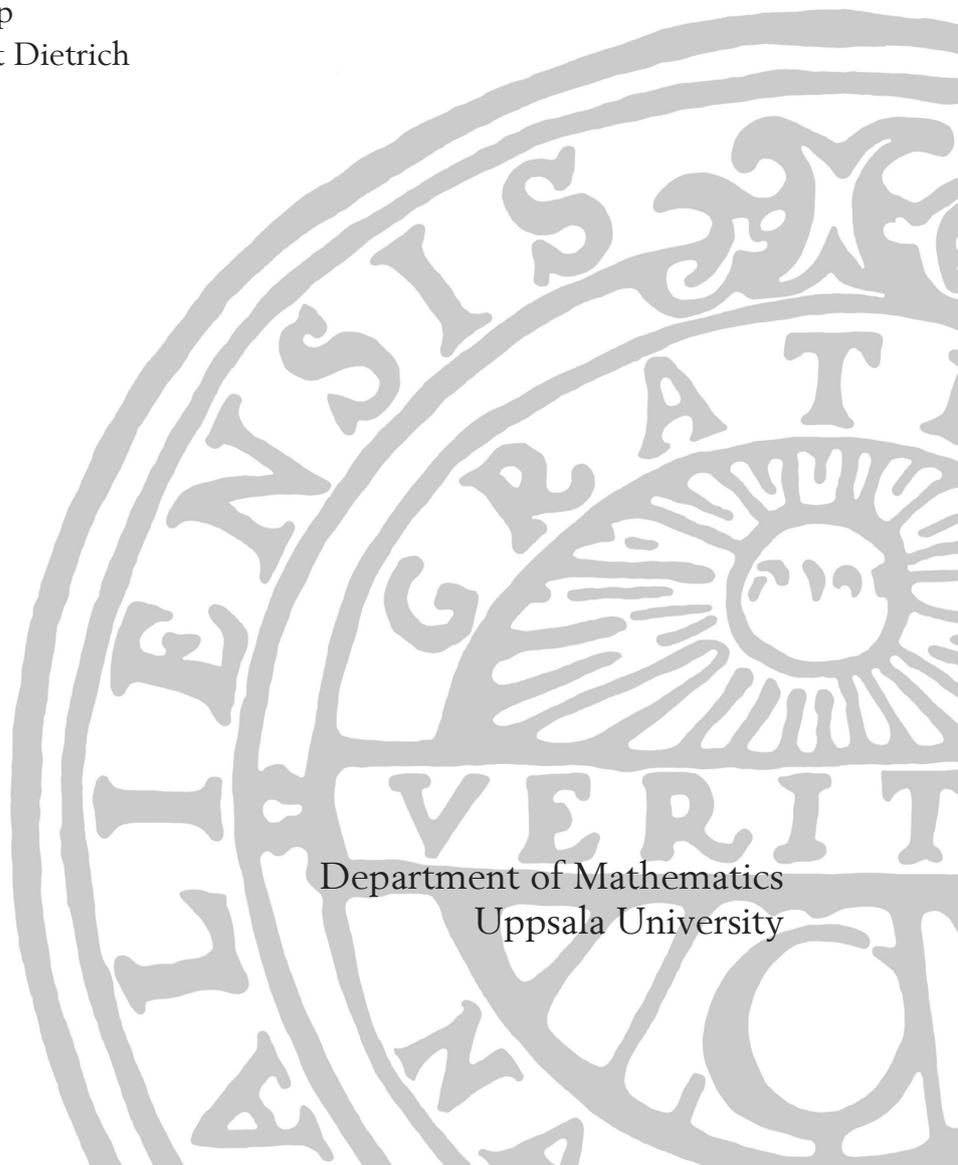UPPSALA
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Description and classification of the category of two-dimensional real commutative division algebras

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Abstract

A description of the category of two-dimensional real commutative division algebras is found using the method of double sign decomposition. This result is then used to classify these division algebras.

Contents

Chapter 1

Introduction

This report is a presentation of some recent results on the topic of finite-dimensional real commutative division algebras, aimed to be as streamlined as possible while still being self-contained enough to be accessible to undergraduate students. The original results, by Dieterich and Darpö (in turn partially based on earlier contributions of, notably, Benkart *et al.*[4]), can be found in [5] and [7].

1.1 Background and goal

The story of real division algebras begins with the real numbers, $\mathbb{R}$, continues with the complex numbers, $\mathbb{C}$, and then with with the quaternions, $\mathbb{H}$, (Hamilton 1843) and the octonions, $\mathbb{O}$, (Graves 1843). As far as many mathematicians are concerned, this is also where it ends. However, while it's true that the first three of these form a classification of all *associative* real division algebras (Frobenius [8], published in 1878), and that the set of all four of these real division algebras classifies those which are *alternative* (Zorn [13], published in 1931), the category of real division algebras in general is to this day far from being classified (let alone the category of the division algebras over an arbitrary field).

The interest in this area of mathematics was for a rather long time around the middle of the last century quite low, the perhaps most important achievement of this period and area being the result that every finite-dimensional real division algebra must have dimension 1, 2, 4 or 8. For the last 30 years or so, however, efforts have been made towards a complete classification, resulting in progress in the form of several partial results. For the above, and more background, see Dieterich and Darpö [5, p. 180–181 and 195–196].

This report will be concerned with two of the most recent results, the first of which is the classification of the finite-dimensional real *commutative* division algebras, a problem which was finally solved by Dieterich and Darpö [5] by completing an old, but seriously flawed, proof of Benkart *et al.*[4].

The second result is the discovery and formalization of a rather general approach to these classification problems; the "double sign decomposition", see Dieterich [7, p. 78–79]. In this report, the original proof of the first result is modified to fit into the more general scheme of the second one. In addition, a couple of other minor modifications have been made, which the author hopes have a beneficial effect on general simplicity and conceptual coherence.

1.2 Content guide

The aim has been that this report should be accessible to an audience as broad as possible, so several of the first chapters are devoted to providing the necessary background in category theory, the theory of division algebras and some less mainstream tools from linear algebra. As a result, the main parts should be fully accessible to readers with no more than knowledge of rather elementary linear algebra (notably the spectral theorem for real, symmetric matrices) and the very basics of abstract algebra (a little group theory in particular) and topology. The one exception to this is Hopf's theorem, which is a slight digression from the main topic of the report and the proof of which moreover requires extensive machinery from algebraic topology, the construction of which is well beyond the scope of this text. If a solid background in algebraic topology is assumed, however, the proof of this theorem is quite elegant and almost trivial, so it is included in an appendix. A reader who already is familiar with the topics of the introductory chapters may want to read the parts which concern the double sign decomposition and otherwise just skim through them for definitions and notation, before moving on to the final chapter which is the core of this report.

The second chapter contains some preliminaries from basic category theory, including a rigorous definition of the terms "description" and "classification", which together provide a framework suitable for the structural considerations of morphisms between division algebras that follow. In the third chapter, real division algebras are defined, and some basic results concerning these are presented, including the tool of double sign decompositions. In particular we mention Hopf's theorem, which reduces the classification problem for finite-dimensional real commutative division algebras to the corresponding one for the two-dimensional ones. In the fourth, it is proved that all two-dimensional real commutative division algebras are, up to isomorphism, "isotopes" of the real algebra of complex numbers. This reduces our problem further, to a matter of real 2×2-matrices and complex multiplication. The rest of the chapter is then devoted to some matrix algebra, and to the application of the double sign decomposition to the two-dimensional real commutative division algebras. Finally, in chapter five, a description of this category is found, and from this a classification of it is derived.

1.3   Acknowledgements

My deep thanks goes to my advisor, Ernst Dieterich, for providing me with the topic and material for this report, as well as patient and thorough guidance during its realization.

Chapter 2

Category-theoretical preliminaries

This chapter presents some general algebraic terminology and definitions which will be extensively used in the following treatment of division algebras. The second section defines the concepts of description (in the rigorous sense recently defined by Dieterich [7]) and classification (which is in widespread use, though with somewhat shifting meanings) using the language of basic category theory, which is in turn introduced in the first section.

2.1 Categories and functors

Category theory may be viewed as one of the most recent leaps towards higher levels of abstraction that mathematics in general, and algebra in particular, has repeatedly undergone historically [9, p. 581]. The original motivation behind the development of category theory was the study of homomorphisms between algebraic structures [2], and this will, for the purpose of this report, indeed be the relevant interpretation of the today abstractly defined concepts of "morphisms" and "objects" respectively. The content of the following definitions is taken from [9, p. 581–588] and [1, p. 8–33], to which the reader is referred for more information concerning category theory.

In the following definitions the concept of a "class" is recurrent. Readers who are unfamiliar with this word may without substantial loss exchange it for "set", as long as they don't give too much thought to Russel's paradox and the like.

Definition 1. *A category $\mathcal{C}$ consists of:*

1. *A class $Ob(\mathcal{C})$ of objects.*

2. *A class $Mor(\mathcal{C})$ of morphisms.*

3. Two class functions from $Mor(\mathcal{C})$ to $Ob(\mathcal{C})$ assigning to each morphism of $\mathcal{C}$ a domain and codomain respectively (one says that a morphism goes from its domain to its codomain).

4. A class function defined on each pair of morphisms (g, f) such that the domain of g equals the codomain of f, which takes such a pair to a morphism $g \circ f = gf$, the composition of f and g, with the same domain as f and the same codomain as g, and which furthermore fulfills:

   (a) $(fg)h = f(gh)$, for all f, g and h in $\mathcal{C}$, whenever the involved compositions are defined.

   (b) For each object C in $\mathcal{C}$, there is an identity morphism e_C, with C being both its domain and codomain, such that $e_C f = f$ and $g e_C = g$, whenever the involved compositions are defined.

More loosely speaking, a category consists of composable morphisms between objects. This composition is associative and has an identity (per object).

Definition 2. A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is a category whose objects and morphisms are objects and morphisms of $\mathcal{C}$, with domain, codomain and composition class functions being the same as the ones in $\mathcal{C}$, but restricted to the morphisms in $\mathcal{D}$. The subcategory $\mathcal{D}$ is called full if all morphisms of $\mathcal{C}$ which have domains and codomains in $\mathcal{D}$ are morphisms of $\mathcal{D}$ too.

