then

$$\frac{\partial C_t}{\partial x_t} = \frac{\partial C_t}{\partial S_t} \frac{\partial S_t}{\partial x_t} = \frac{\partial C_t}{\partial S_t} S_t \text{ and } \frac{\partial^2 C_t}{\partial x_t^2} = S_t^2 \frac{\partial^2 C_t}{\partial S_t^2}. \text{ Therefore, (3.13) becomes}$$

$$\begin{aligned}
& \frac{\partial C_t}{\partial t} - rC_t + r \frac{\partial C_t}{\partial x_t} + [\kappa(\theta - v_t) - \lambda(x_t, v_t, t)] \frac{\partial C_t}{\partial v_t} \\
& + \frac{1}{2} v_t \frac{\partial^2 C_t}{\partial x_t^2} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 C_t}{\partial v_t^2} + \rho \sigma v_t \frac{\partial^2 C_t}{\partial x_t \partial v_t} = 0
\end{aligned} \tag{3.14}$$

In equation (3.14) one can see that both r and the term $[\kappa(\theta - v_t) - \lambda(x_t, v_t, t)]$ are playing the same role for stochastic stock price and stochastic volatility, respectively. In the Black-Scholes case, r is the rate at which the stock price must grow in order that the option price grows at the riskless rate. Similarly, $[\kappa(\theta - v_t) - \lambda(x_t, v_t, t)]$, which is the risk neutral drift of the volatility process, has the same role for volatility.

The task from now on is to solve the above equation. The first things therefore we need are the boundary conditions for the European call option with strike price K and maturity T

$$\begin{aligned}
C(S_T, v_t, r, K, T, t) &= \max(S_T - K, 0) \\
C(0, v_t, r, K, T, t) &= 0 \\
\frac{\partial C_t}{\partial S_t}(\infty, v_t, r, K, T, t) &= 1
\end{aligned} \tag{3.15}$$

The first two conditions are obvious but the third one means that the changes in the value of the call option is equivalent to changes in the value of the underlying given the option is deeply in the money.

Recall that we left the function $\lambda(x_t, v_t, t)$ totally general. We suffice to say that it can be interpreted as volatility risk premium but devote the next section to get more insight on this function.

3.2.2 Market Price of Volatility Risk

The term market price of volatility risk refers to $\lambda(S_t, v_t, t)$ which appeared in the pricing equation (3.14). Here we explain why we call it market price of volatility risk. Suppose that, in the presence of stochastic volatility, we construct a portfolio of the option C which is hedged with the underlying asset only, $\Pi_t = C_t - \alpha S_t$. What does this mean? Simply, this means that of the two sources of randomness that must be hedged only the random source due to the asset price is hedged. Therefore, we expect that the unhedged risk due to the stochastic volatility attracts some reward from the market. Using (3.3)-(3.5) we get

$$\begin{aligned}
 d\Pi_t &= dC_t - \alpha dS_t \\
 &= \left[\frac{\partial C_t}{\partial t} + \mu_s \frac{\partial C_t}{\partial S_t} + \mu_v \frac{\partial C_t}{\partial v_t} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 C_t}{\partial S_t^2} + \frac{1}{2} \sigma_v^2 \frac{\partial^2 C_t}{\partial v_t^2} + \rho \sigma_s \sigma_v \frac{\partial^2 C_t}{\partial S_t \partial v_t} \right] dt \\
 &\quad + \sigma_s \frac{\partial C_t}{\partial S_t} dB_t^1 + \sigma_v \frac{\partial C_t}{\partial v_t} dB_t^2 - \alpha dS_t \\
 &= \left[\frac{\partial C_t}{\partial t} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 C_t}{\partial S_t^2} + \frac{1}{2} \sigma_v^2 \frac{\partial^2 C_t}{\partial v_t^2} + \rho \sigma_s \sigma_v \frac{\partial^2 C_t}{\partial S_t \partial v_t} \right] dt \\
 &\quad + \frac{\partial C_t}{\partial v_t} dv_t + \left[\frac{\partial C_t}{\partial S_t} - \alpha \right] dS_t
 \end{aligned} \tag{3.16}$$

Since we are delta hedging ($\alpha = \frac{\partial C_t}{\partial S_t}$) the last term is zero. Therefore

$$\begin{aligned}
 d\Pi_t - r\Pi_t dt &= \left[\frac{\partial C_t}{\partial t} + rS_t \frac{\partial C_t}{\partial S_t} - rC_t + \frac{1}{2} \sigma_s^2 \frac{\partial^2 C_t}{\partial S_t^2} + \right. \\
 &\quad \left. \frac{1}{2} \sigma_v^2 \frac{\partial^2 C_t}{\partial v_t^2} + \rho \sigma_s \sigma_v \frac{\partial^2 C_t}{\partial S_t \partial v_t} \right] dt + \frac{\partial C_t}{\partial v_t} dv_t
 \end{aligned} \tag{3.17}$$

