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Radial drift invariant in long-thin mirrors

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Abstract: In omnigenous systems, guiding centers are constrained to move on magnetic surfaces. Since a magnetic surface is determined by a constant radial Clebsch coordinate, omnigenicity implies that the guiding center radial coordinate (the Clebsch coordinate) is a constant of motion. Near omnigenicity is probably a requirement for high quality confinement and in such systems only small oscillatory radial banana guiding center excursions from the average drift surface occur. The guiding center radial coordinate is then the leading term for a more precise radial drift invariant I_r , corrected by oscillatory “banana ripple” terms. An analytical expression for the radial invariant is derived for long-thin quadrupolar mirror equilibria. The formula for the invariant is then used in a Vlasov distribution function. Comparisons are first made with Vlasov equilibria using the adiabatic parallel invariant. To model radial density profiles, it is necessary to use the radial invariant (the parallel invariant is insufficient for this). The results are also compared with a fluid approach. In several aspects, the fluid and Vlasov system with the radial invariant give analogous predictions. One difference is that the parallel current associated with finite banana widths could be derived from the radial invariant.

I Introduction

It is essential for plasma confinement that the collisionless particle motion is restricted to the confining region of the magnetic field. Oscillatory “banana” excursions from a magnetic surface is tolerable, but an average radial drift would quickly ruin the confinement. The concept of omnigenity was introduced by Hall and McNamara [1] to analyze the radial particle confinement in mirror machines by a bounce averaged procedure. In their approach, omnigenity is determined in an average sense [2], allowing for oscillatory guiding center excursions from a given magnetic surface. Such banana orbit effects are unavoidable in tokamaks, and are also present in mirror machines and other devices. Conditions for complete vanishing of the radial drift component are harder to satisfy than average omnigenity. Catto and Hazeltine [2], Ryutov and Stupakov [3] and Skovoroda [4] have analyzed such local omnigenous systems with an emphasis on non-axisymmetric mirror machines.

A magnetic surface can be labeled by

$$r_0(x, y, z) = \text{constant}$$

where r_0 is a radial Clebsch coordinate for the magnetic field. In locally omnigenous systems, the radial coordinate $\bar{r}_0$ of the guiding centers is thus constant along the guiding center orbit (bars are used throughout this manuscript to denote guiding center quantities). When this constant is expressed in the form $\bar{r}_0(\mathbf{x}, \mathbf{v}) = \text{constant}$, where $\mathbf{x}$ and $\mathbf{v}$ are the position and velocity of the particle, a constant of motion is identified for locally omnigenous systems. An expression for a radial drift invariant $I_r(\mathbf{x}, \mathbf{v}) = \text{constant}$ is derived in this paper for quadrupolar mirror machines, which reduces to the guiding center radial coordinate $\bar{r}_0$ in locally omnigenous systems. The radial invariant, in combination with the energy ε and the magnetic moment μ , is thereafter used to determine properties of Vlasov equilibria with a distribution function $F(\varepsilon, \mu, I_r)$. A result is that the radial invariant is required to model radial profiles for the density and pressure tensor components. Comparisons are made with approaches based on the parallel

invariant. Comparisons with a fluid model are also made. Several of the fluid model assumptions can be associated with the existence of a radial drift invariant in near omnigenous systems.

The concept of a radial invariant has a bearing on virtually all magnetic confinement schemes. Screw pinches and axisymmetric mirrors are omnigenous and $\bar{r}_0(\mathbf{x}, \mathbf{v})$ is then a constant of motion for these systems. The quadrupolar straight field line mirror vacuum field has four identified constants of motion, which has been used to determine omniguity properties of finite β equilibria [5]. In tokamaks, an approximate radial invariant Ψ_* , essentially the guiding center value of the toroidal invariant, is often used to label drift surfaces and is a basis for the neoclassical transport theory in Ref. [6]. The quantity Ψ_* neglects a small correction associated with the drift and the poloidal field, of little importance for typical axisymmetric toroidal systems. A closed form expression is then obtained for Ψ_* which makes further calculations less demanding. For isomorphic systems with only one helicity, Ψ_* was introduced by Boozer as a constant of motion [15]. A radial invariant for tokamaks has also been derived in Refs. [6-7], which includes the poloidal field drift effects in a Fourier sum, but suffers from a somewhat more complex expression. Presumably, a radial invariant could in principle also be identified for stellarators (compare Appendix A). One purpose of this paper is to introduce a method to derive a radial invariant which potentially could be extended to more general magnetic configurations.

II. Formulas for long-thin mirror equilibria

In this section, some standard formulas (see also Refs. 2-4) are presented which are required for the derivation of the radial invariant. Near the z axis, a quadrupolar long-thin expansion for a vacuum magnetic field $\mathbf{B}_v = \nabla \phi_v$ can be written

$$\phi_v = \tilde{\phi}(z) + A_1(z) \frac{x^2}{2} + A_2(z) \frac{y^2}{2} + \dots, \quad (1a)$$

where $\tilde{\phi}(z) = \int_0^z \tilde{B}(z) dz$ and the condition $\nabla \cdot \mathbf{B}_v = 0$ yield to leading order

$$0 = \tilde{B}'(z) + A_1 + A_2 \quad (1b)$$

For an arbitrary function $g(z)$, which determines the strength of the quadrupolar field, Eq. (1b) is satisfied with

$$A_1 = \frac{g(z) - \tilde{B}'(z)}{2}, \quad A_2 = -\frac{g(z) + \tilde{B}'(z)}{2} \quad (1c,d)$$

where the case $g(z)=0$ corresponds to an axisymmetric field. We obtain

$$B = \tilde{B}(z) + \frac{x^2}{2} \left(A_1' + \frac{A_1'^2}{\tilde{B}} \right) + \frac{y^2}{2} \left(A_2' + \frac{A_2'^2}{\tilde{B}} \right) \quad (2a)$$

where smaller terms are neglected and

$$\hat{\mathbf{B}}_v = \hat{\mathbf{z}} \left[1 - \frac{x^2}{2} \left(\frac{A_1}{\tilde{B}} \right)^2 - \frac{y^2}{2} \left(\frac{A_2}{\tilde{B}} \right)^2 \right] + \hat{\mathbf{x}} \frac{x A_1}{\tilde{B}} + \hat{\mathbf{y}} \frac{y A_2}{\tilde{B}} \quad (2b)$$

The field line equations [i.e. a parameterized curve $\mathbf{x}_B(z) = x_B(z)\hat{\mathbf{x}} + y_B(z)\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ with a tangent parallel to $\mathbf{B}_v$] can be written

$$x_B(z) = [1 + \gamma_1(z)]x_0 + \dots, \quad y_B(z) = [1 + \gamma_2(z)]y_0 + \dots \quad (3a,b)$$

where $x_0(\mathbf{x})$ and $y_0(\mathbf{x})$ are constant along field $\mathbf{B}_v$. Hereby $x_0(\mathbf{x}) = x / (1 + \gamma_1)$ and