Intuitively, a subcategory is a category being the result of (possibly) removing some objects and morphisms from some category.

Definition 3. A category $\mathcal{C}$ is said to decompose into categories $\{\mathcal{C}_i\}_{i \in I}$, where I is some indexing set, if all the $\mathcal{C}_i$ are (necessarily full) subcategories of $\mathcal{C}$ such that every object and morphism of $\mathcal{C}$ belongs to precisely one $\mathcal{C}_i$.

Abusing terminology from set theory, one might say that decomposing a category is to write it as a "disjoint union" of other categories.

One very particular kind of morphism, the isomorphism, will be used throughout the rest of the report (in the form of its common interpretation in more elementary algebraic contexts, certainly already known to the reader), and deserves its own definition.

Definition 4. An isomorphism in a category is a morphism f with domain A and codomain B, which has an inverse morphism f^{-1} with domain B and codomain A, fulfilling $f \circ f^{-1} = e_B$ and $f^{-1} \circ f = e_A$.

Categories may relate to each other through certain structure-preserving correspondences of their objects and morphisms, called functors.

Definition 5. *A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$, written $F : \mathcal{C} \to \mathcal{D}$, consists of two class functions (both denoted by F)*

$$Ob(\mathcal{C}) \ni C \mapsto F(C) \in Ob(\mathcal{D})$$
$$Mor(\mathcal{C}) \ni f \mapsto F(f) \in Mor(\mathcal{D})$$

such that the following holds:

1. *If the morphism f in $\mathcal{C}$ has domain A and codomain B, then $F(f)$ has domain $F(A)$ and codomain $F(B)$.*

2. *If f and g are composable morphisms in $\mathcal{C}$, then $F(fg) = F(f)F(g)$.*

3. *If e_C is the identity morphism of an object C in $\mathcal{C}$, then $F(e_C) = e_{F(C)}$ is the identity morphism of $F(C)$.*

It is easily seen that functors preserve isomorphisms. Functors are also clearly composable, and it is straightforward to define an identity functor (in the case that the involved classes of objects and morphisms are sets, functors are functions, and one may therefore consider the category having the class of all such categories as the class of objects, and the class of functors between them as morphisms).

This suggests a definition of isomorphisms between categories which is consistent with the intuition that isomorphic categories should be structurally identical; an isomorphism between two categories is an invertible functor (that is, in analogy with the case of an inverse morphism, there exists a functor in the opposite direction whose left and right composition with the isomorphism, respectively, equals the identity functor of the respective categories) from one of the categories to the other.

However, the requirements in this definition are too strong for the notion of "structurally identical" that we will want to capture in this report (and in many other applications of category theory for that matter). Indeed, it is common that one thinks of different, but isomorphic, objects within a category as instances of the same object, and that the number of "guises" each object appears in is not of much interest. One may therefore argue that two categories which would be isomorphic if it were not for some isomorphic (now with regard to morphisms between objects!) copies of some objects inside the categories "should" be recognized as structurally identical, but in general they are not so by the isomorphism definition. The following definition fills this gap.

Definition 6. *An equivalence, F, of categories $\mathcal{C}$ and $\mathcal{D}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$, fulfilling:*

1. *For any $A, B \in \mathcal{C}$, the class function from the class of morphisms from A to B to the class of morphisms from $F(A)$ to $F(B)$ induced by F is surjective. One says that F is full.*

2. *For any $A, B \in \mathcal{C}$, the class function from the class of morphisms from A to B to the class of morphisms from $F(A)$ to $F(B)$ induced by F is injective. One says that F is* faithful.

3. *If D is an object of $\mathcal{D}$, then there exists an object C of $\mathcal{C}$ such that $F(C)$ is isomorphic to D. One says that F is* dense.

2.2 The concepts of description and classification

The two-dimensional real commutative division algebras studied in this report, and defined in the next chapter, will be shown to constitute the objects of a category, denoted $\mathcal{C}_2$. The goal of this report is to discern which different (that is, non-isomorphic) kinds of such division algebras there are.

In order to accomplish this goal, we will first find another category of a more familiar kind, but equivalent to $\mathcal{C}_2$. This category will be easier to work with, so that we in particular will be able to select a maximal collection of non-isomorphic objects of it, which will immediately yield exactly one object of each isomorphism class in $\mathcal{C}_2$. In this section, the two parts of this strategy are made precise.

Proposition 1. *Let G be a group acting (on the left) on a set M, and let $x, y \in M$. Then the elements of M are the objects of a category, which we will denote by $_G M$, whose morphisms from x to y are given by the set $\{(g, x, y) : g \in G \wedge gx = y\}$, and where the composition of two morphisms (g, x, y) and (h, y, z) is defined to be (hg, x, z).*

Proof. That proper domains and codomains are assigned when morphisms are composed is clear from the definition. Associativity of compositions follows from the associativity in the group G, and the action of the identity element of G on an object gives rise to the identity morphism on that object. $\square$

In the situation of the previous proposition, we say that $_G M$ is the category of the group action of G on M. The next definition is a slightly generalized version of the one given by Dieterich [7, p. 79].

Definition 7. *A* description *of a category $\mathcal{D}$ is an equivalence of categories from $\mathcal{C}$ to $\mathcal{D}$, where $\mathcal{C}$ decomposes into categories of some group actions.*

It should be noted that, since a group element always has an inverse, every morphism in a group action category is invertible and therefore an isomorphism. Thus descriptions can only exist of categories where all morphisms are isomorphisms.

Definition 8. *A class of objects of a category is said to* exhaust *the category if every object in the category is isomorphic to some object in that class.*

A class of objects of a category is said to be irredundant *if no two different objects of the class are isomorphic.*

A class of objects which is irredundant with respect to a category it exhausts is said to classify *or be a* classification *of that category. In other words, a classification of a category is a class of objects of the category, such that for each object of the category, the class contains precisely one object isomorphic to it.*

Proposition 2. *Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories, and let $\mathcal{E}$ be a classification of $\mathcal{C}$. Then $\mathcal{F}(\mathcal{E})$ is a classification of $\mathcal{D}$.*

Proof. We must show that $\mathcal{F}(\mathcal{E})$ exhausts $\mathcal{D}$ and is irredundant.

Let D be an arbitrary object of $\mathcal{D}$. Since $\mathcal{F}$ is dense, there is some $C \in Ob(\mathcal{C})$ such that we have an isomorphism $g : \mathcal{F}(C) \to D$. Also, since $\mathcal{E}$ is a classification of $\mathcal{C}$, there must be an $E \in \mathcal{E}$ with an isomorphism $f : E \to C$. Functors preserve isomorphisms though, so $\mathcal{E}$ is isomorphic to D via the composition $g \circ \mathcal{F}(f) : \mathcal{E} \to D$. Therefore $\mathcal{F}(\mathcal{E})$ exhausts $\mathcal{D}$.

Next, let $E, E' \in \mathcal{E}$ such that there is an isomorphism $g : \mathcal{F}(E) \to \mathcal{F}(E')$. Since $\mathcal{F}$ is full, there exist $f, h \in Mor(\mathcal{C})$ such that $\mathcal{F}(f) = g$ and $\mathcal{F}(h) = g^{-1}$. Thus $\mathcal{F}(h \circ f) = g^{-1} \circ g = e_{\mathcal{F}(E)} = \mathcal{F}(e_E)$, so that the faithfulness of $\mathcal{F}$ implies that $h \circ f = e_E$. Analogously, $f \circ h = e_{E'}$. These results combined imply that f is an isomorphism (with h being its inverse). But $\mathcal{E}$ is irredundant, so we have that $E = E'$, so that $\mathcal{F}(E) = \mathcal{F}(E')$. Therefore $\mathcal{F}(\mathcal{E})$ is irredundant. $\qquad\square$

If a description of a category has been found, the corresponding classification problem is thus reduced to the problem of finding precisely one member of every orbit of the description's group actions.

Chapter 3

Basic results on real division algebras

In this chapter, the objects and morphisms of the category of real division algebras are defined, and a few basic observations concerning these are given. Also, the double sign block decomposition is introduced.

3.1 Introduction to real division algebras

The content of this section is taken from [7, p. 74-76]. Though this report is concerned solely with real division algebras, the results and definitions of this section are readily generalized to division algebras over arbitrary fields.

Definition 9. *A* real algebra *A* *is a real vector space endowed with a* multiplicative structure, *which is a bilinear map $A \times A \to A$, $(a, b) \mapsto ab$. This operation is also called* the multiplication *of A.*

On a real division algebra, an additional requirement is imposed.

Definition 10. *A* real division algebra, *D,* *is a non-zero real algebra in which, for all $a \in D \backslash \{0\}$,* *both the* left *and* right multiplication maps $\lambda_a : D \to D$, $x \mapsto ax$ *and* $\rho_a : D \to D$, $x \mapsto xa$, *are bijective.*

Thus, for $a, b \in D$, solving the equation $a = \lambda_b(x)$ can be seen as left-dividing a by b.

Observe that at this stage, many of the usual conditions required in algebraic structures, such as associativity of the multiplication, or the existence of an identity element, are *not* assumed. Indeed, as noted by Dieterich [6], unmotivated additional assumptions seem to be a root of confusion even among renowned authors on the subject.

Definition 11. *A* morphism of real division algebras $f : A \to B$ is a nonzero linear map from A to B, such that $f(xy) = f(x)f(y)$, for all $x, y \in A$.