Using (3.4) and (3.12) we get

$$\begin{aligned}
d\Pi_t - r\Pi_t dt &= \frac{\partial C_t}{\partial v_t} \lambda(S_t, v_t, t) dt - \mu_v \frac{\partial C_t}{\partial v_t} dt + \frac{\partial C_t}{\partial v_t} dv_t \\
&= \frac{\partial C_t}{\partial v_t} [\lambda(S_t, v_t, t) dt - \mu_v dt + dv_t] \\
&= \frac{\partial C_t}{\partial v_t} [\lambda(S_t, v_t, t) dt - \mu_v dt + \mu_v dt + \sigma_v dB_t^2] \\
&= \sigma_v \frac{\partial C_t}{\partial v_t} \left[\frac{\lambda(S_t, v_t, t)}{\sigma_v} dt + dB_t^2 \right]
\end{aligned} \tag{3.18}$$

The terms in the bracket in equation (3.18) shows that for one unit of risk due to the random source of volatility we get λ/σ_v unit of reward for an investmen in the market rather than a bank account and thats why λ is called market price of risk (note that we could have λ' rather than λ/σ_v because we could define $\lambda = \lambda' \sigma_v$). Interested readers may want to consult Rebonato (2004) page 395-6 for a discussion on this function from a different perspective. We now come back to the pricing problem by discussing how the closed-form solution can be derived.

3.2.3 Characteristic Function and the Closed-form Solution

We now have almost everything required to price the European call option. In this section we closely follow the procedure in Jondeau, Poon and Rockinger (2007). We already know that the price of the European call option in its general form is

$$C(S_t, v_t, K, T, t, \theta) = e^{-r\tau} \int_0^\infty \max(S_T - K, 0) p(S_T | S_t, v_t) dS_T \tag{3.19}$$

where $\tau = T - t$ and θ is the vector of parameter, and $p(S_T | S_t, v_t)$ is the conditional probability density function of the price process, conditioned on the earlier information about the price and volatility. Using the change in variable $x_t = \ln(S_t)$ we get

$$\begin{aligned}
C(S_t, v_t, K, T, t, \theta) &= e^{-r\tau} \int_{-\infty}^{\infty} \max(e^{x_T} - K, 0) p(x_T | x_t, v_t) dx_T \\
&= e^{-r\tau} \int_{\log K}^{\infty} e^{x_T} p(x_T | x_t, v_t) dx_T \\
&\quad - e^{-r\tau} K \int_{\log K}^{\infty} p(x_T | x_t, v_t) dx_T
\end{aligned} \tag{3.20}$$

Heston then uses the martingale relation

$$S_t = e^{x_t} = e^{-r\tau} \int_{-\infty}^{\infty} e^{x_T} p(x_T | x_t, v_t) dx_T$$

and then (3.20) turns out to be

$$\begin{aligned}
C(S_t, v_t, K, T, t, \theta) &= S_t \int_{\log K}^{\infty} \frac{e^{x_T} p(x_T | x_t, v_t)}{\int_{y_T = -\infty}^{\infty} e^{y_T} p(y_T | x_t, v_t) dy_T} dx_T \\
&\quad - e^{-r\tau} K \int_{\log K}^{\infty} p(x_T | x_t, v_t) dx_T
\end{aligned} \tag{3.21}$$

The first integrand in (3.21) is positive and defines a probability measure that we denote by q

$$\begin{aligned}
C(S_t, v_t, K, T, t, \theta) &= S_t \int_{\log K}^{\infty} q(x_T | x_t, v_t) dx_T \\
&\quad - e^{-r\tau} K \int_{\log K}^{\infty} p(x_T | x_t, v_t) dx_T \\
&= S_t P_1(X_T > \log K | x_t, v_t) \\
&\quad - e^{-r\tau} K P_2(X_T > \log K | x_t, v_t)
\end{aligned} \tag{3.22}$$

now recall from the Fourier inversion theorem of chapter 1 we have

$$\begin{aligned}
F(x) &= \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{\exp(-iux) \phi(u)}{iu} \right] du \quad \Rightarrow \\
P[X > x] &= 1 - F(x) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{\exp(-iux) \phi(u)}{iu} \right] du
\end{aligned} \tag{3.23}$$

where ϕ is the characteristic function of some density. Therefore, we can write P_1 and P_2 in (3.22) as

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{\exp(-iu \log K) \phi_{j;x_t, v_t, t}(u)}{iu} \right] du \quad j = 1, 2. \tag{3.24}$$