$y_0(\mathbf{x}) = y / (1 + \gamma_2)$ can be identified as leading order expressions for Clebsch coordinates

determining the magnetic field by $\mathbf{B}_v = B_0 \nabla x_0 \times \nabla y_0$, where B_0 is a constant. This gives

$$\frac{\tilde{B}}{B_0} = \frac{\partial x_0}{\partial x} \frac{\partial y_0}{\partial y} = \frac{1}{(1 + \gamma_1)(1 + \gamma_2)} \quad (4a)$$

Since $dx_B(z)/dz = \gamma_1' x_0 = B_x / B_z$ and $dy_B(z)/dz = \gamma_2' x_0 = B_y / B_z$, where primes refer to

differentiation with respect to z , we also find that $A_{1,2}$ are determined by

$$\frac{A_1}{\tilde{B}} = \frac{\gamma_1'}{1 + \gamma_1}, \quad \frac{A_2}{\tilde{B}} = \frac{\gamma_2'}{1 + \gamma_2} \quad (4b,c)$$

The arc length

$$s = z + b_1(z) \frac{x^2}{2} + b_2(z) \frac{y^2}{2} + \dots, \quad (5a)$$

is defined by the condition $\nabla s \cdot \hat{\mathbf{B}}_v = 1$, which implies that the gradient is of the form

$$\nabla s = \hat{\mathbf{B}}_v + a_0(z)\nabla x_0 + b_0(z)\nabla y_0, \quad (5b)$$

The ansatz for the arc length in the long-thin approximation yield

$$\nabla s = \hat{\mathbf{B}}_v + \hat{\mathbf{z}} \left\{ \frac{x^2}{2} [b'_1 + (\frac{A_1}{\tilde{B}})^2] + \frac{y^2}{2} [b'_2 + (\frac{A_2}{\tilde{B}})^2] \right\} + \hat{\mathbf{x}}(b_1 - \frac{A_1}{\tilde{B}})x + \hat{\mathbf{y}}(b_2 - \frac{A_2}{\tilde{B}})y \quad (5c)$$

Since $\partial x_0 / \partial y = 0$ and $\partial y_0 / \partial x = 0$ to leading orders, Eqs. (5a-c) imply

$$a_0 = \frac{(b_1 - \frac{A_1}{\tilde{B}})x}{\partial x_0 / \partial x}, \quad b_0 = \frac{(b_2 - \frac{A_2}{\tilde{B}})y}{\partial y_0 / \partial y}, \quad a_0 \frac{\partial x_0}{\partial z} + b_0 \frac{\partial y_0}{\partial z} = \frac{x^2}{2} [b'_1 + (\frac{A_1}{\tilde{B}})^2] + \frac{y^2}{2} [b'_2 + (\frac{A_2}{\tilde{B}})^2] \quad (6a-c)$$

Inspection of Eqs. (6a,b) yield the long-thin formulas

$$a_0 = x_0 \tilde{a}_0(s) \quad \text{and} \quad b_0 = y_0 \tilde{b}_0(s) \quad (7a,b)$$

where

$$\tilde{a}_0 = (1 + \gamma_1)^2 b_1 - (1 + \gamma_1) \gamma'_1, \quad \tilde{b}_0 = (1 + \gamma_2)^2 b_2 - (1 + \gamma_2) \gamma'_2 \quad (7c,d)$$

Eqs. (7c,d) are substituted into (6c), providing the formulas

$$\frac{d[(1 + \gamma_1)^2 b_1]}{dz} = (\frac{d\gamma_1}{dz})^2, \quad \frac{d[(1 + \gamma_2)^2 b_2]}{dz} = (\frac{d\gamma_2}{dz})^2 \quad (7e,f)$$

which alternatively can be written

$$\frac{d\tilde{a}_0}{dz} = -(1 + \gamma_1) \frac{d^2 \gamma_1}{dz^2}, \quad \frac{d\tilde{b}_0}{dz} = -(1 + \gamma_2) \frac{d^2 \gamma_2}{dz^2}. \quad (7g,h)$$

From Eqs. (7g,h) and the expansion for B follows the formula

$$B(x_0, y_0, s) = \tilde{B}(s) - \frac{x_0^2}{2} \frac{d(\tilde{B}\tilde{a}_0)}{ds} - \frac{y_0^2}{2} \frac{d(\tilde{B}\tilde{b}_0)}{ds} \quad (8a)$$

We mention in passing that, when $s=0$ is part of the flux coordinate domain, general expressions

for a_0 and b_0 are (see Appendix A)

$$Ba_0 = \Gamma_1(x_0, y_0) + \int_0^s ds' \left[-\frac{\partial B(x_0, y_0, s')}{\partial x_0} - h_y \right] \quad (8b)$$

$$Bb_0 = \Gamma_2(x_0, y_0) + \int_0^s ds' \left[-\frac{\partial B(x_0, y_0, s')}{\partial y_0} + h_x \right] \quad (8c)$$

where $h_x = \mu_0 \mathbf{j} \cdot \nabla x_0 / (B / B_0)$, $h_y = \mu_0 \mathbf{j} \cdot \nabla y_0 / (B / B_0)$, $h_s = \mu_0 \mathbf{j} \cdot \nabla s / (B / B_0)$ are associated with the plasma current and $\partial \Gamma_1 / \partial y_0 - \partial \Gamma_2 / \partial x_0 = h_s|_{s=0}$. A check shows that Eqs. (7a,b) and (8a) are consistent with (8b,c).

III Guiding center motion

We now turn over to determine properties of the guiding center motion. Drift-equations in the guiding center potential $U = q\phi + \mu B$, where $\mu B = Mv_\perp^2 / 2$ is the perpendicular kinetic energy of the particle and M is the mass, can be written [7]

$$M\Omega_0 \frac{d\bar{x}_0}{dt} = -\frac{\partial \bar{U}}{\partial \bar{y}_0} + 2\sqrt{\varepsilon - \bar{U}} \frac{\partial [\sqrt{\varepsilon - \bar{U}} \bar{b}_0]}{\partial s}, \quad M\Omega_0 \frac{d\bar{y}_0}{dt} = \frac{\partial \bar{U}}{\partial \bar{x}_0} - 2\sqrt{\varepsilon - \bar{U}} \frac{\partial [\sqrt{\varepsilon - \bar{U}} \bar{a}_0]}{\partial s}$$

where $M\Omega_0 = qB_0$, bars denote guiding center values and the arc length motion of the guiding center is determined by

$$\bar{v}_\parallel = \frac{d\bar{s}}{dt} - \bar{a}_0 \frac{d\bar{x}_0}{dt} - \bar{b}_0 \frac{d\bar{y}_0}{dt}$$

and in stationary fields the energy conservation for the guiding center motion

$$\varepsilon = U + \frac{M}{2} \bar{v}_\parallel^2 = \text{constant}$$

These guiding center equations yield the standard form of the drift [7]

$$\mathbf{v}_{\perp, \text{drift}} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\mu}{q} \frac{\mathbf{B} \times \nabla B}{B^2} + \frac{Mv_\parallel^2}{q} \frac{\mathbf{B} \times [(\mathbf{B} \cdot \nabla) \times \mathbf{B}]}{B^4}$$

To leading order in a long-thin expansion with the electric potential $\phi = 0$, the perpendicular drift for a particle bouncing in a long-thin quadrupolar mirror field is determined by

$$d\bar{x}_0 / dt = \bar{y}_0 \omega_2(\bar{s}) \text{ and } d\bar{y}_0 / dt = -\bar{x}_0 \omega_1(\bar{s}) \text{ where } \omega_i(s) = -\frac{2\varepsilon - \mu\tilde{B}}{m\Omega_0} (1 + \gamma_i) \gamma_i'' \text{ with } i = 1, 2 \text{ and}$$

double prime denote second order differentiation with respect to s .