Lemma 1. *Every morphism of real division algebras is injective.*

Proof. Assume that $f : A \to B$ is a morphism of real division algebras which is not injective. Then some nonzero $a \in A$ is in the kernel of f. Then $(f \circ \lambda_a)(x) = f(ax) = f(a)f(x) = 0$, for all $x \in A$. Since λ_a is surjective, f must be the zero map, which contradicts the definition of a real division algebra morphism. $\qquad\square$

Proposition 3. *The real division algebras and their morphisms form a category, which we will denote by $\mathcal{D}$.*

Proof. The morphisms clearly compose associatively when viewed merely as functions, and the composition is therefore associative. The identity function on each real division algebra is obviously an identity morphism (using that real division algebras are non-zero). Furthermore, the composition of two non-zero, injective functions is non-zero. It remains to be shown that the composition of two morphisms preserves multiplication: Let f and g be real division algebra morphisms, such that $h = g \circ f$ is defined. Then, for any x and y in the domain of f, $h(xy) = g \circ f(xy) = g(f(x)f(y)) = g(f(x))g(f(y)) = (g \circ f(x))(g \circ f(y)) = h(x)h(y)$, as desired. $\qquad\square$

One could also have chosen to include the zero-maps in the definition of morphisms, but if so, the next proposition would fail to hold, putting an early end to any hopes of finding a description of the categories with which this report is concerned. The exclusion of the zero-maps is no real loss anyway, since precisely one such map clearly exists between any two division algebras, and therefore these aren't very interesting.

From this point, any morphisms discussed will, unless otherwise explicitly stated, be understood to be division algebra morphisms.

Proposition 4. *The morphisms of the full subcategory $\mathcal{D}_n$ of $\mathcal{D}$ formed by the real division algebras of dimension $n \in \mathbb{N}$ are isomorphisms.*

Proof. By Lemma 1, the morphisms must be injective, and an injective linear tranformation between vector spaces of the same finite dimension must be a bijection. The inverse of this bijection is clearly a morphism too, since it is nonzero and preserves the multiplicative structure of its domain, so there is an inverse morphism, which is what is required of a morphism for being an isomorphism. $\qquad\square$

The following proposition classifies the real division algebras of dimension one.

Proposition 5. $Ob(\mathcal{D}_1) = [\mathbb{R}]$ *(where $[\mathbb{R}]$ denotes the isomorphism class of the real division algebra $\mathbb{R}$).*

Proof. By the previous proposition, it suffices to find a morphism $f : \mathbb{R} \to D$, for an arbitrary $D \in \mathcal{D}_1$. Towards this end, let $d \in D\backslash\{0\}$. Then, since D has dimension one, $d^2 = rd$, for some $r \in \mathbb{R}$. Now let the map $f : \mathbb{R} \to D$ be defined by $f(x) = xr^{-1}d$, for all $x \in \mathbb{R}$. This map is clearly nonzero, and for any $x, y \in \mathbb{R}$, we have (using bilinearity of the multiplication in D for the third equality) $f(xy) = xyr^{-1}d = xyr^{-2}d^2 = (xr^{-1}d)(yr^{-1}d) = f(x)f(y)$, so f is indeed a morphism. $\qquad\square$

The main concern of this report is not real division algebras in general however, but rather the finite-dimensional commutative ones (and in particular the two-dimensional commutative ones). The restrictions of the results of this section to these division algebras are readily derived from their more general counterparts, and are given, without proof, in the proposition to follow.

Definition 12. *1. The full subcategory of $\mathcal{D}$ whose objects are all those in which the multiplication is commutative is denoted by $\mathcal{C}$.*

  2. For all $n \in \mathbb{N}$, the full subcategory of $\mathcal{C}$ whose objects are all those with dimension n is denoted by $\mathcal{C}_n$.

Proposition 6. *1. The morphisms of $\mathcal{C}_n$ are isomorphisms, for any $n \in \mathbb{N}$.*

  2. $Ob(\mathcal{C}_1) = [\mathbb{R}]$.

The previous proposition and the next theorem motivates our interest in $\mathcal{C}_2$.

Theorem 1. *(Hopf's theorem.) Let $2 < n \in \mathbb{N}$. Then $\mathcal{C}_n$ is empty.*

The proof of this theorem, though quite short, uses machinery from algebraic topology that is well beyond the scope of this report, and is therefore omitted from the main text. The interested reader may instead find it in the appendix, or in [10, p. 173-174].

Since there can not be any linear bijections between vector spaces of different finite dimension, and in particular no division algebra isomorphisms, we now see that all that remains in order to classify the category of all finite-dimensional real commutative division algebras is to classify $\mathcal{C}_2$. Doing this is what the greater part of the report is devoted to.

3.2   The double sign decomposition

In the following proposition, the full subcategories of $\mathcal{D}$ formed by all real division algebras of an arbitrary finite dimension greater than one will be shown to decompose in accordance with the signs of the determinants of the

left and right multiplication maps of its objects. This decomposition is a general tool, used for instance in [7, p. 78-79] (from which the material of this section is taken), for studying the structures of these categories, and will in particular be a cornerstone of the description of the two-dimensional real commutative division algebras given in this report.

Definition 13. *For any $n \in \mathbb{N}$, $D \in \mathcal{D}_n$ and $d \in D$, the maps λ and ρ are defined by $\lambda(d) = \lambda_d$ and $\rho(d) = \rho_d$ (where λ_d and ρ_d are the left and right multiplication maps from Definition 10).*

Proposition 7. *Let $2 \leq n \in \mathbb{N}$, and consider an arbitrary subcategory $\mathcal{A}$ of $\mathcal{D}_n$. Then $\mathcal{A}$ decomposes into four full subcategories, where two objects A and B belong to the same subcategory if and only if $\mathrm{sgn} \circ \det \circ \lambda(a) = \mathrm{sgn} \circ \det \circ \lambda(b)$ and $\mathrm{sgn} \circ \det \circ \rho(a) = \mathrm{sgn} \circ \det \circ \rho(b)$, for all $a \in A \backslash \{0\}$ and $b \in B \backslash \{0\}$.*

Proof. To begin with, the maps $\mathrm{sgn} \circ \det \circ \lambda$ and $\mathrm{sgn} \circ \det \circ \rho$ are well-defined because the left and right multiplication maps (the multiplications being with non-zero elements) are both injective for the given a and b, so their determinants are non-zero. For $\mathrm{sgn} \circ \det \circ \lambda(a) = \mathrm{sgn} \circ \det \circ \lambda(b)$ (respective $\mathrm{sgn} \circ \det \circ \rho(a) = \mathrm{sgn} \circ \det \circ \rho(b)$) to either hold or not hold for all $a \in A$ and $b \in B$ simultaneously, the map $\mathrm{sgn} \circ \det \circ \lambda$ (respective $\mathrm{sgn} \circ \det \circ \rho$) must be constant on each of the real division algebras, so this needs to be proved. In this case, the image of each real division algebra under $\mathrm{sgn} \circ \det \circ \lambda$ is either $\{1\}$ or $\{-1\}$, and the same holds for $\mathrm{sgn} \circ \det \circ \rho$, so there are in total four different possibilities for each of the algebras — each possibility giving rise to a candidate for one of the subcategories. We also need to show that there are no morphisms between these different would-be subcategories.

The first part uses a simple topological argument: Let $C \in \mathcal{A}$ and consider $C \backslash \{0\}$, $GL(C)$ and $\mathbb{R} \backslash \{0\}$ viewed as $\mathbb{R}^n \backslash \{0\}$, $GL_n(\mathbb{R})$ (say embedded in $\mathbb{R}^{n^2}$) and $\mathbb{R} \backslash \{0\}$ respectively, equipped with the usual, Euclidean, topology, and $\{1, -1\}$ equipped with the discrete topology. Now, sgn, $\det$, λ and ρ are continuous ($\det$ is a polynomial in the entries of $GL_n(\mathbb{R})$ in the standard basis, and λ and ρ are $\mathbb{R}$-linear). Therefore, the compositions $\mathrm{sgn} \circ \det \circ \lambda$ and $\mathrm{sgn} \circ \det \circ \rho$ are continuous too. Since $n \geq 2$, $C \backslash \{0\}$ is connected, and a continuous map from a connected space to a discrete one must be constant, $\mathrm{sgn} \circ \det \circ \lambda$ and $\mathrm{sgn} \circ \det \circ \rho$ are indeed constant.

As for the second part, let $A, B \in \mathcal{A}$, and let $f : A \to B$ be a morphism (and therefore an isomorphism). Let us now pick any basis of A, as well as the corresponding basis of B obtained as its image under f. This allows us to define the determinant of any linear map between the different (albeit isomorphic) vector spaces A and B as the determinant of the matrix of the map in the chosen bases. In particular, $\det(f)$ and $\det(f^{-1})$ are then defined. For any $a \in A \backslash \{0\}$, $\lambda_{f(a)} = f \lambda_a f^{-1}$, so we see that

$$\det(\lambda_{f(a)}) = \det(f) \det(\lambda_a) \det(f^{-1}) = \det(f) \det(\lambda_a) \det(f)^{-1} = \det(\lambda_a)$$

which implies that $\mathrm{sgn} \circ \det \circ \lambda$ must be the same constant function on both A and B. The corresponding proof for ρ is entirely analogous. This implies that A and B indeed do lie in the same subcategory. $\square$

Definition 14. *Let $a, b \in \{1, -1\}$, $n \in \mathbb{N}$, and let $\mathcal{A}$ be a full subcategory of $\mathcal{D}_n$. Then the full subcategory of $\mathcal{A}$ which consists of the division algebras with images $\{a\}$ and $\{b\}$ under $\mathrm{sgn} \circ \det \circ \lambda$ and $\mathrm{sgn} \circ \det \circ \rho$ respectively is denoted by $\mathcal{A}^{a,b}$. This will often be written as $\mathcal{A}^{1,1} = \mathcal{A}^{++}$, $\mathcal{A}^{1,-1} = \mathcal{A}^{+-}$, $\mathcal{A}^{-1,1} = \mathcal{A}^{-+}$ and $\mathcal{A}^{-1,-1} = \mathcal{A}^{--}$ for short. Subcategories of this form are called* double sign blocks.

For the rest of this report we will only consider the case $\mathcal{A} = \mathcal{C}_2$. Then, because of commutativity, $\lambda = \rho$ so that $\mathcal{C}_2^{+-}$ and $\mathcal{C}_2^{-+}$ are in fact empty, and the only interesting double sign blocks are $\mathcal{C}_2^{++}$ and $\mathcal{C}_2^{--}$.