The characteristic function ϕ_j for $j = 1$ and 2 , is associated with the transition probabilities q and p , respectively. The last step is to identify the characteristic function of these two transition probabilities to be able to derive a closed-form solution for the option price. To do so, we use the approach of Feynman-Kac which is the classical tool for the case of Black-Scholes model. Recall that from previous section we arrived at equation (3.14) by changing variable $x_t = \ln(S_t)$. This partial differential equation which is rewritten again below is enough for p

$$\begin{aligned}
&\frac{\partial C_t}{\partial t} - rC_t + r \frac{\partial C_t}{\partial x_t} + [\kappa(\theta - v_t) - \lambda(x_t, v_t, t)] \frac{\partial C_t}{\partial v_t} \\
&+ \frac{1}{2} v_t \frac{\partial^2 C_t}{\partial x_t^2} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 C_t}{\partial v_t^2} + \rho \sigma v_t \frac{\partial^2 C_t}{\partial x_t \partial v_t} = 0
\end{aligned} \tag{3.25}$$

but we do not have yet the partial differential equation needed for q . So, to derive it, we put the solution (3.22) into (3.25) and then regroup terms in P_1 and P_2 . We then have

$$\begin{aligned}
& (r + u_j v_t) \frac{\partial P_j}{\partial x} + \frac{1}{2} v_t \frac{\partial^2 P_j}{\partial x^2} + \sigma \rho v_t \frac{\partial^2 P_j}{\partial x \partial v_t} + \\
& (a - b_j v_t) \frac{\partial P_j}{\partial v_t} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 P_j}{\partial v_t^2} + \frac{\partial P_j}{\partial t} = 0 \quad j = 1, 2.
\end{aligned} \tag{3.26}$$

where, following the assumption by Heston (1993), the volatility risk premium is replaced by a linear function of variance, that is $\lambda(S_t, v_t, t) = \lambda v_t$, and also

$$a = \kappa \theta, \quad b_1 = \kappa + \lambda - \sigma \rho, \quad b_2 = \kappa + \lambda, \quad u_1 = 0.5, \quad u_2 = -0.5.$$

Having (3.26) we can use the Feynman-Kac technique (see the appendix of Heston (1993) for more clarification). Here the goal is to obtain the characteristic function ϕ_j that satisfies (3.26).

Given that the coefficients in (3.26) are linear, the characteristic function has the form

$$\phi_{j;x_t,v_t,t}(u) = \exp[A_j(\tau, u) + B_j(\tau, u)v_t + C_j(\tau, u)x_t] \tag{3.27}$$

such that $\phi_{j;x_t,v_t,t}(u) \xrightarrow{t \rightarrow T} \exp(iux_T)$, and therefore, $A_j(0, u) = B_j(0, u) = 0, \quad C_j(0, u) = iu$.

Substituting (3.27) and its partial derivatives into (3.26) we get

$$\begin{aligned}
& (r + u_j v_t) C_j + \frac{1}{2} v_t C_j^2 + \rho \sigma v_t B_j C_j + (a - b_j v_t) B_j + \frac{1}{2} \sigma^2 v_t B_j^2 \\
& = \frac{\partial A_j}{\partial t} + \frac{\partial B_j}{\partial t} v_t + \frac{\partial C_j}{\partial t} x_t \quad j = 1, 2.
\end{aligned} \tag{3.28}$$

which must hold for for all v_t and x_t . Therefore, we get three ODEs by setting

$$(x_t, v_t) = (0, 0), \quad (x_t, v_t) = (1, 0), \quad (x_t, v_t) = (0, 1)$$

For the case of C_j we have $\partial C_j / \partial t = 0$ and by using its boundary condition we get $C_j(\tau, u) = iu$.

So, what we have to solve is a system of two ODEs, called Riccati equations

$$\begin{aligned}
\frac{\partial B_j}{\partial t} &= -\frac{1}{2}u^2 + \rho\sigma iuB_j + \frac{1}{2}\sigma^2 B_j^2 - b_j B_j + iu^2 \\
\frac{\partial A_j}{\partial t} &= aB_j + riu
\end{aligned} \tag{3.29}$$

By solving this system, finally, we have the closed-form solution of Heston option pricing for the European call option as follow

$$\begin{aligned}
C(S_t, v_t, K, T, t, \theta) &= S_t P_1 - K e^{-r\tau} P_2 \\
\left\{ P_j &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[\frac{\exp(-iu \log K) \phi_{j;x_t, v_t, t}(u)}{iu} \right] du \quad j=1,2 \right. \\
\left. \left\{ \phi_{j;x_t, v_t, t}(u) &= \exp[A_j(\tau, u) + B_j(\tau, u)v_t + iux_t] \right. \right. \\
\left. \left\{ A_j(\tau, u) &= rui\tau + \frac{\kappa\theta}{\sigma^2} \left[(b_j - \rho\sigma iu + d_j)\tau - 2 \log \left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j} \right) \right], \right. \right. \\
\left. B_j(\tau, u) &= \frac{b_j - \rho\sigma iu + d_j}{\sigma^2} \times \frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}, \right. \\
\left. \left\{ g_j &= \frac{b_j - \rho\sigma iu + d_j}{b_j - \rho\sigma iu - d_j}, \right. \right. \\
\left. d_j &= \sqrt{(\rho\sigma iu - b_j)^2 - \sigma^2(2u_j iu - u^2)}, \\
u_1 &= 0.5, \quad u_2 = -0.5, \\
b_1 &= \kappa + \lambda - \sigma\rho, \quad b_2 = \kappa + \lambda.
\end{aligned} \tag{3.30}$$

This is the first closed-form solution for an option valuation model based on continuous stochastic volatility model which is very popular in the academia and industry. An important feature of this model is that it allows for the correlation between the asset spot price and volatility.

