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Empirical evaluation of a stochastic model for order book dynamics

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Abstract

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Abstract A stochastic model for order book dynamics is proposed in Cont et al. (2010) and empirically evaluated in this thesis. Arrival rates of limit, market and cancellation orders are described in terms of a Markov chain where the arrival rates are exponentially distributed. The model not only considers the best bid and ask queues but also additional price levels of the order book. Methods for computing several quantities important to high frequency trading are proposed using Laplace transforms and continued fractions. These quantities include conditional probabilities such as the probability of a price increase depending on the profile of the order book. Computing these probabilities are supposed to be easy enough to compute analytically. However this was not the case. We failed in the inversion of the Laplace transform methods and the main reason is that the instructions in Cont et al. (2010) are not adequate when it comes to perform the inversion. Hence we draw the conclusion that the method is no good for predicting short term behavior of limit order books. For long term applications the model can be used to simulate the order book with good results.

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Chapter 1

Summary in Swedish

En stokastisk modell för orderboksdynamik framställs i Cont et al. [2010]. Syftet med detta examensarbete var att implementera modellen för att evaluera dess användbarhet samt prestanda. Nästa steg var att eventuellt förbättra modellen och till sist konstruera en strategi för högfrekvenshandel baserat på modellen. Ankomstfrekvensen hos limiterade-, marknads- och annuleringsordrar beskrivs med hjälp av en Markov kedja där ankomstfrekvenserna är expopnentiellt fördelade. Modellen tar inte bara hänsyn till bästa köp och sälj ordrarna men även ytterligare prisnivåer i orderboken. Flertalet viktiga kvantiteter ur högfrekvenshandelssynpunkt kan beräknas med metoder som tillämpar Laplacetransformering kombinerat med så kallade "continued fractions" eller fortsatta fraktioner. Bland dessa kvantiteter utmärker sig sannolikheter baserade på orderbokens nuvarande tillstånd, dessa kallas "conditional probabilities", som kan användas för att förutspå prisändringar hos den underliggande värdehandlingen. Enligt källan ska dessa sannolikheter vara lätta att räkna ut analytiskt, tyvärr var det inte möjligt att genomföra. Detta på grund av bristande instruktioner i källan men även brist på tid. En följd av detta är att inga förbättringar kunde göras av modellen, möjligheter att skapa en handelsstrategi fanns inte heller. Slutsatsen blir på grund av detta att metoden inte är så bra som det påstås i Cont et al. [2010] eftersom huvudsyftet, vilket är att man kortsiktigt ska kunna prediktera priset, inte uppfylls. För applikationer där ett längre tidsperspektiv är intressant kan dock modellen användas för att simulera orderboken.

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Chapter 2

Introduction

In general *High Frequency Trading* (HFT) refers to the buying and selling of stocks, or other securities, where the speed is crucial for success. A delay of a few milliseconds could be the difference between a profit or a loss. Obviously even the fastest human can't keep up with these kind of speeds, hence automated trading is needed. The high frequency trader has developed from the more traditional market maker whose essential profit is the spread between the prices at which he bought and then sold. These spreads have gone from a size of a fraction of a dollar to just a penny or less. This, combined with the fact that technology has improved over the last 10 years, has lead to that HFT-firms have to settle for much smaller spreads. To compensate they operate in massive scale. In 2005 the average daily trading volume of New York Stock Exchange (NYSE)-listed stocks were 2.1 billion shares and four years later the same quantity had almost tripled to 5.9 billion shares. In the same period the average number of daily trades went up from 2.9 to 22.1 million trades which implies the decrease of the average trade size from 724 shares per trade to 268 trades per share. These increases can be explained by the fact that HFT is becoming increasingly common. However the main indicator that more automated trading is taking place is the average speed of execution which has dropped from 10.1 seconds in 2005 to just 0.7 seconds in 2009. All of the previously named facts can be found in Durbin [2010].

Studies of financial assets in the past have mainly focused on quote-driven markets, where a market maker centralizes buy and sell orders and provides liquidity. One example of such a system is the NYSE specialist system. An alternative to the traditional quote-driven market is the electronic order-driven market where all outstanding limit orders are assembled in a limit order book is available to market participants. Market orders are executed the best possible prices available. Many established stock exchanges such as the NYSE, NASDAQ, the Tokyo Stock Exchange and the London Stock Exchange have either fully or partially implemented electronic order-driven platforms.

The aim of this thesis will be to implement and evaluate a model for order book dynamics proposed in Cont et al. [2010]. When the implementation has been done eventual improvements will be done and hopefully a trading strategy will be created.

Order-driven markets have become an interesting candidate for stochastic modeling due to all the data that is available but also the dynamics of a limit order book, which in many ways resembles that of a queuing system. A limit order arrive and wait in a queue to either get canceled or executed against a market order. Hence a limit order can be modeled as a continuous time Markov process that keeps track of how many limit orders there are at each price level in the order book. This model has supposedly three preferable attributes. It can be estimated easily with high frequency data, empirical values of order books can be obtained and it is analytically manageable. This means that is possible to predict the short term behavior of the order book based on its current state by using Laplace transform methods. Focus will be on *conditional probabilities* of events given the state of the order book. These include the probability of an increase in midprice in the next move, the probability of a bid order being executed before the ask quote moves and the probability of both a bid and ask order being executed before the price moves.

Chapter 3

A Continuous-Time Model for a Stylized Limit Order Book

3.1 Limit Order Books

Consider a stock in an order-driven market. Market participants have the possibility to make four types of orders:

1. A limit buy order
2. A limit sell order
3. A market buy order
4. A market sell order

A *limit order* is an order to buy or sell a particular amount of a stock at a given price. It is posted to an electrical trading system where the state of the outstanding limit orders can be obtained by summing up the quantities at each price level. This is called the *limit order book*. The highest price associated with an outstanding limit buy order is called the *bid price* and the lowest sell price is called the *ask price*.

A *market* order is an order to buy or sell a particular amount of the stock at the best available price in the limit order book. An incoming market order is matched with the best available price in the limit order book and the trade takes place. The quantity at that price level decreases and if it is depleted the next price level will become the new bid/ask price.

A limit order stays in the order book until it is either canceled or executed against a market order. The chance of a limit order being executed is larger if it corresponds to a price close to the bid and the ask, in that case it will most likely be executed very quickly. On the other hand it may take quite some time before a limit order gets executed if the requested price is too far from the ask/bid or if the requested price moves away from the requested price. A limit order can also be canceled at any time.

In theory a limit order can be placed as far away from the ask/bid price as one could want, although this would probably mean that it would not get executed. To prevent this the model only considers market where limit orders can be placed on a price grid $\{1, ..., n\}$ representing multiples of a price tick. The upper boundary n is chosen so that it is highly unlikely that any incoming order will be larger than n within the time frame being studied. Introducing a continuous time process $X(t) \equiv (X_1(t), ..., X_n(t))_{t\geq 0}$, where $|X_p(t)|$ is the number of limit orders at price p, $1 \leq p \leq n$. If $X_p(t) < 0$, then there are $-X_p(t)$ bid orders at price p. If $X_p(t) > 0$ then there are $X_p(t)$ ask orders at price p.

The ask price $p_A(t)$ is the lowest sell price in the order book. If there are no ask orders in the order book an ask price of $n + 1$ is forced. The ask price $p_A(t)$ is defined by

$$p_A(t) = \min\left(\inf\{p = 1, ..., n,\ X_p(t) > 0\},\ (n + 1)\right).$$

As for the ask price a bid price has to be forced when there are no bid orders in the order book. Hence the bid price $p_B(t)$ is defined by

$$p_B(t) = \max\left(\sup\{p = 1, ..., n,\ X_p(t) < 0\},\ 0\right).$$

The *bid-ask spread* $p_S(t)$ and the *midprice* $p_M(t)$ are defined by

$$p_S(t) = p_A(t) - p_B(t)$$

and

$$p_M(t) = \frac{p_B(t) + p_A(t)}{2}.$$

To highlight the depth of the order book relative to the best quotes it can be useful to use a different notation, thus the number of buy orders at a distance i from the ask price is defined by

$$Q_i^B(t) = \begin{cases} X_{p_A(t)-i}(t) & 0 < i < p_A(t) \\ 0 & p_A(t) \le i < n \end{cases}$$

and the number of sell orders at a distance i from the bid price is defined by

$$Q_i^A(t) = \begin{cases} X_{p_B(t)+i}(t) & 0 < i < n - p_B(t) \\ 0 & n - p_B(t) \le i < n \end{cases}.$$

3.2 Dynamics of the Order Book

Let us take a look at how incoming orders changes the order book. For a state $x \in \mathbb{Z}^n$and $1 \le p \le n$, define

$$x^{p\pm1} \equiv x \pm (0, ..., 1, ..., 0),$$

where the 1 in the vector is in the pth component. Assuming that all orders are of unit size

- a limit sell order at level $p > p_B(t)$ increases the quantity at level $p:\ x \to x^{p+1}$
- a limit buy order at level $p < p_A(t)$ increases the quantity at level $p:\ x \to x^{p-1}$
- a market sell order decreases the quantity at the bid price: $x \to x^{p_B(t)+1}$
- a market buy order decreases the quantity at the ask price: $x \to x^{p_A(t)-1}$

- a cancellation of a limit sell order at level $p > p_B(t)$ decreases the quantity at level $p: \ x \to x^{p-1}$
- a cancellation of a limit buy order at level $p < p_A(t)$ decreases the quantity at level $p: \ x \to x^{p+1}$

Hence the development of the order book is driven by the flow of incoming limit orders, market orders and cancellations at each price level. The limit orders can be represented as a counting process, the same is true for both the market orders and the cancellations. Incoming orders arrive more frequently closer to the current ask/bid price and the rate of arrivals depend on the distance from the ask/bid. This has been observed empirically in Bouchaud et al. [2002].

To acquire these empirical attributes in a model that is analytically manageable and allows computations of interesting quantities a stochastic model is proposed. Modeling the events above with independent Poisson processes gives, for $i \geq 1$,

- Limit sell (respectively buy) orders arrive at a distance of i ticks from the opposite best quote at independent, exponential times with rate $\lambda(i)$,
- Market sell (respectively buy) orders arrive at independent, exponential times with rate μ,
- Cancellations of limit orders at a distance i ticks from the opposite best quote occur at a rate proportional to the number of orders at that level. If the number of orders are x, then the cancellation rate is $\theta(i)x$. This can be interpreted as follows: if we have a batch of x orders, each of which can be canceled at an exponential time with rate $\theta(i)$, then the total cancellation rate for the entire batch is $\theta(i)x$.

All of the events above are mutually independent.

Given the assumptions above, X is a continuous-time Markov chain with state space $\mathbb{Z}^n$ with transition rates given by:

$x \to x^{p+1}$ with rate $\lambda(p - p_B(t))$ for $p > p_B(t)$,

$x \to x^{p-1}$ with rate $\lambda(p_A(t) - p)$ for $p < p_A(t)$,

$x \to x^{p_A(t)-1}$ with rate μ,

$x \to x^{p_B(t)+1}$ with rate μ,

$x \to x^{p-1}$ with rate $\theta(p - p_B(t))|x_p|$ for $p > p_B(t)$,

$x \to x^{p+1}$ with rate $\theta(p_A(t) - p)|x_p|$ for $p < p_A(t)$.

In the real world the ask price is always greater than the bid price, thus a state is *admissible* if it fulfills

$$\mathscr{A} \equiv \{x \in \mathbb{Z}^n | \exists k,\, l \in \mathbb{Z} \text{ s.t. } 1 \le k \le l \le n,\, x_p \ge 0 \text{ for } p \ge l,\, x_p = 0 \text{ for } k \le p \le l,\, x_p \le 0 \text{ for } p \le k\}.$$

If the order books initial state is admissible, then it remains admissible with probability one. This is shown in Cont et al. [2010]. The following proposition and proof are also from Cont et al. [2010].

