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WOLFF-POTENTIAL ESTIMATES AND DOUBLING OF SUBELLIPTIC p -HARMONIC MEASURES

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ABSTRACT. Let $\mathbf{X} = \{X_1, \dots, X_m\}$ be a system of C^∞ vector fields in $\mathbb{R}^n$ satisfying Hörmander's finite rank condition and let Ω be a non-tangentially accessible domain with respect to the Carnot-Carathéodory distance d induced by $\mathbf{X}$. We prove the doubling property of certain boundary measures associated to non-negative solutions, which vanish on a portion of $\partial\Omega$, to the equation

$$L_{A,p}u = \sum_{i=1}^m X_i^* A_i(x, \mathbf{X}u) = 0.$$

Given p , $1 < p < \infty$, fixed, we impose conditions on the function $A = (A_1, \dots, A_m) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, which imply that the equation is a quasi-linear partial differential equations of p -Laplace type structured on vector fields satisfying the classical Hörmander condition. In the case $p = 2$ and for linear equations, our result coincides with the doubling property of associated elliptic measures. To prove our result we establish, and this is of independent interest, a Wolff potential estimate for subelliptic equations of p -Laplace type.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, i.e., a bounded, open and connected set. The study of the boundary behaviour of non-negative solutions in Ω to linear equations of the type

$$(1.1) \quad Lu = \sum_{i,j=1}^m X_i^*(a_{ij}X_j u) = 0,$$

where $\mathbf{X} = \{X_1, \dots, X_m\}$ is a general system of C^∞ vector fields in $\mathbb{R}^n$, $\mathbf{X}u = (X_1 u, \dots, X_m u)$ and X_i^* is the formal adjoint of X_i , is by now an established field of research within the area of partial differential equations. In this context $\mathbf{X} = \{X_1, \dots, X_m\}$ is assumed to satisfy the Hörmander's finite rank condition, see [H],

$$(1.2) \quad \text{rank Lie } [X_1, \dots, X_m] \equiv n,$$

and concerning the $m \times m$ matrix-valued function $A(x) = \{a_{ij}(x)\}$ it is assumed that $A(x)$ is real, symmetric, for every $x \in \mathbb{R}^n$, with bounded and measurable entries, and that

$$(1.3) \quad \beta^{-1}|\xi|^2 \leq \sum_{i,j=1}^m a_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2$$

for all $x \in \mathbb{R}^n$, and $\xi \in \mathbb{R}^m$ and for some constant β , $1 \leq \beta < \infty$. Let in the following $d(x, y)$ be the Carnot-Carathéodory distance, between $x, y \in \mathbb{R}^n$, induced by $\{X_1, \dots, X_m\}$ and let

$$(1.4) \quad B_{cc}(x, r) = \{y \in \mathbb{R}^n : d(x, y) < r\}$$

be the Carnot-Carathéodory metric ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$. For the operator L in (1.1), assuming (1.2) and (1.3), weak solutions to the Dirichlet problem

$$(1.5) \quad Lu = 0 \text{ in } \Omega, \quad u = f \text{ on } \partial\Omega,$$

can be introduced in a standard manner, and using the Lax-Milgram theorem the existence and uniqueness of weak solutions can be established. Concerning the domain Ω it is natural to assume that Ω is a **X**-NTA-domain (non-tangentially accessible domain), with parameters M, r_0 , in the sense of [CG], see Definition 2 below. We note that the notion of **X**-NTA-domain is a subelliptic version of NTA-domains introduced in [JK] in the standard Euclidean setting, i.e., in the case $m = n$, $\{X_1, \dots, X_m\} = \{\partial_{x_1}, \dots, \partial_{x_n}\}$. Assuming that Ω is a bounded **X**-NTA-domain one can prove that all points on $\partial\Omega$ are regular for the Dirichlet problem for the operator L in (1.1) assuming (1.2) and (1.3). In particular, one can conclude that there exists, for any $f \in C(\partial\Omega)$, a unique weak solution $u = u_f \in C(\overline{\Omega})$ to the Dirichlet problem in (1.5). Furthermore, one can conclude that there exists, for every $x_0 \in \Omega$, a unique probability measure $\omega(\cdot, x_0)$ on $\partial\Omega$ such that

$$(1.6) \quad u(x_0) = \int_{\partial\Omega} f(y) d\omega(y, x_0).$$

Given $E \subseteq \partial\Omega$, $\omega(E, x_0)$ is referred to as the L -elliptic measure of E relative to Ω and x_0 . For the above results, and the following theorem, we refer to [CG].