Chapter 4

Isotopes, and some matrix algebra

From this point on, all division algebras considered will be members of $\mathcal{C}_2$, and any division algebra discussed will be assumed to satisfy this additional property even when this is not explicitly stated.

4.1 Isotopes of the real algebra of complex numbers

The first step towards the description of the double sign blocks of $\mathcal{C}_2$ will be the exploration of how division algebras can be modified into so-called "isotopes". Clearly $\mathbb{C}$ may be seen as a division algebra, with basis $\{1, i\}$ (for instance), and the isotopes of this division algebra will prove to be of particular interest, since it will be shown that every division algebra of $\mathcal{C}_2$ can, up to isomorphism, be written on that form! Calculations involving several different division algebras will follow, so some more precise (though inevitably also a bit more cluttered) notation ought to be established in order to avoid confusion: Let us from here on denote the multiplication operation of a general division algebra C by "$\star^C$", and in the special case $C = \mathbb{C}$, let us also write $\star^{\mathbb{C}}$ as "$\cdot$". The content of this section is, except for Lemma 2, taken from [5, p. 183-184].

Definition 15. *If C is a division algebra, and $\alpha \in GL(C)$, then the algebra C_α is defined to be the real vector space C equipped with the multiplication operation $C \times C \to C$, $(a, b) \mapsto \alpha(a) \star^C \alpha(b)$. C_α is called the* isotope *of C with respect to α, and is easily seen to be a division algebra. We also denote the operation $\star^{C_\alpha}$ by $\star^C_\alpha$ (and if $C = \mathbb{C}$, $\cdot^{\mathbb{C}_\alpha}$ by $\cdot^{\mathbb{C}}_\alpha$).*

Proposition 8. *Let $C \in Ob(\mathcal{C}_2)$. Then there exists an automorphism $f \in GL(\mathbb{C})$ such that $C \simeq \mathbb{C}_f$.*

Proof. As a first step, we prove that there exists a $\phi \in GL(C)$, such that $C_\phi \simeq \mathbb{C}$:

Take any $c \in C \backslash \{0\}$ and let $\phi = \rho_c^{-1}$. For all $x \in C$ we then have that

$$x \star_\phi^C (c \star^C c) = \phi(x) \star^C c = \rho_c(\phi(x)) = x$$

so C_ϕ has a unity element. The first step is then an immediate consequence of the following lemma, taken from [10, p. 174].

Lemma 2. *Every division algebra in $\mathcal{C}_2$ which also has a unity is isomorphic to $\mathbb{C}$.*

Proof. Assume that $C \in \mathcal{C}_2$ is a division algebra with unity, denoted by e, and pick any element $j_1 \in C$ which is not a scalar multiple of e (possible since C is two-dimensional). Let $j_1^2 = ae + bj_1$ for some $a, b \in \mathbb{R}$. Now let $j_2 = j_1 - be/2$, so that $j_2^2 = (a + b^2/4)e$. By our assumption on j_1, j_2 is not a scalar multiple of e either. This implies that $j_2^2 = -c^2 e$, for some $c \in \mathbb{R}$, since otherwise we would have $j_2^2 = c^2 e \implies (j_2 + ce)(j_2 - ce) = 0 \implies j_2 \in \{ce, -ce\}$, for some $c \in \mathbb{R}$. Finally, let $j = j_2/c$, so that $j^2 = -e$. Obviously the linear map from C to $\mathbb{C}$ taking e to 1 and j to i is an isomorphism. $\square$

The second and last step of the proof of Proposition 8 requires the next lemma, which is the first of two "isomorphism juggling" lemmas, which will be used several times in this chapter and the next to obtain isomorphisms between different isotopes of a division algebra. The second "isomorphism juggling" lemma is treated in the next section.

Lemma 3. *Let $C, D \in \mathcal{C}_2$, $\phi \in GL(C)$ and let $f : C_\phi \to D$ be an algebra isomorphism. Then $f : C \to D_{f\phi^{-1}f^{-1}}$ is an algebra isomorphism too.*

Proof. The map $f : C \to D_{f\phi^{-1}f^{-1}}$ is of course still a linear bijection, but we need to show that it preserves multiplication: For any $x, y \in C$,

$$\begin{aligned} f(x \star^C y) &= f(\phi^{-1}(x) \star_\phi^C \phi^{-1}(y)) \\ &= f\phi^{-1}(x) \star^D f\phi^{-1}(y) \\ &= f\phi^{-1}f^{-1}(f(x)) \star^D f\phi^{-1}f^{-1}(f(y)) \\ &= f(x) \star_{f\phi^{-1}f^{-1}}^D f(y) \end{aligned}$$

as required. $\square$

Take $D = \mathbb{C}$ in the above lemma to see that there is even an isomorphism between C and some isotope of $\mathbb{C}$, as promised. $\square$

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4.2 Conjugation and rotation matrices

The first three lemmas in this section are taken from [5, p. 183]. As we have just seen, the isotopes of the real algebra of complex numbers constitute, up to isomorphism, all two-dimensional real commutative division algebras, so the study of these has, in essence, been reduced to a matter of complex and 2×2-matrix multiplication. In order to simplify the calculations to follow, a few definitions and conventions should be established. Let us agree that matrix multiplication binds stronger than complex multiplication, and let us allow abusing notation by identifying complex numbers with their real coordinate columns in the basis $\{1, i\}$, so that complex numbers can in particular be multiplied by real 2×2-matrices in the usual way. The following definitions will also prove convenient.

Definition 16. *For any $A \in GL_2(\mathbb{R})$, define the automorphism $\underline{A} \in GL(\mathbb{C})$ by $\mathbb{C} \ni x \mapsto Ax$.*

Definition 17. *For any $A \in GL_2(\mathbb{R})$, define $\mathbb{C}_A$ to be $\mathbb{C}_{\underline{A}}$.*

In proceeding from here, two kinds of matrices which interact in a nice way with complex multiplication will be of particular interest, namely the conjugation and rotation matrices. In this section, a few lemmas concerning these matrices will be presented. The easy three first of these lemmas will be used without further comment throughout the rest of this report.

Definition 18. *The* conjugation matrix, K, *is defined by*

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Lemma 4. *For any $x, y \in \mathbb{C}$, $K(x \cdot y) = Kx \cdot Ky$.*

Proof. It is easy to see that K acts on complex numbers by conjugating them, and conjugation of complex numbers is multiplicative. $\square$

Note also that $K^{-1} = K$.

Definition 19. *The* rotation matrix of angle $\alpha \in \mathbb{R}$, R_α, *is defined by*

$$R_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

A matrix being a rotation matrix of some angle is called a rotation matrix, or simply a rotation.

Noting that matrix multiplication of R_α with a complex number is the same as complex multiplication of $e^{\alpha i}$ with that number, we obtain the following lemma.

Lemma 5. *For any $x, y \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$, $R_{\alpha+\beta}(x \cdot y) = R_\alpha x \cdot R_\beta y$.*

Proof. $R_{\alpha+\beta}(x \cdot y) = e^{(\alpha+\beta)i} \cdot (x \cdot y) = e^{\alpha i} \cdot x \cdot e^{\beta i} \cdot y = R_\alpha x \cdot R_\beta y.$ $\qquad\square$

Lemma 6. *Let $R = R_\alpha$ be an arbitrary rotation matrix. Then $RK = KR^{-1}$.*

Proof. This follows immediately from

$$KRK = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{pmatrix} = R_{-\alpha} = R^{-1}.$$

$\qquad\square$

The following lemma is the second of the "isomorphism juggling" lemmas, to be used several times in the next chapter.

Lemma 7. *Let $G \in GL_2(\mathbb{R})$ such that $Gw = z \cdot w$ for some z and all w in $\mathbb{C}$. Then, for any $A \in GL_2(\mathbb{R})$, $\underline{G}^{-2} : \mathbb{C}_A \to \mathbb{C}_{G^{-1}AG^2}$ is an isomorphism.*

Proof. $\underline{G}^{-2}$ is clearly a linear bijection, and for any $x, y \in \mathbb{C}$, we have $G^{-2}(x \cdot_A y) = z^{-2} \cdot (Ax \cdot Ay) = z^{-1} \cdot Ax \cdot z^{-1} \cdot Ay = G^{-1}Ax \cdot G^{-1}Ay = G^{-1}AG^2G^{-2}x \cdot G^{-1}AG^2G^{-2}y = G^{-2}x \cdot_{G^{-1}AG^2} G^{-2}y$, so the division algebra isomorphism conditions are fulfilled. $\qquad\square$

One may note that, by polar decomposition of complex numbers (not to be confused with the, albeit somewhat similar, polar decomposition of matrices introduced in the next section), the matrix G in the above lemma must be of the form rR, for some rotation matrix R, and $r \in \mathbb{R}$.

4.3 The polar decomposition

As explained in the previous section, among the invertible real 2×2-matrices, rotation matrices are in this setting of particular interest due to how nicely they interact with complex multiplication. In light of this, the reader probably won't find it hard to believe that it would be of use to be able to factor out the "rotation part" of a real invertible 2×2-matrix. The main content of this section is a proof, if somewhat sketchy, which shows us a way to do precisely this, a so called "polar decomposition", for those 2×2-matrices which have positive determinant (the result may be generalised in several ways if required, see [11, p. 412]).

When the "rotation part" of a matrix is factored out, the other factor turns of to be of the special kind defined as follows.

Definition 20. *A real $n \times n$-matrix P is called* symmetric *if $P = P^T$, and* positive definite *if for any $x \in \mathbb{R}^n\backslash\{0\}$, $x^T Px > 0$. The set of all real 2×2-matrices fulfilling both conditions will be denoted by Pds_2.*

The proof of the main result hinges on a couple of well known facts from elementary linear algebra, which are stated here without proof, as well as a few perhaps less well known ones, given with proof.