Proposition 1. If $\theta \equiv \min_{1 \le i \le n} \theta(i) > 0$, then X is an ergodic Markov process. In particular, X has a proper stationary distribution.

Proof. Let $N \equiv (N(t), t \ge 0)$, where $N(t) \equiv \sum_{p=1}^{n} |X_p(t)|$, and let $\tilde{N}$ be a birth-death process with birth rate given by $\lambda \equiv 2\sum_{p=1}^{n} \lambda(p)$ and death rate in state i, $\mu_i \equiv 2\mu + i\theta$. Notice that N increases by one at a rate bounded from above by λ and decreases by one at a rate bounded from below by $\mu_i \equiv 2\mu + i\theta$ when in state i. Thus, for all $t \ge 0$, N is stochastically bounded by $\tilde{N}$. For $k \ge 1$, let T_0^k and T_{-0}^k denote the duration of the kth visit to 0 and the duration between the $(k-1)$th and kth visit to 0 of the process N, respectively. Define random variables $\tilde{T_o^k}$ and $\tilde{T_{-0}^k}$, $k \ge 1$, for process $\tilde{N}$ similarly. Then the point process with interarrival times T_{-0}^1, T_0^1, T_{-0}^2, $T_0^2, \ldots$ and the point process with interarrival times $\tilde{T_{-0}^1}$, $\tilde{T_0^1}$, $\tilde{T_{-0}^2}$, $\tilde{T_0^2}, \ldots$ are alternating renewal processes. By theorem VI.1.2 of Asmussen [2003] and the fact that N is stochastically dominated by $\tilde{N}$, we then have for each $k \ge 1$,

$$\frac{E\left[T_0^k\right]}{E\left[T_0^k\right] + E\left[T_{-0}^k\right]} = \lim_{t\to\infty} P\left[N(t) = 0\right] \ge \lim_{t\to\infty} P\left[\tilde{N}(t) = 0\right] = \frac{E\left[\tilde{T_0^k}\right]}{E\left[\tilde{T_0^k}\right] + E\left[\tilde{T_{-0}^k}\right]}. \tag{3.1}$$

Notice that in state 0 both N and $\tilde{N}$ have birth rate λ. Thus,

$$E\left[T_0^k\right] = E\left[\tilde{T_0^k}\right] = \frac{1}{\lambda}. \tag{3.2}$$

Combining 3.1 and 3.2 gives us

$$E\left[T_{-0}^k\right] \leq E\left[T_{-0}^{\tilde{k}}\right]. \tag{3.3}$$

To show $\tilde{N}$ is ergodic, notice the inequalities

$$\sum_{i=1}^{\infty} \frac{\lambda^i}{\mu_1 \cdots \mu_i} < \sum_{i=1}^{\infty} \frac{1}{i!}\left(\frac{\lambda}{\theta}\right)^i = e^{\lambda/\theta} - 1 < \infty, \tag{3.4}$$

and

$$\sum_{i=1}^{\infty} \frac{\mu_1 \cdots \mu_i}{\lambda^i} > \sum_{i=1}^{M} \frac{\mu_1 \cdots \mu_i}{\lambda^i} + \sum_{i=M+1}^{\infty} \left(\frac{2\mu + M\theta}{\lambda}\right)^i = \infty, \tag{3.5}$$

for $M > 0$ chosen large enough so that $2\mu + M\theta > \lambda$.Therefore, by Corollary 2.5 of Asmussen [2003], $\tilde{N}$ is ergodic so that $E[T_{-0}^{\tilde{k}}] < \infty$. Combining this with the bound 3.3 and the fact that for each $t \geq 0$ $X(t) = (0, ..., 0)$ if and only if $N(t) = 0$ shows that X is positive recurrent. Because X is clearly irreducible, it follows that X is ergodic. $\square$

In a theoretical point of view the ergodicity of X is a favorable feature since it allow us to compare time averages of different quantities in simulations to unconditional expectations of the same quantities computed in the model. A couple of examples of these quantities are the average shape of the order book and the average price impact.

Chapter 4

Parameter Estimation and Order Book creation

4.1 Description of the Data Set

The data contains detailed information about the Ericsson B stock on October 7th, 2011 and was provided by NASDAQ OMX Group. There is three separate files for the different types of events. One for incoming orders, one for cancellations and one for executions. Small extractions from these files can be seen in tables 4.1, 4.2 and 4.3. Note that not all information are presented in these tables, some of the omitted information are trader ID, stock ID-number, etc. The most important columns are described here,

- $refdate$ - the date of the trade,
- *mykey* - a unique key to keep track of events in case of timestamp being the same, used for sorting,
- *mstime* - time after midnight in nanoseconds,
- $ordersequence$ - an identifier, used to match inserted orders with cancellations or executions,

- *side* - Bid or Sell order (B/S),
- *quantity* - number of shares,
- *price* - divide with 10000 to acquire the price in SEK i.e. 684000 represent 68.40 SEK,
- *liquidity* - this column show if the entire order was depleted or not. “R” means that it did and “A” means that it did not. The executions with “R” are called market orders, the other ones are just called executions.

Table 4.1: Extraction from the incoming order file for Ericsson B.

refdate	mykey	mstime	ordersequence	side	quantity	price
2011-10-07	11610797	4,38178E+13	5247323	B	1000	684000
2011-10-07	11615151	4,38223E+13	5249299	S	540	685500
2011-10-07	11666693	4,39044E+13	5272636	B	600	684500
2011-10-07	11647306	4,38622E+13	5263819	S	630	685000
2011-10-07	11647393	4,38622E+13	5263851	S	1000	685500

Table 4.2: Extraction from the cancellation file for Ericsson B.

refdate	mykey	mstime	ordersequence	quantity
2011-10-07	11574939	4,37933E+13	5231295	900
2011-10-07	11575436	4,37952E+13	5197162	1000
2011-10-07	11575488	4,37952E+13	5197744	702
2011-10-07	11594617	4,38075E+13	5197130	200
2011-10-07	11595651	4,38078E+13	5240013	1

Table 4.3: Extraction from the execution file of Ericsson B.

refdate	mykey	mstime	ordersequence	quantity	price	liquidity
2011-10-07	18143670	5,24786E+13	8211285	1000	700000	R
2011-10-07	26255781	5,76369E+13	11904308	175	692500	A
2011-10-07	26255784	5,76369E+13	11905733	25	692500	A
2011-10-07	26255796	5,76369E+13	11905733	200	692500	A
2011-10-07	26255811	5,76369E+13	11922133	250	692500	R

4.2 Creation of the Order Book

As mentioned in Limit Order Books the data is divided in to three separate files. To create the order book these files have to be combined in to a single file with all the information needed. This can be done in several different manners, where the primary difference is the time between updates. Updating the order book every second saves a lot of computational time compared to update say every tenth of a second. However since several orders can come in during a very small time interval one could lose valuable information. The only way to prevent this is to update every time a new event occurs, i.e. for every new incoming order, cancellation and execution. As mentioned previously this is the most computational heavy alternative but the accuracy benefits makes the additional computational time tolerable.

Note that not all of the trades in the original data are added to the order book. Some of the trades are not visible to traders, thus called $non-displayed$ orders. These orders were deleted from the data set before the order book creation began. In the incoming order file this was an easy task since there was a label telling you whether or not they were visible. In the files for cancellations and executions however, this information did not exist. This problem can be solved by matching the order sequence number of the non-displayed order with the corresponding cancellation or execution. When a match is found the trades are removed from the files they belong to.

After all non-displayed orders have been removed it is time to begin creating the order book. All of the events from the three files are combined and sorted on *mykey*, that is unique. Then the following algorithm is applied to all of the sorted data:

1. Choose the first event.

2. Determine the type of the event, if it is

   (a) an *incoming order*. Determine if it is

      i. an *ask order*. After that compare the price with the ask price levels in the order book. If a match is found increase the quantity at that level with the amount of the incoming order. Otherwise place the new order so that the ask queue is sorted from the smallest to the largest price.

Table 4.4: Extraction from the complete order book.

Bid price 2	Bid queue 2	Bid price 1	Bid queue 1	Ask price 1	Ask queue 1	Ask price 2	Ask queue 2
689000	20757	689500	12009	690500	1579	691000	22179
689500	12009	690000	130	690500	1579	691000	22179
689000	20757	689500	12009	690000	500	690500	1579

ii. a *bid order*. Compare the price with the bid price levels in the order book. If a match is found increase the quantity at that level with the amount of the incoming order. Otherwise place the new order so that the bid queue is sorted from the largest to the smallest price.

(b) a *cancellation order*. Check the order sequence number and locate the corresponding order in the order book. Use the cancellations quantity to reduce the queue size at the correct price level. If the entire queue was depleted resort the price levels to close the gap that has been created in the order book.

(c) an *execution order*. Check the sequence number and locate the corresponding order in the order book. Reduce the queue size at that price level by the execution quantity. As for the cancellations the price levels need to be resorted if the entire queue was depleted and created a hole.

3. Choose the next event and go to 2.

This proceedings repeated until the total order book has been created.

4.3 Estimation Procedure

In this section the estimations used for modeling the order book will be presented. They can also be found in Cont et al. [2010] with the exception that they only consider a maximum distance of 5 ticks from the opposite best quote whereas here a maximum distance of 20 ticks will be considered. Recall that in Dynamics of the Order Book all orders were assumed to be of unit size. The average size of market orders S_m, limit orders S_l, and canceled orders S_c can be computed from the data set. The unit size is chosen to be the average size of limit orders S_l. The arrival rate of the limit orders can be estimated by the function

$$\hat{\lambda}(i) = \frac{N_l(i)}{T_*},$$

for $1 \leq i \leq 20$, where $N_l(i)$ is the total number of limit orders that arrived at distance i from the opposite best quote, and $T_*$is the total trading time in the sample. The total number of limit orders that arrived is obtained by enumerating the number of times a quote increases in size at a distance $1 \leq i \leq 20$ ticks from the opposite best quote. In Cont et al. [2010] a power law function is used to obtain the limit order arrival rate for distances larger than 5 ticks from the opposite best quote. The power law function

$$\hat{\lambda}(i) = \frac{k}{i^\alpha}$$

was suggested by Bouchaud et al. [2002] and Zovko and Farmer [2002]. The parameters k and α are acquired by solving the least-square fit problem

$$\min_{k,\alpha} \sum_{i=1}^{5} \left(\hat{\lambda}(i) - \frac{k}{i^\alpha} \right)^2.$$

Since we already have the arrival rates for distances up to 20 ticks from the opposite best quote this power law is redundant. Nonetheless the estimated arrival rates from the power law function are displayed in figure 4.1 together with the first five observed arrival rates from the data. All the limit order arrival rates observed from the data are displayed in figure 4.2.

We estimate the arrival rate of market orders, μ, by simply counting the number of incoming market order and then divide with the total trading time. Market orders matched with hidden orders are ignored.

The cancellation rate is given by

$$\hat{\theta}(i) = \frac{N_c(i) S_c}{T_* Q_i S_l}$$

for $i \leq 20$, where Q_i is the the steady state shape of the order book i.e. the average number of orders at distance i from the opposite best quote. N_c is the number of cancellations and is obtained by enumerating the number of times that a quote decreases in size, except the decreases caused by

Figure 4.1: The limit order arrival rate estimated by a power law.

