Price sensitivity to the exponent in the CEV model

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Abstract
Since the CEV model captures the negative link between a stock price and its volatility of return, it partly corrects the assumption that volatility is constant in the Black-Scholes model. This thesis presents the option price sensitivity to the exponent parameter in the CEV model, and the roles played by strike price, time to expiration and barriers (for Barrier options) in determining this sensitivity. Using closed-form solutions, finite difference, binomial tree and Monte-Carlo methods, the impact of those model parameters are analyzed and presented.
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Chapter 1

Introduction

1.1 Overview

Since the 1987 market crash, several market observations have implied an association between stock price and volatility. This phenomenon is called 'leverage effect', which is not reconciled with the results given by option pricing formula [1] under the assumption of constant volatility, as in the classical Black-Scholes model.

This financial leverage effect was first discussed in a paper of Black [4] and has enjoyed support from many empirical evidences [5]. The effect can be explained from the perspective of economics: If a firm's stock price falls, it will increase the debt-equity ratio of the firm and therefore increases the variance of the stock's return. The constant elasticity of variance (CEV) class of stock price distributions establishes a theoretical framework within which this inverse relationship can be captured.

The family of constant elasticity of variance diffusion processes is described by the following stochastic differential equation:

$$dS = (\mu - q)Sdt + \delta S^{\alpha/2}dW,$$

(1)

where $\alpha$, $\mu$ and $q$ are the exponent parameter, local mean rate of return and dividend yield, and $W$ is a Wiener process. The instantaneous variance of the percentage price return is given by the equation:

$$\sigma = \delta S^{\alpha/2-1},$$

(2)

the elasticity of return variance with respect to price equals to $\alpha - 2$. If $\alpha < 2$ ( $\alpha > 2$), volatility and price are inversely (positively) related. Since the financial market often exhibit volatility skews of negative slope, the situation $\alpha > 2$ is rarely considered in research. When $\alpha$ equals 2, prices are lognormally distributed and the variance of returns is constant, as is assumed in the Black-Scholes model.

In 1975, Cox and Ross [2] [3] derived an option pricing formula for European call options which holds if the stock price follows CEV diffusion:

$$C(S_t, \tau) = S_t e^{-q \tau} \sum_{n=0}^{\infty} g(n + 1, x) \left( n + 1 + \frac{1}{2 - \alpha}, kK^{2-\alpha} \right) G(n + 1, kK^{2-\alpha}) - Ke^{-r \tau} \sum_{n=0}^{\infty} g(n + 1 + \frac{1}{2 - \alpha}, x) G(n + 1, kK^{2-\alpha}),$$

(3)

where
\[ k = \frac{2r}{\sigma^2(2 - \alpha)(e^{r(2 - \alpha)\tau} - 1)} \]
\[ x = kS_t^2 - e^{(r - q)(2 - \alpha)\tau}, \]
and \( g(m, v) = \frac{e^{-vm}v^{m-1}}{\Gamma(m)} \) is the gamma density function; and \( G(m, v) \) denotes the complimentary gamma distribution. Strike price and time to expiration are indicated by \( K \) and \( \tau = T - t \).

In 1989, Schroder [6] established a connection between the above formula and the non-central chi-squared distribution, and then obtained a pricing formula based on Cox and Ross’ work for European call options:

\[ C(S_t, \tau) = S_t e^{-q\tau} \left( 1 - F \left( 2y; 2 + \frac{2}{2-\alpha}, 2x \right) \right) - Ke^{-r\tau}F \left( 2x; \frac{2}{2-\alpha}, 2y \right), \]

where

\[ x = kS_t^2 - e^{(r - q)(2 - \alpha)\tau}, \]
\[ y = kK^{2-\alpha}, \]
\[ k = \frac{2r}{\sigma^2(2 - \alpha)(e^{r(2 - \alpha)\tau} - 1)} \]

and \( F \) is non-central chi-squared cumulative distribution function.