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a **X**-NTA-domain with constants M and r_0 . Let L be as in (1.1) and assume (1.2) and (1.3). Let $w \in \partial\Omega$, $0 < r < r_0$. Then there exists $c = c(n, \beta, \mathbf{X}, M)$, $1 \leq c < \infty$, such that if $x_0 \in \Omega \setminus B_{cc}(w, cr)$ then*

$$\omega(\partial\Omega \cap B_{cc}(w, 2r), x_0) \leq c\omega(\partial\Omega \cap B_{cc}(w, r), x_0).$$

Problems concerning the doubling property of elliptic measures and Theorem 1.1 have a long and rich history. In particular, for the uniformly elliptic case, i.e., the case $m = n$, $\{X_1, \dots, X_m\} = \{\partial_{x_1}, \dots, \partial_{x_n}\}$, we refer to [Ca], [Da], [CFMS], [JK] and the references in these papers. When $\mathbf{X} = \{X_1, \dots, X_m\}$ satisfies Hörmander's finite rank condition Theorem 1.1 is proved in [CG]. Also degenerate versions of Theorem 1.1 are available in the literature. Indeed, in the degenerate setting (1.3) is replaced by the condition

$$(1.7) \quad \beta^{-1}\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^m a_{ij}(x)\xi_i\xi_j \leq \beta\lambda(x)|\xi|^2,$$

for all $x \in \mathbb{R}^n$, and $\xi \in \mathbb{R}^m$, and for some constant β , $1 \leq \beta < \infty$. Now the weight $\lambda = \lambda(x)$ is assumed to be a non-negative measurable function satisfying a Muckenhoupt type condition, λ an A_2 -weight, with respect to the metric introduced by the system of vector fields $\mathbf{X} = \{X_1, \dots, X_m\}$. In this setting, the case $m = n$, $\{X_1, \dots, X_m\} = \{\partial_{x_1}, \dots, \partial_{x_n}\}$, and λ an A_2 -weight is treated in [FKS], [FJK], [FJK1]. The case when $\mathbf{X} = \{X_1, \dots, X_m\}$ satisfies Hörmander's finite rank condition and λ is an A_2 -weight is treated in [GNy]. We also note that establishing the doubling property of elliptic measures is, in general, the starting point for far reaching developments concerning the boundary behaviour for non-negative solutions as outlined in [Ca], [Da], [CFMS], [JK], [CG], [FKS], [FJK], [FJK1], [GNy].

The purpose of this paper is to establish an appropriate version of Theorem 1.1 for non-negative solutions to general equations of the form

$$(1.8) \quad L_{A,p}u = \sum_{i=1}^m X_i^* A_i(x, \mathbf{X}u) = 0,$$

where again $\mathbf{X} = \{X_1, \dots, X_m\}$ is a system of C^∞ vector fields in $\mathbb{R}^n$, $\mathbf{X}u = (X_1u, \dots, X_mu)$, X_i^* is the formal adjoint of X_i , and $\mathbf{X} = \{X_1, \dots, X_m\}$ satisfies (1.2). Given p , $1 < p < \infty$, fixed, we impose conditions on the function $A = (A_1, \dots, A_m) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, see Definition 1 below, which imply that (1.8) is a quasi-linear partial differential equations of p -Laplace type structured on vector fields satisfying the classical Hörmander condition. A weak solution to (1.8), see (1.13) below, will be referred to as an A -harmonic function and as discussed below the classical Dirichlet problem, with continuous boundary data, for the equation in (1.8) is well-posed in $\mathbf{X}$ -NTA-domains. We next state our main result in this context and we refer the reader to the bulk of the paper for the exact definition of A -harmonic functions, $\mathbf{X}$ -NTA-domains and the local homogeneous dimension (of $\mathbf{X}$) relative to $B_{cc}(0, R_0)$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a $\mathbf{X}$ -NTA-domain with constants M and r_0 . Assume (1.19) below and let Q be the local homogeneous dimension (of $\mathbf{X}$) relative to $B_{cc}(0, R_0)$. Let p , $1 < p < \infty$, be fixed and assume that $A = (A_1, \dots, A_m) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is as in Definition 1 for some α, β . Let $w \in \partial\Omega$, $0 < r < r_0$ and suppose that u is a non-negative A -harmonic function in $\Omega \cap B_{cc}(w, 2r)$ such that $u = 0$ continuously on $\partial\Omega \cap B_{cc}(w, 2r)$. Extend u to $B_{cc}(w, 2r)$ by defining $u \equiv 0$ on $B_{cc}(w, 2r) \setminus \Omega$. Then there exists a unique locally finite positive Borel measure μ on $\partial\Omega \cap B_{cc}(w, 2r)$, such that*