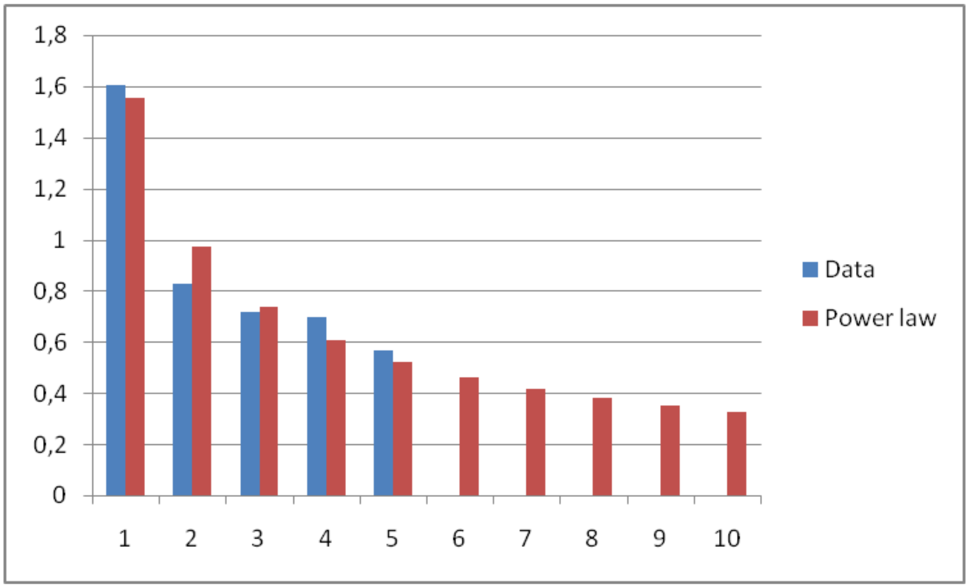


Figure 4.2: The limit order arrival rate as a function of the distance from the opposite best quote.

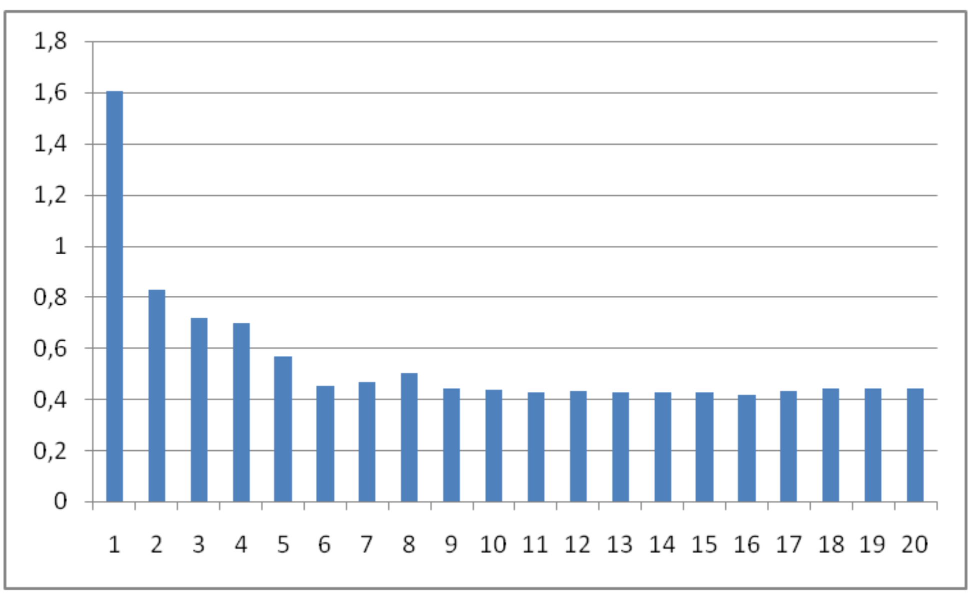


Figure 4.3: The cancel order arrival rate as a function of the distance from the opposite best quote.

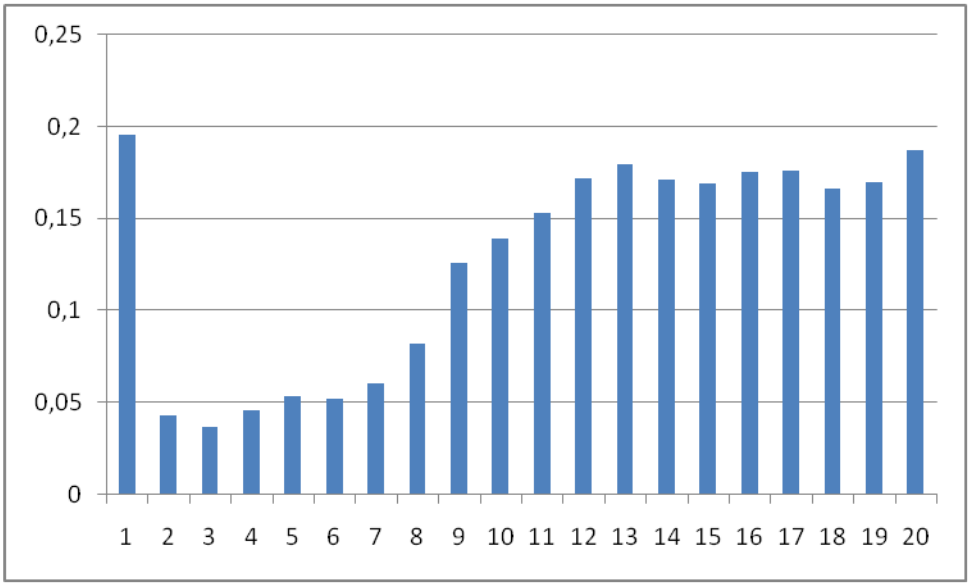


Table 4.5: Estimated Parameters: Ericsson B.

i	1	2	3	4	5
$\hat{\lambda}(i)$	1.6029	0.8296	0.7167	0.6991	0.5674
$\hat{\theta}(i)$	0.1959	0.0431	0.0371	0.0460	0.0533
$\hat{\mu}$	0.2783				
k	1.5537				
α	0.6765				

market orders. S_c is the average size of cancellation orders and similarly S_l is the average size of limit orders. As before T_* is the total trading time. The cancellation arrival rates can be seen in figure 4.3.

All of the estimated parameters are shown in table 4.5.

Chapter 5

Laplace Transform methods for Computing Conditional Probabilities

A motivation for modeling high frequency dynamics of order books is to use the information provided for predicting short-term behavior of different quantities useful in trade executions and algorithmic trading. These quantities can be expressed as *conditional probabilities* given the current state of the order book and include, among others the probability of an increase in midprice. In this section we will show that our model allows conditional probabilities to be computed analytically using Laplace methods.

5.1 Laplace Transforms and First-Passage Times of Birth-Death Processes

Before we start we need to go through some basic facts about Laplace transforms and Laplace transforms for first-passage times of birth-death processes (Abate and Whitt [1999], Cont et al.

[2010]). Given a function $f : \mathbb{R} \to \mathbb{R}$, its two-sided Laplace transform is given by

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt,$$

where s is a complex number. If f is probability density function (pdf) of a random variable X, $\hat{f}$ is the two-sided Laplace transform of the random variable X. The reason for using *two-sided* Laplace transforms is that our function f will normally correspond to the pdf of a random variable with both negative and positive support. For convenience the two-sided Laplace transform will simply be denoted Laplace transform from now on. If X and Y are independent random variables with well-defined Laplace transforms, then

$$\hat{f}_{X+Y}(s) = E[s^{-s(X+Y)}] = E[e^{-sX}]E[e^{-sY}] = \hat{f}_X(s)\hat{f}_Y(s). \tag{5.1}$$

If for some $\gamma \in \mathbb{R}$ we have $\int_{-\infty}^{\infty} |\hat{f}(\gamma + i\omega)| d\omega < \infty$ and $f(t)$ is continuous at t, then the inverse transform is given by the Bromwich contour integral

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{ts} \hat{f}(s) ds. \tag{5.2}$$

5.1.1 Continued Fractions

A *continued fraction* is an expression obtained through an iterative process and is well described in Abate and Whitt [1999]. Here we will make a short summary of what a continued fraction is and how it can be used.

An (infinite) continued fraction (CF) associated with a sequence $\{a_n : n \geq 1\}$ of partial numerators and a sequence $\{b_n : n \geq 1\}$ of partial denominators, which are complex numbers with $a_n \neq 0$ for all n, is the sequence $\{w_n : n \geq 1\}$, where

$$w_n = t_1 \circ t_2 \circ ... \circ t_n(0), \quad n \geq 1,$$

and

$$t_k(u) = \frac{a_k}{b_k + u}, \quad k \geq 1,$$

i.e. w_n is the n-fold composition the mappings $t_k(u)$ applied to 0. If $w \equiv \lim_{n\to\infty} w_n$, the CF is convergent and the limit w is said to be the value of the CF. We write

$$w = \Phi_{n=1}^{\infty} \frac{a_n}{b_n}$$

or

$$w = \frac{a_1 \; a_2 \; a_3}{b_1 + b_2 + b_{3+}} \cdots .$$

5.1.2 First-Passage Times in Birth-Death Processes

Now we will show that CFs can be used to compute the Laplace transform of a first-passage time pdf in a birth-death (BD) process (Abate and Whitt [1999]). Let T_b be a random variable representing the first-passage time from state b to state 0. Such first-passage times can be expressed in terms of first-passage times to neighboring states,

$$T_b = T_{b,b-1} + T_{b-1,b-2} + \cdots + T_{1,0}, \tag{5.3}$$

where the random variables on the right hand side are mutually independent and $T_{i,i-1}$ denotes the first-passage time of the BD from state i to state $i-1$. Let $f_{i,i-1}$ be the pdf of $T_{i,i-1}$ and let $\hat{f}_{i,i-1}$ be its Laplace transform, i.e.,

$$\hat{f}_{i,i-1}(s) = \int_0^\infty e^{-st} f_{i,i-1}(t)dt \equiv Ee^{-sT_{i,i-1}}. \tag{5.4}$$

From 5.1 and 5.3, we have

$$\hat{f}_b(s) = \prod_{i=1}^{b} \hat{f}_{i,i-1}(s). \tag{5.5}$$

Hence, in order to compute the Laplace transform $\hat{f}_b$, it suffices to be able to compute the Laplace transform of the first-passage time to a neighboring state.

It is also possible to construct CFs representing the Laplace transforms of first-passage times with an infinite time space. Consider a BD with constant birth rate λ and death rates μ_i in state $i \geq 1$. By considering the first transition from state i, we obtain the recursion

$$\hat{f}_{i,i-1}(s) = \frac{\mu_i}{\lambda + \mu_i + s} + \frac{\lambda \hat{f}_{i+1,i}(s) \hat{f}_{i,i-1}(s)}{\lambda + \mu_i + s} \tag{5.6}$$

from which we obtain

$$\hat{f}_{i,i-1} = \frac{\mu_i}{\lambda + \mu_i + s - \lambda \hat{f}_{i+1,i}(s)}. \tag{5.7}$$

A CF is acquired by iterating on 5.7 and is displayed here

$$\hat{f}_{i,i-1}(s) = -\frac{1}{\lambda} \Phi_{k=i}^{\infty} \frac{-\lambda \mu_k}{\lambda + \mu_k + s}. \tag{5.8}$$

Combining 5.5 and 5.8 yields

$$\hat{f}_b(s) = \left(-\frac{1}{\lambda}\right)^b \left(\prod_{i=1}^{b} \Phi_{k=1}^{\infty} \frac{-\lambda \mu_k}{\lambda + \mu_k + s}\right). \tag{5.9}$$

5.2 Direction of Price Moves

This section will be dedicated to computing the probability of an increase in the midprice when it changes. This occurs either at the first-passage time of the bid or ask queue to zero or, assuming that the spread between the bid and ask is greater than one tick, the first time a limit order arrives inside the spread. Let $X_A \equiv X_{p_A(\cdot)}(\cdot)$ and $X_B \equiv |X_{p_{B(\cdot)}}(\cdot)|$. Moreover, let $W_B \equiv \{W_B(t),\ t \geq 0\}$, where W_B is the number of orders remaining at the bid queue at time t of the initial $X_B(0)$ orders, similarly W_A is the number of orders remaining at the ask queue. Let ϵ_B and ϵ_A be the first-passage time of W_B and W_A to 0 respectively, and let T be the time of the first change in midprice:

$$T \equiv \inf\left\{t \geq 0,\ p_M(t) \neq p_M(0)\right\}.$$

Given the assumptions made and the configuration of the order book, the probability of an increase in midprice at the next price change can be written as

$$P\left[p_M(T) > p_M(0) \mid X_A(0) = a,\ X_B(0) = b,\ p_S(0) = S\right], \tag{5.10}$$

where $S > 0$ (Cont et al. [2010]).