We specialize this to a mirror field satisfying $\tilde{B}(-s) = \tilde{B}(s)$, where $\tilde{B}(s)$ has a local minimum at $s=0$. Then Eq. (4a) shows that $\gamma_2(-s) = \gamma_1(s) \equiv \gamma(s)$, where we may introduce the even and odd functions $\gamma_e(s)$ and $\gamma_o(s)$ and write

$$\gamma(s) = 1 + \gamma_e(s) + \gamma_o(s) \quad (9a)$$

where we choose $\gamma_e(0) = \gamma_o(0) = 0$. Eq (4a) can now be written

$$\tilde{B}(z) = \frac{B_0}{[1 + \gamma_e(z)]^2 - \gamma_o^2(z)} \quad (9b)$$

For the special case of axisymmetric mirrors, we remark that $\gamma_o(z) \equiv 0$ and $\gamma_e(z)$ is negative in the confinement region. Since

$$\omega_2(-s) = \omega_1(s) \equiv \omega(s) = -\frac{2\varepsilon - \mu\tilde{B}}{m\Omega_0}(1 + \gamma)\gamma'',$$

the guiding center motion is determined by the three equations

$$\frac{d\bar{x}_0}{dt} = \bar{y}_0\omega(-\bar{s}) \quad , \quad \frac{d\bar{y}_0}{dt} = -\bar{x}_0\omega(\bar{s}) \quad , \quad \varepsilon = \mu\tilde{B}(\bar{s}) + \frac{M}{2}\left(\frac{d\bar{s}}{dt}\right)^2 \quad (9c,d,e)$$

where it is sufficient for the considered order to limit the analysis to the z axis for the parallel motion. We can reformulate Eqs. (9c,d) by introducing the even and odd functions (compare Refs. [3,4])

$$\omega_e(s) \equiv \frac{\omega(s) + \omega(-s)}{2} = -\frac{2\varepsilon - \mu\tilde{B}}{m\Omega_0} [(1 + \gamma_e)\gamma_e'' + \gamma_o\gamma_o''] \quad (10a)$$

$$\omega_o(s) \equiv \frac{\omega(s) - \omega(-s)}{2} = -\frac{2\varepsilon - \mu\tilde{B}}{m\Omega_0} [\gamma_o\gamma_e'' + (1 + \gamma_e)\gamma_o''] \quad (10b)$$

and cylindrical Clebsch coordinates by $x_0 = r_0 \cos \theta_0$ and $y_0 = r_0 \sin \theta_0$. Eqs. (9c,d) thus yield

$$\frac{d\bar{r}_0}{dt} = -\bar{r}_0\omega_0(\bar{s})\sin(2\bar{\theta}_0) \quad , \quad \frac{d\bar{\theta}_0}{dt} = -\omega_e(\bar{s}) - \omega_o(\bar{s})\cos(2\bar{\theta}_0) \quad (10c,d)$$

IV Exactly omnigenous equilibria

Exactly omnigenous equilibria (within the limits for the long-thin expansion), i.e. $\bar{r}_0 = \text{constant}$, arise if $\omega_0(z) \equiv 0$, i.e. from Eq. (10b)

$$\gamma_o \gamma_e'' + (1 + \gamma_e) \gamma_o'' = 0 \quad (11a)$$

which is the condition by Stupakov and Ryutov [3]. This condition is also equivalent to the condition given by Catto and Hazeltine [their Eq (85)], as well as the formulas by Skovoroda [4].

The guiding center angle coordinate is for an omnigenous case determined by $\bar{\theta}_0 = -\int^t \omega_e dt$.

The SFLM (straight field line mirror) corresponds to the case $\gamma_e = 0$ and $\gamma_o = z/c$, where the constant c determines the scale length of the longitudinal variations of the magnetic field. There are no guiding center drifts in the straight nonparallel magnetic mirror field lines of the SFLM ($\omega_e = \omega_0 = 0$), and both $\bar{x}_0$ and $\bar{y}_0$ are constant, implying that each guiding center bounce back and forth on a single field line. Axisymmetric fields (without a parallel current) are also known to be omnigenous and correspond to $\gamma_o = 0$ and $1 + \gamma_e(z) = \sqrt{B_0 / \tilde{B}(z)}$.

As pointed out in earlier work, see Eq. (85) in Ref. [2], the omnigenicity condition (11a) could be satisfied for a quadrupolar field with an arbitrary $\tilde{B}(z)$. This can be achieved by substituting Eq. (9a) into Eq. (11a) and solving the resulting equation for $\gamma_o(z)$. An alternative direct method to construct an omnigenous quadrupolar equilibrium is to choose a finite odd function $\gamma_o(z) \neq 0$ (finite except at $z=0$) and then solve the initial value problem for $\eta(z) = 1 + \gamma_e$:

$$\eta'' + \frac{\gamma_o''}{\gamma_o} \eta = 0, \text{ with } \eta(0) = 1 \text{ and } \eta'(0) = 0 \quad (11b)$$

Instead of choosing $\gamma_o(z)$, we could alternatively choose an even function $\gamma_e(z)$, where after the odd function is determined from the Cauchy problem

$$\gamma_o'' + \frac{\gamma_e''}{1 + \gamma_e} \gamma_o = 0, \text{ with } \gamma_o(0) = 0 \text{ and } \gamma_o'(0) = 1/c \quad (11c)$$

With $\gamma_o'(0) \neq 0$, an omnigenous quadrupolar field is obtained from (11c) (the case $\gamma_o'(0) = 0$ results in $\gamma_o(z) \equiv 0$, which corresponds to an axisymmetric field, see Ref. [2]).

After an omnigenous equilibrium has been determined, gross MHD stability could be checked with some appropriate criterion, such as the average minimum B criterion or the flute criterion (which is a pressure weighted condition on the field line curvature [9]). It is possible to find equilibria satisfying both the omnigenuity and the flute stability criterion [2]. The SFLM field is an omnigenous marginal minimum B field, see [10]. Skovoroda [4] has earlier pointed out that this omnigenous field is flat in the radial direction. The SFLM field has a minimizing property for the flux tube fanning (the ellipticity of the magnetic surface) [10], but the field gradients near the ends of the confinement region becomes inconveniently large for a coil design corresponding to the ideal SFLM field. More convenient fields for the coil design, resembling the SFLM field in the major part of the confining region, are possible to construct [11], but exact omnigenuity is then hard to combine with gross MHD stability. It is possible to construct realistic coil systems resulting in finite (but small) banana widths and with a resulting field providing flute stability. A price for flute stabilization is a somewhat inconveniently high but still tolerable flux tube ellipticity [11].