3.3 Limitations and further developments

Despite its popularity, the Heston model has some limitations. This is mainly due to the unobservability of volatility. According to Heston and Nandi (2000), this makes it impossible to exactly filter a volatility variable from discrete observations of spot asset prices. Therefore, it is not possible to compute out-of-sample options valuation errors from the history of asset returns. Moreover, Mikhailov and Nogel (2003) showed that the integrals in the solution do not always have a convenient convergence behavior. Besides, the model might need to be extended (such as time dependent parameters) to better its performance across large time intervals of maturities. It also fails to create short term skew of a magnitude observed in the market and is unable to fit inverse yield curves.

In the next chapter, we discuss the GARCH option valuation model of Heston and Nandi (2000) which provides a closed-form solution for European options and address the main problems due to Heston model as well. Briefly, the variance of spot asset follows a GARCH process and is correlated with the asset returns. The single lag version of this process converges to the Heston stochastic volatility model as the time intervals shrink.

Chapter 4

Heston-Nandi's Option Pricing Model

4.1 Overview

In the previous chapter we described the stochastic volatility model of Heston (1993) which extended the Black-Scholes formula to the Heston option pricing model. In this chapter we study the option pricing framework based on a GARCH diffusion model presented by Heston and Nandi (2000). Similar to the former model, this model accounts for the stochastic nature of volatility as well as the correlation between the volatility and the spot returns. The model addresses the criticism of the SV model of Heston (see section 3 in the previous chapter and also Heston and Nandi (2000) for more details). Moreover, compared to the existing GARCH option models at that time (e.g. Engle and Mustafa (1992), Amin and Ng (1993) and Daun (1995)) which are simulation based and computationally intensive, this model provides the option value with a closed-form solution. The resulting option valuation model differs from the Black-Scholes and Heston's formula in the sense that the option values depend upon the current and lagged asset prices. Moreover, this model includes the Heston's SV model as its continuous time limit.

4.2 Heston-Nandi GARCH diffusion model

In the Heston-Nandi (hencforth HN) model the one period rate of return on S_t is assumed to be conditionally log-normally distributed under physical probability measure \mathbb{P}

$$R_t = r + \lambda v_t + \varepsilon_t, \quad \varepsilon_t = \sqrt{v_t} z_t, \quad \varepsilon_t | \mathcal{F}_{t-1} \sim N(0, v_t) \quad (4.1)$$

where $R_t = \ln(S_t / S_{t-1})$ is the log-return between $t-1$ and t , with v_t as the conditional variance. \mathcal{F}_{t-1} is the information set of all information up to and including time t , λ is a constant, r is the continuously compounded riskless interest rate for a period of length one, and z_t has the standard normal distribution. The mean equation in (4.1) shows that the average spot return depends on the level of risk. Under the normality, the conditional expected rate of return equals $r + \lambda v_t$, therefore λ can be interpreted as the unit risk premium. It is also evident that the return premium per unit of risk is proportional to the volatility, as in the Cox, Ingersoll and Ross (1985) model.

Heston and Nandi also assume that, under the measure \mathbb{P} , ε_t follows a particular GARCH(1,1) process

$$v_t = \omega + \beta v_{t-1} + \alpha \left(\varepsilon_{t-1} / \sqrt{v_{t-1}} - \gamma \sqrt{v_{t-1}} \right)^2 \quad (4.2)$$

where $\omega > 0, \alpha \geq 0, \beta \geq 0$ and γ are constant, and α and γ control the kurtosis and skewness of the distribution of the log-returns, respectively.

Following Foster and Nelson (1994), who derived the continuous time version of the variance process in the typical GARCH(1,1) model, one can show that the variance process in (4.2)

converges weakly to the continuous time variance process of Heston (1993) (see Appendix B of Heston and Nandi (2000)).

The variance process can be expressed in terms of the spot returns by substituting the expression for ε_t into the variance equation (4.2)

$$v_t = \omega + \beta v_{t-1} + \frac{\alpha}{v_{t-1}} (R_{t-1} - r - (\lambda + \gamma)v_{t-1})^2 \quad (4.3)$$

Similar to the continuous case, the price process and volatility process are negatively correlated

$$\text{Cov}(v_{t+1}, \ln S_t) = -2\alpha\gamma v_t$$

for positive α and γ .

4.3 The risk neutral GARCH process and the closed form formula

In order to value an option we first need to state the physical process in terms of risk-neutral measure \mathbb{Q} . Heston and Nandi propose that for the processes in (4.1)-(4.2), the risk neutral process takes the same GARCH form as before with $\lambda^* = -1/2$ and $\gamma^* = \gamma + \lambda + 1/2$. This means that under measure \mathbb{Q}

$$\begin{aligned} R_t &= r - \frac{1}{2}v_t + \varepsilon_t^*, \quad \varepsilon_t^* | \mathcal{F}_{t-1} \sim N(0, v_t) \\ v_t &= \omega + \beta v_{t-1} + \alpha \left(\varepsilon_{t-1}^* / \sqrt{v_{t-1}} - \gamma^* \sqrt{v_{t-1}} \right)^2, \quad \gamma^* = \gamma + \lambda + 1/2 \end{aligned} \quad (4.4)$$

One can see from (4.4) that $\frac{S_t}{S_{t-1}} | \mathcal{F}_{t-1}$ is log-normally distributed under risk neutral measure \mathbb{Q} .