The expression (5.10) can be computed by using a coupling argument (Cont et al. [2010]).

Lemma 3. Let $p_S(0) = S$. Then

1. There exist independent birth-death processes $\tilde{X}_A$ and $\tilde{X}_B$ with constant birth rates $\lambda(S)$ and death rates $\mu + i\theta(S)$, $i \geq 1$, such that for all $0 \leq t \leq T$, $\tilde{X}_A(t) = X_A(t)$, and $\tilde{X}_B(t) = X_B(t)$.

2. There exist independent pure death processes $\tilde{W}_A$ and $\tilde{W}_B$ with death rate $\mu + i\theta(S)$ in state $i \geq 1$, such that for all $0 \leq t \leq T$, $\tilde{W}_A(t) = W_A(t)$ and $\tilde{W}_B(t) = W_B(t)$. Furthermore, $\tilde{W}_A$ is independent of $\tilde{X}_B$, $\tilde{W}_B$ is independent of $\tilde{X}_A$, $\tilde{W}_A \leq \tilde{X}_A$, and $\tilde{W}_B \leq \tilde{X}_B$.

Proof. We prove Part 1. Part 2 can be proven analogously. X is a continuous-time Markov chain, with transition rates given by Section 3.2. For $0 \leq t \leq T$, $p_A(t) = p_A(0)$ and $p_B(t) = p_B(0)$, so substituting in Section 3.2 yields that $X_A(t)$ and $X_B(t)$ have the following (identical) transition rates for $0 \leq t \leq T$

$$\begin{cases} n \to n+1 & \textit{with rate}\ \lambda(S) \\ n \to n-1 & \textit{with rate}\ \mu + n\theta(S). \end{cases} \tag{5.11}$$

Define $\tilde{X}_A$ and $\tilde{X}_B$ such that

- $\tilde{X}_A(t) = X_A(t)$ and $\tilde{X}_B(t) = X_B(t)$ for $t \leq T$ and
- $\tilde{X}_A(t)$, $\tilde{X}_B(t)$, $t \geq T$ follow independent birth-death processes with rates given by (5.11).

The above remarks show that in fact $\left(\tilde{X}_A(t)\right)_{t\geq 0}$ (respectively $\left(\tilde{X}_B(t)\right)_{t\geq 0}$) has the same law as a birth-death process with rates (5.11). To show that $\tilde{X}_A$ and $\tilde{X}_B$ are independent, we note that because the transition rates of X_A (respectively X_B) do not depend on $(X_p(t),\, p \neq p_A(0))$ (respectively $(X_p(t),\, p \neq p_B(0))$) for $0 \leq t \leq T$, we have, in particular, conditional independence of $X_A(t)$ and $X_B(t)$ given $X(0)$ and $\{t \leq T\}$. $\square$

From here onward we let σ_A and σ_B denote the first-passage time of $\tilde{X}_A$ and $\tilde{X}_B$ to 0, respectively. Before we can compute the conditional probability (5.10) we need the following result (Cont et al. [2010]).

Lemma 5. Let Z be an exponentially distributed random variable with parameter Λ. Then the Laplace transform of the random variable $\sigma_B \wedge Z$ is given by

$$\hat{f}_b^1(\Lambda + s) + \frac{\Lambda}{\Lambda + s}\left(1 - \hat{f}_b^1(\Lambda + s)\right),$$

where $\hat{f}_b^1$ is given in (5.12).

Proof. We first compute the density $f_{\sigma_B \wedge Z}$ of the random variable $\sigma_B \wedge Z$ in terms of the density f_b of the random variable σ_B. Because Z is exponential with rate Λ, we have for all $t \geq 0$,

$$\begin{aligned} P\left[\sigma_B \wedge Z < t\right] &= 1 - P\left[\sigma_B > t\right] P\left[Z > t\right] \\ &= 1 - \left(1 - F_{\sigma_B}(t)\right) e^{-\Lambda t}. \end{aligned}$$

Taking derivatives with respect to t gives

$$f_{\sigma_B \wedge Z}(t) = f_b^1(t) e^{-\Lambda t} + \Lambda\left(1 - F_b^1(t)\right) e^{-\Lambda t},$$

for $t \geq 0$, where $F_b^1(t)$ $(f_b^1(t))$ is the cdf (pdf) of σ_B. Also, $f_{\sigma_B \wedge Z}(t) = 0$ for $t < 0$. The Laplace transform of $\sigma_B \wedge Z$ is thus given by

$$\begin{aligned}
\hat{f}_{\sigma_B \wedge Z}(s) &= \int_{-\infty}^{\infty} e^{-st} f_{\sigma_B \wedge \sigma_B^{\Sigma}}(t) dt \\
&= \int_{0}^{\infty} e^{-st} \left(f_b^1(t) e^{-\Lambda t} + \Lambda \left(1 - F_b^1(t)\right) e^{-\Lambda t} \right) ds \\
&= \int_{0}^{\infty} e^{-t(s+\Lambda)} f_b^1(t) dt + \Lambda \int_{0}^{\infty} \left(1 - F_b^1(t)\right) e^{-t(s+\Lambda)} dt \\
&= \hat{f}_b^1(s+\Lambda) + \frac{\Lambda}{\Lambda + s} \Big(a - \hat{f}_b^1(s+\Lambda) \Big),
\end{aligned}$$

where the last equality follows from integration by parts. □

Now we can take a look at proposition 4 from Cont et al. [2010] which are used to compute (5.10).

Proposition 4 (Probability of Increase in Midprice). Let $\hat{f}_j^S$ be given by

$$\hat{f}_j^S(s) = \left(-\frac{1}{\lambda(S)} \right)^j \left(\prod_{i=1}^{j} \Phi_{k=i}^{\infty} \frac{-\lambda(S)\left(\mu + k\theta(S)\right)}{\lambda(S) + \mu + k\theta(S) + s} \right), \tag{5.12}$$

for $j \geq 1$, and let $\Lambda_S \equiv \sum_{i=1}^{S-1} \lambda(i)$. Then (5.10) is given by the inverse Laplace transform of

$$\begin{aligned}
\hat{F}_{a,b}^S(s) &= \frac{1}{s} \left(\hat{f}_a^S(\Lambda_S + s) + \frac{\Lambda_S}{\Lambda_S + s} \Big(1 - \hat{f}_a^S(\Lambda_S + s) \Big) \right) \\
&\quad \cdot \left(\hat{f}_b^S(\Lambda_S + s) + \frac{\Lambda_S}{\Lambda_S + s} \Big(1 - \hat{f}_b^S(\Lambda_S - s) \Big) \right),
\end{aligned} \tag{5.13}$$

evaluated at 0. When $S = 1$, (5.13) reduces to

$$\hat{F}_{a,b}^1(s) = \frac{1}{s} \hat{f}_a^1(s) \hat{f}_b^1(-s). \tag{5.14}$$

Proof. We will start with the special case when $S = 1$ and then extend the analysis to the case when $S > 1$, using Lemma 5 above. Construct the independent birth-death processes $\tilde{X}_A$ and $\tilde{X}_B$ as in Lemma 3. When $S = 1$, the price changes for the first time exactly when one of the two processes $\tilde{X}_A$ and $\tilde{X}_B$ reaches the state 0 for the first time. Thus, given our initial conditions, the distribution of T is given by the minimum of the independent first-passage times σ_A and σ_B. Furthermore, the quantity (5.10) is given by $P[\sigma_A < \sigma_B]$. By (5.9), the conditional Laplace transform of $\sigma_A - \sigma_B$ given the initial conditions is given by $\hat{f}_a^1(s)\hat{f}_b^1(-s)$ so that the conditional Laplace transform of the cumulative distribution function (cdf) of $\sigma_A - \sigma_B$ is given by (5.14). Thus, our desired probability is given by the inverse Laplace transform of (5.14) evaluated at 0.

We now move on to the case where $S > 1$. Let σ_A^i denote the first time an ask order arrives at distance i ticks from the bid and σ_B^i denote the first time a bid order arrives at distance i from the ask, for $i = 1, \ldots, S-1$. The time of the first change in midprice is now given by

$$T = \sigma_A \wedge \sigma_B \wedge \min\left\{\sigma_A^i,\, \sigma_B^i,\, i = 1, \ldots, S-1\right\}.$$

Notice that $\tilde{X}_A$ and $\tilde{X}_B$ are independent of the mutually independent arrival times σ_A^i, σ_B^i, for $i = 1, \ldots, S-1$. Also notice that σ_A^i and σ_B^i are exponentially distributed with rates $\lambda(i)$ for $i = 1, \ldots, S-1$. The first change in midprice is an increase if there is an arrival of a limit bid order within $S-1$ ticks of the best ask or $\tilde{X}_A$ hits zero, before there is an arrival of a limit ask order within $S-1$ ticks of the best bid or $\tilde{X}_B$ hits zero. Thus, the quantity (5.10) can be written as

$$P\left[\sigma_A \wedge \sigma_B^1 \wedge \cdots \wedge \sigma_B^{S-1} < \sigma_B \wedge \sigma_A^1 \wedge \cdots \wedge \sigma_A^{S-1}\right] = P\left[\sigma_A \wedge \sigma_B^\Sigma < \sigma_B \wedge \sigma_A^\Sigma\right], \tag{5.15}$$

where σ_A^Σ and σ_B^Σ are independent exponential random variables, both with rate Λ_S. To compute (5.15), we first need to compute the conditional Laplace transform of the minimum $\sigma_B \wedge \sigma_A^\Sigma$. This is given in Lemma 5, substituting σ_A^Σ for Z. The conditional Laplace transform of the random variable $\sigma_B \wedge \sigma_A^\Sigma - \sigma_A \wedge \sigma_B^\Sigma$ can then be computed using (5.1), and the probability (5.10) can be computed by inverting the conditional Laplace transform of the cdf of this random variable and evaluating at 0 as in the case $S = 1$. □

To sum up this section proposition 4 can be used to compute the probability of a price increase given that the price changes. However, in order to obtain the probability an inversion of the Laplace transform has to be made. More on this implementation is discussed in Inverse Laplace Transform.

5.3 Executing an Order Before the Midprice Moves

When placing an order the trader has two choices, either he can place a market order or a limit order. At a given time placing a limit order gives a better price than placing a market order at the same time, this is due to the fact that a limit order faces a risk of never being executed. A market order is executed almost instantaneously but a limit order stays in the order book until either the order is canceled or a matching order is inserted. This means that the midprice could move away rendering the limit order useless. Hence it makes sense talking about the probability of a limit order being executed before the price moves since it is a quantity that is useful when choosing between a limit order and a market order. We will now shoe how to compute the probability of an order placed at the bid price is executed before the midprice moves in any direction, given that it is not canceled. The results holds for $S \equiv p_S(0) \geq 1$, however note that in the case when $S = 1$ the probability we are looking at is equal to the probability of the order being executed before the midprice moves *away* from the desired price, given that the order is not canceled. The model is symmetric in bids and asks which means that the results holds for orders placed at both the ask and bid price.