$$\int A(x, \mathbf{X}u) \cdot \mathbf{X}\phi \, dx = - \int \phi \, d\mu,$$

whenever $\phi \in C_0^\infty(B_{cc}(w, 2r))$. Furthermore, there exists $c = c(p, n, \alpha, \beta, \mathbf{X}, Q, M)$, $1 \leq c < \infty$, such that if $\tilde{r} = r/c$, then

$$\mu(B_{cc}(w, \tilde{r})) \leq c\mu(B_{cc}(w, \tilde{r}/2)).$$

Remark 1.1. Note that the boundary measure μ in Theorem 1.2, is the Riesz measure, as defined in (1.14), associated to the extended function u in $B_{cc}(w, 2r)$. This is a consequence of the fact that the extended function u is an A -subsolution in $B_{cc}(w, 2r)$, see the proof of Theorem 1.4 in section 5.

Consider the more restrictive structure

$$(1.9) \quad \tilde{L}_{A,p}u = \mathbf{X}^* \cdot (\langle A(x)\mathbf{X}u, \mathbf{X}u \rangle^{p/2-1} A(x)\mathbf{X}u),$$

where $A(x) = \{a_{ij}(x)\}$ is a matrix-valued function and $\cdot$ denotes the Euclidean scalar product in $\mathbb{R}^m$. In this case it follows that Theorem 1.2 essentially coincides, when $p = 2$ and u is the Green function associated to $\tilde{L}_{A,2}$ with pole at $x_0 \in \Omega$, with Theorem 1.1 as in this case $\mu(\cdot) = \omega(\cdot, x_0)$. In particular, Theorem 1.2 is a non-linear version of Theorem 1.1 valid for non-negative solutions to general equations of the form stated in (1.8).

1.1. Subelliptic and quasi-linear pdes of p -Laplace type. Given an open set $O \subset \mathbb{R}^n$, and $1 \leq q \leq \infty$, we in the following let $W_{\mathbf{X}}^{1,q}(O)$ denote the Folland-Stein Sobolev space of equivalence classes of functions f with horizontal distributional derivatives $X_i f$, for $i \in \{1, \dots, m\}$, all of which are integrable to the q -th power on O . Given $f \in W_{\mathbf{X}}^{1,q}(O)$ we let

$$\|f\|_{1,q} = \|f\|_q + \sum_{i=1}^m \|X_i f\|_q,$$

be the norm in $W_{\mathbf{X}}^{1,q}(O)$ where $\|\cdot\|_q$ denotes the Lebesgue q norm on O . We let $W_{\mathbf{X},0}^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W_{\mathbf{X}}^{1,q}(O)$. In this paper we consider the equation in (1.8) assuming that $\mathbf{X} = \{X_1, \dots, X_m\}$ satisfies (1.2).

Given p , $1 < p < \infty$, fixed, we impose conditions on the function $A = (A_1, \dots, A_m) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ according to the following definition.

Definition 1. Let p , $1 < p < \infty$, be fixed. Let $A = (A_1, \dots, A_m) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be such that $x \rightarrow A(x, \eta)$ is measurable for all $\eta \in \mathbb{R}^m$ and such that the mapping $\eta \rightarrow A(x, \eta)$ is continuous for almost all $x \in \mathbb{R}^n$. Furthermore, there exist $\alpha \in (1, \infty)$, $\beta \in \mathbb{R}_+$, such that

$$(1.10) \quad |A(x, \eta)| \leq \alpha |\eta|^{p-1},$$

$$(1.11) \quad (A(x, \eta) - A(x, \xi)) \cdot (\eta - \xi) \geq \beta (|\eta| + |\xi|)^{p-2} |\eta - \xi|^2,$$

for all $\eta, \xi \in \mathbb{R}^m$ not both identically to 0.