Since $R_t | \mathcal{F}_{t-1} \sim N(r - \frac{1}{2}v_t, v_t)$, therefore,

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{S_t}{S_{t-1}} \middle| \mathcal{F}_{t-1}\right) = \mathbb{E}^{\mathbb{Q}}\left(e^{R_t} \middle| \mathcal{F}_{t-1}\right) = e^{r - \frac{1}{2}v_t + \frac{1}{2}v_t} = e^r \quad (4.5)$$

We also need to show that the conditional variances under two measures are equal, this means that the variance process in (4.4) must be equal to

$$\text{Var}^{\mathbb{Q}}(R_t | \mathcal{F}_{t-1}) = \text{Var}^{\mathbb{P}}(R_t | \mathcal{F}_{t-1}) \quad (4.6)$$

This is desirable because it enables us to observe and estimate the conditional variance under \mathbb{P} .

To show (4.6) holds, we first compare the two physical and risk neutral mean equations and get

$\varepsilon_t = \varepsilon_t^* - (\lambda + \frac{1}{2})v_t$. By substituting such ε_t into the variance equation in (4.2) we get

$$\begin{aligned} v_t &= \omega + \beta v_{t-1} + \alpha \left(\frac{\varepsilon_{t-1}}{\sqrt{v_{t-1}}} - \gamma \sqrt{v_{t-1}} \right)^2 \\ &= \omega + \beta v_{t-1} + \alpha \left(\frac{\varepsilon_{t-1}^* - (\lambda + \frac{1}{2})v_{t-1}}{\sqrt{v_{t-1}}} - \gamma \sqrt{v_{t-1}} \right)^2 \\ &= \omega + \beta v_{t-1} + \alpha \left(\frac{\varepsilon_{t-1}^*}{\sqrt{v_{t-1}}} - (\gamma + \lambda + \frac{1}{2})\sqrt{v_{t-1}} \right)^2 \\ &= \omega + \beta v_{t-1} + \alpha \left(\frac{\varepsilon_{t-1}^*}{\sqrt{v_{t-1}}} - \gamma^* \sqrt{v_{t-1}} \right)^2 \end{aligned} \quad (4.7)$$

which proves (4.6).

Finally, the value of the call option is the discounted expected value of the payoff calculated using the risk-neutral probabilities. To develop the pricing formula we first solve for the generating function of the GARCH process. Let $f(u)$ denotes the conditional generating function of the stock price

$$f(u) = f(u; t, T) = \mathbb{E}^{\mathbb{P}} \left[S_T^u \right] \quad (4.8)$$

which is equivalently the moment generating function of the $\ln S_T^u$, i.e.

$$f(u) = \mathbb{E}^{\mathbb{P}} \left[e^{u \ln S_T} \right]$$

For their GARCH process, Heston and Nandi showed that the moment generating function takes the log-linear form (see Appendix A in Heston and Nandi (2000))

$$f(u) = S_t^u \exp \left(A(u; t, T) + B(u; t, T) v_{t+1} \right) \quad (4.9)$$

where

$$A(u; t, T) = A(u; t+1, T) + ur + B(u; t+1, T) \omega - \frac{1}{2} \ln \left(1 - 2\alpha B(u; t+1, T) \right) \quad (4.10)$$

and

$$B(u; t, T) = u(\lambda + \gamma) - \frac{1}{2} \gamma^2 + \beta B(u; t+1, T) + \frac{\frac{1}{2}(u - \gamma)^2}{1 - 2\alpha B(u; t+1, T)} \quad (4.11)$$

can be calculated recursively from the terminal condition $A(u; T, T) = B(u; T, T) = 0$.

Since the generating function of the spot price, $f(u)$, is the moment generating function of the logarithm of the spot price, the characteristic function of the logarithm of the spot price is simply $f(iu)$. Therefore, to use this characteristic function we have to replace u by iu in (4.9)-(4.11).

The option pricing formula can be obtained by recovering the risk neutral probabilities from the characteristic function of the log spot price. We have the closed-form solution of HN option pricing formula for the European call option as follow

$$C(S_t, v_{t+1}, K, T) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} [\max(S_T - K, 0)] = S_t P_1 - K e^{-r(T-t)} P_2$$

$$\begin{cases} P_1 = \frac{1}{2} + \frac{e^{-r(T-t)}}{\pi S_t} \int_0^{\infty} \operatorname{Re} \left[\frac{K^{-iu} f^*(iu+1)}{iu} \right] du \\ P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \operatorname{Re} \left[\frac{K^{-iu} \phi(u)}{iu} f^*(iu) \right] du \end{cases} \quad (4.12)$$