Some new notations are introduced. Let NC_b (NC_a) denote the event that an order that never is canceled is placed at the bid (ask) at time 0. The probability that an order placed at the bid price is executed before the midprice moves is given by

$$P\left[\epsilon_B < T \mid X_B(0) = b,\ X_A(0) = a,\ p_S(0) = S,\ NC_b\right], \tag{5.16}$$

and can be computed with proposition 6 from Cont et al. [2010].

Proposition 6 (Probability of Order Execution Before Midprice Moves). Define $\hat{f}_a^S(s)$ as in (5.9), let $\hat{g}_J^S$ be given by

$$\hat{g}_j^S(s) = \prod_{i=1}^{j} \frac{\mu + \theta(S)(i-1)}{\mu + \theta(S)(i-1) + s}, \tag{5.17}$$

for $j \geq 1$, and let $\Lambda_S \equiv \sum_{i=1}^{S-1} \lambda(i)$. Then the quantity (5.16) is given by the inverse Laplace transform of

$$\hat{F}_{a,b}^S(s) = \frac{1}{s}\hat{g}_b^S(s)\left(\hat{f}_a^S(2\Lambda_S - s) + \frac{2\Lambda_S}{2\Lambda_S - s}\left(1 - \hat{f}_a^S(2\Lambda_S - s)\right)\right), \tag{5.18}$$

evaluated at 0. When $S = 1$, (5.18) reduces to

$$\hat{F}_{a,b}^1(s) = \frac{1}{s}\hat{g}_b^1(s)\hat{f}_a^1(-s). \tag{5.19}$$

Proof. Construct $\tilde{X}_A$ and $\tilde{W}_B$ using Lemma 3. Let us first consider the case $S = 1$. Let $T' \equiv \epsilon_B \wedge T$ denote the first time when either the process $\tilde{W}_B$ hits 0 or the midprice changes. Conditional on an infinitely patient order being placed at the bid price at time 0, T' is the first time when either that order gets executed or the midprice changes. Notice that conditional on our initial conditions, ϵ_B is given by a sum of b independent exponentially distributed random variables with parameters $\mu + (i-1)\theta(1)$, for $i = 1, \ldots, b$, and independent of $\tilde{X}_A$. Thus, the conditional Laplace transform of ϵ_B given our initial conditions is given by (5.17). Because in the case $S = 1$ the midprice can change before time ϵ_B if and only if $\sigma_A < \epsilon_B$, the quantity (5.16) can be written simply as $P[\epsilon_B < \sigma_A]$. Using (5.1) with the conditional Laplace transforms of ϵ_B and σ_A, given in (5.17) and (5.9), respectively, we obtain (5.19).

This analysis can be extended to the case where $S > 1$ just as in the proof of Proposition 4. When $S > 1$, our desired quantity can be written as $P[\epsilon_B < \sigma_A \wedge \sigma_B^\Sigma \wedge \sigma_A^\Sigma]$. Because the conditional distribution of $\sigma_B^\Sigma \wedge \sigma_A^\Sigma$ is exponential with parameter $2\Lambda_S$, Lemma 5 then yields the result. □

5.4 Making the Spread

Arbitrage is explained in Durbin [2010] as: "The simultaneous buying of a security at one price and selling it (or an equivalent security or portfolio) at another, higher price in order to earn risk-free profit". In other words free money without any risk. This can be achieved by placing two orders, one at the ask price and one at the bid price, and hoping that the orders will be executed before the midprice moves given that the orders are not canceled. If both orders execute before the price move the strategy has paid off, we refer to this as "making the spread". Otherwise, losses may be reduced by placing a market order and losing the bid-ask spread. In this section we will show how to compute the probability that two orders, placed at the ask and bid price respectively, are executed before the midprice moves. We will only consider the case where the initial spread is one tick: $S = 1$. The probability of making the spread can be expressed as

$$P\left[\max\left\{\epsilon_A, \epsilon_B\right\} < T \mid X_B(0) = b,\ X_A(0) = a,\ p_S(0) = 1,\ NC_a,\ NC_b\right]. \tag{5.20}$$

The following result, which can be found in Cont et al. [2010], can be used to compute this probability:

Proposition 7. The probability (5.20) of making the spread is given by $h_{a,b} + h_{b,a}$, where

$$h_{a,b} = \sum_{i=0}^{\infty}\sum_{j=1}^{a} P\left[\epsilon_j < \sigma_i\right] \int_0^{\infty} \mathcal{P}_{0,i}^{X}(t)\mathcal{P}_{a,j}^{W}(t) g_b^1(t)dt, \tag{5.21}$$

where

$$\mathcal{P}_{0,i}^{X}(t) \equiv \frac{e^{-\lambda^X(t)}\lambda^X(t)^i}{i!}, \quad \lambda^X(t) \equiv \frac{\lambda}{\theta}(1 - e^{-\theta t}), \tag{5.22}$$

$$\mathcal{P}_{a,j}^{X}(t) \equiv \left(e^{Q_a^W t}\right)_{a,j} \equiv \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\left(Q_a^W\right)^k\right)_{a,j}, \tag{5.23}$$

$$Q_a^W \equiv \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \mu & -\mu & 0 & \cdots & 0 \\ 0 & \mu+\theta & -\mu-\theta & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu+(a-1)\theta & -\mu-(a-1)\theta \end{bmatrix}, \tag{5.24}$$

and g_b^1 is the inverse Laplace transform of g_b^1, which is given in (5.17).

Proof. Because $S = 1$, $T = \min\{\sigma_A, \sigma_B\}$, and the quantity (5.20) can be written as

$$P\left[\max\{\epsilon_B, \epsilon_A\} < \min\{\sigma_B, \sigma_A\}\right]. \tag{5.25}$$

Construct $\tilde{X}_A$, $\tilde{X}_B$, $\tilde{W}_A$ and $\tilde{W}_B$ using Lemma 3. Let $T^{'} = \max\{\epsilon_A, \epsilon_B\} \wedge T$ denote the first time when either both of the processes $\tilde{W}_A$ and $\tilde{W}_B$ have hot 0, or the midprice has changed. Conditional on infinitely patient orders being placed at the best bid and ask prices at time 0, $T^{'}$ is the first time when either both the orders get executed or the midprice changes. Furthermore, by Lemma 3, $\tilde{W}_A$ and $\tilde{W}_B$ are independent pure death processes with death rate $\mu + i\theta(1)$ in state $i \geq 1$, and $\tilde{W}_A(t) \leq \tilde{X}_A(t)$ and $\tilde{W}_B(t) \leq \tilde{X}_B(t)$. This implies that ϵ_A and ϵ_B are independent of each other and σ_A and σ_B are independent of each other with $\epsilon_A \leq \sigma_A$ and $\epsilon_B \leq \sigma_B$. Using these properties, we obtain

$$\begin{aligned} P\left[\max\{\epsilon_B, \epsilon_A\} < \min\{\sigma_B, \sigma_A\}\right] &= P\left[\epsilon_B < \sigma_A,\ \epsilon_A < \sigma_B,\ \epsilon_B < \epsilon_A\right] \\ &\quad + P\left[\epsilon_B < \sigma_A,\ \epsilon_A < \sigma_B,\ \epsilon_A < \epsilon_B\right] \\ &= P\left[\epsilon_A < \sigma_B,\ \epsilon_B < \epsilon_A\right] \\ &\quad + P\left[\epsilon_B < \sigma_A,\ \epsilon_A < \epsilon_B\right] \\ &= h_{a,b} + h_{b,a}, \end{aligned} \tag{5.26}$$

where we define $h_{a,b} \equiv P\left[\epsilon_B < \epsilon_A < \sigma_B\right]$, the probability that the order placed at the bid is executed before the order placed at the ask, and the order at the ask is executed before the bid quote disappears. Focus will now be on computing $h_{a,b}$. Conditioning on the value of ϵ_B gives

$$h_{a,b} = \int_0^\infty P\left[\epsilon_B < \epsilon_A < \sigma_B \mid \epsilon_B = t\right] g_b^1(t)dt. \tag{5.27}$$

Focusing on the first factor in the integrand in (5.27) and conditioning on the values of $\tilde{X}_B(t)$ and $\tilde{W}_A(t)$ gives us

$$\begin{aligned} P\left[\epsilon_B < \epsilon_A < \sigma_B \mid \epsilon_B = t\right] &= \sum_{i=0}^{\infty}\sum_{j=0}^{a} P\left[\epsilon_B < \epsilon_A < \sigma_B \mid \epsilon_B = t,\ \tilde{X}_B(t) = i,\ \tilde{W}_A(t) = j\right] \\ &\quad \cdot P\left[\tilde{X}_B(t) = i,\ \tilde{W}_A(t) = j \mid \epsilon_B = t\right]. \end{aligned} \tag{5.28}$$

The first conditional probability on the right hand side of (5.28) can now be simplified. For $i = 0$ or $j = 0$ it is simply 0. For $i,\ j \geq 1$, under the condition of the probability, at time t there are j orders in the ask queue that have been placed before time 0 that have yet to be executed, and there are a total of i orders in the bid queue. Thus, the probability of interest is simply the probability that the j ask orders get executed before the number of orders in the bid queue hits 0. Thus,

$$P\left[\epsilon_B < \epsilon_A < \sigma_B \mid \epsilon_B = t,\ \tilde{X}_B(t) = i,\ \tilde{W}_A(t) = j\right] = P\left[\epsilon_j < \sigma_i\right]. \tag{5.29}$$

Moreover, the second probability on the right hand side of (5.28) can be written as

$$\begin{aligned} P\left[\tilde{X}_B(t) = i,\ \tilde{W}_A(t) = j \mid \epsilon_B = t\right] &= P\left[\tilde{X}_B(t) = i \mid \epsilon_B = t\right] P\left[\tilde{W}_A(t) = j \mid \epsilon_B = t\right] \\ &= P\left[\tilde{X}_B(t) = i \mid \epsilon_B = t\right] P\left[\tilde{W}_A(t) = j\right]. \end{aligned} \tag{5.30}$$

Combining (5.26)-(5.28) and using Tonelli's theorem to interchange the integral and the summation gives us

$$h_{a,b} = \sum_{i=0}^{\infty} \sum_{j=1}^{a} P\left[\epsilon_j < \sigma_i\right] \int_0^{\infty} P\left[\tilde{X}_B(t) = i \mid \epsilon_B = t\right] \cdot P\left[\tilde{W}_A(t) = j\right] g_b^1(t)dt.$$

The quantity $P\left[\tilde{X}_B(t) = i \mid \epsilon_B = t\right]$ can be computed using an analogy with the $M/M/\infty$ queue. The number of orders in the bid queue at the time when the bid order placed at time 0 has executed is simply the number of customers at time t in an initially empty $M/M/\infty$ queue with arrival rate λ and service rate θ, which has a Poisson distribution with mean given by $\lambda^X(t)$ in (5.22).

The quantity $P\left[\tilde{W}_A(t) = j\right]$ is the probability that a pure death process with death rate $\mu + (k-1)\theta(1)$ in state $k \geq 1$ is in state j at time t, given that it begins in state a. The infinitesimal generator of this pure death process is given by (5.24). Hence, by Corollary II.3.5 of Asmussen [2003], $P\left[\tilde{W}_A(t) = j\right]$ is given by (5.23). $\square$

We now have all the necessary results to compute the conditional probabilities we are interested in, all that remains are to do some inversion on the Laplace transform to acquire the probabilities. However this this turned out to be easier said than done, which you will see later.

Chapter 6

Inverse Laplace Transform

Numerically inverting Laplace transforms can be done by using the Fourier-series method which has many different variants. Here we will mainly look at two of these, i.e. the EULER method, based on Euler summation, and POST-WIDDER. Both methods are found in Abate and Whitt [1995].