Remark 1.2. Note that that much of the analysis in [HKM] and [D] is developed under the assumptions on A stated in Definition 1. However, in these references the authors also assume that A should satisfy the homogeneity condition

$$(1.12) \quad A(x, t\eta) = t|t|^{p-2} A(x, \eta) \quad t \in \mathbb{R} \setminus \{0\}.$$

This condition is not assumed in [TW]. To us it is not clear why the authors in [HKM] and [D] also assume (1.12) and hence we, in analogy with [TW], will only use the properties stated in Definition 1.

Let $G \subset \mathbb{R}^n$ be a domain, i.e., an open and connected set, and let A be as in Definition 1 for some p , $1 < p < \infty$, fixed. We say that u is a weak solution (A -harmonic) in G provided that $u \in W_{\mathbf{X},0}^{1,p}(G)$ and

$$(1.13) \quad \int A(x, \mathbf{X}u) \cdot \mathbf{X}\phi \, dx = 0,$$

whenever $\phi \in W_{\mathbf{X},0}^{1,p}(G)$. We note that (1.8) is the Euler-Lagrange equation associated to the functional

$$J_p(u) = \frac{1}{p} \int A(x, \mathbf{X}u) \cdot \mathbf{X}u \, dx.$$

We say that $u \in W_{\mathbf{X}}^{1,p}(G)$ is an A -subsolution (A -supersolution) in G if (1.13) holds with $=$ replaced by $\leq$ ($\geq$) whenever $\phi \in W_{\mathbf{X},0}^{1,p}(G)$, $\phi \geq 0$. Finally, an A -subharmonic function (A -superharmonic function) is a function which is upper (lower) semicontinuous and which satisfies the standard comparison principle with respect to A -harmonic functions.

Given an A -subsolution u to (1.8) in a domain G it follows from the Riesz representation theorem there exists a unique locally finite positive Borel measure μ on $\mathbb{R}^n$, called the Riesz measure associated to u , such that

$$(1.14) \quad \int A(x, \mathbf{X}u) \cdot \mathbf{X}\phi \, dx = - \int \phi \, d\mu,$$

whenever $\phi \in C_0^\infty(G)$. Following Theorem 3.1 in [TW] we also see that a Riesz measure can be associated to an A -subharmonic function as the weak limit of the Riesz measures associated to a sequence of A -subsolutions converging to u . Furthermore, this can be done independently of the choice of convergent subsequence.

1.2. A Wolff potential estimate. Let $d(x, y)$ be the Carnot-Carathéodory distance, between $x, y \in \mathbb{R}^n$, induced by $\{X_1, \dots, X_m\}$ and recall the definition of the metric ball $B_{cc}(x, r)$, centered at $x \in \mathbb{R}^n$ with radius $r > 0$, in (1.4). It is well known that if $U \subset \subset \mathbb{R}^n$, then there exists a constant $C(U) \geq 1$ such that

$$(1.15) \quad |B_{cc}(x, 2r)| \leq C|B_{cc}(x, r)| \text{ whenever } B_{cc}(x, 2r) \subset U,$$

and where $|E|$ denotes the Lebesgue measure of a measurable set E . The above doubling inequality gives rise to the existence of $Q(U) \in \mathbb{R}_+$ such that

$$(1.16) \quad |B_{cc}(x, r)| \leq Ct^{-Q}|B_{cc}(x, tr)| \quad \text{for all } 0 < t < 1,$$

whenever $B_{cc}(x, r) \subset U$ and we call the number Q the local homogeneous dimension (of $\mathbf{X}$) relative to U . We note that by Chow's accessibility theorem it follows, see [Ch], that the metric space $(\mathbb{R}^n, d)$ is locally compact and that there exists $R_0 > 0$ such that the closure of any ball $B_{cc} \subseteq B_{cc}(0, R_0)$ is compact. We stress that, in general, metric balls of large radii fail to be compact, see [GN1]. In view of these observations, we will in the following always assume that R_0 is such that

$$(1.17) \quad \overline{B_{cc}(0, R_0)} \text{ is compact,}$$

and we let Q denote the local homogeneous dimension (of $\mathbf{X}$) relative to $B_{cc}(0, R_0)$.

Let μ be a non-negative measure on $\mathbb{R}^n$. Given $w \in \mathbb{R}^n$, $r > 0$, we introduce the subelliptic Wolff potential

$$(1.18) \quad \mathbf{W}_{p, \mathbf{X}}^\mu(w, r) = \int_0^r \left(\frac{\mu(B_{cc}(w, t))}{t^{-p}|B_{cc}(w, t)|} \right)^{1/(p-1)} \frac{dt}{t}.$$

We prove the following theorem.