where $f^*(iu)$ is the characteristic function with λ and γ in (4.10) and (4.11) replaced by $\lambda^* = -1/2$, and $\gamma^* = \lambda + \gamma + 1/2$, respectively. P_1 and P_2 are the risk-neutral probabilities calculated by inversion of the characteristic function $f^*(iu)$ of the logarithm of the spot asset price. The valuation formula can also be expressed as follow, which makes the computation faster

$$C(S_t, t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} [\max(S_T - K, 0)] = \frac{1}{2} (S_t - K e^{-r(T-t)}) + P_{1,2}$$

$$P_{1,2} = \frac{e^{-r(T-t)}}{\pi} \int_0^{\infty} \left\{ \operatorname{Re} \left(\frac{K^{-iu} f^*(iu+1)}{iu} \right) - K \operatorname{Re} \left(\frac{K^{-iu} f^*(iu)}{iu} \right) \right\} du \quad (4.13)$$

It is important to note that, contrary to the Black-Scholes and Heston's SV formula, the valuation formula (4.13) is a function of the current asset price and the conditional variance v_t which is itself a function of the observed path of the asset price (see (4.3)). Therefore, in contrast to the continuous case (see (3.30)) there is no need for the volatility to be calculated by other methods.

4.4 Limitations and further developments

In chapter 2 we discussed the limitation of the Heston's SV model which addressed by the Heston-Nandi model in the current chapter. Yet another problem to be tackled in option valuation models, including the latter model, is due to the pricing kernel which was a function of

the index return only. Christofesson, Heston and Jacobs (2011) discuss that existing models use the risk-neutral valuation relationship of Rubinstein (1976) and Brennan (1979) which produces the Black and Scholes formula (1973) for one-period options, and limits the models' ability to explain longer-term option prices. Recently, by incorporating variance risk premium in the pricing kernel, Christofesson et al (2011) have solved this problem. They developed an option pricing model that incorporates the aversion to variance risk in the pricing kernel while the dynamics of the price process are governed by the Heston-Nandi's GARCH framework. We will discuss this in the next chapter.

Chapter 5

Christoffersen-Heston-Jacobs's Option

Pricing Model

5.1 Overview

In chapter 3 we described the option valuation based on the continuous time SV model of Heston (1993). This approach however is difficult to implement and test for the volatility is not observable. This problem was addressed by computing implied volatilities, for example in Bates (1996b, 2000), or by estimating volatilities from the cross section of observed option prices, for example in Bakshi, Cao and Chen (1997). These methods also proved to be computationally inefficient. In chapter 4, we discussed the option valuation framework of Heston and Nandi (2000) which shows that the difficulty in the estimation of SV models can be best treated by using GARCH models. Since volatilities are observable in the GARCH model, a GARCH option model provides a framework to price option without using implied volatilities. In their paper, they developed a closed-form option valuation formula with a GARCH process ruling the dynamics of the variance of the asset.

In the last two decades, many researchers (e.g. Duan (1995) and Heston and Nandi (2000) for the case of conditionally normal innovations, and Barone-Adesi, Engle and Mancini (2008) and Christoffersen, Heston, and Jacobs (2006)) employed the GARCH framework for pricing European options. According to Christoffersen et al (2011) however, “all existing models are characterized by the same limitation. The filtering problem in these models is straightforward because the distribution of one-period returns has a known conditional variance. This does not severely restrict variance modeling, but it has implications for option pricing. Because the models do not contain an independent adjustment for variance risk, they do not offer much flexibility in the modeling of variance risk premia.” In their paper, they offer a variance-dependent pricing kernel combined with the dynamic of HN, and obtain a closed-form solution for the option. In this chapter we study this framework with a particular attention on the physical and risk-neutral processes as well as the new pricing kernel.

5.2 Theoretical Consideration

In this section, we provide a more technical review on the foundation and ideas behind the Christoffersen-Heston-Jacobs’s option pricing model. One could roughly consider the statement from Bates (1996a) as one of the key questions Christoffersen et al (2011) have tried to answer:

“the central empirical issue in option research is whether the distributions implicit in option prices are consistent with the time series properties of the underlying asset prices.”

Moreover, we know that time series models with dynamic volatility results in the physical distribution whereas recalibration of option pricing models using option data gives us the risk neutral distribution. Therefore, trying to solve the inconsistency between option prices and the time series of underlying index returns, stated by Bates (1996a), leads us to carefully investigate

the structure of the pricing kernel. There has not been much research done on the specification of the pricing kernel. While the physical distribution has been well investigated, the research on the risk neutral distribution has been mainly directed to require the form of the distribution in such a way that a closed-form formula for the option could be attainable. Investigation of the literature on this issue is not our aim, however, we channel our discussion in two ways: generally, how the pricing kernel should look like to be able to answer the argument of Bates, and how to prepare the right framework (discrete vs. continuous) for the pricing kernel in order to develop an option valuation model.