6.1 Euler Method

The first method is called Euler simply because we use Euler summation, based on the Bromwich contour inversion integral (5.2), which can be written as the integral of a real-valued function of a real variable by choosing a specific contour. If the contour is a vertical line $s = a$ with a large enough such that $\hat{f}(s)$ has no singularities on the line or to the right of it, we obtain

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} e^{st}\hat{f}(s)ds = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} e^{(a+iu)t}\hat{f}(a+iu)du \\ &= \frac{e^{at}}{2\pi}\int\limits_{-\infty}^{\infty}(\cos(ut)+i\sin(ut))\hat{f}(a+iu)du \\ &= \frac{e^{at}}{2\pi}\int\limits_{-\infty}^{\infty}\left[Re\left(\hat{f}(a+iu)\right)\cos(ut) - Im\left(\hat{f}(a+iu)\right)\sin(ut)\right]du \\ &= \frac{2e^{at}}{\pi}\int\limits_{0}^{\infty} Re\left(\hat{f}(a+iu)\right)\cos(ut)du, \end{aligned} \tag{6.1}$$

where $Re(s)$ and $Im(s)$ are the real and imaginary parts of s. The integral (6.1) is then calculated approximately by using the Fourier- series method to replace the integral by a series, corresponding to the trapezoidal rule, with specified discretization error. Instead of calculating an infinite sum we apply Euler summation, thus accelerating the convergence. Another motivation for the need of faster convergence is that the series is nearly alternating which makes the convergence rate a lot slower.

As mentioned above (6.1) is evaluated with the trapezoidal rule. A step size h gives

$$f(t) \approx f_h(t) \equiv \frac{he^{at}}{\pi}Re\left(\hat{f}(a)\right) + \frac{2he^{at}}{\pi}\sum_{k=1}^{\infty}Re\left(\hat{f}(a+ikh)\right)\cos(kht). \tag{6.2}$$

By letting $h = \pi/2t$ and $a = A/2t$ we obtain the nearly alternating series

$$f_h(t) = \frac{e^{A/2}}{2t}Re\left(\hat{f}(\frac{A}{2t})\right) + \frac{e^{A/2}}{t}\sum_{k=1}^{\infty}(-1)^k Re\left(\hat{f}(\frac{A+2k\pi i}{2t})\right). \tag{6.3}$$

We shall now take a look at the discretization error related with (6.3). We begin by replacing the damped function $g(t) \equiv e^{-bt}f(t)$ for $b > 0$ by the *periodic function*

$$g_p(t) = \sum_{k=-\infty}^{\infty} g\left(t + \frac{2\pi k}{h}\right) \tag{6.4}$$

with period $2\pi/h$. We then representing (6.4) by its complex Fourier series

$$g_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikht}, \tag{6.5}$$

where c_k is the kth Fourier coefficient of g_p, i.e.,

$$\begin{aligned} c_k &= \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} g_p(t)e^{-ikht}dt = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \sum_{k=-\infty}^{\infty} g\left(t+\frac{2k\pi}{h}\right)e^{-kht}dt \\ &= \frac{h}{2\pi}\int_{-\infty}^{\infty} g(t)e^{-ikht}dt \\ &= \frac{h}{2\pi}\int_{0}^{\infty} e^{-bt}f(t)e^{-ikht}dt \\ &= \frac{h}{2\pi}\hat{f}(b+ikh). \end{aligned} \tag{6.6}$$

Combining (6.4)-(6.6) yields a version of the Poisson summation formula

$$\begin{aligned} g_p(t) &= \sum_{k=-\infty}^{\infty} g\left(t+\frac{2k\pi}{h}\right) = \sum_{k=-\infty}^{\infty} f\left(t+\frac{2k\pi}{h}\right)e^{-b(t+2\pi k/h)} \\ &= \frac{h}{2\pi}\sum_{k=-\infty}^{\infty} \hat{f}(b+ikh)e^{ikht}. \end{aligned} \tag{6.7}$$

Letting $h = \pi/t$ and $b = A/2t$ in (6.7) gives

$$f(t) = \frac{e^{A/2}}{2t}\sum_{k=-\infty}^{\infty}(-1)^k Re\left(\hat{f}\left(\frac{A+2k\pi i}{2t}\right)\right) - \sum_{k=1}^{\infty} e^{-kA}f((2k+1)t). \tag{6.8}$$

Note that the first term on the right in (6.8) coincides with the trapezoidal approximation in (6.3), hence the second term on the right in (6.8) gives the discretization error for the trapezoidal rule, i.e.,

$$e_d = \sum_{k=1}^{\infty} e^{-kA}f((2k+1)t). \tag{6.9}$$

In probability applications $|f(t)| \leq 1$ for all t which means that the error is bounded by

$$|e_d| \leq \frac{e^{-A}}{1 - e^{-A}}$$

and is approximately equal to e^{-A} when e^{-A} is small enough. Hence if $A = \gamma \log 10$ we have at most a discretization error of $10^{-\gamma}$.

So now we know how large the discretization error will be, however we still need to calculate (6.3) numerically. Since it contains an infinite sum, which also is an alternating series when $Re\left(f((A + 2k\pi i)/2t)\right)$ has constant sign for all k, it makes sense to consider acceleration methods for alternating series. One of the more elementary techniques, Euler summation, are suggested due to its simplicity. According to Abate and Whitt [1995] Euler summation provides adequate computational efficiency for practical purposes.

The method can be described as the weighted average of the last m partial sums by a binomial probability distribution with parameters m and $p = 1/2$. More precisely, let

$$s_n(t) = \frac{e^{A/2}}{2t} Re\left(\hat{f}\left(\frac{A}{2t}\right)\right) + \frac{e^{A/2}}{t} \sum_{k=1}^{n} (-1)^k a_k(t), \tag{6.10}$$

where

$$a_k(t) = Re\left(\hat{f}\left(\frac{A + 2k\pi i}{2t}\right)\right). \tag{6.11}$$

Euler summation is applied to m terms after an initial n so the Euler sum (approximation to (6.3)) is given by

$$E(m, n, t) = \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(t), \tag{6.12}$$

for $s_n(t)$ in (6.10). Thus (6.12) is the binomial average of s_n, s_{n+1}, $\ldots$, s_{n+m} and the overall computation is specified by (6.10)-(6.12).

If you are interested in estimating the error associated with Euler summation you can simply check the difference between successive terms, i.e. $E(m, n+1, t) - E(m, n, t)$, since it is often a good error estimate.

Note that in order for the Euler summation to be effective the coefficients a_k in (6.11) should preferably be of constant sign for sufficiently large k. However this condition is not necessary for the method to work but is significantly speeds up the convergence rate.

6.2 Post-Widder Method

This method is based on the Post-Widder Theorem, which refers to $f(t)$ as the pointwise limit as $n \to \infty$ of

$$f_n(t) = \frac{(-1)^n}{n!} \left(\frac{n+1}{t} \right)^{n+1} f^{\hat{(n)}}((n+1)/t), \tag{6.13}$$

where $f^{\hat{(n)}}$ is the nth derivative of the Laplace transform $\hat{f}$ at s. By differentiating the transform it is clear that $f_n(t) = E\left[f(X_{n,t}\right]$, where $X_{n,t}$ is a random variable with a gamma distribution on $(0, \infty)$ with mean t and variance $t/(n+1)$. Thus $X_{n,t}$ converges in probability to X_t, as $n \to \infty$, with $P\left(X_t = t\right) = 1$. This means that $f_n(t) \to f(t)$ as $n \to \infty$ for all bounded real-valued f continuous at t.

We calculate $f_n(t)$ numerically via a generating function

$$G(z) \equiv \sum_{n=0}^{\infty} a_n(t) z^n = \frac{n+1}{t} \hat{f} \left(\frac{(n+1)}{t}(1-z) \right), \tag{6.14}$$

whose nth coefficient is $f_n(t)$, i.e., $a_n(t) = f_n(t)$. By using the Cauchy contour integral we get

$$f_n(t) = \frac{1}{2\pi i} \int\limits_C \frac{G(z)}{z^{n+1}} dz, \tag{6.15}$$

where C is a circle with radius r. Substituting variables ($z = re^{iu}$) yields the inversion integral

$$\begin{aligned} f_n(t) &= \frac{1}{2\pi r^n} \int\limits_0^{2\pi} G(re^{iu}) e^{-inu} du \\ &= \frac{n+1}{t} \frac{1}{2\pi r^n} \int\limits_0^{2\pi} \hat{f} \left(\frac{n+1}{t}(1-re^{iu}) \right) e^{-inu} du. \end{aligned} \tag{6.16}$$

As for the Euler method we apply the Fourier-series method to acquire the trapezoidal-rule approximation to (6.16) with an explicit error bound. Unlike the Euler method the integral in (6.16) has a finite interval, hence the resulting sum will also be finite and no truncation is necessary.

We apply the discrete Poisson summation formula to (6.16) for any n and obtain the trapezoidal-rule approximation with step size π/n

$$\begin{aligned} f_n(t) &= \frac{n+1}{2tnr^n}\sum_{k=1}^{2n}(-1)^k Re\left(\hat{f}\left(\frac{n+1}{t}(1-re^{\pi ik/n})\right)\right) - e_d \\ &= \frac{n+1}{2tnr^n}\{\hat{f}\left((n+1)(1-r)/t\right) + (-1)^n\hat{f}\left((n+1)(1+r)/t\right) \\ &\quad +2\sum_{k=1}^{n-1}(-1)^k Re\left(\hat{f}\left(\frac{n+1}{t}(1-re^{\pi ik/n})\right)\right)\} - e_d, \end{aligned} \tag{6.17}$$

where the associated error bound e_d can be written as

$$e_d = \sum_{j=1}^{\infty} f_{n+jm}\left(t + \frac{tj2m}{n+1}\right) r^{2jm}. \tag{6.18}$$

Assuming that $|f(t)| \leq 1$ for all t, which is a valid assumption for probability applications, we also have $|f_n(t)| \leq 1$ for all n and t so that

$$|e_d| \leq \frac{r^{2n}}{1-r^{2n}} \approx r^{2n}. \tag{6.19}$$

Given that

$$\phi(u) = \sum_{-\infty}^{\infty} a_k e^{iku} \quad \text{and} \quad a_n = \frac{1}{2\pi}\int_0^{\infty} \phi(u)e^{-inu}du, \tag{6.20}$$

as in this case where $a_n = f_n(t)r^n$ and $\phi(u) = G(re^{iu})$, we can assemble the periodic sequence

$$a_k^p = \sum_{j=-\infty}^{\infty} a_{k+jm} \tag{6.21}$$

with period m. If $|f(t)| \leq 1$ for all t, then $|f_n(t)| \leq 1$ for all n and t, so that $\sum_{k=-\infty}^{\infty} |a_k| < \infty$. The next step is to create the discrete Fourier transform of $\{a_k^p\}$ to acquire

$$\begin{aligned}
\hat{a}_k^p &= \frac{1}{m}\sum_{j=0}^{m-1} a_j^p e^{i2\pi kj/m} \\
&= \frac{1}{m}\sum_{j=0}^{m-1}\sum_{l=-\infty}^{\infty} a_{j+lm} e^{i2\pi jk/m} \\
&= \frac{1}{m}\sum_{j=-\infty}^{\infty} a_j e^{i2\pi jk/m} = \frac{1}{m}\phi\left(\frac{2\pi k}{m}\right) \qquad (6.22)
\end{aligned}$$

and then apply the inversion formula for discrete Fourier transforms. This yields

$$\begin{aligned}
a_k^p &= \sum_{j=0}^{m-1} \hat{a}_j^p e^{-i2\pi jk/m} \\
&= \frac{1}{m}\sum_{j=0}^{m-1} \phi\left(\frac{2\pi j}{m}\right) e^{-2\pi jk/m}. \qquad (6.23)
\end{aligned}$$

The discrete Poisson summation formula is acquired by combining (6.21) and (6.23)

$$\sum_{k=1}^{m} g\left(\frac{2\pi k}{m}\right) e^{-ik2\pi n/m} = m\sum_{k=-\infty}^{\infty} a_{n+km}, \qquad (6.24)$$

which together with (6.16) implies that

$$f_n(t) = \frac{n+1}{tmr^n}\sum_{k=1}^{m} \hat{f}\left(\frac{n+1}{t}(1-re^{ikh})\right) e^{-inkh} - e_d, \qquad (6.25)$$

with $h = 2\pi/m$ and

$$e_d = \sum_{j=1}^{\infty} f_{n+jm}\left(t + \frac{tjm}{n+1}\right) r^{jm}. \qquad (6.26)$$

Let $m = 2n$ in (6.25) and you will get (6.17).