### 1.2 Parameters in the CEV model

Two specific examples will give us a glimpse of what real options might be like. The first one is an equity index option based on S&P 500 index. The S&P 500 index is a capitalization-weighted index founded since 1957 of the prices of 500 large-cap common stocks actively traded in the United States. The second option is built on United States Oil Fund, which is an exchange-traded security designed to track changes in crude oil prices. The common ground of these two options is that they are all European-style call/put options with large trading volume.

When pricing European options under CEV model, five factors should be considered: strike price, expiration time, implied volatility, risk-free rate, and dividend rate.

1) The strike prices for most active contracts in the market are mainly located in 80% ~ 120% of the underlying stock price.
2) The time to expirations are usually within 12 months in most exchanges.
3) Volatility can be observed by some indicators, for example, VIX is a popular measure of the implied volatility of S&P 500 index options, it varied from 15 to 45 in the year 2011.

The formula for VIX is quite different from the implied volatility in the Black-Scholes model. The index is calculated as 100 times the square root of the expected 30-day variance of the S&P 500 rate of return. The variance is
annualized and VIX expresses volatility in percentage points:

\[ VIX = 100\sqrt{\text{var}}. \]  

4) The choice of a risk-free rate can be a short-term government debt rate, such as a 30-day T-bill rate. In the following analysis, we use the annualized US 30-day Treasury bill rate: \( r=0.06. \)

5) For simplicity, the dividend rate is set to zero in this article.

1.3 Valuation of the exponent parameter

Chart 1 and Chart 2 show the motion of S&P 500 index and VIX index from Mar 2011 to Mar 2012, respectively:
The two charts show a clear inverse relationship between volatility and underlying assets.

Since accumulating evidence suggests the assumption of constant volatility is unrealistic, stochastic volatility models are gradually accepted by the practitioners in the financial industry. Many research articles indicate that option pricing formulas based on the constant elasticity of variance diffusion could fit the actual market prices better than the Black-Scholes model.

As a deterministic volatility model, a question has been put forward about the CEV model: what is the optimal value for the exponent? In earlier researches, $\alpha = 0$ and $\alpha = 1$ are generally used for simplicity. Some people attempted to give a numerical solution, but so far there is no commonly agreed method.

It seems that the approximations of $\alpha$ and $\delta$ are easy to calculate if we have enough data. With a set of index during a period of time, people can use computer to solve the nonlinear curve-fitting problem in least-squares sense. However the results are not ideal, the value of $\alpha$ fluctuates violently and far away from the interval $[0,2)$, which is one reflection of the CEV model’s imperfection in fitting the actual market movement: it couldn’t fully explain the inverse relationship between volatility and price.

The aim of this paper is to analyze the sensitivity of the price with respect to the changes in the exponent parameter, when underlying price process follows the CEV model. Several studies support the CEV pricing model instead of the Black-Scholes pricing model, but few of them focus on the influence brought by the selection of exponent parameter. This paper provides analysis on this problem both on vanilla and exotic options using analytical and numerical methods.

The rest of the paper is organized as follows: Chapter 2 presents the analysis on European call options using closed-form solution. In Chapter 3, two different numerical methods, the binomial tree method and the finite difference method, are compared and used to test the price sensitivity on Barrier options, which are exotic and weak path-dependent options. In Chapter 4, the Monte-Carlo method is adopted in pricing the Asian options, which are exotic but strong path-dependent options. In Chapter 5, the integrated effect of model parameters has been checked. Finally, Chapter 6 concludes.
Chapter 2

European options

2.1 Price difference caused by exponent parameter

Since it is impossible to determine an exact value for the exponent $\alpha$, people have to use an approximation value instead. But that brings a new question: To which extent would the difference in the value of $\alpha$ affect pricing? This question has practical significance, which would help people better assess the applicability of the CEV model.

In the following, we attempt to look deeper into this question by analyzing European call options using closed-form solution.

A critical point for pricing under the CEV model is the selection of the volatility. When we try to use the CEV model to predict option prices, the value of $\alpha$ and $\delta$ in Formula (1) should be constant. At least in the short term, we assume the volatility of stock price doesn’t change. That means if we try to figure out the price sensitivity by varying the exponent $\alpha$, a corresponding $\delta$ must be updated continually to make sure the volatility remains unchanged. To investigate the impact brought by $\alpha$, in this part we fix all other parameters like the time to expiration.