Theorem 1.3. *Assume (1.17). Let p , $1 < p < \infty$, be fixed and assume that $A = (A_1, \dots, A_m) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is as in Definition 1 for some α, β . Let u be a non-negative A -subharmonic function in $\mathbb{R}^n$ and let μ be the Riesz measure associated to u . Then there exist $\delta = \delta(p, n, \alpha, \beta, \mathbf{X}, Q)$, $1 \leq \delta < \infty$, and $C = C(p, n, \alpha, \beta, \mathbf{X}, Q)$, $1 \leq C < \infty$, such that*

$$u(w) + C^{-1} \mathbf{W}_{p, \mathbf{X}}^\mu(w, r/2) \leq \sup_{B_{cc}(w, r)} u \leq 2u(w) + C \mathbf{W}_{p, \mathbf{X}}^\mu(w, \delta r),$$

whenever $w \in \mathbb{R}^n$, $r > 0$ and $B_{cc}(w, \delta r) \subset B_{cc}(0, R_0/2)$.

Note that our formulation of the Wolff potential is a subelliptic version of the potential studied in [KZ] in the standard Euclidean case. In particular, Theorem 1.3 is a generalization of the main result in [KZ] to the subelliptic setting. We also note that Theorem 1.3 is a nonlinear and subelliptic version of Nevanlinna's first fundamental theorem for classical subharmonic functions associated to the Laplace equation, see [HK] p.126-127, and with respect to a Wolff potential.

1.3. The doubling property of boundary measures. Let $\Omega \subset \mathbb{R}^n$ be a $\mathbf{X}$ -NTA-domain with parameters $M \geq 2, r_0 > 0$, in the sense of Definition 2. It then follows from [D], see also [TW], that every point $w \in \partial\Omega$ is regular for operator $L_{A, p}$ in (1.8) whenever $1 < p < \infty$. In particular, by definition there exists $a_r(w) \in \Omega$ such that $r/M < d(a_r(w), w) \leq r$ and $d(a_r(w), \partial\Omega) > r/M$ whenever $w \in \partial\Omega$ and $r \leq r_0$. Based on the considerations above, we will in the following always assume that

$$(1.19) \quad \Omega \subset B_{cc}(0, R_0/2).$$

Theorem 1.2 is a straightforward consequence of the following theorem proved in Section 5.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a $\mathbf{X}$ -NTA-domain with constants $M \geq 2$ and r_0 . Assume (1.19) and let Q be the local homogeneous dimension (of $\mathbf{X}$) relative to $B_{cc}(0, R_0)$. Let p , $1 < p < \infty$, be fixed and assume that $A = (A_1, \dots, A_m) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is as in Definition 1 for some α, β . Let $w \in \partial\Omega$, $0 < r < r_0$ and suppose that u is a non-negative A -harmonic function in $\Omega \cap B_{cc}(w, 2r)$, continuous in $\overline{\Omega \cap B_{cc}(w, 2r)}$ and $u = 0$ on $\partial\Omega \cap B_{cc}(w, 2r)$. Extend u to $B_{cc}(w, 2r)$ by defining*

$u \equiv 0$ on $B_{cc}(w, 2r) \setminus \Omega$. Then there exists a unique locally finite positive Borel measure μ on $\partial\Omega \cap B_{cc}(w, 2r)$, such that

$$(1.20) \quad \int A(x, \mathbf{X}u) \cdot \mathbf{X}\phi \, dx = - \int \phi \, d\mu,$$

whenever $\phi \in C_0^\infty(B_{cc}(w, 2r))$. Moreover, there exists $C = C(p, n, \alpha, \beta, \mathbf{X}, Q, M)$, $1 \leq C < \infty$, such that if $\tilde{r} = r/C$, then

$$(1.21) \quad C^{-1} \frac{r^p \mu(B_{cc}(w, \tilde{r}))}{|B_{cc}(w, r)|} \leq [u(a_{\tilde{r}}(w))]^{p-1} \leq C \frac{r^p \mu(B_{cc}(w, \tilde{r}/2))}{|B_{cc}(w, r)|}.$$