When calculating (6.17) we can make the error suitably small by choosing r small enough. To obtain an accuracy of $10^{-\gamma}$ simply let $r \approx 10^{-\gamma/2}$, however be careful since round off problems increases as r decreases.

We have now indicated how to calculate the approximate function $f_n(t)$ in (6.13), still it is the function $f(t)$ we arr interested in. Moreover, $f_n(t)$ is known to converge slowly towards $f(t)$ as $n \to \infty$ and the computations gets more complicated for larger n. To enhance the accuracy we use a linear combination of terms

$$\tilde{f}_{j,m}(t) = \sum_{k=1}^{m} w(k,\ m) f_{jk}(t), \tag{6.27}$$

e.g., with $m = 6$ and $j = 10$.

It can be shown that the error in (6.13) has the asymptotic form

$$f_n(t) - f(t) \sim \sum_{j=1}^{\infty} c_j(t) n^{-j}. \tag{6.28}$$

Thus one can easily motivate using (6.27) where the weights are chosen to cancel out the leading coefficients $c_j(t)$ in (6.28). General weights for knocking out these coefficients can be written as

$$w(k, m) = (-1)^{m-k} \frac{k^m}{k!(m-k)!}, \tag{6.29}$$

which can be derived from the combinatorial identity

$$\sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} \frac{k^j}{m!} = \begin{cases} 0, & j = 1,\ 2,\ \ldots,\ m-1 \\ 1, & j = 0 \ \text{ and } \ m. \end{cases} \tag{6.30}$$

Thus the final approximation is (6.27).

Chapter 7

Numerical Results

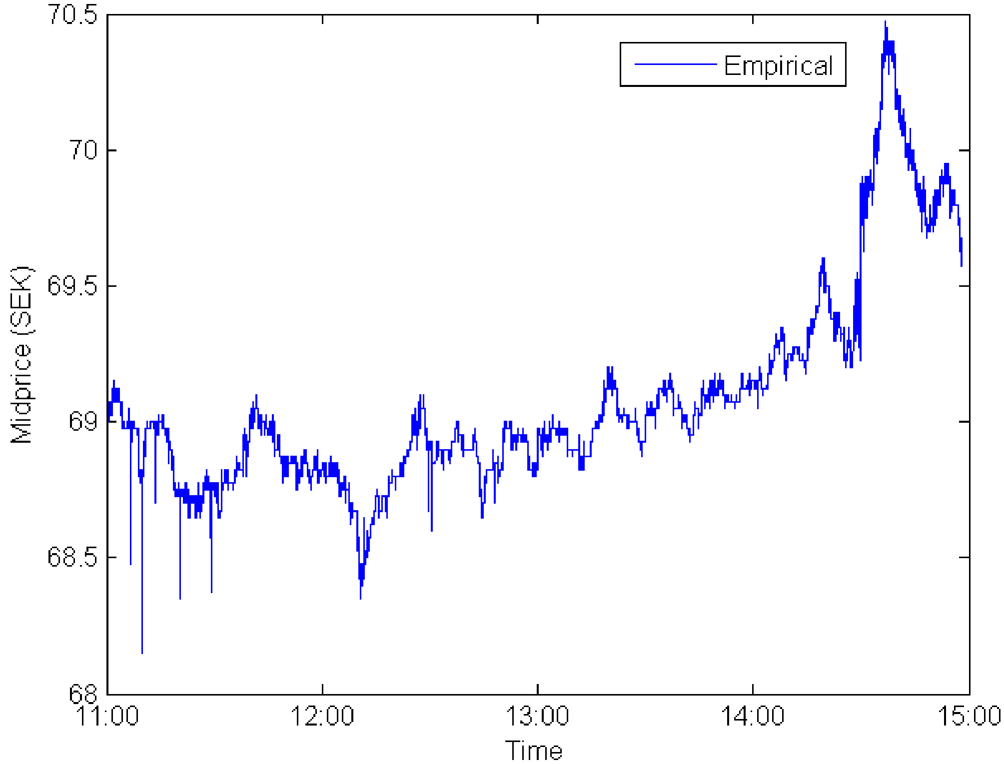
7.1 Long-Term Behavior

The long-term behavior of the order book may not be very important to traders interested in a short time period. However quantities like volatility of the midprice and steady-state shape of the order book are indications on how good the model reproduces the actual data. Some of the previous work done has focused on average properties of the order book (Bouchaud et al. [2002]). The fact that the Markov chain X is ergodic, shown in proposition 1 in section 3.2, implies that expectations such as $E[f(X_\infty)]$ can be computed by simulating over a large horizon and averaging $f(X(t))$ over the simulated path:

$$\frac{1}{T}\int_0^T f(X(t))dt \to E[f(X_\infty)] \quad as\ T \to \infty.$$

To investigate further how the model reproduces the data we chose to consider how the price changes during the day. The empirical data is taken from a time window between 11:00 to 15:00. The model is the simulated for a large number of time steps that will correspond to the same amount of time as in the empirical data.

Figure 7.1: Empirical and simulated midprice for Ericsson B.



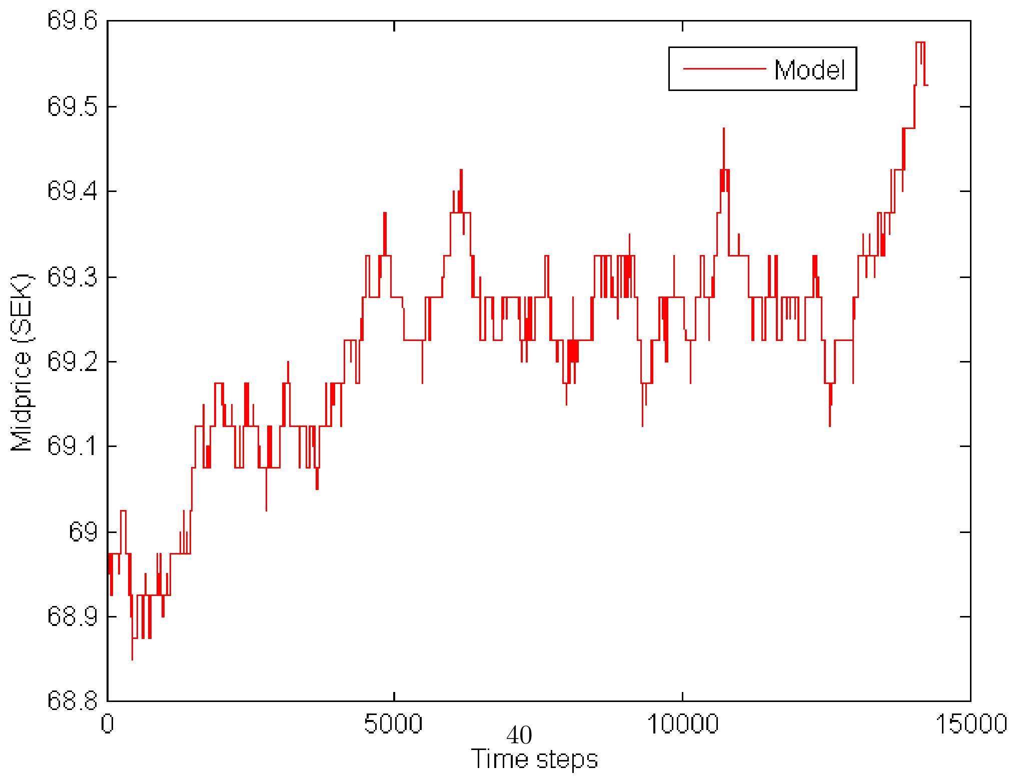
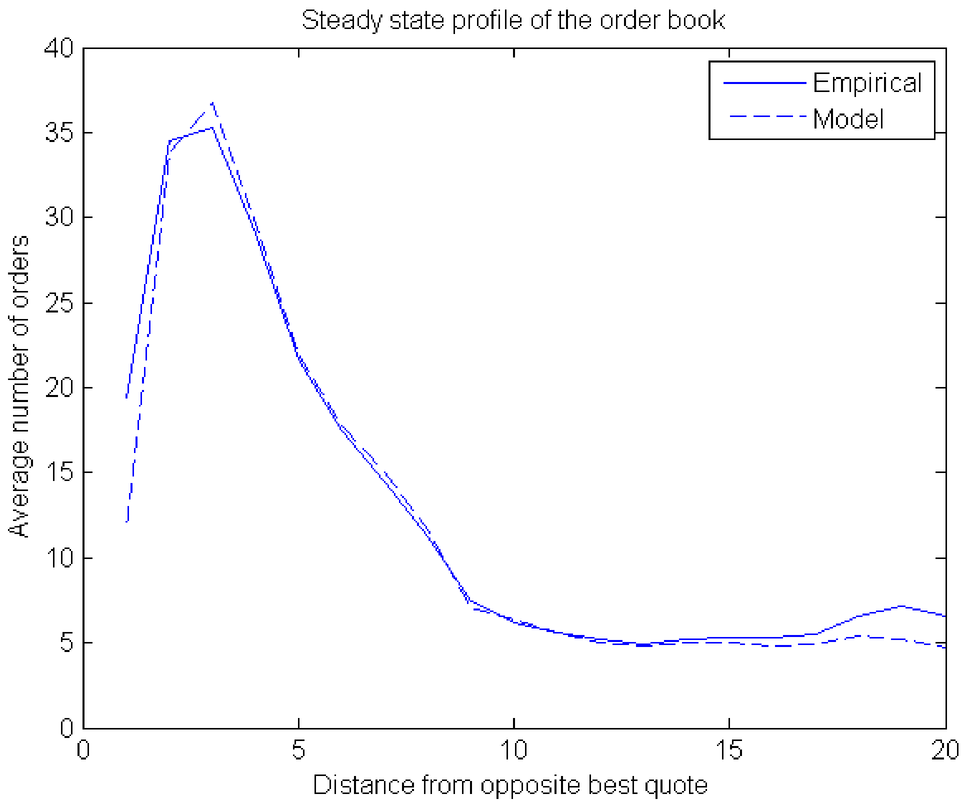


Figure 7.2: Steady state profile of the order book.



7.1.1 Steady-State Shape of the Order Book.

The order book is simulated over a long period of time, i.e. $n = 10^6$ events. The average number of ask (and bid) orders Q_i^A (Q_i^B) at distance i ticks from the opposite best quote is observed and displayed in Figure 7.2 on page 41. Note that figure shows a hump at two ticks from the opposite best quote, the same hump have been observed in other empirical studies (Cont et al. [2010]). This hump is not a result of tuned parameters or additional features such as correlation between order flow or past price fluctuations. Figure 7.2 on page 41 shows both the empirical value as well as the simulated value from the model and they are very similar which indicates that the model is a good approximation for long term behavior.