In order to develop our argument, we need to identify the stylized facts of the relative distributions of index return and option prices, i.e. the pricing kernel. This way, we could form the kernel in such way that it captures these stylized facts. Christoffersen et al (2011) summarize these facts as (1) the logarithm of the pricing kernel is U-shaped (2) option implied volatility is almost always higher than the realized one from index returns and (3) long term options tend to overreact to changes in short term volatility. In this work, we only consider the first one, which is about the shape of the pricing kernel. Using a semiparametric method Christoffersen et al (2011) obtained model-free physical and risk neutral conditional densities, and showed that the conditional pricing kernel is U-shaped, and conclude that, contrary to the hypothesis inherent in the Black-Scholes model, the pricing kernel is not a monotonic function of returns. Starting from the pricing kernel in the SV model of Heston (1993), one can show that such kernel (model-based) is also U-shaped when adjusted for variance risk premia. However, to overcome the difficulties in the latter model it would be interesting to see if we could construct such pricing kernel for the discrete time GARCH model of Heston and Nandi (2000), and then develop a

closed-form solution for European options. We put our discussion in a mathematical framework in the following two sections.

To do so, we start with the continuous time SV model of Heston and discuss why it is important to explicitly determine the pricing kernel when going from the physical process to the risk neutral process. We then redo the same procedure in discrete time GARCH model.

5.3 Continuous Time Risk Neutralization and the Pricing Kernel

Recall from chapter 3 that the physical process in the Heston's SV model with independent Wiener processes has the form

$$\begin{aligned} dS_t &= (r + \tilde{\mu}v_t)S_t dt + \sqrt{v_t}S_t dB_t^1 \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}\left(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2\right) \end{aligned} \quad (5.1)$$

where r is the risk free rate, $\tilde{\mu}$ is the equity premium since $\tilde{\mu} = (\mu - r)/v_t$ (μ is the expected rate of return as in (3.1)) and B_t^1 and B_t^2 are independent Wiener processes. It is important to note that in (5.1) both equity and volatility have separate sources of risk, that is, B_t^1 and B_t^2 . Consequently, there has to be separate premia for these risks in our framework.

Christoffersen et al (2011) assume that the pricing kernel takes the exponential affine form

$$M_t = M_0 \left(\frac{S_t}{S_0}\right)^\phi \exp\left(\delta t + \eta \int_0^t v_s ds + \xi(v_t - v_0)\right) \quad (5.2)$$

where δ and η are the time-preference parameters and ϕ and ξ are aversion to equity risk and aversion to variance risk, respectively, or more generally preference parameters. Transforming

the bivariate diffusion (5.1) into the risk neutral world we get (see appendix A in Christoffersen et al (2011))

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t d\tilde{B}_t^1 \\ dv_t &= (\kappa(\theta - v_t) - \lambda v_t) dt + \sigma \sqrt{v_t} (\rho d\tilde{B}_t^1 + \sqrt{1 - \rho^2} d\tilde{B}_t^2) \end{aligned} \quad (5.3)$$

where λ is the variance premium. The equity premium and variance premium can be expressed in terms of the preference parameters ϕ and ξ as below

$$\begin{aligned} \tilde{\mu} &= -\phi - \xi \sigma \rho \\ \lambda &= -\rho \sigma \phi - \sigma^2 \xi = \rho \sigma \tilde{\mu} - (1 - \rho^2) \sigma^2 \xi \end{aligned} \quad (5.4)$$

One can refer to the simple and intuitive argument in Christoffersen et al (2011) to see how the equity premium is positive while the variance premium is negative (note that $\phi < 0, \rho < 0$ and $\xi > 0$). However, what is important in (5.4) is the form of variance premium λ . It has two components, one is based on ϕ (or $\tilde{\mu}$) and the other based on ξ . Consequently, it is not appropriate to say that (5.3) holds for an arbitrary negative λ because either $\xi = 0$ or $\xi \neq 0$ results in some negative λ in (5.4), whereas, these two situations have different economic implications. Put it differently, the reduced-form (due to λ) dynamic of variance in (5.3) does not tell us whether the variance premium λ is fully coming out from the ϕ or ξ . Christoffersen et al (2011) therefore conclude that in option valuation with SV models, one has to explicitly identify the pricing kernel that links the physical and risk neutral dynamics. They obtained model-free physical and risk neutral conditional densities, and showed that the pricing kernel is U-shaped when $\xi > 0$.

After all, as discussed frequently before, the option valuation based on the SV model of Heston has the filtering problem and therefore awkward in estimation. We therefore switch to the discrete version of this model that is Heston-Nandi's GARCH model.

5.4 Discrete Time Risk Neutralization and the Pricing Kernel

In this section we describe how Christoffersen et al (2011) developed a new version of HN-GARCH model by combining the HN-GARCH dynamics of chapter 4 with the new pricing kernel of previous section. The resulting model is able to produce a U-shaped pricing kernel (or more generally capture the stylized facts). Finally, we also present the formula for the European call option. Having the physical process in hand, we start by constructing the discrete version of the U-shaped pricing kernel, stated in the previous section, and see what could be the risk neutral process.