2. PRELIMINARIES

2.1. Carnot-Carathéodory distance. We consider a system $\mathbf{X} = \{X_1, \dots, X_m\}$ of C^∞ vector fields in $\mathbb{R}^n$, with $n \geq 3$, satisfying Hörmander's finite rank condition (1.2). A piecewise C^1 curve $\gamma : [0, \ell] \rightarrow \mathbb{R}^n$ is called subunitary, see [FP], if

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{j=1}^m \langle X_j(\gamma(t)), \xi \rangle^2,$$

for every $\xi \in \mathbb{R}^n$, whenever $\gamma'(t)$ exists. We note explicitly that the above inequality forces $\gamma'(t)$ to belong to the span of $\{X_1(\gamma(t)), \dots, X_m(\gamma(t))\}$. The subunit length of γ is by definition $l_s(\gamma) = \ell$. If we fix an open set $U \subset \mathbb{R}^n$, then given $x, y \in U$, we denote by $\mathcal{S}_U(x, y)$ the collection of all sub-unitary $\gamma : [0, \ell] \rightarrow U$ which join x to y . The accessibility theorem of Chow and Rashevsky, see [Ch], [Ra], states that, if U is connected, then for every $x, y \in U$ there exists $\gamma \in \mathcal{S}_U(x, y)$. As a consequence, if we define

$$d_U(x, y) = \inf \{l_s(\gamma) \mid \gamma \in \mathcal{S}_U(x, y)\},$$

we obtain the Carnot-Carathéodory distance associated with the system $\mathbf{X}$. When $U = \mathbb{R}^n$, we write $d(x, y)$ instead of $d_{\mathbb{R}^n}(x, y)$. It is clear that $d(x, y) \leq d_U(x, y)$, $x, y \in U$, for every connected open set $U \subset \mathbb{R}^n$. In [NSW] it was proved that, given $U \subset \subset \mathbb{R}^n$, there exist $C \geq 1, \epsilon > 0$ such that

$$(2.1) \quad C^{-1}|x - y| \leq d_U(x, y) \leq C|x - y|^\epsilon, \text{ whenever } x, y \in U.$$

This gives $d(x, y) \leq C|x - y|^\epsilon$, $x, y \in U$, and therefore the inclusion map

$$i : (\mathbb{R}^n, |\cdot|) \rightarrow (\mathbb{R}^n, d) \text{ is continuous.}$$

$|\cdot|$ is the standard Euclidean norm. Furthermore, it is easy to see that the opposite inclusion also holds, see [GN1], and hence the metric and the Euclidean topologies are equivalent.

2.2. $\mathbf{X}$ -balls. The basic properties of the balls in this metric were established in [NSW], where the authors prove that there exist, given $U \subset \subset \mathbb{R}^n$, constants $C \geq 1, R_0 > 0$ such that, for any $x \in U$, and $0 < r \leq R_0$,

$$(2.2) \quad C^{-1} \leq \frac{|B_{cc}(x, r)|}{\Lambda(x, r)} \leq C,$$

where $\Lambda(x, r) = \sum_I |a_I(x)| r^{d_I}$ is a polynomial function with continuous coefficients. In particular the degree of $\Lambda(x, r)$ is bigger than or equal to n and less than or equal to Q . Hence the function $r \rightarrow \Lambda(x, r)/r^2$ is increasing, we let

$$E(x, r) = \frac{\Lambda(x, r)}{r^2},$$

and we let $F(x, r)$ denote the inverse of $E(x, r)$ w.r.t. the variable r . Following [CGL] we define the $\mathbf{X}$ -balls as

$$(2.3) \quad B_{\mathbf{X}}(x, r) = \{y \in \mathbb{R}^n \mid \Gamma(x, y) > 1/E(x, r)\}$$

where $\Gamma(x, y)$ is the fundamental solution associated to the operator $\sum_{i=1}^m X_i^* X_i$. In [CGL] it is proved that the sets $\{B_{\mathbf{X}}(x, r)\}$ can serve as a replacement for the balls $\{B_{cc}(x, r)\}$ in the sense that there exists, given $U \subset\subset \mathbb{R}^n$, a constant $a = a(n, \mathbf{X}, Q, U)$, $1 \leq a < \infty$, such that

$$(2.4) \quad B_{cc}(x, r/a) \subseteq B_{\mathbf{X}}(x, r) \subseteq B_{cc}(x, ar),$$