With no loss of generality, we set parameters as follows: risk-free rate $r = 0.06$, time to expiration $T - t = 1$, dividend yield $q = 0$, current price $S_0 = 1400$, strike price $K = 1400$, volatility $\sigma = 0.16$ (on March 21, 2012, S&P index closed at 1405.52, VIX index closed at 15.58). Then $\delta$ can be calculated by

$$\delta = \sigma S_0^{1-\alpha/2}.$$  

First, let’s check the relationship between option price and $\alpha$:
Chart 3 shows that the varied $\alpha$ does lead to a different option pricing result, but the gap is relatively small compared to the option price. We should notice that this is only the case when the option is at-the-money; in fact the negative relation and the relatively small differences in Chart 3 are not general rules.

2.2 The impact of the strike price on sensitivity

In Section 2.2 and 2.3, the impact of strike price and time to expiration on price sensitivity with respect to the exponent parameter will be checked respectively. Since the interactions between these factors are too complicated to be expressed in the form of functions, specific examples have been given. Although not representative of all circumstances, examples presented in this article are intended to give people a glimpse at the desired relationships.

To get a direct impression of $\Delta \alpha$ ’s influence, we set $\alpha = 2$ as reference and vary $\alpha$ to see how much the option price will change. Repeat this process under each strike price in succession until we get the three-dimensional drawing as in Chart 4. The X-axis represents the change of $\alpha$, Y-axis represents the different strike prices and Z-axis stands for the percent changes in option prices, which is:
\[
\frac{\text{Call}(\alpha S_0, K) - \text{Call}(\alpha - \Delta \alpha S_0, K)}{\text{Call}(\alpha S_0, K)}
\]

(7)

Since the strike prices of most active contracts in financial market are mainly located in 80% ~ 120% of the underlying asset price, we first carry out the analysis in this interval.

From Chart 4, we can see that \(\Delta \alpha\)'s impact is lowest when the strike price \(K\) is less than or equal to the current stock price \(S_0\). But as the strike price approaches \(1.2S_0\), \(\Delta \alpha\)'s impact will increase and the percent change in option price will grow to 20%.

In the above analysis, we examined the interval \(0.8S_0 \leq K \leq 1.2S_0\), but what about the situations in the other parts? This is of little practical significance in financial market, but as a mathematics question, it’s interesting to explore the situations when \(K\) approaches zero or infinity.

Next, the analysis is provided in the extended interval: \(0 \leq K \leq 3S_0\). Chart 5 indicates the impact of \(\Delta \alpha\) is negligible when \(K\) is closed to zero. When \(K\) approaches infinity, however, the percent changes of option price increase rapidly till 100% as \(\alpha\) move towards 0.
It is common knowledge that for European call options, the option prices will be close to zero when the strike price is large enough. In this case, the denominator in Formula (7) will be so small that the results are no longer meaningful.

Now we use Formula (8) to draw another graph in which the Z-axis is diverted from ‘percentage change’ to ‘absolute change’.

\[ \text{Call}(\alpha, S_0, K) - \text{Call}(\alpha - \Delta\alpha, S_0, K), \]  

(8)
From Chart 6 we can see that the varied $\alpha$ will not cause option price change when the strike price $K$ approaches zero or infinity. Only when the strike prices are around current stock price $\Delta \alpha$ will make a difference.

### 2.3 The impact of the time to expiration on sensitivity

Besides the strike price, the time to maturity is another important factor which will influence $\Delta \alpha$'s effect. Just like the above analysis, we begin with a survey in the interval $0 \leq T-t \leq 1$, which covers most cases in financial market.

Chart 7 shows a slight rise in percent change when time to expiration increase. Next, the interval is extended to $0 \leq T-t \leq 100$ for mathematical interest, Chart 8 and 9 reveal the percentage change and absolute change respectively:
Chart 8 and 9 indicate that the percent (absolute) change will turn to zero when the time to expiration approaches infinity. In somewhere near 20 years, the influence of $\Delta \alpha$ reaches its maximum and this point can be easily found using computing program. In Chart 8 and 9, the time to expiration when option price is most sensitive to $\alpha$ are 16 and 22 years, respectively.