V The radial invariant I_r

With a slow drift, i.e. $|\bar{v}_\parallel| \gg |d\bar{r}_0/dt|$ and $|\bar{v}_\parallel| \gg \bar{r}_0 |d\bar{\theta}/dt|$ (apart from a small region around bounce points) and near omnigenuity, a radial invariant can be identified by deriving from Eq. (10c) an expression for the initial value of the radial coordinate

$$\bar{r}_0(0) = \bar{r}_0 e^{\int_0^t \omega_0(\bar{s}) \sin(2\bar{\theta}_0) dt} \approx \bar{r}_0 + \bar{r}_0 \sin(2\bar{\theta}_0) \int_0^t \omega_0[s(t)] dt = \text{constant}$$

where $\dot{s} = \sigma \sqrt{2[\varepsilon - \mu \tilde{B}(s)]/m}$ is the arc length velocity along the magnetic axis and $\sigma = \pm 1$ stands for the sign of the parallel velocity. Based on this and by introducing the bounce time

variable $\tau = \tau(\sigma, s)$, which satisfies $ds / d\tau(\sigma, s) = \dot{s}$, we can identify a radial invariant $I_r(\mathbf{x}, \mathbf{v})$ as a function of the position and velocity of the particle

$$I_r = \bar{r}_0 + \bar{r}_{0,osc}, \quad (12a)$$

and where the last part

$$\bar{r}_{0,osc} = \frac{2\bar{x}_0\bar{y}_0}{\bar{r}_0} \int_0^\tau d\tau' \omega_0[s(\tau')] \quad (12b)$$

is associated with “banana excursions” of the guiding centers. It goes to zero as $\bar{r}_0 \rightarrow 0$ and we choose $\bar{r}_{0,osc}$ to be zero at bounce points. The banana term $\bar{r}_{0,osc}$ is oscillatory, since $\omega_0(s) = -\omega_0(-s)$ is an odd function of the arc length, implying the integral over a longitudinal bounce is zero, i.e. $\langle \bar{r}_{0,osc} \rangle_{bounce} \approx 0$. The oscillatory nature is more transparent in a Fourier representation

$$\bar{r}_{0,osc}(\mathbf{x}, \mathbf{v}) = \frac{2\bar{x}_0\bar{y}_0}{\bar{r}_0} \sum_{n=1}^{\infty} h_n(\varepsilon, \mu) \left[\cos n\pi - \cos\left(\frac{n\pi\tau}{\tau_{\parallel}/4}\right) \right] \quad (12c)$$

where $\tau_{\parallel}$ is the time for a longitudinal bounce back and forth and

$$h_n(\varepsilon, \mu) = \frac{2}{n\pi} \int_0^{\tau_{\parallel}/4} d\tau \omega_0[s(\tau)] \sin\left[\frac{n\pi\tau}{\tau_{\parallel}/4}\right] \quad (12d)$$

For the half bounce $|\tau(\sigma, s)| < \tau_{\parallel}/4$, we choose $s(\pm\tau_{\parallel}/4) \equiv \pm s_t(\varepsilon, \mu)$ as bounce points for the arc length, whereby

$$\tau(\sigma, s) = \begin{cases} \tau_+^+(s) = \int_0^s \frac{ds}{|\dot{s}|} , & 0 < \tau < \frac{\tau_{\parallel}}{4} , s > 0 , \sigma = 1 \\ \tau_-^-(s) = -\int_s^0 \frac{ds}{|\dot{s}|} , & -\frac{\tau_{\parallel}}{4} < \tau < 0 , s < 0 , \sigma = 1 \end{cases} \quad (12e)$$

where $\dot{s} = \sigma|\dot{s}|$ and $\sigma = -1$ for $|\tau(\sigma, s)| < \tau_{\parallel}/4$. The extension of this to the remaining part

$|\tau(\sigma, s)| > \tau_{\parallel}/4$ of the bounce, where $\sigma = -1$, is

$$\tau(\sigma, s) = \begin{cases} \tau_+^-(s) = \frac{\tau_{\parallel}}{2} - \int_0^s \frac{ds}{|\dot{s}|} , & \frac{\tau_{\parallel}}{4} < \tau < \frac{\tau_{\parallel}}{2} , s > 0 , \sigma = -1 \\ \tau_-^-(s) = -\frac{\tau_{\parallel}}{2} + \int_s^0 \frac{ds}{|\dot{s}|} , & -\frac{\tau_{\parallel}}{2} < \tau < -\frac{\tau_{\parallel}}{4} , s < 0 , \sigma = -1 \end{cases} \quad (12.f)$$

At the bounce point $\mu\tilde{B}(s_t) = \varepsilon$, i.e. $s_t(\varepsilon, \mu) = \tilde{B}^{-1}(\varepsilon / \mu)$, and the magnetic field $\tilde{B}$ is assumed to monotonically increase with the distance from the midplane in the confinement region, thereby forcing all guiding centers to cross the midplane. The Fourier expansion of $\omega_o[s(\tau)]$ is then

$$\omega_o[s(\tau)] = \sum_{n=1}^{\infty} \frac{4n\pi}{\tau_{\parallel}} h_n \sin(4n\pi \frac{\tau}{\tau_{\parallel}})$$

from which Eq. (12.a) can be derived with the chosen boundary condition. Eq. (12.c) shows that $\bar{r}_{0,osc}$ and $d\bar{r}_{0,osc} / dt$ are zero (thus continuous) at bounce points. For passing particles in the confinement region, defined here by $|s| \leq s_{max}$ where $\tilde{B}(s_{max}) = \sup \tilde{B}$, we can use the same formulas with the identification $s_t \rightarrow s_{max}$.

The N th order approximation to the radial invariant is the sum with N terms,

$$I_r^N(\mathbf{x}, \mathbf{v}) = \bar{r}_0 + \frac{2\bar{x}_0\bar{y}_0}{\bar{r}_0} \sum_{n=1}^N h_n(\varepsilon, \mu) [\cos n\pi - \cos(\frac{n\pi\tau}{\tau_{\parallel}/4})]$$

where $I_r^0 = \bar{r}_0$ for $N=0$. Fig. 1 shows a representative case (a perturbed SFLM field) of

$I_r^N[\mathbf{x}(t), \mathbf{v}(t)]$ versus $s(t)$, with the initial position $\bar{r}_0(0) = 40$ cm and a device length $2c = 25$ m, and a pitch angle chosen to yield a bounce point at $\tilde{B}(s_t) = 2B_0$. The banana width is about 2.5 cm as revealed by the variation of $I_r^0 = \bar{r}_0[s(t)]$ in Fig. 1, yielding a 6% relative variation for the lowest order invariant, i.e. the guiding center radial coordinate $I_r^0 \equiv \bar{r}_0$. The relative variation for $I_r^N(\mathbf{x}, \mathbf{v})$ decreases with increasing N , and already $N=1$ gives a substantial reduction. For $N=500$, the relative variation (not visible in Fig. 1) are reduced by a factor of 100 compared to the lowest order invariant, i.e. very close to an exact invariant. Some (but negligibly small)

variation in $I_r^N(\mathbf{x}, \mathbf{v})$ remains even for very large N , since the derivation of the expression for the invariant is based on an expansion in the slow drift, i.e. the expression for the invariant relies on the approximation that $x_0(t)$ and $y_0(t)$ vary more slowly than $s(t)$ and $\omega_0[s(t)]$ (which is not the case near bounce points). Due to a small perpendicular drift, the orbit $\bar{r}_0[s(t)]$ deviates slightly from a closed orbit in Fig. 1 (this apply also to all I_r^N , although the deviations from closed curves for $I_r^N[s(t)]$ decrease in a similar manner as the relative variation of $I_r^N(t)$ decreases with N .)

