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Horizon-unbiased Utility of Wealth and Consumption

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Abstract

This thesis studies dynamically consistent utility functions related to finding the optimal time to sell an asset. We consider a risk-averse utility maximizing agent who owns an asset and wants to choose the optimum time to sell it. The asset which forms part of the agent's wealth is indivisible, non-traded and the asset sale is irreversible. She has access to a financial market to invest in and also consumes a part of her wealth at each instant.

We formulate the sale of the real asset as a mixed optimal stopping/optimal control problem with respect to the agent's utility pair of wealth and consumption. We will show that in order to eliminate biases in the choice of the optimal selling time of the asset, the agent's utility pair must be horizon-unbiased. It turns out that, by appealing to the dynamic programming principle and taking the utility of wealth as the solution to the associated Hamilton-Jacobi-Bellman (HJB) equation, we can find the agent's utility of consumption such that her utility pair of wealth and consumption is horizon-unbiased.

We also reduce the asset sale problem to a free-boundary problem and state and prove a verification theorem. Our interpretation is that, it is optimal for the agent to sell the real asset the first time the ratio of the real asset to wealth exceeds the free boundary.

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Contents

1 Introduction

In recent times, the concept of utility functions satisfying certain consistency conditions has received a lot of interest in the mathematical finance literature. Motivated by Merton's portfolio problem, Berrier et al [1] and Berrier and Tehranchi [2] studied related objects called "forward utility functions" in order to capture the dynamically changing preferences of an investor. Their studies were done in terms of convex duality. Henderson in [8] independently defined and Henderson and Hobson [10], Evans et al [7] also studied related objects in the context of finding the optimal time to sell an indivisible, non-traded asset using the dynamic programming approach.

In this thesis, we extend the works of Henderson and Hobson [10] and Evans et al [7] by introducing consumption into the problem setup. This is natural, since in the real real world, an investor endowed with a certain wealth will consume a part of it. In that sense, not only does our agent invest in a frictionless financial market, but he also consumes a part of his wealth at each instant. To be precise, we consider a risk-averse utility maximizing agent who owns a single unit of an asset and wants to choose the optimum time to sell it. The preferences of the agent are modelled by a constant relative risk aversion (CRRA) utility. The price process of the asset - which we call a real asset - is given by the stochastic process $(Y_t)_{t\geq 0}$. Since the asset is not traded, the agent faces an incomplete market. Access to the financial market, which includes assets which are partially correlated with the real asset, enables the agent to eliminate systematic risk by trading. However, the idiosyncratic part of risk associated with the real asset still remains. Trading takes place in the infinite horizon. The assumptions of indivisible asset, irreversible sale and an infinite horizon, though uncommon, are both realistic and natural. For details and examples that fall within the framework of this work, the interested reader is referred to Dixit and Pindyck [5] and the real options literature. Again, by considering the problem in the infinite horizon, the time dependence on the optimal stopping rule is completely eliminated.

Consider that we can write the agent's problem as one of mixed optimal stopping/control problem of the form:

$$\sup_{\tau} \sup_{C,\pi\in\mathcal{A}_\tau} \mathbb{E}\left[U_X(\tau, X_\tau^{\pi,C} + Y_\tau) + \int_t^\tau U_C(s, C_s)\,\mathrm{ds}\right] \tag{1}$$

$\mathbb{E}$ is the expectation operator, U_X and U_C are respectively the utility derived by the agent from wealth and consumption. The stopping rule, τ, belongs to the class of all admissible stopping times, $\mathcal{T}$; $\mathcal{A}_\tau$ is the set of admissible strategies defined up to the stopping time; X parameterized by π and C is an element of the set of admissible wealth process for the agent; and Y_τ is the amount the agent receives at the time of sale, τ^*. If we denote respectively by $X = X_t^{\pi,C} \in \{\mathcal{A}_t : t \in [0,\tau)\}$ and $X = (X_t^{\pi,C} + Y_t) \in \{\mathcal{A}_t : t \in (\tau,\infty]\}$, the wealth of the agent before and after the sale of the real asset, then except at the optimal stopping time, $t = \tau^*$, the agent's wealth process is self-financing.

We will argue that in order to eliminate the possibility of the agent preferring certain stopping times over others at which to sell the asset, her utility functions of wealth and consumption cannot be chosen arbitrarily, but must instead satisfy certain consistency conditions so that the mathematical problem has the desired economic interpretation. With our agent's utility functions satisfying the right properties, we are sure that the only motivation available to her is the existence of the right to sell the real asset. We shall call such utility functions "horizon-unbiased". This idea has been used previously in the works of Davis and Zariphopoulou [5] for pricing American options under transaction costs, and Oberman and Zariphopoulou [15] for pricing finite horizon American options in an incomplete market. Some of the other existing literature that assume market incompleteness are Henderson [8], Henderson and Hobson [10], Evans et al [7], Miao and Wang [18] and Ekström and Lu [19]. Ekström and Lu [19] consider an agent who wants to liquidate an asset with unknown drift and believes that the drift takes one of two values. They demonstrate that the optimal strategy is to "liquidate the first time the asset price falls below a certain time-dependent boundary". Henderson and Hobson [10], Evans et al [7] consider the following optimal stopping/optimal control problem facing a risk-averse utility maximizing agent:

$$\sup_{\tau} \sup_{\pi \in \mathcal{A}_\tau} \mathbb{E}\left[U(\tau, X_\tau^\pi + Y_\tau)\right] \tag{2}$$

where U is a CRRA utility given by $U(t,x) = e^{-\beta t}\frac{x^\alpha}{\alpha}$, and β, a subjective discount factor. They argue that for the problem (2), the subjective discount factor, β, cannot be chosen arbitrarily in order for the problem to be internally consistent. They say that " in the infinite horizon case, the problem (2) has no preferred horizon if its solution is a supermartingale in general and a martingale for optimal investment strategy." In that sense, when the agent faces the problem without the real asset, $(Y_t)_{t\geq 0}$, she should not be biased over the choice of stopping times at which to measure her utility. By choosing such a utility function, there is the assurance that conclusions about the optimal sale time are not influenced by artificial incentives for the agent to prefer one horizon over another. In fact, the requirement of no preferred horizon forces the discount factor, β, to be $\frac{\alpha\lambda^2}{2(1-\alpha)} + r\alpha$.