The peak value in Chart 8 and 9 seems very small but it’s not a general rule. When the strike price is no longer around the current price as in our examples, $\Delta \alpha$’s influence will become stronger to an extent which must not be ignored.

You may find that the situation $T - t \to 0$ is not discussed in the last two paragraphs. This is a complication which need to be stated separately: if $K \leq S_0$, which means the European call options are in- or at-the-money, price sensitivity to exponent will fade gradually until nothing as $T - t$ approach zero (which is the case for the Chart 8); if $K > S_0$, the European call options are out-the-money. Price sensitivity to exponent will grow as $T - t$ approach zero. This situation is illustrated in Chart 10, where current price $S_0 = 1400$ and strike price $K = 1500$.
Chapter 3
Barrier options

3.1 Price difference caused by exponent parameter

For a Barrier option, the contract either becomes void (out option) or comes into existence (in option) as the price of the underlying reaches a barrier. The latter only have a payoff if the barrier level is reached before expiry and the former only have a payoff if the barrier is not reached before expiry. These contracts are weakly path dependent, meaning that the price depends only on the current level of the asset and the time to maturity.

For example, in the case of a down-and-out call option, the payoff is given by

\[ V_T = \begin{cases} (S(T) - K)^+, & S(t) > B \text{ for all } t \in [0, T] \\ 0, & S(t) \leq B \text{ for some } t \in [0, T] \end{cases} \] (9)

where B is the barrier.

European Puts and Calls are known as vanilla options, that is, they are the most basic type of options, with relatively simple features and payoffs. For these options, a closed-form solution exists as we have just shown. However, for most exotic options, (for example, American or Asian options), a closed-form solution does not exist.

Although Barrier call options under Black-Scholes model can be found to have a closed-form solution, when it comes to the CEV model the analytical solution has not been discovered. Thus we have to use numerical methods to get an approximation.

To price Barrier call options (down-and-out) under CEV model, we still use the parameters’ settings as above: current price $S_0 = 1400$, strike price $K = 1400$, risk-free rate $r = 0.06$, time to expiration $T - t = 1$, dividend rate $q = 0$, volatility $\sigma = \delta S^{\alpha/2 - 1} = 0.16$. An additional parameter in this part is the barrier, we set barrier $B = 1200$.

There are two main numerical techniques used for pricing options: the finite difference method and the binomial tree method.

For finite difference methods, there are three different schemes: the explicit/implicit methods and the Crank-Nicolson methods. It seems very natural to modify the algorithm to make it work on a Barrier option by adding one more boundary condition, but it’s not the case. Because the additional boundary condition is no longer trivial as it is in vanilla call/put options. You will find the result very
unstable if using the implicit finite difference method, the reason has been discussed by Fabien in his working paper [8]. The exact same thing will happen to Crank-Nicolson method as Crank-Nicolson is just a mix of explicit and implicit methods. For this reason, we price Barrier options with the explicit method.

The binomial tree method for Vanilla options also needs some change to fit in the CEV model, due to the fact that the volatility is not constant but varies with the level of the underlying price. When the volatility shifts, the probability of an upward move has to be recomputed at each node. For this reason, the algorithm used for Black-Scholes model is not feasible.

In 2004, Richard and Yi-Hwa [7] constructed a binomial process under CEV model to yield a simple and efficient computation procedure (Appendix A) for practical valuation of Vanilla options. They transform the diffusion process to one in which the volatility is constant and then approximate the transformed process by a simple lattice. This computation procedure can be easily modified to price Barrier options.

Here we use the above two methods to price options with different $\alpha$ and compare the results with Chart 3:

The explicit finite difference method and the binomial tree method work well when used to price Barrier options. Compared with Chart 3, instead of negative, a positive relation exists between $\alpha$ and price.
3.2 The impact of the strike price on sensitivity

At the beginning, strike price’s impact on pricing sensitivity is studied in the interval \(0.8S_0 \leq K \leq 1.2S_0\), and the results are shown in Chart 12. We can see the patterns in Chart 12 are very similar to those in Chart 4.