The radial invariant $I_r = \bar{r}_0 - \bar{r}_{0,osc}$ can be simplified to

$$I_r \approx \bar{r}_0 \quad (13a)$$

for nearly omnigenous equilibria, where $\bar{r}_0$ is constant in an average sense, i.e.

$$\langle \bar{r}_0 \rangle_{bounce} = \frac{\oint \bar{r}_0 dt}{\oint dt} = constant \quad (13b)$$

where integrations are over a longitudinal bounce. A neglect of $\bar{r}_{0,osc}$ in the radial invariant corresponds to a zero banana width approximation. The approximation (13a) ought in any case to be accurate if the gyro radii are larger than the banana widths. As a rough estimate

$$h_1(\varepsilon, \mu) \approx \frac{\tau_{\parallel}}{2\pi} \omega_{0,ch}$$

where $\omega_{0,ch}$ is a characteristic value for the odd frequency for $s > 0$. Neglecting the difference between the position for the particle and the guiding center leads to

$$\bar{r}_{0,osc}(\mathbf{x}, \mathbf{v}) \approx -\frac{2}{\pi} \frac{x_0 y_0}{r_0} \omega_{0,ch} \tau_{\parallel} \cos^2\left(\frac{\pi}{2} \frac{4\tau}{\tau_{\parallel}}\right)$$

An estimate of the ratio of the banana width to the gyro radius is therefore

$$\left| \frac{\bar{r}_{0,osc}}{r_g} \right| \approx \frac{r_0}{r_g} \frac{\omega_{0,ch} \tau_{\parallel}}{\pi}$$

which substantially could exceed unity if there is an appreciable $\omega_{0,ch}$.

Occasionally the parallel adiabatic invariant $J_{\parallel}$ has been used for Vlasov equilibria with a distribution function of the form $F(\varepsilon, \mu, J_{\parallel})$, see Ref. [3]. However, near the magnetic axis the parallel invariant can be defined as

$$J_{\parallel} \approx J_{\parallel}^{(0)}(\varepsilon, \mu) \equiv \int_{-s_t(\varepsilon, \mu)}^{s_t(\varepsilon, \mu)} 2\sqrt{\frac{2}{M}} \sqrt{\varepsilon - \mu \tilde{B}(s)} ds$$

Along the magnetic axis, the parallel invariant is thus a dependent invariant (a function of only ε and μ). For this reason, radial density variations cannot conveniently be modeled with a distribution function $F(\varepsilon, \mu, J_{\parallel})$, since a distribution function of the form

$F(\varepsilon, \mu) \approx F[\varepsilon, \mu, J_{\parallel}^{(0)}(\varepsilon, \mu)]$ would to leading order correspond to a density $n(B)$ which only depends on the magnetic field modulus. Another, although minor, concern is that $J_{\parallel}$, as a result of perpendicular drifts, is not exactly conserved away from the magnetic axis.

A flexibility to model radial density profiles can be achieved if the parallel invariant is replaced by the radial invariant in a distribution function of the form $F(\varepsilon, \mu, I_r)$, where (ε, μ, I_r) forms a set of three *independent* invariants.

VI Vlasov equilibria with approximate radial invariant

In this section we illustrate how Vlasov equilibria could be constructed with the approximation

$$I_r \approx \bar{r}_0 \quad (14a)$$

This ought to be reasonable accurate if the radial banana widths are smaller than the density gradient scale length. We consider distribution functions of the form in a small gyro radius expansion

$$F = F(\bar{r}_0, \varepsilon, \mu) = F_c - r_{0,gyro} \frac{\partial F_c}{\partial r_0} \quad (14b)$$

where $F_c = F(r_0, \varepsilon, \mu)$, $\bar{r}_0 = r_0 - r_{0,gyro}$, or more precisely

$$\bar{r}_0(\mathbf{x}, \mathbf{v}) = r_0 - \dot{x}_0 \frac{y_0 |\nabla x_0|^2 + x_0 \nabla x_0 \cdot \nabla y_0}{\Omega r_0 B / B_0} + \dot{y}_0 \frac{x_0 |\nabla y_0|^2 + y_0 \nabla x_0 \cdot \nabla y_0}{\Omega r_0 B / B_0} \quad (15a)$$

The perpendicular part $\mathbf{v}_\perp$ of the velocity $\mathbf{v} = v_\parallel \hat{\mathbf{B}} + \mathbf{v}_\perp$ is, see [7]

$$\begin{aligned} \mathbf{v}_\perp &= \frac{(\dot{x}_0 \nabla y_0 - \dot{y}_0 \nabla x_0) \times \hat{\mathbf{B}}}{B / B_0} \\ &= -\frac{q}{|q|} \frac{v_\perp}{\cos \alpha} \left[\frac{\nabla y_0}{|\nabla y_0|} \times \hat{\mathbf{B}} \sin(\varphi_g + \alpha) + \hat{\mathbf{B}} \times \frac{\nabla x_0}{|\nabla x_0|} \cos \varphi_g \right] \end{aligned} \quad (15b)$$

where we have made the substitutions

$$\dot{x}_0 = -\frac{q}{|q|} v_\perp |\nabla x_0| \sin(\varphi_g + \alpha) \quad , \quad \dot{y}_0 = -\frac{q}{|q|} v_\perp |\nabla y_0| \cos \varphi_g \quad (15c,d)$$

where q is the charge, φ_g is the gyro angle and

$$\sin \alpha = \frac{\nabla x_0 \cdot \nabla y_0}{|\nabla x_0| |\nabla y_0|} \quad (15e)$$

Using $\int d^3\mathbf{v} \equiv \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty dv_\perp v_\perp \oint d\varphi_g$ and that $\dot{x}_0$ and $\dot{y}_0$ are oscillatory in the gyro angle φ_g result in

$$n(r_0, B) = \int d^3\mathbf{v} F_c = \pi \left(\frac{2}{M}\right)^{3/2} B \int_0^\infty d\mu \int_{\mu B}^\infty \frac{d\varepsilon}{\sqrt{\varepsilon - \mu B}} F_c(\varepsilon, \mu, r_0) \quad (16a)$$

$$P_\perp(r_0, B) = \int d^3\mathbf{v} \mu B F_c \quad (16b)$$

where it can be noticed that the density reduces to a function of the form $n(B)$ if $\partial F_c / \partial r_0 = 0$,

i.e. a distribution function $F(\varepsilon, \mu)$ implies a density that only depends on the magnetic field modulus. Formulas for the current are $j_\parallel = 0$ and

$$\mathbf{j}_\perp = \frac{\hat{\mathbf{B}}}{B} \times \frac{\partial P_\perp}{\partial r_0} \nabla r_0 \quad (17a)$$