7.1.2 Volatility.

We define the realized volatility as in Cont et al. [2010]:

$$RV_n = \sqrt{\sum_{i=1}^{n} (log(\frac{P_{i+1}}{P_i}))^2}, \tag{7.1}$$

where n is the number quotes in the time window and P_i represents the midprice of the stock for $i = 1, ..., n$.

A volatility of approximately 5 % was acquired when applying (7.1) to the empirical data, however for the simulated data the same quantity only was 1 %. This can be seen in Figure 7.1 on page 40 where the empirical data shows a lot more fluctuations than the simulated data.

7.2 Conditional Distributions

Conditional distributions are the primary quantities of concern for applications regarding high frequency trading. If you have conditional distributions for the variables characterizing the order book you are able to predict these variables in the short term, thus giving you the opportunity to create trading strategies and optimize trade executions.

7.2.1 One-Step Transition Probabilities.

In order to estimate how well our model predicts the order books short term behavior we will take a look at one-step transition probabilities. Empirical data is compared with transition probabilities. When the number of orders at a given price level changes, the probability of the change being an increase will be inspected.

The probability that the number of orders at distance i from the best ask/bid quote changes from n to $n+1$ when it changes can be found in Cont et al. [2010] and is given by

$$P_i(n) \equiv P[Q_i^A(T_{m+1}) = n+1 \mid Q_i^A(T_m) = n,\ Q_i^A(T_{m+1}) \neq n] = \begin{cases} \frac{\lambda(1)}{\lambda(1)+\mu+n\theta(1)}, & i = 1, \\ \frac{\lambda(i)}{\lambda(i)+n\theta(i)}, & i > 1, \end{cases}$$

where T_m is the time of the mth event of the order book. This expression can be motivated by considering the case $i = 1$. An increase in Q_1^A occurs if a limit order arrives before any of the existing limit orders get canceled or a market order occurs. The arrival rate of a limit order is given by $\lambda(1)$ and the arrival rate of a cancellation or a market order is given by $\mu + n\theta(1)$. Hence the probability of an increase is given by $\lambda(1)/(\lambda(1) + \mu + n\theta(1))$.

The corresponding empirical probability, denoted with a hat to indicate that it is empirical, is given by

$$\hat{P}_i(n) \equiv \frac{\hat{B}_{up} + \hat{A}_{up}}{\hat{B}_{change} + \hat{A}_{change}},$$

where $\hat{B}_{up}$ is the number of times the bid orders increased from one event to the next. $\hat{B}_{change}$ represents the number of times the bid orders changes from one event to the next. Similarly $\hat{A}_{up}$ and $\hat{A}_{change}$ represents the same quantities but for the ask orders.

$P_i(n)$ and $\hat{P}_i(n)$ for $1 \leq i \leq 5$ for Ericsson B are displayed in Figure 7.3 on page 44.

7.2.2 Direction of Price Moves

This subsection was supposed to display the results from using the Laplace transform methods described earlier. Unfortunately, due to lack of time and perhaps knowledge, we were unable to implement the required inversion of the Laplace transforms. According to Cont et al. [2010] the inversion are performed by shifting the random variable X under study by a constant c such that $P[X + c \geq 0] \approx 1$, then inverting the corresponding one-sided Laplace transform by using the methods from Abate and Whitt [1992, 1995]. To find a good shift c use the fact that when the number of orders at the ask is equal to the number of orders at the bid the probability of an increase in midprice is 0.5. The same shift used will then be used for cases when this condition is not satisfied. However, the inversion of the Laplace transform was not conceivable. The results of the simulations can be seen in 7.1 along with the empirically computed probabilities.

Figure 7.3: Probability of an increase in the number of orders at a distance i from the opposite best quote in the next change, for $i = 1, ..., 5$.

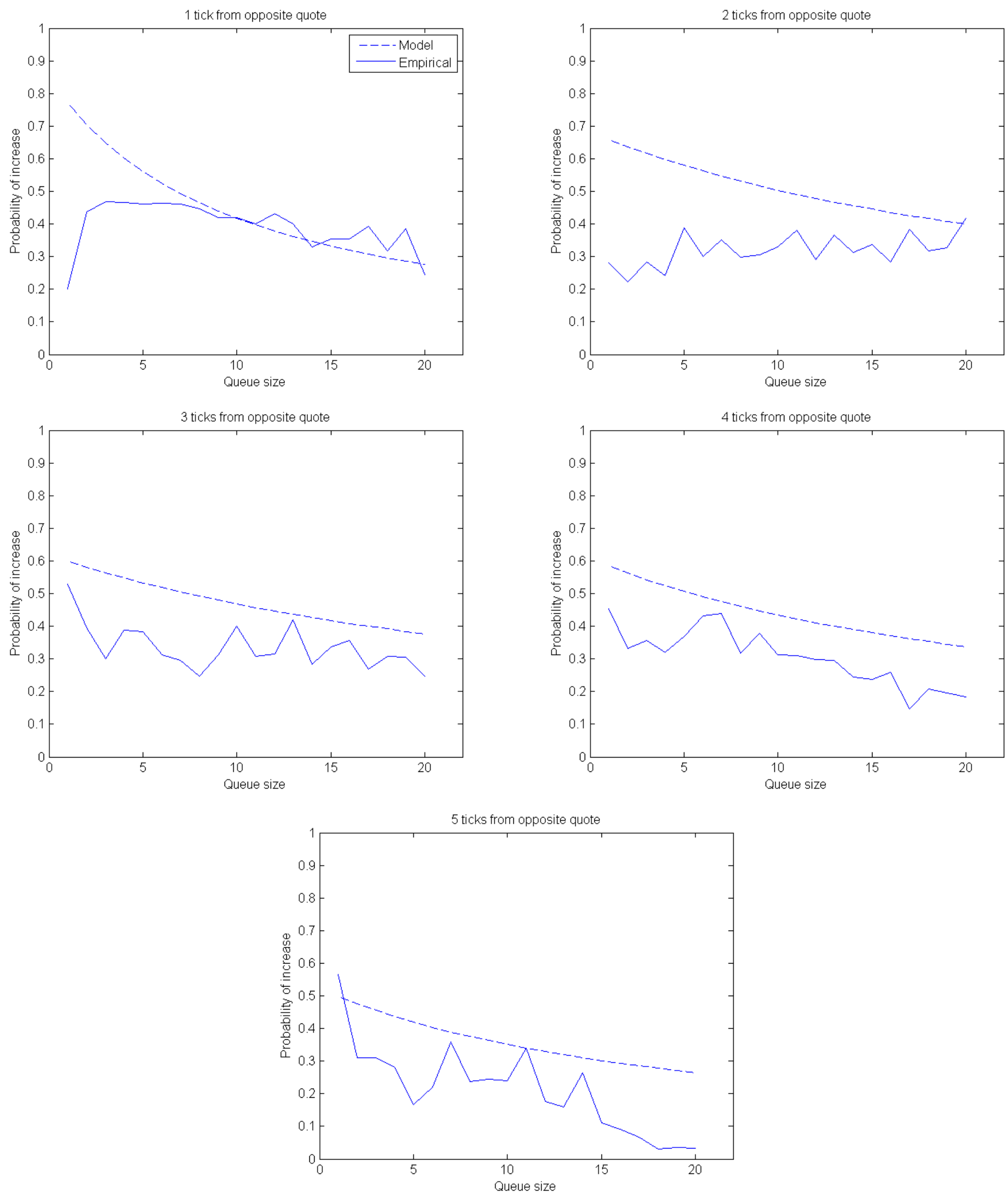


Table 7.1: Probability of an increase in midprice: empirical frequencies (i), simulation results (ii). The numbers in the edge of the table is the size of the bid/ask queue, i.e. position 1-1 means there was one bid order and one ask order.

i).

b \ a	1	2	3	4	5	6	7	8	9	10
1	0.5185	0.4003	0.4732	0.4812	0.5963	0.5411	0.6064	0.6952	0.6100	0.5878
2	0.5292	0.3878	0.6241	0.6561	0.7366	0.6290	0.5758	0.5238	0.5881	0.5701
3	0.4615	0.4847	0.5548	0.5789	0.6281	0.5164	0.5142	0.5782	0.5545	0.6455
4	0.4591	0.4342	0.3986	0.5368	0.5714	0.5650	0.5882	0.5488	0.5355	0.6687
5	0.4671	0.3352	0.3309	0.4886	0.4804	0.4623	0.4384	0.5561	0.5840	0.6376
6	0.4338	0.3261	0.3022	0.4675	0.4826	0.4444	0.4957	0.5615	0.5251	0.6261
7	0.4858	0.3921	0.3515	0.4302	0.3626	0.4737	0.5670	0.5495	0.5169	0.4871
8	0.3763	0.4030	0.4293	0.4852	0.4637	0.4412	0.6039	0.4920	0.7050	0.5289
9	0.4137	0.3875	0.3710	0.4670	0.4545	0.5388	0.5223	0.6138	0.6257	0.5263
10	0.3333	0.4185	0.4934	0.4592	0.6170	0.5503	0.4654	0.5000	0.5255	0.4876

ii).

b \ a	1	2	3	4	5	6	7	8	9	10
1	0.6897	0.6943	0.6136	0.5218	0.5971	0.6737	0.7115	0.7349	0.7111	0.6652
2	0.2818	0.4481	0.5227	0.4666	0.5479	0.6284	0.5293	0.5751	0.5762	0.6087
3	0.2203	0.3067	0.4286	0.4866	0.4753	0.4862	0.5410	0.5020	0.5368	0.4905
4	0.1515	0.2922	0.3804	0.4477	0.4393	0.5085	0.5142	0.4598	0.4848	0.4974
5	0.1542	0.3415	0.4105	0.4041	0.4042	0.4341	0.4652	0.4127	0.4949	0.4276
6	0.2485	0.3451	0.3795	0.3970	0.4384	0.4703	0.4646	0.4808	0.4630	0.4335
7	0.2606	0.3405	0.3503	0.4013	0.4512	0.4510	0.4461	0.4172	0.5251	0.5731
8	0.1818	0.2821	0.2936	0.3695	0.4702	0.4001	0.4722	0.4635	0.4588	0.6298
9	0.1408	0.2532	0.2304	0.3955	0.4361	0.4655	0.4934	0.4384	0.5053	0.4748
10	0.1270	0.3256	0.2929	0.4448	0.4523	0.4526	0.4690	0.4150	0.6143	0.6418

Chapter 8

Conclusions

Since we were unable to implement the Laplace transform methods described in Cont et al. [2010] our general conclusion must be that the method is no good. One of the reasons for using this specific model is that it is easy to calculate the conditional probabilities analytically and that was not possible. It is not obvious how the inversion of the Laplace transforms should be done. Two methods for the inversion are considered in Chapter 6 but it is not clear how they are to be combined with the continued fractions. However, the long term behavior of the order book the model recreates some major properties from the empirical data, namely the steady state shape of the order book but also the midprice follows the same basic upward movement. This means that the model is not all bad although it does not fulfill the most important requirements. In order to predict the short term behavior of the order book, hence being able to earn money on high frequency trading, you have to calculate conditional probabilities, e.g. the probability of an increase in midprice. This can be done by simulating the order book but means a lot of extra computational time compared to analytical Laplace transforms. The main reason for the failure in implementing the model is the lack of instructions on how to do the previously named inversion of the Laplace transformation. Cont et al. [2010] just states that the inversion is performed using the methods presented here in 6. However the methods don't take in to consideration the existence of the continued fractions which is not at all trivial.

To sum up one could claim that this is a fairly good model since it shows the same steady state shape as the empirical data and the short term behavior can be predicted fairly well by simulation. However the aim with this model is to analytically compute conditional probabilities, hence predicting the short term behavior. Since this was not possible the model, or perhaps more precisely the method should be considered unfavorable.

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