In this thesis, our concept of horizon-unbiased utility functions is to take the solution to the associated HJB equation as the definition of the utility of wealth and then find a utility of consumption such that the pair (U_X, U_C), is horizon-unbiased. To be exact, let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a given filtered probability space satisfying the usual conditions. By considering (1) and taking the solution of the associated HJB equation as the definition of the utility function of wealth, we define a horizon-unbiased utility of wealth and consumption as a pair (U_X, U_C), such that U_X is a supermartingale in general and a martingale for optimal portfolio-consumption pair.

This thesis is organized as follows: In section 2, we give mathematical concepts and ideas needed to understand the content. Section 3 is devoted to introducing assumptions on the utility functions and a description of our model. In section 4, we state the main result of this thesis and necessary and sufficient conditions for a utility pair to be horizon-unbiased. This is our main contribution to the already existing literature on the subject. In section 4, we will reduce the asset sale problem to a free-boundary problem and state and prove a verification theorem. The last section is devoted to giving conclusions about the work in this thesis.

2 Mathematical Preliminaries

In this section, we give a brief overview of the mathematical concepts and ideas needed to understand the subsequent sections of this thesis. The interested reader is referred to the monographs by Peskir and Shiryaev [16] and Karatzas and Shreve [12] for details.

2.1 Probability Spaces

Definition 2.1. A filtration $(\mathcal{F}_t)_{t\geq 0}$ is a nondecreasing and right continuous family of sub-σ-algebras of $\mathcal{F}$. $\mathcal{F}_t$ is interpreted as the information available up to time t.

Definition 2.2. A filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$.

Throughout this thesis, we make use of the following assumption:

Assumption 2.1. The filtered probability space in Definition 2.2 satisfies the following conditions:

(i) the σ-algebra $\mathcal{F}$ is $\mathbb{P}$-complete

(ii) every $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets from $\mathcal{F}$

2.2 Martingale theory

One of the fundamental mathematical properties which underpins many important results in finance is the martingale property. As a necessary condition for an efficient market, a martingale basically assumes that tomorrow's price of a financial asset is expected to be today's and therefore it is its best forecast. For example, the First Fundamental Theorem of Asset Pricing states that, given a fixed numéraire process, a financial market is arbitrage free if and only if there exist an equivalent martingale measure.

We give formal definitions of martingale, supermartingale and submartingale.

Definition 2.3. Let $X = (X_t)_{t\geq 0}$ be a continuous time stochastic process adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We say that X is a

(i) martingale if $\mathbb{E}|X_t| < \infty \ \forall t$ and $\mathbb{E}\left(X_t|\mathcal{F}_s\right) = X_s \ \forall s \leq t$;

(ii) supermartingale if $\mathbb{E}\left(X_t|\mathcal{F}_s\right) \leq X_s \ \forall s \leq t$;

(iii) submartingale if $\mathbb{E}\left(X_t|\mathcal{F}_s\right) \geq X_s \ \forall s \leq t$.

2.3 Optimal Stopping in continuous time

In this subsection, we give basic results of optimal stopping in continuous time. Once a problem of interest has been setup as an optimal stopping problem, then one needs to consider the exact solution techniques to use in dealing with the problem. There are two important approaches to optimal stopping problems - the martingale approach and the Markovian approach. Since our problem uses Markov models, we only treat the Markovian approach.

2.3.1 Markovian approach

Consider the a Markov process $X = (X_t)_{t\geq 0}$ defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ and taking values in a measurable space $\left(\mathbb{R}^d, \mathcal{B}\right)$ for some $d \geq 1$. $\mathcal{B} = \mathcal{B}\left(\mathbb{R}^d\right)$ is the Borel σ-algebra on $\mathbb{R}^d$. It is assumed that the process X starts at x under $\mathbb{P}_x$ for $x \in \mathbb{R}^d$ and that the sample paths of X are right-continuous and left-continuous over stopping times. $(\mathcal{F}_t)_{t\geq 0}$ is also assumed to be right-continuous.

Definition 2.4. A random variable $\tau : \Omega \to [0, \infty)$ is called a stopping time $\mathbb{P}$-almost surely if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Let $G : \mathbb{R}^d \to \mathbb{R}$ be a measurable function, called the gain function, satisfying the following integrability condition (with $G(X_\tau) = 0$ if $T = \infty$):

$$\mathbb{E}_x\Big(\sup_{0\leq t\leq T} |G(X_t)| \Big) < \infty \tag{3}$$

for all $x \in \mathbb{R}^d$. Consider the optimal stopping problem:

$$V(x) = \sup_{0\leq \tau\leq T} \mathbb{E}G(X_\tau) \tag{4}$$

$x \in \mathbb{R}^d$ and the supremum is taken over stopping times τ with respect to $(\mathcal{F}_t)_{t\geq 0}$. V is called the value function.

In order to solve the optimal stopping problem (4), we need to find a stopping time τ^* at which the supremum is attained and also compute the value $V(x)$ for $x \in \mathbb{R}^d$ as explicitly as possible. Evaluating $G\big(X_t(\omega)\big)$ for the sample path $X_t(\omega)$ where $\omega \in \Omega$ is given and fixed, one will be able to optimally decide either to continue with the observation of X or to stop it. It therefore natural to split the state space, $\mathbb{R}^d$, into 2 region; one where it is optimal to continue called the continuation region C, and one where it is optimal to stop called the stopping region $D = \mathbb{R}^d \backslash C$ as soon as $X_t(\omega)$ enters D.

Consider the infinite horizon problem i.e. when $T = \infty$:

$$V(x) = \sup_{\tau} \mathbb{E}_x G(X_\tau) \tag{5}$$

where τ is a stopping time with respect to the $(\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{P}_x(X_0 = x) = 1$. Define the continuation and the stopping regions respectively by:

$$C = \left\{x \in \mathbb{R}^d : V(x) > G(x)\right\}$$

$$D = \left\{x \in \mathbb{R}^d : V(x) = G(x)\right\}$$

and let

$$\tau_D = \inf\{t \geq 0 : X_t \in D\} \tag{6}$$

be the first entry of X into D.
We remark that if V is lsc (lower semicontinuous) and G is usc (upper semicontinuous), then C is open and D is closed. The usc of G ensures that we don't stop outside the stopping region.

Definition 2.5. A measurable function $F : \mathbb{R}^d \to \mathbb{R}$ is said to be superharmonic if

$$\mathbb{E}_x F(X_\sigma) \leq F(x)$$

for all stopping times σ and all $x \in \mathbb{R}^d$. (This requires $F(X_\sigma) \in \mathcal{L}^1(\mathbb{P}_x)$ for $x \in \mathbb{R}^d$).