The formulas above apply to all omnigenous systems with a distribution function $F(\bar{r}_0, \varepsilon, \mu)$

when the scalar potential variations are neglected and the parallel current can be neglected (they

can therefore have a bearing on stellarators). Specializing to long-thin quadrupolar mirrors, with

$\beta = 2\mu_0 P_\perp / B_v^2$, the current can be written

$$\mu_0 \mathbf{j} = \nabla \times \left(-\frac{\beta}{2} \mathbf{B}_v \right) + \frac{1}{2} \frac{\partial \beta}{\partial B} \nabla B \times \mathbf{B}_v \approx \nabla \times \left(-\frac{\beta}{2} \mathbf{B}_v \right) \quad (17b)$$

This approximation is relevant for long-thin mirrors since ∇B is almost parallel to $\mathbf{B}_v$. Thus

$$\mathbf{B} = \left(1 - \frac{\beta}{2}\right) \mathbf{B}_v + \nabla \phi_{pl} \quad (17c)$$

where the condition $\nabla \cdot \mathbf{B} = 0$ yields the Poisson equation $\nabla^2 \phi_{pl} = (\nabla \beta \cdot \mathbf{B}_v) / 2$. The Coulomb integral solution is, see Refs. [5,14],

$$\phi_{pl}(\mathbf{x}) = -\frac{1}{8\pi} \int \frac{dV'}{|\mathbf{x} - \mathbf{x}'|} B(\mathbf{x}') \frac{\partial \beta(\mathbf{x}')}{\partial s'} = -\frac{B_0}{8\pi} \int \frac{ds' dx'_0 dy'_0}{|\mathbf{x} - \mathbf{x}'|} \frac{\partial \beta(\mathbf{x}')}{\partial s'} \quad (17d)$$

with the integration domain $|s'| < s_{\max}$, $x_0'^2 + y_0'^2 < a^2$, where a is the midplane radius and

$a \ll s_{\max}$. Since $B = B(x_0^2, y_0^2, s)$ and $\beta = \beta(x_0^2, y_0^2, s)$ from Eqs. (8a) and (16b), a paraxial

expansion of (17d) can be derived in the form (compare Ref. [12])

$$\phi_{pl}(\mathbf{x}) = \phi_{pl}(s) \Big|_{x_0=y_0=0} + \frac{x_0^2}{2} \cdot \frac{\partial^2 \phi_{pl}(s)}{\partial x_0^2} \Big|_{x_0=y_0=0} + \frac{y_0^2}{2} \cdot \frac{\partial^2 \phi_{pl}(s)}{\partial y_0^2} \Big|_{x_0=y_0=0} + \dots \quad (17e)$$

In case the contribution from $\nabla \phi_{pl}$ to the magnetic field is small, it could be reasonable to use

the approximation $\mathbf{B} \approx \sqrt{1 - \beta} \mathbf{B}_v$ [9], a procedure linked to approximating $B^2 + 2\mu_0 P_\perp$ with a constant.

The flexibility to model radial density profiles with the radial invariant can be illustrated with a distribution function resembling a local Maxwellian distribution:

$$F(\varepsilon, \mu, \bar{r}_0) = n_0(\bar{r}_0) \left[\frac{M}{2\pi k_B T_0(\bar{r}_0)} \right]^{3/2} e^{-\varepsilon/k_B T_0(\bar{r}_0)} g_t(\tilde{B}_t) \quad (18a)$$

Here, $\tilde{B}_t(\varepsilon, \mu) = \varepsilon / \mu$ is the magnetic field modulus at the bounce points, the positive (but

otherwise arbitrary) function $n_0(r_0)$ can be chosen as the density at the $s = 0$, and the function

$T_0(r_0)$ determines radial profiles for the temperature. The role of the function $g_t(\tilde{B}_t)$ is to model variations along B . The density is in this case given by the Abel transform (compare Ref. [13])

$$n(\bar{r}_0, B) = n_0(\bar{r}_0) \frac{B}{2} \int_B^\infty \frac{dB_t}{\sqrt{B_t - B}} \frac{g_t(\tilde{B}_t)}{\tilde{B}_t^{3/2}} \quad (18b)$$

which has the inversion

$$\frac{g_t(\tilde{B}_t)}{\tilde{B}_t^{3/2}} = -\frac{2}{\pi} \frac{d}{dB} \int_{B_t}^\infty \frac{dB}{\sqrt{B - B_t}} \frac{\bar{n}(B)}{B} \quad (18b)$$

where $\bar{n}(B) = n(\bar{r}_0, B) / n_0(\bar{r}_0)$. A property of the Abel transform (18b) is that if $g_t(\tilde{B}_t) = 0$ for $\tilde{B}_t > \tilde{B}_{\max}$, then $n(\bar{r}_0, B) = 0$ for $B > \tilde{B}_{\max}$. It is thus straightforward to model loss cone distributions with confined particles restricted to a region $B < \tilde{B}_{\max}$. There is also flexibility to model various profiles, for instance sloshing ions with a stationary double humped “camel” density distribution along B , see Ref. [13]. With a sufficiently slow dependence on B , the Abel transforms approaches $\bar{n}(B) \approx g_t(B)$.

VII Equilibria with finite banana widths

We now use the more complete formula $I_r = \bar{r}_0 + \bar{r}_{0,osc}$, i.e. with a small gyro radius

$$I_r = r_0 - r_{0,gyro} + \bar{r}_{0,osc}$$

and with a small but finite banana widths as well as small gyro radii we obtain

$$F(\varepsilon, \mu, I_r) = F_c + (\bar{r}_{0,osc} - r_{0,gyro}) \frac{\partial F_c}{\partial r_0}$$

This give as before

$$n(r_0, B) = \int d^3v F_c$$

$$P_\perp(r_0, B) = \int d^3v \mu B F_c$$

$$\mathbf{j}_\perp = \frac{\hat{\mathbf{B}}_v}{B} \times \frac{\partial P_\perp}{\partial r_0} \nabla r_0$$

No finite banana effects appear in the formulas above. However, from Eq. (12c) results a parallel current from the finite banana widths,

$$j_{\parallel}(r_0, \theta_0, s) = \frac{2x_0 y_0}{r_0} \int d^3 v \ q v_{\parallel} h \frac{\partial F_c}{\partial r_0} \quad (19)$$

where

$$h = \sum_{n=1}^{\infty} h_n(\varepsilon, \mu) [\cos n\pi - \cos(\frac{n\pi\tau}{\tau_{\parallel}/4})]$$

and $\tau(\mathbf{x}, \mathbf{v})$ is specified in Eqs. (12e,f). Along the x and y axes, where $x_0 y_0 = 0$ with the given orientation of the magnetic field, the parallel banana current vanishes for a long-thin mirror equilibrium.