Next, the analysis is carried out in the extended interval: \(0 \leq K \leq 3S_0\), and the results are shown in Chart 13:
Also, the trends in Chart 13 are similar with those in Chart 5.

3.3 The impact of the time to expiration on sensitivity

It seems that for Barrier options, the strike’s impact on sensitivity make no difference to that of European options. What about the time to Expiration’s impact? Here we try to figure out this question by drawing Chart 14.

Compared with Chart 7, the direction of the trend is totally changed. What caused this change? The ‘only’ difference between European call options and Barrier call options is the addition of the ‘barrier’, it’s obviously that the closer ‘barrier’ is to current price, the larger its influence will be. In Chart 14, the current price and barrier is set to 1400 and 1200, what if we draw down the barrier to 1000? The question has been answered in Chart 15, which looks the exact same as in Chart 7. Therefore, when the barrier is far away from the current price, its influence will become smaller.
3.4 The impact of the barriers on sensitivity

Barrier’s impact is another interesting thing we want to know. In this section, we set $S_0 = 1400$, $K = 1400$ and $T - t = 1$, then vary the barrier to see what will happen.

For down-and-out Barrier options, it will be meaningless if $B \geq S_0$. For this reason our analysis focuses on the part where $B < S_0$. Also, when barrier is small enough, its influence will be trivial. Therefore, we use the interval $0.8S_0 \leq B < S_0$ to make our investigation:
The disadvantage of the binomial tree method is exposed in Chart 16, we can see the trend in the left graph is vague when the barrier is large, and this problem cannot be resolved by increasing the number of time steps in the constructed binomial tree.

On the other hand, it’s easy to draw a conclusion using finite difference method. As it approaches $S_0$, the barrier’s impact on sensitivity first rise and then drop back to zero. But this impact is limited within 4%, will it become larger if we use other combinations of $S_0$, $K$, and $T - t$? This question needs further research.
Chapter 4

Asian options

4.1 Price difference caused by exponent parameter

In Chapter 2 and 3, we have discussed European and Barrier options by using analytical and numerical methods respectively. European options are non-path-dependent options and Barrier options are weak path-dependent options, now it’s natural to analyze the strong path-dependent options. These contracts have payoffs that depend on some property of the asset price path in addition to the value of the underlying at the present moment in time.

Asian options are typical strong path-dependent options; they have payoffs that depend on the average value of the underlying asset from inception to expiry. We must keep track of more information about the asset price path than simply its present position.

Since we have already use finite difference and binomial tree methods to price options in the Chapter 3, here we adopt a new numerical method to price Asian options: the Monte-Carlo method.

Pricing via Monte-Carlo simulation is simple in principle: The value of an option is the present value of the expected payoff under a risk-neutral random walk. The advantage and disadvantage are both very obvious: the Monte-Carlo method is easy to code and to implement, but it can be slow since tens of thousands of simulations are needed to get an accurate answer.

Here we follow the steps in Chapter 2 and 3 to find the sensitivity of a Asian option’s price with respect to the exponent factor. First, we want to know the Asian option’s prices with different α:
It seems that we achieve the third kind of patterns in addition to those in Chart 3 and 11, in which the trends are negative and positive. We can see the option prices first decrease and then increase as $\alpha$ arise. Will this new type yield different results than those obtained above?

### 4.2 The impact of the strike price on sensitivity

Again, strike price’s impact is first examined in the interval: $0.8S_0 \leq K \leq 1.2S_0$. 

![Chart 17](chart17.png)
We have to say the result is rather disappointing: there is no significant difference from Chart 4 and 11. In Chart 19, the interval is extended to $0 \leq K \leq 2S_0$:
4.3 The impact of the time to expiration on sensitivity

Now it’s the time to look into the impact of time to expiration. Chart 20 is obtained when we fix the current stock price $S_0$, strike price $K$ and then vary the time to expiration $\tau$ on the interval $0 \leq T - t \leq 1$.