The following theorem constitutes necessary conditions for the existence of an optimal stopping time.

Theorem 2.1. *Suppose there exists an optimal stopping time* $\tau*$ *with* $\mathbb{P}(\tau* < \infty) = 1$ *so that*

$$V(x) = \mathbb{E}_x G(X_\tau*)$$

for all $x \in \mathbb{R}^d$. *Then*

(i) V *is the smallest superharmonic function dominating* G *on* $\mathbb{R}^d$*;*
In addition, if V *is lsc and* G *is usc, then also*

(ii) $\tau_D \leq \tau* \ \mathbb{P}_x$*-a.s for all* $x \in \mathbb{R}^d$*, and* τ_D *is optimal;*

(iii) the stopped process $\left(V(X_{t\wedge\tau_D})\right)_{t\geq 0}$ *is right-continuous* $\mathbb{P}_x$*-martingale for every* $x \in \mathbb{R}^d$*.*

Now, we consider sufficient conditions for the existence of an optimal stopping time. This is the main theorem of this subsection.

Theorem 2.2. *Consider the optimal stopping problem* (5) *and assume that the integrability condition* (3) *is satisfied. Suppose there exists a function* $\hat{V}$*, which is the smallest superharmonic function dominating* G *on* $\mathbb{R}^d$*. suppose also* $\hat{V}$ *is lsc and* G *is usc. Let* $D = \left\{x \in \mathbb{R}^d : \hat{V} = G(x)\right\}$ *and suppose* τ_D *is defined by* (6)*. Then*

(i) if $\mathbb{P}_x(\tau_D < \infty)$*, for* $x \in \mathbb{R}^d$*, then* $\hat{V} = V$ *and* τ_D *is optimal in* (5)*;*

(ii) if $\mathbb{P}_x(\tau_D < \infty) < 1$*, for some* $x \in \mathbb{R}^d$*, then there is no optimal stopping time* τ *in* (5)*.*

2.4 Free-boundary problems

In recent times, there has been a great interest in the theory of free boundaries. Motivated purely by an interest in financial application, we give a brief review of reduction of an optimal stopping problem to a free-boundary problem.

A free-boundary problem deals with solving a PDE (partial differential equation) in a domain, with part of the domain an unknown free boundary. In addition to standard conditions required to the PDE, an additional condition must be imposed at the free boundary. What one does from there is to find both the free boundary and the solution to the PDE.

Throughout this subsection, we will adopt the settings and notation of the previous Sub subsection 2.3.2. We consider a strong Markov process $X = (X_t)_{t\geq 0}$ defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ and taking values in a measurable space $\mathbb{R}^d$ for some $d \geq 1$.

Let $G : \mathbb{R}^d \to \mathbb{R}$ be a measurable function satisfying needed regularity conditions. Consider the optimal stopping problem:

$$V(x) = \sup_{\tau} \mathbb{E}_x G(X_\tau) \tag{7}$$

where the supremum is taken over all stopping times τ of X and $\mathbb{P}_x(X_0 = x) = 1$, for $x \in \mathbb{R}$.
It has already been seen in the previous Subsection 2.3.2 that the problem (7) is equivalent to the problem of finding the smallest superharmonic function $\hat{V} : \mathbb{R}^d \to \mathbb{R}$ which dominates G on $\mathbb{R}^d$. Denote by τ_D the first entry time of X into the stopping set. Recall that we split the domain into the

stopping region $D = \{\hat{V} = G\}$ which is optimal and the continuation region $C = \{\hat{V} > G\}$.

$\hat{V}$ and C should solve the free-boundary problem:

$$\mathbb{L}_X \hat{V} \leq 0 \ (\hat{V} \text{ minimal}), \tag{8}$$

$$\hat{V} \geq G \ (\hat{V} > G \text{ on } C \text{ and } \hat{V} = G \text{ on } D) \tag{9}$$

where $\mathbb{L}_X$ is the infinitesimal generator of X. $\hat{V}$ and C have to be determined as they are both unknown in the system.

Under sufficient conditions and identifying $V = \hat{V}$, we can write

$$V(x) = \mathbb{E}_x G(X_{\tau_D})$$

for some $x \in \mathbb{R}^d$ where τ_D is as defined in (6).

It follows that

$$\mathbb{L}_X V = 0 \text{ in } C,$$

$$V|_D = G|_D$$

Assume that G is smooth in a neighbourhood of ∂C. If X after starting at ∂C enters $\text{int}(D)$ immediately, then (8) leads to the smooth-fit condition:

$$\left.\frac{\partial V}{\partial x}\right|_{\partial C} = \left.\frac{\partial G}{\partial x}\right|_{\partial C} \tag{10}$$

where the 1-dimensional case is assumed.

3 Set-up

3.1 Utility functions

Assumption 3.1. We shall call a function $U : (0, \infty) \to \mathbb{R}$ a utility function if it satisfies the following properties:

(i) $U(x) > 0$ for all $x > 0$

(ii) U is strictly increasing, strictly concave and twice-continuously differentiable

(iii) $U'(0) := \lim_{x \downarrow 0} U'(x) = \infty$ and $U'(\infty) := \lim_{x \uparrow \infty} U'(x) = 0$

Throughout this thesis, we will use the power utility function, $U(x) = \frac{x^\alpha}{\alpha}$, $\alpha > 0$ where α is the degree of relative risk aversion of the agent. The main motivation for the choice of this utility function lies in the fact that it is the only utility function that has the property of constant relative risk aversion and therefore has the implication that utility is only defined for positive wealth. Therefore, by considering an agent investing in a financial market consisting of a stock and a bank account, the proportion of wealth optimally invested in the stock is independent of his initial wealth.