Definition 21. *A real matrix $n \times n$-matrix B is called* orthogonal *if $BB^T = I$ (where I is the $n \times n$ identity matrix).*

Theorem 2. *(Spectral theorem for real matrices.) Let S be a real, symmetric $n \times n$-matrix. Then there is an orthonormal eigenbasis of S. Equivalently, there is an orthogonal real $n \times n$-matrix U such that $U^{-1}SU = D$, where D is diagonal. (See [11, p. 104].)*

Proof. Proof omitted. $\square$

Proposition 9. *The members of $SO_2(\mathbb{R})$ (the set of all real 2×2-matrices which are orthogonal and have determinant 1) are precisely the rotations. (See [11, p. 68].)*

Proof. Proof omitted. $\square$

Lemma 8. *Let $P \in Pds_2$ and let B be a real, orthogonal 2×2-matrix. Then the following hold.*

1. *P has only positive eigenvalues (and thus in particular positive determinant).*

2. *$\det(B) \in \{1, -1\}$.*

3. *$B^{-1}PB \in Pds_2$, and $P^{-1} \in Pds_2$.*

Proof.   1. Let μ be an arbitrary eigenvalue of P, and v a corresponding member of an orthonormal eigenbasis. Then $\mu = v^T P v > 0$.

2. $(\det(B))^2 = \det(B)\det(B^T) = \det(I) = 1 \implies \det(B) \in \{1, -1\}$.

3. $B^{-1}PB$ is symmetric because $(B^{-1}PB)^T = B^T P^T B^{-1^T} = B^{-1}PB$, and positive definite because for any $x \in \mathbb{R}^2 \backslash \{0\}$, $x^T B^{-1}PBx = (Bx)^T P(Bx) > 0$, where the inequality follows from P being positive definite.

   The inverse P^{-1} is symmetric because $P(P^{-1})^T = (P^{-1}P)^T = I$. It is positive definite because for any $x \in \mathbb{R}^2 \backslash \{0\}$, $x^T P^{-1} x = (P^{-1}x)^T P(P^{-1}x) > 0$.
   $\square$

In particular the third part of the previous lemma will be used several times also in the next chapter. The proof of part one of the next lemma is an elaborated version of the one found in [3].

Lemma 9. *Let $P \in Pds_2$. Then:*

1. *There is a unique positive definite symmetric square root of P, $\sqrt{P} \in Pds_2$, fulfilling $\sqrt{P}^2 = P$. Also $\sqrt{P} = B^T\sqrt{D}B$, for some real diagonal matrix $\sqrt{D}$ and orthogonal matrix B.*

2. $\sqrt{P}^{-1} = \sqrt{P^{-1}}$.

Proof. 1. **Existence:** By the spectral theorem, we may write $P = B^{-1}DB$, where $D = (d_{i,j})$ is diagonal and B is orthogonal. The lemma is easily seen to hold for those matrices which are diagonal — just take the square roots of each of the diagonal entries to obtain the square root matrix. Thus let $\sqrt{D} = (\sqrt{d_{i,j}})$ be the matrix with entries the square roots of the corresponding entries of D. Clearly $\sqrt{P} = B^{-1}\sqrt{D}B$ fulfills $\sqrt{P}^2 = P$.

Uniqueness: Let $B_1^{-1}\sqrt{D_1}B_1$ and $B_2^{-1}\sqrt{D_2}B_2$ be any two positive definite symmetric matrices, decomposed in accordance with the spectral theorem, fulfilling the square root property with respect to P, and let μ^2, with $\mu > 0$, be an arbitrary eigenvalue of P (possible by lemma 9.1), with v being an arbitrary member of its eigenspace. Clearly the eigenspace of μ^2 with respect to $D_{1,2}$ equals that of μ with respect to $\sqrt{D_{1,2}}$, which implies that the eigenspace of μ^2 with respect to $B_{1,2}^{-1}D_{1,2}B_{1,2}$ equals that of μ with respect to $B_{1,2}^{-1}\sqrt{D_{1,2}}B_{1,2}$. That, by hypothesis, $B_1^{-1}D_1B_1 = P = B_2^{-1}D_2B_2$ then implies that the two square roots agree on the eigenspace of μ^2 with respect to P. It's easily seen that the eigenspaces with respect to P together span $\mathbb{R}^2$, so, since μ^2 was arbitrary, it must be that the square roots are equal.

2. Let $P = B^TDB$, where $D = (d_{i,j})$ is diagonal, and B is orthogonal. Then $\sqrt{P}^{-1} = B^{-1}(\sqrt{d_{i,j}})^{-1}B = B^{-1}(\sqrt{d_{i,j}^{-1}})B = \sqrt{P^{-1}}$. $\qquad\square$

Proposition 10. *(Polar decomposition.) Let $A \in GL_2(\mathbb{R})$ be a matrix with positive determinant. Then $A = PR$, for uniquely determined matrices $P \in Pds_2$ and $R \in SO_2(\mathbb{R})$.*

Proof. This proof is taken from [12, p. 4].

Existence: We have $AA^T \in Pds_2$ because $(AA^T)^T = (A^T)^TA^T = AA^T$, and for any non-zero $x \in \mathbb{R}^2$, $x^TAA^Tx = (A^Tx)^T(A^Tx) = \langle A^Tx, A^Tx \rangle > 0$. Therefore, by the previous lemma, $P = \sqrt{AA^T} \in Pds_2$ is well-defined, so by Lemma 8.1, P^{-1} exists.

The matrix $R = P^{-1}A$ is orthogonal because

$$
\begin{aligned}
R^T R &= (P^{-1}A)^T (P^{-1}A) \\
&= (\sqrt{AA^T}^{-1} A)^T \sqrt{AA^T}^{-1} A \\
&= A^T \sqrt{(AA^T)^{-1}} \sqrt{(AA^T)^{-1}} A \\
&= A^T (AA^T)^{-1} A \\
&= I,
\end{aligned}
$$

and since A is assumed to have positive determinant, lemma 8.2 gives that $\det(R) = 1$ so that R is indeed a rotation by Proposition 9.

Uniqueness: If $A = PR$ is any polar decomposition of A, then $AA^T = PRR^T P = P^2$, so that $P = \sqrt{AA^T}$, which implies that $A = PR$ must be the polar decomposition found above.

$\square$

4.4 Decomposition of the category of isotopes of the real algebra of complex numbers

We will proceed to classify $\mathcal{C}_2$ by finding a description of it. To find the description, we will decompose the category of isotopes of $\mathbb{C}$ into subcategories, each of which will in the next chapter be shown to be equivalent to a group action-category. It is here that the double sign decomposition comes into play.

Proposition 11. *Let $A \in GL_2(\mathbb{R})$. Then*

$$
\mathbb{C}_A \in \left\{ \begin{array}{ll} \mathcal{C}_2^{++} & \text{if } \det(A) > 0 \\ \mathcal{C}_2^{--} & \text{if } \det(A) < 0 \end{array} \right.
$$

Proof. Let $x \in \mathbb{C}_A$, $y \in \mathbb{C}_A \backslash \{0\}$ and $Ay = \begin{pmatrix} a \\ b \end{pmatrix}$. Then $\lambda_y(x) = y \cdot_A x = Ay \cdot Ax = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} Ax$, so $\det(\lambda_y) = (a^2 + b^2)\det(A)$. Since $a^2 + b^2 > 0$, $\mathrm{sgn} \circ \det(\lambda_y) = \mathrm{sgn} \circ \det(A)$. $\square$

Chapter 5

Description and classification of the double sign blocks

5.1 Descriptions of the double sign blocks

This section contains a theorem, heavily inspired by proofs of similar results by Dieterich and Darpö [5], providing the two similar descriptions of the respective double sign blocks of $\mathcal{C}_2$. Fortunately it is possible to treat the two cases simultaneously (by letting an index, here labelled k, keep track of the separate cases), but this main result will still require a fairly long proof.

First let us present the two group actions of the description.

Definition 22. *Throughout this chapter, let $R = R_{2\pi/3}$. We define $\mathcal{K} = \langle K \rangle = \{I, K\}$ to be the group generated by K, and $\mathcal{R} = \langle K, R \rangle$ to be the group generated by K and R.*

Let also $\mathcal{P}$ be the set of all elements of Pds_2 which have determinant 1.

Lemma 10. *Let $\mathcal{G} \in \{\mathcal{K}, \mathcal{R}\}$. Then*

$$\mathcal{G} \times \mathcal{P} \to \mathcal{P}$$
$$(F, P) \mapsto FPF^{-1}$$

defines a group action of $\mathcal{G}$ on $\mathcal{P}$.

Proof. Clearly $\mathcal{K}$ is a subgroup of $\mathcal{R}$, so it suffices to show the case where $\mathcal{G} = \mathcal{R}$.

To begin with, $\mathcal{R}$ is a subgroup of the orthogonal group $O(2, \mathbb{R})$, so each element of $\mathcal{R}$ is orthogonal. By lemma 9.3, $\mathcal{P}$ is closed under conjugation with such elements. It's obvious that conjugation with the identity of $\mathcal{R}$ leaves elements of $\mathcal{P}$ unchanged. Lastly, associativity holds because for arbitrary $F_1, F_2 \in \mathcal{R}$, we have that $F_2(F_1 P F_1^{-1})F_2^{-1} = (F_2 F_1)P(F_2 F_1)^{-1}$. $\square$

Theorem 3. *Let $_{\mathcal{K}}\mathcal{P}$ and $_{\mathcal{R}}\mathcal{P}$ be the categories corresponding to the group actions defined in the previous lemma. Then the maps*

$$\mathcal{F}^{++} : {}_{\mathcal{K}}\mathcal{P} \to \mathcal{C}_2^{++},$$
$$\mathcal{P} \ni P \mapsto \mathbb{C}_P,$$
$$Mor({}_{\mathcal{K}}\mathcal{P}) \ni (F, P, Q) \mapsto \underline{F} : \mathbb{C}_P \to \mathbb{C}_Q$$

and

$$\mathcal{F}^{--} : {}_{\mathcal{R}}\mathcal{P} \to \mathcal{C}_2^{--},$$
$$\mathcal{P} \ni P \mapsto \mathbb{C}_{KP},$$
$$Mor({}_{\mathcal{R}}\mathcal{P}) \ni (F, P, Q) \mapsto \underline{F} : \mathbb{C}_{KP} \to \mathbb{C}_{KQ}$$

are descriptions of $\mathcal{C}_2^{++}$ and $\mathcal{C}_2^{--}$ respectively.