The graph in Chart 20 is really ugly; one can hardly get any useful information from it. It seems that the Monte-Carlo method is not suitable for analyzing the impact of time to expiration. In Chart 20, we use the number of simulation equal to 5000, what if we increase the number to 50000?
After a relative long runtime, Chart 21 can be drawn. From the graph, we can say a large number of simulations indeed help improve the performance of Monte-Carlo method. Although not perfect, it’s clearly enough to recognize a ‘down-up-down’ tendency from Chart 21, and the amplitude of variation is limited within 0.1%.
Chapter 5

Integrated effect of model parameters

In Chapter 2, 3 and 4, we have examine the price sensitivity to the exponent in the CEV model, and the roles played by model parameters like strike price and time to expiration in determining this sensitivity. Since all these analyses are conducted separately, it would have been a natural thing for us to survey the combined effect of model parameters in this chapter.

We set up the same environment as in Chapter 2, and try to find the combined effect of strike price and time to expiration. For simplicity, we only focus on the option prices’ difference between two extreme situations: $\alpha=0$ and $\alpha=2$. By doing this, the axis for $\Delta\alpha$ in previous three-dimensional drawings can be replaced by other parameters. In our case, the X-axis and Y-axis are used to denote strike price and time to expiration, and the Z-axis still stands for the percent change of option price. For a realistic purpose, the range of strike price and time to expiration is limited to $[0.8S_0, 1.2S_0] \times [0,1]$.

From Chart 22, we can see the extreme point lies on the boundary, and this result is consistent with Chart 5 and 10. However the peak in Chart 8 isn’t reflected in Chart 22, it may because the impact of the time to expiration is extremely weak in Chart 8.
Chapter 6

Conclusion

The aim of this article is to describe the option price sensitivity to the exponent in the CEV model. The degree of sensitivity is mainly affected by three factors: strike price, time to expiration and barriers (for Barrier options). Using closed-form solutions, finite difference, binomial tree and Monte-Carlo methods, the influence of those model parameters are presented in a simple and intuitive way.

We should keep in mind that our goal is to investigate price sensitivity to the exponent but not the volatility, therefore we should ensure the volatility $\delta S^{\alpha/2 - 1}$ is not changed at the current stock price. There are two ways to achieve this target: one way is to make the current stock price $S_0$ equal to 1, and then the volatility at $S_0$ will not depend on the exponent $\alpha$. The other way is to adjust $\delta$ continuously to make sure the volatility remains constant at $S_0$. The latter method is more complicated but has no requirement for $S_0$, so it’s adopted in our analysis.

To present the study results explicitly, we build a three-dimensional model. One axis is for $\Delta \alpha$ and one axis is for model parameters like strike price, the third axis is for changes in option price. These changes could be absolute or relative. Furthermore, to show the general validity of the conclusion, three different options and four different pricing methods are adopted in our research.

The analysis results reflect two distinguishing features: One is that the types of options and pricing methods play only a small role in determining the price sensitivity. When it comes to Barrier or Asian options, most results for European call options still remain. On the other hand, the analysis results strongly depend on the value of model parameters. For example, no matter how trivial the change is, varying the strike price from $K \leq S_0$ to $K > S_0$ will totally reshape the patterns of price sensitivity we observe.

Suppose we price a European call option under such an environment: risk-free rate $r = 0.06$, dividend yield $q = 0$, current price $S_0 = 1400$, volatility $\sigma = 0.16$. If $K \leq S_0$, the European call option is in- or at-the-money. The impact of the time to expiration on sensitivity is shown in Chart 8. As $\tau$ approaches zero or infinity, the price sensitivity will turn to zero. At some place between zero and infinity, price sensitivity reaches its maximum. In our analysis of Chapter 2, the strike price is set equal to the current stock price, the largest percent change resulted from $\Delta \alpha = 2$ is about 0.45% when $\tau \approx 16$. On the other hand, if $K > S_0$, which means the European call option is out-the-money, the impact of the time to expiration on sensitivity is
shown in Chart 10. As before, the price sensitivity turns to zero as $\tau$ approaches infinity. When $\tau$ approaches zero, however, the price sensitivity will increase instead of turning to zero. It's worth noting that in the case of $K \leq S_0$, the impact of $\Delta \alpha$ is very weak.