3.2 The Model

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$ be a given filtered probability space satisfying the usual conditions. We consider a financial market consisting of a risk-free asset called a savings account with price process $(I_t)_{t \geq 0}$, and also acting as the numéraire process. $(I_t)_{t \geq 0}$ is absolutely continuous, strictly positive and governed by the stochastic differential equation

$$dI_t = rI_t dt \tag{11}$$

and a risky asset called a stock with price process $(S_t)_{t \geq 0}$. $(S_t)_{t \geq 0}$ is absolutely continuous, strictly positive and $\mathcal{F}_t$-adapted semimartingale satisfying the stochastic differential equation

$$dS_t = \mu S_t + \sigma S_t dW_t \tag{12}$$

Let Y_t with $Y_0 = y$ be the price process of the real asset with the following dynamics:

$$dY_t = \nu Y_t + \Sigma Y_t dB_t \tag{13}$$

$\{(W_t, B_t)\,; 0 \leq t \leq T\}$ are 1-dimensional Brownian motions defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\{\mathcal{F}_t; 0 \leq t \leq T\}$ is the $\mathbb{P}$ augmentation of the filtration $\left\{\left(\mathcal{F}_t^{W_t}, \mathcal{F}_t^{B_t}\right); 0 \leq t \leq T\right\} = \sigma\left\{(W_s, B_t)\,; 0 \leq s \leq t\right\}$ generated by the two correlated Brownian motions driving the price processes in our model. The correlation between the Brownian motions driving the price processes (12) and (13) is purely for the motive of hedging $(Y_t)_{t \geq 0}$.

Let $dW_t dB_t = \rho dt$ and $\bar{W}_t$ a further Brownian motion uncorrelated with W_t. Writing as a linear combination of the three Brownian motions, we have that

$$dB_t = \rho dW_t + \bar{\rho} d\bar{W}_t \tag{14}$$

where $\rho^2 + \bar{\rho}^2 = 1$ with $(\rho, \bar{\rho}) \in [-1, 1] \times [-1, 1]$

Remark 3.1. We assume that, the interest rate r, the drifts μ and ν, and the volatilities σ and Σ, of the Markov models (11), (12) and (13) respectively, are constant throughout.

The financial market consisting of the bank account and the stock form a complete market. By the Girsanov Theorem, we can find risk neutral measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the discounted price processes $(S_t)_{t \geq 0}$ is a martingale. By the fundamental theorem of asset pricing, the existence of a risk neutral measure, is essentially that, the financial market is free of arbitrage opportunities.

Denote by

$$\lambda = \frac{\mu - r}{\sigma} \tag{15}$$

and

$$\eta = \frac{\nu - r}{\Sigma} \tag{16}$$

the instantaneous Sharpe ratios of the stock and the real asset respectively. (12) and (16) can be rewritten as

$$\frac{dS_t}{S_t} = (\sigma\lambda + r)\, dt + \sigma dW_t \tag{17}$$

and

$$\frac{dY_t}{Y_t} = \nu dt + \Sigma\left(\rho dW_t + \bar{\rho} d\bar{W}_t\right) \tag{18}$$

respectively. The terms in (18) has the interpretation that $\Sigma\rho$ is the systematic component of volatility associated with the real asset and W_t is the Brownian motion describing the systematic risk; $\Sigma\bar{\rho}$ is the idiosyncratic volatility and $\bar{W}_t$ is the Brownian motion describing the idiosyncratic risk.

Let $X_t = x$ denote the wealth of the agent at time t and θ_t his holdings of the stock. An $\mathcal{F}_t$ - adapted process $\pi = \pi_t = \theta_t S_t$ represents the agent's proportion of wealth invested in the stock and the remaining $1-\pi$ invested in the bank account. Her proportions of wealth invested in the stock and bank account respectively sum to 1. Her consumption rate $C = c_t$, is positive and also an $\mathcal{F}_t$ - adapted process.

Definition 3.1. A portfolio-consumption pair (π, C) is self-financing if

$$dX_t = X_t \left(\pi \frac{dS_t}{S_t} + (1-\pi) \frac{dI_t}{I_t} \right) - C dt \tag{19}$$

After a slight rearrangement, the dynamics of the agent's wealth is given by the stochastic differential equation

$$dX_t^{\pi,C} = \left[\left(\pi\lambda\sigma \ + r \right) X_t^{\pi,C} - C \right] dt + \pi\sigma X_t^{\pi,C} dW_t \tag{20}$$

4 Horizon-unbiased utility functions

4.1 Statement of the problem

Consider (1) without the real asset. The objective of our agent is to maximize her expected utility of wealth and consumption up to a stopping time. That is

$$\sup_{\tau} \sup_{C,\pi\in\mathcal{A}_\tau} \mathbb{E}\left[U_X(\tau, X_\tau^{\pi,C}) + \int_t^\tau U_C(s, C_s)\,\mathrm{ds}\right] \tag{21}$$

The main aim of this section is to show that in order to eliminate biases over the choice of stopping times for the agent solving (21), her utility preferences of wealth and consumption must satisfy certain properties. We therefore seek to find the agent's utility pair of wealth and consumption, (U_X, U_C), such that the pair is horizon-unbiased. In that sense, when we finally introduce the real asset into the problem, we are sure that man-made incentives that will influence the agent's decision to sell are eliminated.

Remark 4.1. The notation U_t for $\frac{\partial U_X}{\partial t}$, U_c for $\frac{\partial U_C}{\partial c}$, U' for $\frac{\partial U_X}{\partial x}$ and U'' for $\frac{\partial^2 U_X}{\partial x^2}$ will be used throughout. This is to ensure a simplification of the formulae.

4.2 Deriving the horizon-unbiased utility pair

Consider (21) and the agent's wealth dynamics given by (20). Let $U_X(x,t) = \frac{x^\alpha}{\alpha}e^{-\beta t}$ and $U_C(c,t) = f(t)\frac{c^\alpha}{\alpha}$ denote the utility function of wealth and consumption respectively. Now, the key step in finding the horizon-unbiased utility pair, is to take the solution to (21) as the utility of wealth. In this way, we can find the utility of consumption such that the pair (U_X, U_C), is horizon-unbiased.

By appealing to the dynamic programming principle, we can write the associated HJB equation as:

$$U_t + \sup\left[U_C(t,c) + (\pi\lambda\sigma x + r - C)\,U' + \frac{\sigma^2\pi^2x^2}{2}U''\right] = 0 \tag{22}$$

The static optimization problem to solve is to maximize

$$U_C(t,c) + (\pi\lambda\sigma x + r - C)\,U' + \frac{\sigma^2\pi^2x^2}{2}U'' \tag{23}$$

over C and π.