Proof. We want to show that $\mathcal{F}^{++}$ and $\mathcal{F}^{--}$ are equivalences of categories. That is, to show that they are functors (with the stated ranges), which are faithful, full and dense.

Throughout this proof, let $k \in \{0, 1\}$ be fixed, let m range over $\{0, 1\}$ and n over $\{0, 1, 2\}$. The elements of $\mathcal{K}$ clearly are of the form K^m, and the element of $\mathcal{R}$ of the form $R^n K^m$ ($\mathcal{R}$ is a representation of the dihedral 3-group, which is known to have 6 elements, and the 6 possible elements of the form $R^n K^m$ are easily seen to be distinct). Now let $\mathcal{G}(0) = \mathcal{K}$ and $\mathcal{G}(1) = \mathcal{R}$, and define the maps

$$\mathcal{F}_k : {}_{\mathcal{G}(k)}\mathcal{P} \to \mathcal{C}_2^{(-1)^k, (-1)^k}$$
$$\mathcal{P} \ni P \mapsto \mathbb{C}_{K^k P},$$
$$Mor({}_{\mathcal{G}(k)}\mathcal{P}) \ni (R^{kn} K^m, P, Q) \mapsto \underline{R^{kn} K^m} : \mathbb{C}_{K^k P} \to \mathbb{C}_{K^k Q}.$$

Then clearly $\mathcal{F}_0 = \mathcal{F}^{++}$ and $\mathcal{F}_1 = \mathcal{F}^{--}$.

Therefore it is sufficient to show that $\mathcal{F}_0$ is a description of $\mathcal{C}_2^{++}$ and that $\mathcal{F}_1$ is a description of $\mathcal{C}_2^{--}$, that is, that $\mathcal{F}_k$ is a functor (with the stated range), which is faithful, full and dense.

$\mathcal{F}_k$ **is a functor to the appropriate category:** If $P \in \mathcal{P}$, $\det(P) = 1$ and $\det(KP) = -1$, so by proposition 11, $\mathcal{F}_0$ maps objects to objects of $\mathcal{C}_2^{++}$, and $\mathcal{F}_1$ maps objects to objects of $\mathcal{C}_2^{--}$. We also need to show that morphisms are mapped to the appropriate morphisms. That the would-be morphisms are non-zero and linear is already clear, but the multiplication preserving property remains to be shown. Towards this end, let $(R^{kn} K^m, P, Q) \in Mor({}_{\mathcal{G}(k)}\mathcal{P})$ (so that $P = K^m R^{-kn} Q R^{kn} K^m$).

Then for any $x, y \in \mathbb{C}$,

$$R^{kn} K^m (x \cdot_{K^k P} y)$$
$$= R^{kn} K^m (K^k P x \cdot K^k P y)$$
$$= R^{-2kn} K^m (K^k P x \cdot K^k P y)$$
$$= R^{-2kn} (K^{k+m} P x \cdot K^{k+m} P y)$$
$$= R^{-kn} K^{k+m} P x \cdot R^{-kn} K^{k+m} P y$$
$$= R^{-kn} K^{k+m} P (R^{kn} K^m)^{-1} R^{kn} K^m x \cdot R^{-kn} K^{k+m} P (R^{kn} K^m)^{-1} R^{kn} K^m y$$
$$= R^{kn} K^m x \cdot_{R^{-kn} K^{k+m} P (R^{kn} K^m)^{-1}} R^{kn} K^m y$$
$$= R^{kn} K^m x \cdot_{K^k Q} R^{kn} K^m y,$$

where the last equality holds because

$$R^{-kn} K^{k+m} P (R^{kn} K^m)^{-1}$$
$$= R^{-kn} K^{k+m} K^m R^{-kn} Q R^{kn} K^m (R^{kn} K^m)^{-1}$$
$$= R^{-kn} K^k R^{-kn} Q$$
$$= R^{kn(-1 - (-1)^k)} K^k Q$$
$$= K^k Q,$$

so $\underline{R^{kn} K^m} : \mathbb{C}_{K^k P} \to \mathbb{C}_{K^k Q}$ is indeed a morphism in the appropriate category.

As for $\mathcal{F}_k$ preserving composition of morphisms, it follows trivially from

$$\mathcal{G}(k) \ni F \mapsto \underline{F} \in GL(\mathbb{R}^2)$$

clearly being a group morphism that for any $A, B, C \in \mathcal{P}$ and $F_1 : A \to B, F_2 : B \to C$ in $Mor(_{\mathcal{G}(k)}\mathcal{P})$, we have $\mathcal{F}_k((F_2, B, C) \circ (F_1, A, B)) = \underline{F_2 F_1} = \mathcal{F}_k((F_2, B, C)) \circ \mathcal{F}_k((F_1, A, B))$.

Finally, $\mathcal{F}_k$ obviously maps identity morphisms to identity morphisms. Thus all functoriality conditions are fulfilled.

$\mathcal{F}_k$ **is faithful:** If $F_1, F_2 \in \mathcal{G}(k)$ give different morphisms from an object of $\mathcal{P}$ to another, then they are different matrices, in which case $\mathcal{F}_k(F_1) = \underline{F_1}$ and $\mathcal{F}_k(F_2) = \underline{F_2}$ are different linear maps, and therefore different morphisms in $\mathcal{C}_2$.

$\mathcal{F}_k$ **is full:** Let $P, Q \in \mathcal{P}$ and $F \in GL_2(\mathbb{R})$ be such that $\underline{F} : \mathbb{C}_{K^k P} \to \mathbb{C}_{K^k Q}$ is a morphism. We must show that F satisfies $Q = FPF^{-1}$ and is of the form $F = R^{kn} K^m$. Set $G = K^k Q F (K^k P)^{-1} F^{-1}$.

Lemma 11. G *is a rotation, and, for some* m, $F = G^{-2} K^m$.

Proof. It is an immediate consequence of the first "isomorphism juggling" lemma (Lemma 3) (for $C = \mathbb{C}$, $D = \mathbb{C}_{K^k Q}$, $\phi = \underline{K^k P}$ and $f = \underline{F}$) that

$$\underline{F} : \mathbb{C} \to \mathbb{C}_G$$

is a morphism too. This implies that G acts like a complex number, in the sense that

$$z = \underline{F}(1 \cdot \underline{F}^{-1}(z)) = \underline{F}(1) \cdot_G z = (GF1) \cdot Gz, \forall z \in \mathbb{C}_G$$

so that $\underline{G}$ is the linear map corresponding to complex multiplication with $(GF1)^{-1}$. From the definition of G, it's easily seen that $\det(G) = 1$, so by the last paragraph of section 4.2, G is a rotation.

Now, the second "isomorphism juggling" lemma (Lemma 7) can be applied to obtain yet another morphism:

$$\underline{G^{-2}} : \mathbb{C} \to \mathbb{C}_G.$$

These newly found (iso)morphisms together yield that

$$\underline{F}^{-1}\underline{G^{-2}} : \mathbb{C} \to \mathbb{C}$$

is an automorphism. As such, it must preserve 1, and by linearity also $(\underline{F}^{-1}\underline{G^{-2}}(i))^2 = \underline{F}^{-1}\underline{G^{-2}}(i^2) = -1 \cdot 1 = -1$, which implies $\underline{F}^{-1}\underline{G^{-2}}(i) = \pm i$. This choice of sign determines the entire automorphism, so $\underline{F}^{-1}\underline{G^{-2}} \in \{\underline{I}, \underline{K}\}$. Thus $F \in \{G^{-2}, G^{-2}K\}$, from which the desired result immediately follows. $\square$

We now have two equations involving F, G and some fixed m:

$$\left.\begin{array}{l} G = K^k Q F (K^k P)^{-1} F^{-1} \\ F = G^{-2} K^m \end{array}\right\}$$

Substitute the second into the first to obtain

$$G = K^k Q G^{-2} K^m P^{-1} K^k K^m G^2$$

and then solve for $Q = QI$:

$$\begin{aligned} QI &= K^k G^{-1} K^{k+m} P K^m G^2 \\ &= (G^{(-1)^{k+1}} K^m P K^m G^{(-1)^k}) G^{2+(-1)^{k+1}} \end{aligned}$$

where the second equality holds because of Lemma 6 and the fact that G is a rotation.

Note that, by Lemma 8.3,

$$G^{(-1)^{k+1}} K^m P K^m G^{(-1)^k} = (K^m G^{(-1)^k})^{-1} P K^m G^{(-1)^k} \in \mathcal{P},$$

so the right hand side is polarly decomposed. But so is the left hand side, so, by uniqueness of the polar decomposition, it follows that $Q = G^{(-1)^{k+1}} K^m P K^m G^{(-1)^k}$ and $I = G^{2+(-1)^{k+1}}$. The second of these equations gives that $G^2 = G^{(-1)^k}$ holds independently of k, which, together with the first equation and Lemma 11, gives that $Q = FPF^{-1}$. Thus our first goal is accomplished.

The second equation furthermore implies that

$$\begin{cases} G^3 = I & \text{if } k = 1 \\ G = I & \text{if } k = 0 \end{cases}$$

so that there exists an n such that

$$\begin{cases} G = R^n & \text{if } k = 1 \\ G = I & \text{if } k = 0 \end{cases}$$

and, by Lemma 11, n and m such that

$$\begin{cases} F = R^n K^m & \text{if } k = 1 \\ F = K^m & \text{if } k = 0 \end{cases}$$

and we are done.

$\mathcal{F}_k$ **is dense:** Let C be an arbitrary object of $\mathcal{C}_2^{(-1)^k,(-1)^k}$. We need to show that there is a $P \in \mathcal{P}$ such that $C \simeq \mathbb{C}_{K^k P}$. We have already seen (in Proposition 8) that we may pick $M \in GL_2(\mathbb{R})$ such that $C \simeq \mathbb{C}_M$, so a lot of the work is already done.