Put the above results all together, we can say the role of $\Delta \alpha$ in option pricing can be ignored under one of the following circumstances:

1) $K \leq S_0$,  
2) $K > S_0$ and $\tau$ is large enough.

However, in other cases, the value of $\alpha$ has material impact on option pricing. When it comes to Barrier, Asian or any other options, a more complex analysis may be needed.

We should remember that the analysis above is built on European call options with instance-specific parameter setting. The conclusions we got may alter according to different situations. As a result, a comprehensive analysis of alternative options, pricing methods and parameter settings would be vast in scope and beyond the bounds of this paper. After all, our goal is to find a proper way to document how $\Delta \alpha$ affect the pricing performance and the parameters’ effect on it, especially from a viewpoint of the financial market. When it comes to a concrete condition, we should carry out a concrete analysis.
Appendix A

To price options under CEV model using binomial tree methods, the interval \([T, t]\) is divided into \(n\) equal pieces, each of width \(\Delta t\). Over each time increment, the stock price can either increase to a particular level or decrease to another level.

For the CEV model, the volatility is not constant but varies according to the level of the underlying price, which means the probability of an upward move has to be recomputed at each node. Under CEV model, the diffusion process is as (A.1):

\[
dS = \mu S dt + \delta S^{\alpha/2} dW,
\]

we assume \(\beta = 1 - \alpha/2\), then we can get the process \(x = S^\beta/\alpha\sigma\) and applying Ito’s Lemma on (A.1):

\[
dx = \frac{\partial x}{\partial S} dS + \frac{\partial x}{\partial t} dt + \frac{1}{2} \frac{\partial^2 x}{\partial S^2} \sigma^2 (\sigma S^{1-\beta})^2 dt
\]

\[
= \frac{s^{\beta-1}}{\sigma} dS + \frac{1}{2} (\beta-1)s^{\beta-2} (\sigma S^{1-\beta})^2 dt
\]

\[
= \frac{s^{\beta-1}}{\sigma} dS + \frac{(\beta-1)\sigma}{2} dt,
\]

since \(x = S^\beta/\alpha\sigma\), we know \(S = (x\beta\sigma)^{1/\beta}\), and the equation (A.2) turn to:

\[
dx = \frac{s^{\beta-1}}{\sigma} dS + \frac{(\beta-1)\sigma}{2} dt
\]

\[
= \frac{(x\beta\sigma)^{1-1/\beta}}{\sigma} dS + \frac{(\beta-1)\sigma}{2} dt
\]

\[
= \left(x\beta\mu + \frac{(\beta-1)\sigma}{2}\right) dt + dw,
\]

we can see from (A.3) that the volatility in original diffusion process (A.1) has been transformed to 1. Based on this transformed process, a new binomial tree method is created:

\[
x^+ = x + \sqrt{\Delta t}
\]

\[
x^- = x - \sqrt{\Delta t}
\]

\[
x^{++} = x^+ + \sqrt{\Delta t}
\]

\[
x^{++} = x^+ + \sqrt{\Delta t}
\]

\[
x^{--} = x^- + \sqrt{\Delta t}
\]

\[
x^{--} = x^- + \sqrt{\Delta t}
\]
After calculating the values on the \( x \) lattice, values of \( S \) are easy to solve:

\[
S = f(x) \quad \Rightarrow \quad S^+ = f(x^+) \quad \Rightarrow \quad S^{++} = f(x^{++})
\]
\[
S^- = f(x^-) \quad \Rightarrow \quad S^{+-} = f(x^{+-})
\]
\[
S^- = f(x^-) \quad \Rightarrow \quad S^{--} = f(x^{--})
\]

The risk-neutral probability of an upward move is calculated by:

\[
P_u = \frac{S_u e^{r \Delta t} - S^-}{S^+ - S^-}
\]
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