Performing an optimization over C and π, we have the following optimal consumption and investment strategies:

$$C^* = x\left(\frac{e^{-\beta t}}{f}\right)^{-\frac{1}{1-\alpha}} \tag{24}$$

$$\pi^* = \frac{\lambda}{(1-\alpha)\sigma^2} \tag{25}$$

Setting (24) and (25) and the required partial derivatives into (22) yields

$$\begin{aligned}
0 &= -\beta\frac{x^\alpha}{\alpha}e^{-\beta t} + f\frac{x^\alpha}{\alpha}\left(\frac{e^{-\beta t}}{f}\right)^{-\frac{\alpha}{1-\alpha}} - \frac{1}{2}\frac{x^\alpha}{\alpha}\frac{\alpha\lambda^2}{1-\alpha}e^{-\beta t} \\
&\quad + \left[\left(\frac{\lambda^2}{1-\alpha}+r\right)x - x\left(\frac{e^{-\beta t}}{f}\right)^{-\frac{1}{1-\alpha}}\right]x^{\alpha-1}e^{-\beta t} \\
&= e^{-\beta t}\frac{x^\alpha}{\alpha}\left[\left(\frac{\lambda^2}{1-\alpha}+r\right)\alpha - \frac{\alpha\lambda^2}{2(1-\alpha)} - \alpha e^{-\frac{\beta}{1-\alpha}t}f^{\frac{1}{1-\alpha}}\right] - \beta e^{-\beta t}\frac{x^\alpha}{\alpha} \\
&\quad + e^{\frac{\beta\alpha}{1-\alpha}t}\frac{x^\alpha}{\alpha}f^{\frac{1}{1-\alpha}}
\end{aligned}$$

which reduces to

$$0 = e^{-\beta t}\frac{x^\alpha}{\alpha}\left\{-\beta + e^{\frac{\beta}{1-\alpha}t}f^{\frac{1}{1-\alpha}} + \frac{\alpha\lambda^2}{2(1-\alpha)} + r\alpha - \alpha e^{\frac{\beta}{1-\alpha}t}f^{\frac{1}{1-\alpha}}\right\} \tag{26}$$

Solving for f in (26), we have

$$f = e^{-\beta t}\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left[\beta - \left(\frac{\alpha\lambda^2}{2(1-\alpha)}+r\alpha\right)\right]^{1-\alpha} \tag{27}$$

Hence our utility of wealth and consumption is given respectively by

$$U_X(x,t) = e^{-\beta t}\frac{x^\alpha}{\alpha} \tag{28}$$

and

$$U_C(c,t) = e^{-\beta t}\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left[\beta - \left(\frac{\alpha\lambda^2}{2(1-\alpha)}+r\alpha\right)\right]^{1-\alpha}\frac{c^\alpha}{\alpha} \tag{29}$$

where it is required that $\beta > \frac{\alpha\lambda^2}{2(1-\alpha)} + r\alpha$.

Remark 4.2. The requirement that the discount factor, $\beta > \frac{\alpha\lambda^2}{2(1-\alpha)} + r\alpha$, ensures that there exists a horizon-unbiased pair. $\frac{\alpha\lambda^2}{2(1-\alpha)}$ has the interpretation that it is the cost the agent incurs by foregoing the opportunity to invest when the sale of asset is delayed and $r\alpha$ discounts future wealth into current amounts. Whenever $\beta = \frac{\alpha\lambda^2}{2(1-\alpha)} + r\alpha$, then there is no consumption motive for the agent and the problem reduces to that in [10]. However, whenever $\beta < \frac{\alpha\lambda^2}{2(1-\alpha)} + r\alpha$, no horizon-unbiased utility pair exists.

Remark 4.3. Consider (21) and a finite horizon, T. A utility pair (U_X, U_C), of wealth and consumption satisfying Assumption 3.1 is horizon-unbiased for the model described in section 3.2, if the solution to (21) does not depend on T.

We are now ready to state and prove the main result of this thesis.

Theorem 4.1. *A utility pair of wealth and consumption* (U_X, U_C)*, defined in* (28) *and* (29)*, is horizon-unbiased for the model described in Section* 3.2*.*

Proof. To show that the pair (U_X, U_C), is horizon-unbiased, we need to check the following:

(i) that the solution to (21), U_X, is a supermartingale for any admissible portfolio-consumption pair, (π, C).

(ii) that the solution to (21), U_X, is a martingale for optimal portfolio-consumption pair, (π^*, C^*).

To simplify the calculations in the proof of Theorem 4.1, consider a slight modification of the utility of consumption as follows:
Let

$$U_C(c,t) = Ae^{-\beta t}\frac{c^\alpha}{\alpha} \tag{30}$$

where

$$A = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left[\beta - \left(\frac{\alpha\lambda^2}{2(1-\alpha)} + r\alpha\right)\right]^{1-\alpha} \tag{31}$$

Consider (21) and let

$$V_t = U_X(t, X_t^{\pi,C}) + \int_0^t U_C(s, C_s)\,\mathrm{ds}$$

Applying Ito's lemma to V_t and considering (20), we have

$$dV = \left\{\frac{\partial V}{\partial t} + [(\lambda\pi + r)x - C]\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\pi^2\frac{\partial^2 V}{\partial x^2}\right\}dt + \sigma\pi\frac{\partial V}{\partial x}dW \tag{32}$$

Inserting the required partial derivatives into (32) yields

$$dV = \left\{ -\beta \frac{x^\alpha}{\alpha} e^{-\beta t} + A e^{-\beta t} \frac{x^\alpha}{\alpha} + \left[(\lambda \pi + r) x - C \right] x^{\alpha - 1} e^{-\beta t} \right.$$
$$\left. + \frac{1}{2} (\alpha - 1) \sigma^2 \pi^2 x^{\alpha - 2} e^{-\beta t} \right\} dt + (\alpha - 1) \sigma \pi x^{\alpha - 2} e^{-\beta t} dW \qquad (33)$$

It is straight forward to check that the drift term in (33) is non-positive for any π and any C. That is

$$0 \geq -\beta \frac{x^\alpha}{\alpha} e^{-\beta t} + A e^{-\beta t} \frac{x^\alpha}{\alpha} + \left[(\lambda \pi + r) x - C \right] x^{\alpha - 1} e^{-\beta t}$$
$$+ \frac{1}{2} (\alpha - 1) \sigma^2 \pi^2 x^{\alpha - 2} e^{-\beta t}$$

This proves (i).