VIII Relation to fluid equilibria

A common fluid approach is to assume that the spatial variations of the pressure tensor components are of the form [9]

$$P_{\perp}(r_0, B) \quad (20a)$$

and $P_{\parallel}(r_0, B)$. In a fluid approach, this can be motivated by “gyro isotropy” and an assumption of zero radial component of the current, see Ref. [2], without relying explicitly on the existence of a radial invariant for the guiding center motion.

We have seen that the existence of a radial invariant is consistent with (20a), at least if the electric potential and radial component of the plasma current are neglected. Our analysis can be extended to include an electric potential of the form

$$\phi(r_0, B) \quad (20b)$$

in case the radial component of the current could be neglected. A radial invariant $I_r(\mathbf{x}, \mathbf{v})$ can then for a long-thin quadrupolar system be derived in the same manner as before. The distribution function is assumed to be of the form $F(\varepsilon, \mu, I_r)$, and with small gyro radii and

banana widths, the expressions for the densities of the particles with subscript α (ions or electrons) are of the form

$$n_{\alpha}(r_0, B, \phi) \quad (20c)$$

Quasi neutrality then leads to an implicit function of the form (20b). For such quasi neutral systems we can make the identifications $n_{\alpha}(r_0, B) \equiv n_{\alpha}[r_0, B, \phi(r_0, B)]$, $P_{\parallel}(r_0, B) \equiv P_{\parallel}[r_0, B, \phi(r_0, B)]$ and

$$P_{\perp}(r_0, B) \equiv P_{\perp}[r_0, B, \phi(r_0, B)] \quad (20d)$$

The fluid approach [9] with pressure tensor components of the form (20a) can thus be justified with the existence of a radial invariant and an electric potential of the form (20b), in case the radial current can be neglected and the three independent invariants (ε, μ, I_r) are used for the distribution functions. The Vlasov system is in several aspects similar to the fluid model. A difference is that the banana width effects can be calculated from the radial invariant.

IX Summary and discussion

A time averaged gyro center radial drift across magnetic surfaces for the collisionless motion should be avoided for a magnetic confinement scheme. It seems intuitively clear that there has to exist a radial invariant for particles confined in a magnetic field, but a concern is that the derivation of the invariant is rather involved if the banana excursion should be included. A main reason for this is the complexity (except for some exceptional cases) to determine flux coordinates analytically. A pragmatic method could be to first identify conditions for exact omnigenous equilibria (such as the helical field with the zero radial component of the plasma current), and based on that derive and identify magnetic configurations which are nearly omnigenous. It may be plausible that such a procedure could lead to a radial invariant of the form $I_r = \bar{r}_0 + \bar{r}_{0,osc}$, where $\bar{r}_0$ is the radial flux coordinate of the guiding center at the mean drift surface, with an oscillatory correction $\bar{r}_{0,osc}$ to the mean drift surface. In such a case the guiding

center orbit average $\langle \bar{r}_0 \rangle$ would be constant, and if the gyro center excursions from the mean drift surface are small, the basis for the approximate invariant

$$I_r \approx \bar{r}_0 \quad (21)$$

would have a similar ground as the approximate conservation of the magnetic moment (which is nearly constant after gyro averaging). With the approximation (21), several conclusions can be drawn without a laborious explicit derivation of flux coordinates and gyro center drifts. One conclusion is that a distribution function $F(\varepsilon, \mu, \bar{r}_0)$ models the diamagnetic current arising from pressure gradients, but these three invariants could not model any parallel current. A parallel plasma current is required in axisymmetric tori for the rotational transform. Although a parallel current arise from finite banana widths, there is little freedom to model the parallel current from this, and a fourth independent invariant seems necessary for the distribution function in axisymmetric tori (the toroidal invariant can be used for this). For some other type of magnetic schemes (mirror machines and probably also stellarators where the plasma current can be neglected in the lowest order analysis), it seems sufficient to restrict the equilibrium analysis to a distribution function $F(\varepsilon, \mu, \bar{r}_0)$ with only three invariants.

We have in this paper carried out derivations for the radial invariant for quadrupolar mirrors. The derivations are based on a slow drift and provide an almost exact determination of the radial invariant in the paraxial approximation. A Vlasov treatment with the radial invariant is compared with methods based on the parallel invariant and a fluid approach. It is straightforward to construct nearly local Maxwellian distributions functions with the aid of the radial invariant, and the radial invariant provide a mean to show the correspondence between Vlasov and fluid equilibria. In the appendix, a general result for the arc length variable of the flux coordinate system is obtained, which is useful for the determination of the guiding center motion in a general flux coordinate system. In the appendix, it is also derived a general condition, for any form of the pressure tensor, for local omnigenuity for the case when there is no radial component

of the plasma current. The resulting omnigenous helical system may have bearing on existence of a radial invariant in helical stellarator systems.

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Fig. 1 (O.Ågren et al)

Fig.1 Plot showing variations of N th order radial invariants versus the arc length $s(t)$ over a longitudinal bounce for $N=0,1$ and 500 . The curves are for a long-thin system (the variations along the y axis are in cm while the x axis extension is about $20\ m$). The plot for the guiding center radial coordinate $\bar{r}_0(t)$ reveals a $2.5\ cm$ banana width, corresponding to a relative oscillatory variation in $\bar{r}_0(t)$ of $\delta\bar{r}_0 / \bar{r}_0(0) \approx 6\%$. The variations of the successive approximations for the radial invariant decreases with N , approaching diminishingly small values for large N .

Appendix A General formulas for $\mathbf{a}_0$ and $\mathbf{b}_0$

The arc length is defined by the condition $\nabla s \cdot \hat{\mathbf{B}} = 1$, which implies that ∇s is of the form

$$\nabla s = \hat{\mathbf{B}} + a_0(z)\nabla x_0 + b_0(z)\nabla y_0, \quad (1a)$$

where $\mathbf{B} / B_0 = \nabla x_0 \times \nabla y_0$. From this follows by using $\nabla \times \nabla s = 0$ and Amperes law $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$

$$\begin{aligned} \mu_0 \mathbf{j} = & \left[\frac{\partial(Ba_0)}{\partial s} + \frac{\partial B}{\partial x_0} \right] \nabla x_0 \times \hat{\mathbf{B}} + \left[\frac{\partial(Bb_0)}{\partial s} + \frac{\partial B}{\partial y_0} \right] \nabla y_0 \times \hat{\mathbf{B}} \\ & + \frac{\hat{\mathbf{B}}}{B_0} \left[Bb_0 \frac{\partial(Ba_0)}{\partial s} - Ba_0 \frac{\partial(Bb_0)}{\partial s} + B^2 \frac{\partial a_0}{\partial y_0} - B^2 \frac{\partial b_0}{\partial x_0} \right] \end{aligned} \quad (A1b)$$