whenever $x \in U \subset\subset \mathbb{R}^n$. In the forthcoming sections we will make use of the Carnot-Carathéodory metric balls as well as the $\mathbf{X}$ -balls. Note that from (2.4) and (1.15) we see that (1.15) holds with $B_{cc}(x, 2r)$, $B_{cc}(x, r)$ replaced by $B_{\mathbf{X}}(x, 2r)$, $B_{\mathbf{X}}(x, r)$ and with C replaced by a constant $\tilde{C} = \tilde{C}(C, a) > 0$. Assuming (1.19) we will often write $a(R_0)$ for the constant a in case we apply the above with $U = B_{\mathbf{X}}(0, R_0)$. The main reason to also work with $\mathbf{X}$ -balls is that in the setting of these balls we can make use of the cutoff functions constructed in Lemma 3.6 in [CGL]. These cutoff functions are needed as technical tools in several interior 'ball in ball' estimates.

2.3. NTA-domains adapted to $\mathbf{X}$. Given a bounded open set $\Omega \subset \mathbb{R}^n$, we recall that a ball $B_{cc}(x, r)$ is M -non-tangential in Ω (with respect to the metric d), see [CG], if

$$Mr < d(x, \partial\Omega) < M^2 r.$$

Note that although this definition differs from the one in [CG], they are equivalent. The reason for using this definition instead is to have the Harnack inequality not depending on M . Furthermore, given $x, x' \in \Omega$ a sequence of M -non-tangential balls in Ω , $B_{cc}(x_1, r_1), \dots, B_{cc}(x_k, r_k)$, is called a M -Harnack chain of length k , joining x to x' , if $x \in B_{cc}(x_1, r_1)$, $x' \in B_{cc}(x_k, r_k)$, and $B_{cc}(x_i, r_i) \cap B_{cc}(x_{i+1}, r_{i+1}) \neq \emptyset$ for $i \in \{1, \dots, k-1\}$. We note that in this definition consecutive balls have comparable radii.

Definition 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that Ω is a *non-tangentially accessible domain* in $\mathbb{R}^n$ with respect to $\mathbf{X}$, a $\mathbf{X}$ -NTA-domain or simply NTA-domain when we suppress $\mathbf{X}$, hereafter, if there exist $M \geq 2$, $r_0 > 0$ such that the following holds.

- (i) (Interior corkscrew condition) For any $w \in \partial\Omega$ and $r \leq r_0$ there exists $a_r(w) \in \Omega$ such that $r/M < d(a_r(w), w) \leq r$ and $d(a_r(w), \partial\Omega) > r/M$.
- (ii) (Exterior corkscrew condition) $\Omega^c = \mathbb{R}^n \setminus \bar{\Omega}$ satisfies property (i).
- (iii) (Harnack chain condition) Whenever $\epsilon > 0$ and $x_1, x_2 \in \Omega$ are such that $d(x_i, \partial\Omega) > \epsilon$, $i \in \{1, 2\}$, and $d(x_1, x_2) \leq C\epsilon$, for some constant $C \geq 1$, then there exists a M -Harnack chain joining x_1 to x_2 whose length depends on C but not on ϵ .

Remark 2.1. Based on (1.19) we see that there exists $r_0 = r_0(\Omega) > 0$ such that the closure of balls $B_{cc}(x, r)$ with $x \in \Omega$ and $0 < r < r_0$ are compact. In view of this, and in consistency with (1.17) and (1.19), we in the following always assume, for a given $\mathbf{X}$ -NTA-domain $\Omega \subset \mathbb{R}^n$ with constant M and r_0 and following [CG], that the constant r_0 has been adjusted in such a way that the closure of balls $B_{cc}(x, r)$, with $x \in \bar{\Omega}$ and $0 < r < r_0$, are compact. In fact, we will always assume that r_0 is adjusted so that $B_{cc}(x, 2r) \subset B_{cc}(0, R_0)$, whenever $x \in \bar{\Omega}$ and $0 < r < r_0$, and where R_0 was introduced in (1.17).

Remark 2.2. Recall that when d is the standard Euclidean distance then the notion of NTA-domain was introduced in [JK] in connection with the study of the boundary behaviour of non-negative harmonic functions. The first study of NTA-domains in a sub-Riemannian context was conducted in [CG], where a large effort was devoted to the nontrivial problem of constructing examples, see also [CGN2].