To prove (ii), we require that from (33),

$$0 = \sup \left\{ -\beta \frac{x^\alpha}{\alpha} e^{-\beta t} + A e^{-\beta t} \frac{x^\alpha}{\alpha} + \left[(\lambda \pi + r) x - C \right] x^{\alpha - 1} e^{-\beta t} \right.$$
$$\left. + \frac{1}{2} (\alpha - 1) \sigma^2 \pi^2 x^{\alpha - 2} e^{-\beta t} \right\}$$
$$= \sup \mathbf{RHS} \qquad (34)$$

Inserting (24) and (25) into (34), we have that

$$\sup \mathbf{RHS} = -\beta \frac{x^\alpha}{\alpha} e^{-\beta t} + A e^{-\beta t} \frac{x^\alpha}{\alpha} \left(\frac{e^{-\beta t}}{f} \right)^{-\frac{\alpha}{1-\alpha}} - \frac{1}{2} \frac{x^\alpha}{\alpha} \frac{\alpha \lambda^2}{1 - \alpha} e^{-\beta t}$$
$$+ \left[\left(\frac{\lambda^2}{1 - \alpha} + r \right) x - x \left(\frac{e^{-\beta t}}{f} \right)^{-\frac{1}{1-\alpha}} \right] x^{\alpha - 1} e^{-\beta t}$$
$$= e^{-\beta t} \frac{x^\alpha}{\alpha} \left(-\beta + \frac{\alpha \lambda^2}{2(1 - \alpha)} + r\alpha + (1 - \alpha) A^{\frac{1}{1-\alpha}} \right)$$
$$= 0$$

which proves (ii)

□

5 The asset sale problem

Having found the horizon-unbiased utility pair, we now return to the asset sale problem (1) . Finding the optimal selling time of the real asset directly from the problem is difficult. As a result, we will use the approach of dynamic programming to reduce the problem into a partial differential equation (PDE) with a single free-boundary. Once we know the solution to this PDE, construction of the optimal stopping rule becomes easier.

5.1 Reduction to a free-boundary problem

Consider the mixed optimal stopping/optimal control problem described in Section 1 and the horizon-unbiased utility pair defined in (28) and (29). To be precise, let

$$V(x,y) = \sup_{\tau} \sup_{C,\pi\in\mathcal{A}_\tau} \mathbb{E}\left[U_X(\tau, X_\tau^{\pi,C} + Y_\tau) + \int_0^\tau U_C(s, C_s)\,\mathrm{ds}\right] \quad (35)$$

where $V(x,y)$ is the value function.

To reduce the number of variables in the problem and make it both less complicated and more tractable, we define $Z_t = \frac{Y_t}{X_t}$, the proportion of real asset to wealth.

Applying Ito's lemma to Z_t yields

$$dZ_t = \frac{1}{X_t}dY_t - \frac{Y_t}{X_t^2}dX_t + \frac{Y_t}{X_t^3}(dX_t)^2 - \frac{1}{X_t^2}dX_t dY_t \quad (36)$$

By inserting (18) and (20) into (36), we can rewrite the wealth dynamics of the agent as

$$\frac{dZ_t}{Z_t} = \left(\eta\Sigma - \pi\sigma\lambda + \pi^2\sigma^2 - \pi\Sigma\sigma\rho + \frac{C}{X_t}\right)dt + (\Sigma\rho - \pi\sigma)dW_t + \Sigma\bar{\rho}d\bar{W}_t \quad (37)$$

The value function can also be rewritten as

$$V(x,y) = \frac{(x+y)^\alpha}{\alpha} = x^\alpha H(z) \quad (38)$$

where

$$H(z) \geq \frac{(1+z)^\alpha}{\alpha} \quad (39)$$

We are now ready to derive the Hamilton-Jacobi-Bellman equation for the solution to the problem above. Before proceeding, we make the following assumptions:

Assumption 5.1. (i) there is a free-boundary;

(ii) the value function satisfies the needed regularity conditions and that Ito's formula is applicable;

(iii) and that the condition of smooth-fit applies.

Let

$$Q_t = e^{-\beta t} x^\alpha H(z) + \int_0^t U_C(c, s) ds$$

Applying Ito's lemma to Q_t yields

$$\begin{aligned}
dQ_t &= \frac{\partial Q_t}{\partial t} dt + \frac{\partial Q_t}{\partial x} dX_t + \frac{\partial Q_t}{\partial z} dZ_t + \frac{1}{2} \frac{\partial^2 Q_t}{\partial x^2} (dX_t)^2 + \frac{1}{2} \frac{\partial^2 Q_t}{\partial z^2} (dZ_t)^2 \\
&\quad + \frac{\partial^2 Q_t}{\partial x \partial z} dX_t dZ_t \\
&= -\beta e^{-\beta t} x^\alpha H dt + U_C(c, t) + (\alpha e^{-\beta t} x^{\alpha-1} H) dX + e^{-\beta t} x^\alpha H' dZ_t \\
&\quad + \frac{1}{2} \alpha(\alpha - 1) e^{-\beta t} x^{\alpha-2} H (dX_t)^2 + \frac{1}{2} e^{-\beta t} x^\alpha H'' (dZ_t)^2 \\
&\quad + \alpha e^{-\beta t} x^{\alpha-1} H' dX_t dZ_t
\end{aligned} \tag{40}$$

Inserting (20), (29) and (37) into (40) and grouping like terms, we have

$$\begin{aligned}
dQ_t = &\left\{ A \frac{c^\alpha}{\alpha} + \left[-\beta + \alpha(\pi\lambda\sigma + r) - \frac{\alpha c}{x} + \frac{1}{2}\alpha(\alpha-1)\pi^2\sigma^2 \right] x^\alpha H \right. \\
&+ \left[\alpha\pi\sigma(\Sigma\rho - \pi\sigma) z + (\eta\Sigma - \pi\sigma\lambda + \pi^2\sigma^2 - \pi\sigma\Sigma\rho) z + \frac{c}{x} z \right] x^\alpha H' \\
&\left. + \frac{1}{2} \left[\Sigma^2 \bar{\rho}^2 + (\Sigma\rho - \pi\sigma)^2 \right] x^\alpha z^2 H'' \right\} e^{-\beta t} dt \\
&+ e^{-\beta t} x^\alpha \left[\alpha\pi\sigma H + (\Sigma\rho - \pi\sigma) z H' \right] dW + \Sigma\bar{\rho} e^{-\beta t} x^\alpha z H' d\bar{W}
\end{aligned}$$