The following equations are discrete time analog of the physical process in (5.1)

$$\begin{aligned} R_t &= r + (\tilde{\mu} - 1/2)v_t + \varepsilon_t, & \varepsilon_t &= \sqrt{v_t} z_t \\ v_t &= \omega + \beta v_{t-1} + \alpha \left(\varepsilon_{t-1} / \sqrt{v_{t-1}} - \gamma \sqrt{v_{t-1}} \right)^2 \end{aligned} \tag{5.5}$$

where r is the daily continuously compounded interest rate and z_t is the standard normal random variable. It can be shown that

$$Cov(R_t, v_{t+1}) = -2\alpha\gamma v_t \tag{5.6}$$

Empirical results reveal a negative relation between returns and the variance making the parameter γ positive. Now we have to identify the pricing kernel, which links the physical process to the risk neutral process. So far, GARCH option valuation models such as Duan (1995)

and Heston and Nandi (2000) have used the power pricing kernel of Rubinstein (1976) which is special case of the general pricing kernel in (5.2) assuming constant variance. Christoffersen et al (2011), however, use the discrete time version of (5.2) as the pricing kernel

$$M_t = M_0 \left(\frac{S_t}{S_0} \right)^\phi \exp \left(\delta t + \eta \sum_{s=1}^t v_s + \xi (v_{t+1} - v_1) \right) \quad (5.7)$$

and uses this to risk neutralize (5.5) as follow

$$\begin{aligned} R_t &= r - \frac{1}{2} v_t^* + \varepsilon_t^*, & \varepsilon_t^* &= \sqrt{v_t^*} z_t^* \\ v_t^* &= \omega^* + \beta v_{t-1}^* + \alpha^* \left(\varepsilon_{t-1}^* / \sqrt{v_{t-1}^*} - \gamma^* \sqrt{v_{t-1}^*} \right)^2 \end{aligned} \quad (5.8)$$

$$v_t^* = \frac{v_t}{1-2\alpha\xi}, \quad \omega^* = \frac{\omega}{1-2\alpha\xi}, \quad \alpha^* = \frac{\alpha}{(1-2\alpha\xi)^2}, \quad \gamma^* = \gamma - \phi \quad (5.9)$$

See the proof in appendix B of Christoffersen et al (2011). We note that in the previous HN-GARCH model $\xi=0$ and only γ changes across measures. Similar to the continuous time the pricing kernel here is also U-shaped when $\xi > 0$. Fitting such model using the option data and underlying return data via the maximum likelihood estimation, one can obtain the physical and risk neutral densities (via the characteristic function) and see that the resulting model-based pricing kernel is also U-shaped. However, without considering the variance premium in the kernel (i.e. $\xi=0$) the kernel would reveal just a decreasing trend.

Finally, following Heston and Nandi (2000), the closed form formula for the value of a call option at time t with strike price K and maturity T is equal to

$$\begin{aligned}
C(S_t, v_{t+1}, K, T) &= e^{-r(T-t)} \mathbf{E}^* [\max(S_T - K, 0)] = S_t P_1 - K e^{-r(T-t)} P_2 \\
&\left\{ \begin{aligned} P_1 &= \frac{1}{2} + \frac{1}{\pi_0} \int_0^\infty \operatorname{Re} \left[\frac{K^{-iu} \phi(u+1)}{iu} \right] du \\ P_2 &= \frac{1}{2} + \frac{1}{\pi_0} \int_0^\infty \operatorname{Re} \left[\frac{K^{-iu} \phi(u)}{iu} \right] du \end{aligned} \right. \quad (5.10)
\end{aligned}$$

where all definitions are as before and ϕ is the characteristic function for the risk neutral process

(5.8).

6 Conclusion

In this thesis, we investigated the option valuation framework in the stochastic volatility model of Heston (1993), GARCH model of Heston and Nandi (2000) and finally the recent work of Christofesson et al (2011), which combines the former model with a new pricing kernel. We started by focusing on the brilliant work of Heston (1993) which extended the Black-Scholes model to the Heston option pricing model by providing the first closed-form solution compare to its preceding works. The (semi) closed-form solution was due to using characteristic function, which is a special case of the Fourier transform. Despite being successful, the Heston's stochastic volatility model is difficult to estimate because the volatility is not observable. This problem was successfully addressed by a GARCH framework in Heston and Nandi (2000). This model accounts for the stochastic nature of volatility as well as the correlation between the volatility and the spot returns. While providing the option value with a closed-form solution, this model adresses the criticism of the SV model of Heston. The resulting option valuation model differs from the Black-Scholes and Heston's formula in the sense that the option values depend upon the current and lagged asset prices. Moreover, this model includes the Heston's SV model as its continuous time limit.

Yet another problem to be tackled in option valuation models was due to the pricing kernel. Empirical studies have revealed that, contrary to the economic theory, the pricing kernel is not decreasing. Therefore, a successful option pricing model should be able to explain such puzzle. We explained how this could be handled by incorporating variance risk premium in the pricing kernel. As a result, the model-based pricing kernel is U-shape which is consistent with semi-

parametric evidence from returns and options showing that the conditional pricing kernel is U-shaped in returns. Therefore, it can be concluded that in option valuation with SV models, one has to explicitly identify the pricing kernel that links the physical and risk neutral dynamics.

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