This expression splits $\mathbf{j}$ into components parallel and perpendicular to $\mathbf{B}$. The expression for the current must be consistent with $\nabla \cdot \mathbf{j} = 0$. A check shows that this is indeed the case, since we can write

$$\mu_0 \mathbf{j} = \left[\frac{\partial(Bb_0)}{\partial s} + \frac{\partial B}{\partial y_0} \right] \nabla y_0 \times \nabla s - \left[\frac{\partial(Ba_0)}{\partial s} + \frac{\partial B}{\partial x_0} \right] \nabla s \times \nabla x_0 + \left[\frac{\partial(Ba_0)}{\partial y_0} - \frac{\partial(Bb_0)}{\partial x_0} \right] \nabla x_0 \times \nabla y_0 \quad (A1c)$$

By taking the divergence of this equation and using $\nabla s \cdot (\nabla x_0 \times \nabla y_0) = B / B_0$ lead to

$$\frac{\mu_0 \nabla \cdot \mathbf{j}}{B / B_0} = \frac{\partial \left[\frac{\partial(Bb_0)}{\partial s} + \frac{\partial B}{\partial y_0} \right]}{\partial x_0} - \frac{\partial \left[\frac{\partial(Ba_0)}{\partial s} + \frac{\partial B}{\partial x_0} \right]}{\partial y_0} + \frac{\partial \left[\frac{\partial(Ba_0)}{\partial y_0} - \frac{\partial(Bb_0)}{\partial x_0} \right]}{\partial s} \equiv 0 \quad (A1d)$$

which is identically zero. Thus Eq. (A1c) give

$$\frac{\partial(Bb_0)}{\partial s} + \frac{\partial B}{\partial y_0} = \frac{\mu_0 \mathbf{j} \cdot \nabla x_0}{B / B_0} \equiv h_x \quad (A2a)$$

$$\frac{\partial(Ba_0)}{\partial s} + \frac{\partial B}{\partial x_0} = -\frac{\mu_0 \mathbf{j} \cdot \nabla y_0}{B / B_0} \equiv -h_y \quad (A2b)$$

$$\frac{\partial(Ba_0)}{\partial y_0} - \frac{\partial(Bb_0)}{\partial x_0} = \frac{\mu_0 \mathbf{j} \cdot \nabla s}{B / B_0} \equiv h_s \quad (A2c)$$

where the last equation essentially is a dependent equation as a result Eqs. (A1c) and (A1d) and the general solution for $a_0(x_0, y_0, s)$ and $b_0(x_0, y_0, s)$ can be derived from this. When $s=0$ is part of the domain, the expressions for $a_0(x_0, y_0, s)$ and $b_0(x_0, y_0, s)$ are

$$Ba_0 = \Gamma_1(x_0, y_0) + \int_0^s ds' \left[-\frac{\partial B(x_0, y_0, s')}{\partial x_0} - \frac{\mu_0 \mathbf{j} \cdot \nabla y_0}{B/B_0} \right] \quad (\text{A3a})$$

$$Bb_0 = \Gamma_2(x_0, y_0) + \int_0^s ds' \left[-\frac{\partial B(x_0, y_0, s')}{\partial y_0} + \frac{\mu_0 \mathbf{j} \cdot \nabla x_0}{B/B_0} \right] \quad (\text{A3b})$$

where $\partial \Gamma_1 / \partial y_0 - \partial \Gamma_2 / \partial x_0 = h_s|_{s=0}$ and Eqs. (A2a,b,c) yield the relation

$$0 = \frac{\partial h_x}{\partial x_0} + \frac{\partial h_y}{\partial y_0} + \frac{\partial h_s}{\partial s} \quad (\text{A3c})$$

This relation can also be obtained by rewriting (A1c) as

$$\mu_0 \mathbf{j} = h_x \nabla y_0 \times \nabla s + h_y \nabla s \times \nabla x_0 + h_s \nabla x_0 \times \nabla y_0 \quad (\text{A4})$$

whereby (A3c) follows from the condition $\nabla \cdot \mathbf{j} = 0$.

It can be noticed that Eq. (A4a) does not exclude a radial component of the current, since

$$\nabla r_0 \cdot \mu_0 \mathbf{j} = \frac{B}{B_0} \left(\frac{x_0}{r_0} h_x + \frac{y_0}{r_0} h_y \right) \quad (\text{A5})$$

Also, with the choice $x_0 \Gamma_1 + y_0 \Gamma_2 = 0$, the evolution of the guiding center radial coordinate is determined by

$$m \Omega_0 \dot{\bar{r}}_0 = -\frac{1}{\bar{r}_0} \frac{\partial U}{\partial \theta_0} + 2\sqrt{\varepsilon - \bar{U}} \frac{\partial}{\partial s} \left[\frac{\sqrt{\varepsilon - \bar{U}}}{\bar{B}} \int_0^s ds' \left(-\frac{1}{\bar{r}_0} \frac{\partial B}{\partial \theta_0} + \frac{\mu_0 \mathbf{j} \cdot \nabla r_0}{B/B_0} \right) \right] \quad (\text{A6a})$$

This equation applies for any form of the pressure tensor, even to a zero β case. When the radial component of the current vanishes, this simplifies to the Lagrange bracket relation

$$m \Omega_0 \dot{\bar{r}}_0 = \frac{B}{r_0} \left\{ W, \frac{\varepsilon - U}{B^2} \right\}_{s, \theta_0} \quad (\text{A6b})$$

where $W = \int_0^s ds' B$. From this several possibilities for omnigenous equilibria can be derived (compare Catto-Hazeltine, see also Boozer). A special case, which could have bearing on stellarators, is when the electric potential is zero, and then

$$m \Omega_0 \dot{\bar{r}}_0 = -\frac{2\varepsilon - \mu B}{\bar{r}_0} \frac{\partial}{\partial s} \left(\frac{1}{B} \frac{\partial W}{\partial \theta_0} \right) = -\frac{2\varepsilon - \mu B}{\bar{r}_0} \frac{\partial}{\partial s} \left(\frac{\partial W / \partial \theta_0}{\partial W / \partial s} \right)$$

Omnigenuity is thus satisfied either with $\partial W / \partial \theta_0 = 0$, i.e.

$$\frac{\partial B}{\partial \theta_0} = 0 \quad (\text{A7a})$$

which is a condition previously derived in Ref. [2] , see their Eq. (39), or

$$\partial W / \partial s = K(r_0, \theta_0) \partial W / \partial \theta_0, \text{ i.e.}$$

$$\frac{\partial B}{\partial s} - K(r_0, \theta_0) \frac{\partial B}{\partial \theta_0} = 0$$

In case $\partial K / \partial \theta_0 = 0$ the substitution $m(r_0)K(r_0) = k(r_0)$ leads to a helical type of solution

(compare Boozer [15])

$$B = B(r_0, \alpha) \quad , \quad \alpha = k(r_0)s + m(r_0)\theta_0 \quad (\text{A7b})$$

It is possible to extend this result and include an electric potential of the form $\phi = \tilde{\phi}(r_0, B)$.

Further analysis is required to check if omnigenity for a given system is in conflict with existence of nested flux surfaces. Possibly, near omnigenous helical systems (with a radial invariant including banana ripples) could be found for stellarators, but a detailed analysis is needed before that question could be settled.