From the supermartingale property, we require that

$$\begin{aligned}
0 = \sup &\left\{ A \frac{c^\alpha}{\alpha} + \left[-\beta + \alpha(\pi\lambda\sigma + r) - \frac{\alpha c}{x} + \frac{1}{2}\alpha(\alpha-1)\pi^2\sigma^2 \right] x^\alpha H \right. \\
&+ \left[\alpha\pi\sigma(\Sigma\rho - \pi\sigma) z + (\eta\Sigma - \pi\sigma\lambda + \pi^2\sigma^2 - \pi\sigma\Sigma\rho) z + \frac{c}{x} z \right] x^\alpha H' \\
&\left. + \frac{1}{2} \left[\Sigma^2 \bar{\rho}^2 + (\Sigma\rho - \pi\sigma)^2 \right] x^\alpha z^2 H'' \right\} e^{-\beta t}
\end{aligned}$$

which simplifies to

$$0 = \sup \left\{ \frac{A}{\alpha} \left(\frac{c}{x}\right)^{\alpha} + \left[-\beta + \alpha(\pi\lambda\sigma + r) - \frac{\alpha c}{x} + \frac{1}{2}\alpha(\alpha-1)\pi^2\sigma^2 \right] H \right.$$
$$+ \left[\alpha\pi\sigma(\Sigma\rho - \pi\sigma)z + (\eta\Sigma - \pi\sigma\lambda + \pi^2\sigma^2 - \pi\sigma\Sigma\rho)z + \frac{c}{x}z \right] H'$$
$$\left. + \frac{1}{2}\left[\Sigma^2\bar{\rho}^2 + (\Sigma\rho - \pi\sigma)^2\right] z^2 H'' \right\} \tag{41}$$

subject to

$$H(0) = \frac{1}{\alpha} \tag{42}$$

$$H(z) \geq \frac{(1+z)^{\alpha}}{\alpha} \tag{43}$$

and the smooth-fit condition at the free-boundary

$$\frac{\partial H}{\partial z} = (1+z)^{\alpha-1} \tag{44}$$

(41) is precisely the HJB equation for (35) in the continuation region.

Since our aim, finally, is to find the optimal selling time of the real asset, we proceed to derive the HJB equation in the stopping region.

Performing an optimization over π and C yields the following consumption and investment strategies:

$$C^{**} = x\left(\frac{\alpha H - zH'}{A}\right)^{-\frac{1}{1-\alpha}} \tag{45}$$

and

$$\pi^{**} = -\frac{\alpha\lambda H - \left[\lambda + \Sigma\rho(1-\alpha)\right]zH' - \Sigma\rho z^2 H''}{\sigma\left[z^2H'' + 2(1-\alpha)zH' - \alpha(1-\alpha)H\right]} \tag{46}$$

Remark 5.1. (45) and (46) are respectively, the optimal consumption and investment strategies available to the agent before the sale of the real asset. These strategies precisely yield a maximum in (41).

Substituting the optimal values (45) and (46) into (41), we have

$$
\begin{aligned}
0 &= -\beta H + \frac{A}{\alpha}\left(\frac{\alpha H - zH'}{A}\right)^{-\frac{\alpha}{1-\alpha}} + \alpha\left(\frac{\alpha H - zH'}{A}\right)^{-\frac{1}{1-\alpha}} H \\
&\quad + \left(\frac{\alpha H - zH'}{A}\right)^{-\frac{1}{1-\alpha}} zH' + \alpha(\pi\lambda\sigma + r)H - \frac{1}{2}\alpha(1-\alpha)\pi^2\sigma^2 H \\
&\quad + (\eta\Sigma - \pi\lambda\sigma + \pi^2\sigma^2 - \pi\sigma\Sigma\rho)zH' + \alpha\big[\pi\sigma(\Sigma\rho - \pi\sigma)\big]zH' \\
&\quad + \frac{1}{2}\big[\Sigma^2\bar{\rho}^2 + (\Sigma\rho - \pi\sigma)^2\big]z^2H'' \\
&= -\beta H + \frac{A}{\alpha}\left(\frac{\alpha H - zH'}{A}\right)^{-\frac{\alpha}{1-\alpha}} + (zH' - \alpha H)\left(\frac{\alpha H - zH'}{A}\right)^{-\frac{1}{1-\alpha}} \\
&\quad + r\alpha H + \eta\Sigma zH' + \frac{1}{2}\Sigma^2z^2H'' + \pi\sigma\Big[\lambda\alpha H - \big[\lambda + \Sigma\rho(1-\alpha)\big]zH' \\
&\quad - \Sigma\rho z^2H''\Big] + \frac{1}{2}\pi^2\sigma^2\big[z^2H'' + 2(1-\alpha)zH' - \alpha(1-\alpha)H\big]
\end{aligned}
$$

Simplifying further yields

$$
\begin{aligned}
0 &= -\beta H + \frac{A}{\alpha}\left(\frac{\alpha H - zH'}{A}\right)^{-\frac{\alpha}{1-\alpha}} + (zH' - \alpha H)\left(\frac{\alpha H - zH'}{A}\right)^{-\frac{1}{1-\alpha}} \\
&\quad + r\alpha H + \eta\Sigma zH' + \frac{1}{2}\Sigma^2z^2H'' + \Bigg\{\sigma\Big[\lambda\alpha H - \big[\lambda + \Sigma\rho(1-\alpha)\big]zH' \\
&\quad - \Sigma\rho z^2H''\Big]\left[\frac{\lambda\alpha H - \big[\lambda + \Sigma\rho(1-\alpha)\big]zH' - \Sigma\rho z^2H''}{\sigma\big[z^2H'' + 2(1-\alpha)zH' - \alpha(1-\alpha)H\big]}\right]^2\Bigg\} \\
&\quad + \Bigg\{\frac{1}{2}\sigma^2\left[\frac{\lambda\alpha H - \big[\lambda + \Sigma\rho(1-\alpha)\big]zH' - \Sigma\rho z^2H''}{\sigma\big[z^2H'' + 2(1-\alpha)zH' - \alpha(1-\alpha)H\big]}\right]^2\Big[z^2H'' \\
&\quad + 2(1-\alpha)zH' - \alpha(1-\alpha)H\Big]\Bigg\}
\end{aligned}
\tag{47}
$$

After a slight rearrangement, (47) reduces to

$$
\begin{aligned}
0 &= \frac{1-\alpha}{\alpha}A^{-\frac{\alpha}{1-\alpha}}\left(\alpha H - zH'\right)^{-\frac{\alpha}{1-\alpha}} + (r\alpha - \beta)H + \eta\Sigma zH' + \frac{1}{2}\Sigma^2z^2H'' \\
&\quad - \frac{1}{2}\frac{\big[\lambda\alpha H - \big[\lambda + \Sigma\rho(1-\alpha)\big]zH' - \Sigma\rho z^2H''\big]^2}{z^2H'' + 2(1-\alpha)zH' - \alpha(1-\alpha)H}
\end{aligned}
\tag{48}
$$

subject to

$$H(0) = \frac{1}{\alpha} \tag{49}$$

$$H(z^*) = \frac{(1+z^*)^\alpha}{\alpha} \tag{50}$$

and the smooth-fit condition at the free boundary

$$\frac{\partial H}{\partial z^*} = (1+z^*)^{\alpha-1} \tag{51}$$

The results thus far in this section can be summarised in the following theorem.