By the second "isomorphism juggling" lemma (Lemma 7) (for $A = M$ and $G = |\det(M)|^{-1/2} I$, $\mathbb{C}_M \simeq \mathbb{C}_{|\det(M)|^{-1/2} M}$, where $\det(|\det(M)|^{-1/2} M) = (-1)^k$, so let us without loss of generality assume that we chose M such that $\det(M) = (-1)^k$ in the first place. Let $N = K^k M$. Then $\det(N) = 1$, so we may apply polar decomposition to N, and obtain $M = K^k Q R_\alpha$, for some $Q \in \mathcal{P}$ and $\alpha \in \mathbb{R}$. Let G be any rotation fulfilling $G^{(-1)^k - 2} = R_\alpha$. Reapply the second "isomorphism juggling" lemma (Lemma 7) for the above G and $A = K^k Q R_\alpha$. This gives, also applying Lemma 6,

$$\mathbb{C}_M = \mathbb{C}_{K^k Q R_\alpha} \simeq \mathbb{C}_{G^{-1} K^k Q R_\alpha G^2} = \mathbb{C}_{K^k (G^{(-1)^{k+1}} Q G^{(-1)^k})}.$$

Set $P = G^{(-1)^{k+1}} Q G^{(-1)^k}$. By lemma 8.3, P is indeed in Pds_2, and since $Q \in \mathcal{P}$, even $P \in \mathcal{P}$, and we have that $C \simeq \mathbb{C}_{K^k P}$.

The proof is now complete! $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.2 Classification of the double sign blocks

Recall that our final goal is to find a classification of $\mathcal{C}_2$, that is, to list precisely one object of each isomorphism class of $\mathcal{C}_2$. We planned to do this via finding a description of each subcategory in a certain "double sign decomposition", and this was accomplished in the previous section. Now the last substantial part of the outlined argument consists in finding transversals for the orbit sets of the group actions occurring in the two descriptions. This is contained in Lemma 12 below, of which the classification of $\mathcal{C}_2$ is an immediate consequence. As was the case when the descriptions were to be found, I have opted for proving the two cases simultaneously, using an argument which should appeal to geometric intuition. This proof too is based on a very similar one by Dieterich and Darpö [5, p. 188-190].

As before, let $k \in \{0, 1\}$ be fixed, let m range over $\{0, 1\}$, n over $\{0, 1, 2\}$ and furthermore j over $\mathbb{Z}$ (throughout this entire section).

Definition 23. *For $\alpha \in \mathbb{R}$ and $\nu \in \mathbb{R}_{>0}$, let $D_\nu = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$ and $P_{\alpha,\nu} = R_\alpha D_\nu R_\alpha^{-1}$.*

Lemma 12.   *1. Let $\alpha \in \mathbb{R}$ and $\nu \in \mathbb{R}_{>0}$. Then:*

(a) *$P_{\alpha,\nu} \in \mathcal{P}$. Using the same notation as in the proof of Theorem 3, we may (and will) therefore for the rest of this lemma regard $P_{\alpha,\nu}$ as an object of $_{\mathcal{G}(k)}\mathcal{P}$.*

(b) *$P_{\alpha+\pi/2,\nu} = P_{\alpha,\nu^{-1}}$.*

(c) *$P_{\alpha+\pi,\nu} = P_{\alpha,\nu}$.*

(d) *$[P_{\alpha,\nu}] = \{P_{(-1)^m\alpha+2kn\pi/3,\nu} : n \in \{0, 1, 2\}, m \in \{0, 1\}\}$.*

2. $\{P_{\alpha,\nu} : \alpha \in [0, \pi/(2 \cdot 3^k)], \nu \in (0, 1)\} \cup I$ classifies $_{\mathcal{G}(k)}\mathcal{P}$.

Proof.   1.   (a) Follows from lemma 8.3 and the easily checked fact that $D_\nu \in \mathcal{P}$.

(b) $P_{\alpha+\pi/2,\nu} = R_\alpha R_{\pi/2} D_\nu R_{\pi/2}^{-1} R_\alpha^{-1} = R_\alpha D_{\nu^{-1}} R_\alpha^{-1} = P_{\alpha,\nu^{-1}}$.

(c) This follows from applying (b) twice.

(d) $[P_{\alpha,\nu}] = \{R^{kn} K^m P_{\alpha,\nu} K^m R^{-kn} : n \in \{0, 1, 2\}, m \in \{0, 1\}\}$, and for any n and m,

$$
\begin{aligned}
R^{kn} K^m P_{\alpha,\nu} K^m R^{-kn} &= R_{2kn\pi/3} K^m R_\alpha D_\nu R_{-\alpha} K^m R_{-2kn\pi/3} \\
&= R_{2kn\pi/3+(-1)^m\alpha} K^m D_\nu K^m R_{-2kn\pi/3+(-1)^{m+1}\alpha} \\
&= R_{2kn\pi/3+(-1)^m\alpha} D_\nu R_{-2kn\pi/3+(-1)^{m+1}\alpha} \\
&= P_{(-1)^m\alpha+2kn\pi/3,\nu}.
\end{aligned}
$$

2. By 1.(a), $\{P_{\alpha,\nu} : \alpha \in \mathbb{R}, \nu \in \mathbb{R}_{>0}\} \subset \mathcal{P}$, and the spectral theorem gives the opposite inclusion. Clearly $I \in \mathcal{P}$. Also I is non-isomorphic to any member of $\{P_{\alpha,\nu} : \alpha \in [0, \pi/(2 \cdot 3^k)], \nu \in (0,1)\}$, because two isomorphic matrices must, by 1.(d), have the same eigenvalues; ν and ν^{-1}. Therefore it suffices to show that $\{P_{\alpha,\nu} : \alpha \in [0, \pi/(2 \cdot 3^k)], \nu \in (0,1)\}$ classifies the full subcategory of $_{\mathcal{G}(k)}\mathcal{P}$ which is obtained by removing the object I. The proof of this splits up into proving exhaustion and proving irredundancy.

Exhaustion: We want to show that

$$\bigcup\{[P_{\alpha,\nu}] : \alpha \in [0, \pi/(2 \cdot 3^k)], \nu \in (0,1)\} = \{P_{\alpha,\nu} : \alpha \in \mathbb{R}, \nu \in \mathbb{R}_{>0}\}\backslash\{I\}.$$

We have, using part one of the lemma, that:

$$\bigcup\{[P_{\alpha,\nu}] : \alpha \in [0, \pi/(2 \cdot 3^k)], \nu \in (0,1)\} =$$
$$\bigcup\{[P_{\alpha,\nu}] : \alpha \in [-\pi/(2 \cdot 3^k), \pi/(2 \cdot 3^k)], \nu \in (0,1)\} =$$
$$\bigcup\{[P_{\alpha,\nu}] : \alpha \in [-\pi/(2 \cdot 3^k), \pi/(2 \cdot 3^k)]\cup$$
$$[2k\pi/3 - \pi/(2 \cdot 3^k), 2k\pi/3 + \pi/(2 \cdot 3^k)]\cup$$
$$[4k\pi/3 - \pi/(2 \cdot 3^k), 4k\pi/3 + \pi/(2 \cdot 3^k)], \nu \in (0,1)\} =$$
$$\bigcup\{[P_{\alpha,\nu}] : \alpha \in [0, 2\pi], \nu \in (0,1)\} =$$
$$\bigcup\{[P_{\alpha,\nu}] : \alpha \in [0, 2\pi], \nu \in (0,1) \cup (1, \infty)\} =$$
$$\bigcup\{[P_{\alpha,\nu}] : \alpha \in \mathbb{R}, \nu \in \mathbb{R}_{>0}\}\backslash[I] =$$
$$\{P_{\alpha,\nu} : \alpha \in \mathbb{R}, \nu \in \mathbb{R}_{>0}\}\backslash\{I\}.$$

These seemingly quite technical calculations have a significantly more transparent geometric interpretation, as follows.

The non-zero points in the plane correspond to matrices in $\mathcal{P}$ via the surjective mapping $(\alpha, \nu) \mapsto P_{\alpha,\nu}$, where (α, ν) is an arbitrary point written in polar coordinates. The illustrations in figure 5.1 depict, from left to right, the points (colored a light gray) corresponding to matrices obtained by applying in succession 1.(d), 1.(d), 1.(c) and 1.(b) to the $P_{\alpha,\nu}$ with indices $\alpha \in [0, \pi/(2 \cdot 3^k)]$, $\nu \in (0,1)$ (the dark gray sectors). The upper row corresponds to the case $k = 0$, the lower to $k = 1$. We see that we in both cases can go from the respective dark gray sector of the unit disc to the entire plane, except the origin and the unit circle (here drawn dashed), using the results of part one of this lemma. By the surjectivity of the mapping, this means that the matrices with indices in the dark gray sectors exhaust the full subcategory of $_{\mathcal{G}(k)}\mathcal{P}$ which is obtained by removing I.
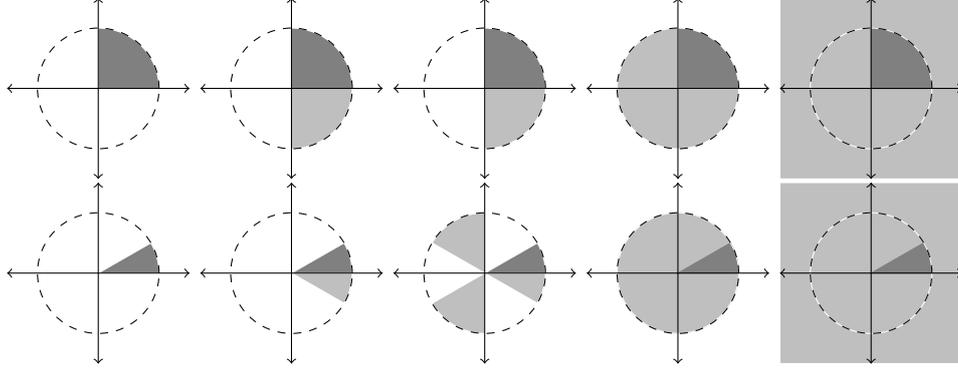
Figure 5.1: Illustration of the matrices isomorphic to those in the would-be classifying set via the mapping $(\alpha, \nu) \mapsto P_{\alpha,\nu}$, where (α, ν) is a nonzero point in the plane, written in polar coordinates.

Irredundancy: Let $P_{\alpha,\nu} \simeq P_{\beta,\mu}$ be isomorphic matrices with $\alpha, \beta \in [0, \pi/(2 \cdot 3^k)]$ and $\nu, \mu \in (0, 1)$. We need to show that $P_{\alpha,\nu} = P_{\beta,\mu}$. By 1.(d) we have that there exist m and n such that

$$P_{\beta,\mu} = P_{(-1)^m \alpha + 2kn\pi/3, \nu}$$

or equivalently

$$D_\mu = R_{(-1)^m \alpha - \beta + 2kn\pi/3} D_\nu R_{(-1)^m \alpha - \beta + 2kn\pi/3}^{-1}.$$

Then in particular these two matrices have the same eigenvalues, so that $\nu = \mu^{\pm 1}$. But $\nu, \mu \in (0, 1)$ implies $\nu \neq \mu^{-1}$, so it must be that $\nu = \mu$. This means that the eigenlines (which are spanned by the standard basis vectors) of D_ν are invariant under the rotation $R_{(-1)^m \alpha - \beta + 2kn\pi/3}$. The rotations which map lines to themselves are precisely those with angle some multiple of π, so it must be that for some j, $(-1)^m \alpha - \beta + 2kn\pi/3 = -j\pi$, which in turn implies that $\beta = (-1)^m \alpha + 2kn\pi/3 + j\pi$. One may now check the different cases obtained from different possible m and n, and different parity of j, to see that α and β must be equal modulo 2π, so that the isomorphic matrices are in fact equal, and we are done. This rather laboreous exercise is again facilitated by considering figure 5.1:

Let (α, ν) be some point in the dark gray sector under consideration. This point gives rise to a redundancy if and only if $P_{\alpha,\nu} \simeq P_{(-1)^m \alpha + 2kn\pi/3 + j\pi, \nu}$, but $P_{\alpha,\nu} \neq P_{(-1)^m \alpha + 2kn\pi/3 + j\pi, \nu}$, where $((-1)^m \alpha + 2kn\pi/3 + j\pi, \nu)$ too is in the dark gray sector. Assume towards a contradiction that it does. That $P_{\alpha,\nu} \neq P_{(-1)^m \alpha + 2kn\pi/3 + j\pi, \nu}$ implies that at least one of m and kn is nonzero. Now observe that this in turn implies that $((-1)^m \alpha +$

$2kn\pi/3 + j\pi, \nu)$ is in some light gray sector too, so it lies where the light and dark gray sectors overlap. They do this precisely at the borderlines of the dark gray sectors, which is easily seen to require one of the following cases to hold:

- $\alpha = 0 = kn$, $m = 1$ and j is even.
- $\alpha = \pi/(2 \cdot 3^k)$, $k(n + (-2)^k) = 0$, $m = 1$ and j is odd.

However, both cases yield matrices $P_{\alpha,\nu}$ and $P_{(-1)^m\alpha+2kn\pi/3+j\pi,\nu}$ which, while trivially isomorphic, are equal, which is a contradiction. Therefore no irredundancy can occur.

$\square$

Theorem 4. *The set*

$$\{\mathbb{C}_{P_{\alpha,\nu}} : \alpha \in [0, \pi/2], \nu \in (0,1)\} \cup \{\mathbb{C}\} \cup \{\mathbb{C}_{KP_{\alpha,\nu}} : \alpha \in [0, \pi/6], \nu \in (0,1)\} \cup \{\mathbb{C}_K\}$$

classifies $\mathcal{C}_2$.

Proof. It's an immediate consequence of the previous lemma, Theorem 3 and Proposition 2 that $\{\mathbb{C}_{P_{\alpha,\nu}} : \alpha \in [0, \pi/2], \nu \in (0,1)\} \cup \{\mathbb{C}\}$ classifies $\mathcal{C}_2^{++}$, and that $\{\mathbb{C}_{KP_{\alpha,\nu}} : \alpha \in [0, \pi/6], \nu \in (0,1)\} \cup \{\mathbb{C}_K\}$ classifies $\mathcal{C}_2^{--}$. The desired result then follows from $\mathcal{C}_2$ decomposing into $\mathcal{C}_2^{++}$ and $\mathcal{C}_2^{--}$. $\square$

Appendix A

Proof of Hopf's theorem

The theorem, labelled theorem 1 in the report, is restated below for reference.

Theorem 5. *Let $2 < n \in \mathbb{N}$. Then $\mathcal{C}_n$ is empty.*

The following proof is taken from [10, 172-174] and requires some familiarity with algebraic topology.

Proof. Assume, towards a contradiction, that $2 < n \in \mathbb{N}$ and that C is an object in $\mathcal{C}_n$. Equip C with the Euclidean topology, and define the map

$$f : S^{n-1} \to S^{n-1}$$
$$x \mapsto x \star^C x / |x \star^C x|.$$

By the bilinearity of $\star^C$, f is continuous. Since $f(x) = f(-x)$ for all $x \in S^{n-1}$, f induces a quotient map

$$\overline{f} : \mathbb{R}P^{n-1} :\to S^{n-1}.$$

The following proves that $\overline{f}$ is injective:

Let $x, y \in S^{n-1}$ such that $f(x) = f(y)$, and let $r = (|x \star^C x| / |y \star^C y|)^{1/2} \in \mathbb{R}$. Then $x \star^C x = r^2 y \star^C y$, which implies that $(x + ry) \star^C (x - ry) = 0$. Since a division algebra clearly doesn't contain any divisors of zero, it must be that $x = \pm ry$, and since x and y have unit length, $x = \pm y$. Therefore x and y correspond to the same point in $\mathbb{R}P^{n-1}$, so $\overline{f}$ is injective.

Being an injective map between compact Hausdorff spaces, $\overline{f}$ must be a homeomorphism onto its image, which must be closed in S^{n-1}. By the invariance of domain theorem, however, the image must be open in S^{n-1} too, and a subset which is both open and closed in a connected set cannot be a proper subset. Thus $\overline{f}$ is in fact surjective, so it's a homeomorphism between $\mathbb{R}P^{n-1}$ and S^{-1}, and homeomorphic sets have isomorphic fundamental groups. However, for $2 < n \in \mathbb{N}$, $\mathbb{R}P^{n-1}$ and S^{-1} are known to have non-isomorphic fundamental groups; contradiction! □

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Nomenclature

$(d_{i,j})$ The matrix with $d_{i,j}$ as entry in row i and column j.

$[C]$ Square brackets in this context denote the isomorphism class of C, but they may also be used to denote a closed interval.

$\cdot_\alpha^{\mathbb{C}}$ The multiplication operation in the isotope of $\mathbb{C}$ given by $\alpha \in \mathbb{C}$.

det   The determinant function, gives the determinant of an $\mathbb{R}$-linear map.

$\lambda(d)$ The function returning the left multiplication map with respect to the argument, here evaluated at d.

λ_a The left multiplication map with respect to a.

$\langle u, v \rangle$ The scalar product of real vectors u and v. Angle brackets are also used to denote the group generated by the elements within.

$\mathbb{C}_A$ An abbreviation of $\mathbb{C}_{\underline{A}}$, whenever $A \in GL_2(\mathbb{R})$.

$\mathbb{R}_{>0}$ The set of all positive real numbers.

$\mathcal{A}^{+,+}$ The twice positive double sign block of the category $\mathcal{A}$. Different signs in the exponent yield different double sign blocks.

$\mathcal{C}_n$ The category of all n-dimensional real commutative division algebras (for $n \in \mathbb{N}$).

$\mathcal{D}$ In the later parts of the text, denotes the category of all real division algebras.

$\mathcal{D}_n$ Denotes the category of all n-dimensional real division algebras.

$\mathcal{P}$ The elements of Pds_2 which have determinant 1.

$\rho(d)$ The function returning the right multiplication map with respect to the argument, here evaluated at d.

ρ_a The right multiplication map with respect to a.

sgn   The sign function, defined by $\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0. \\ -1 & \text{if } x < 0. \end{cases}$, for $x \in \mathbb{R}\backslash\{0\}$.

$\simeq$ The isomorphism relation.

$\sqrt{P}$ The positive definite symmetric square root of a matrix $P \in Pds_2$.

$\star_\alpha^C$ The multiplication operation in the isotope of C given by $\alpha \in C$.

$\underline{A}$ The linear map corresponding (in the standard basis) to the matrix A.

C_α The isotope of $\mathbb{C}$ given by $\alpha \in \mathbb{C}$.

D_ν For $\nu \in \mathbb{R}_{>0}$, $D_\nu = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$.

$GL(C)$ The general linear group of the vector space C.

$GL_2(\mathbb{R})$ The group of invertible real 2×2-matrices.

I The real identity matrix of dimension two.

K $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$Mor(\mathcal{C})$ The class of morphisms of a category $\mathcal{C}$.

$O(2, \mathbb{R})$ The group of all real orthogonal 2×2-matrices.

$Ob(\mathcal{C})$ The class of objects of the category $\mathcal{C}$.

P^T The transpose of the matrix P.

$P_{\alpha,\nu}$ For $\alpha \in \mathbb{R}$ and $\nu \in \mathbb{R}_{>0}$, $P_{\alpha,\nu} = R_\alpha D_\nu R_\alpha^{-1}$.

Pds_2 The set of all real, positive definite symmetric 2×2-matrices.

R In the final chapter denotes the roation matrix $R_{2\pi/3}$.

R_α The matrix in $GL_2(\mathbb{R})$ acting like a rotation of angle α.

$SO_2(\mathbb{R})$ The special orthogonal group of real 2×2-matrices.

$_G M$ Denotes the category of the group action of G on M.