Theorem 5.1. (A Verification Theorem) *Consider* (35) *and the horizon-unbiased utility pair given by* $U_X(x,z,t) = e^{-\beta t}x^\alpha H(z)$ *and* $U_C(c,t) = Ae^{-\beta t}\frac{c^\alpha}{\alpha}$ *where* $H(z)$ *and* A *are given respectively by* (39) *and* (31)*. For* $0 \leq z \leq z^*$*,* H *solves* (48) *subject to* $H(0) = \frac{1}{\alpha}$*,* $H(z^*) = \frac{(1+z^*)^\alpha}{\alpha}$ *and* $\frac{\partial H}{\partial z^*} = (1-z^*)^{\alpha-1}$*. Thus, the value function is given by* $V(x,z,t) = e^{-\beta t}x^\alpha H(z)$

Sketch of proof. $V(x,z,t)$ is a supermartingale for any π and C. Hence for each τ,

$$V(x,z,t) \geq \sup_{C,\pi \in \mathcal{A}_\tau} \mathbb{E}\left[e^{-\beta t}x^\alpha \frac{(1+z)^\alpha}{\alpha} + \int_t^\tau Ae^{-\beta s}\frac{c^\alpha}{\alpha}ds\right]$$

To conclude that for τ^* (optimal τ),

$$V(x,z,t) = \sup_{C,\pi \in \mathcal{A}_\tau} \mathbb{E}\left[e^{-\beta t}x^\alpha \frac{(1+z)^\alpha}{\alpha} + \int_t^\tau Ae^{-\beta s}\frac{c^\alpha}{\alpha}ds\right]$$

is the solution to the (1), it is sufficient to prove that V is a martingale for optimal strategies (45) and (46) .
For a detailed proof, the interested reader is referred to [20] □

Remark 5.2. Assumption 5.1 is needed to ensure that there is a nice solution to (35). So the key step is to determine the value function H. As discusssed in Subsection 2.4, free-boundary problems are characterized by conditions imposed at the free boundary. As long as there is continuity in H and H' for our derived nonlinear PDE (48), the condition of smooth-fit applies. Such PDEs have been studied extensively regarding the obstacle problem. Though very difficult to solve analytical, numerical solutions do exist.

5.2 Optimal selling time of the asset

We are now ready to state the optimal time to sell the asset. We assume that there is a nice solution to (48), and that, we also know the free boundary, z^*.

The instant the ratio of the real asset to wealth, Z_t, is greater than the free boundary, the agent should sell the asset. Thus, the optimal stopping rule, τ^*, is to sell the real asset the first time Z_t exceeds z^*.

Remark 5.3. When $Z_t > z^*$, the return on the real asset is very high and our agent may be tempted to delay sale in order to benefit from this high expected growth. Such a motive is completely artificial and may introduce biases into the optimal selling rule. The elimination of such biases is our main reason for considering utility pair that is horizon-unbiased. Even so, by delaying sale, the unhedgeable part of risk associated with the real asset will eventually outweigh the benefits he may derive from the high expected return. Our agent will therefore be forced to sell the asset the first time its ratio to wealth exceeds the free boundary.

After the sale of the real asset, the problem facing the agent reduces to that of maximizing his expected utility of consumption. This is given by

$$V(x,y) = \sup_{C,\pi\in\mathcal{A}_\tau} \mathbb{E}\left[\int_\tau^\infty U_C(s,C_s)\,\mathrm{ds}\right] \tag{52}$$

(52) is precisely Merton's portfolio problem in the infinite horizon and this can be solved explicitly for the value function. See [14] for details.

6 Conclusion

In this thesis, we considered a risk-averse utility maximizing agent with CRRA utility preferences, who owned a unit of an asset and wanted to choose the optimum time to sell it. Mathematically, the problem facing the agent was to maximize his expected utility of wealth and consumption up to a stopping time. We argued that to eliminate biases in the choice of the optimal selling time for the agent solving the problem

$$\sup_{\tau} \sup_{C,\pi\in\mathcal{A}_\tau} \mathbb{E}\left[U_X(\tau, X_\tau^{\pi,C} + Y_\tau) + \int_t^\tau U_C(s, C_s)\,\mathrm{ds}\right]$$

his utility pair of wealth and consumption must be horizon-unbiased. Using the dynamic programming approach, we found the agent's horizon-unbiased utility pair of wealth and consumption to be given respectively by:

$$U_X(x,t) = e^{-\beta t}\frac{x^\alpha}{\alpha}$$

and

$$U_C(c,t) = e^{-\beta t}\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left[\beta - \left(\frac{\alpha\lambda^2}{2(1-\alpha)} + r\alpha\right)\right]^{1-\alpha}\frac{c^\alpha}{\alpha}$$

where we required that $\beta > \frac{\alpha\lambda^2}{2(1-\alpha)} + r\alpha$.
This was the main results of this thesis and our contribution to the literature.

In finding the optimal selling time of the real asset, we reduced the problem characterizing the asset sale to that of a free-boundary problem with a single free boundary. This was the best approach since once we knew the value function - which in our case we assumed there was - to the problem, we were able to construct the optimal stopping rule. Finding the optimal selling time directly form the stated problem would have been very difficult and therefore our resort to the approach of dynamic programming. It turned out that, the optimal selling time of the real asset was the first time the ratio of the asset to wealth exceeded the free boundary.

An interest would be to find horizon-unbiased utility pair for other utility functions, and for such utility functions what the optimal stopping rules are. Instead of considering diffusion processes that are everywhere continuous, which was the case in this thesis, one may be interested in solving the same problem in the case when there are jumps in the price process of the real asset. This particular scenario arises especially when there is a new competitor in the market. These and finding the solution to the free-boundary problem constructed in Subsection (5.1) are problems left for future research.

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