In [CG] a Fatou theory was also developed and, in particular, the doubling condition for harmonic measure, and the comparison theorem for quotients of non-negative solutions of sub-Laplacians were established. Subsequently, in [CGN1], [CGN3], [CGN4] the notion of $\mathbf{X}$ -NTA-domain was combined with an intrinsic outer ball condition to obtain the solvability of the Dirichlet problem with boundary data in L^p . Concerning NTA-domains in a sub-Riemannian context we also refer to [MM1].

Remark 2.3. In general it is not easy to determine if a domain is a $\mathbf{X}$ -NTA-domain or not. However, in [CG] it was proved that in every Carnot group of step 2 the so called gauge balls are $\mathbf{X}$ -NTA-domains. Furthermore, in the Heisenberg group, $\mathbb{H}^n$, every bounded domain whose boundary is (Euclidean) $C^{1,1}$ is a $\mathbf{X}$ -NTA-domain. This result is proved in [MM2]. In fact in [MM2] this result is proved for every Carnot group of step 2.

2.4. The continuous Dirichlet problem.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a $\mathbf{X}$ -NTA-domain. Then, given $f \in W_{\mathbf{X}}^{1,p}(\Omega) \cap C(\overline{\Omega})$, there exists a unique weak solution $u \in W_{\mathbf{X}}^{1,p}(\Omega) \cap C(\overline{\Omega})$ to the continuous Dirichlet problem for the operator in (1.8). In particular, all points on $\partial\Omega$ are regular for the Dirichlet problem for the operator in (1.8).*

Proof. This follows from Theorem 3.8 in [D], see also [TW], and the properties of NTA-domains. $\square$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Following [LU] we say that Ω satisfies a uniform outer positive d -density condition if there exist constants $C \geq 1, r_1 > 0$ such that for every $w \in \partial\Omega$ and every $0 < r < r_1$ one has,

$$(2.5) \quad |(\mathbb{R}^n \setminus \Omega) \cap B_{cc}(w, r)| \geq C^{-1} |B_{cc}(w, r)|.$$

Then, using the Wiener criteria in Theorem 3.8 in [D] it is easily seen that the following more general result holds.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that Ω satisfies a uniform outer positive d -density condition with parameters $C \geq 1, r_1 > 0$ as stated above. Then all points on $\partial\Omega$ are regular for the Dirichlet problem for the operator in (1.8).*

By condition (ii) in Definition 2 we conclude that if $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain, then Ω satisfies a uniform outer positive d -density condition with parameters $c \geq 1, r_1 > 0$ which only depend on the NTA-parameters of Ω . We next note the following lemma, see Lemma 6.4 in [LU].

Lemma 2.3. *Assume (1.17) and we let Q denote the local homogeneous dimension (of $\mathbf{X}$) relative $B_{cc}(0, R_0)$. Then there exists a positive constant $\epsilon = \epsilon(\mathbf{X}, Q)$, such that if $B_{cc}(w, 2r) \subset B_{cc}(0, R_0/2)$, then*

$$|B_{cc}(x, t) \cap B_{cc}(w, r)| \geq \epsilon |B_{cc}(x, t)|$$

whenever $0 < t \leq r$, $x \in \partial B_{cc}(w, r)$.

Using Lemma 2.3 we see that $\mathbb{R}^n \setminus \overline{B_{cc}(w, r)}$ satisfies a uniform outer positive d -density condition, and thus, by Lemma 2.2, all points on $\partial B_{cc}(w, r)$ are regular for the Dirichlet problem for the operator in (1.8) in $U \setminus \overline{B_{cc}(w, r)}$ where U is any bounded open set such that $\overline{B_{cc}(w, r)}$ is a compact subset of U . Note also that the intersection of two domains, both of which satisfies a uniform outer positive d -density condition, also satisfies this condition. This is the idea behind the following lemma, see Theorem 6.5 in [LU], which we here state and, due to its simple and illustrative proof, also prove.

Lemma 2.4. *Let $D \subset \mathbb{R}^n$ be a bounded open set, $D \subset B_{cc}(0, R_0/2)$. Then, for every $\delta > 0$ there exists a set D_δ such that*

- (i) $\{x \in D : d(x, \partial D) > \delta\} \subset D_\delta \subset D$,
- (ii) $|B_{cc}(w, t) \setminus D_\delta| \geq \epsilon |B_{cc}(w, r)|$ for every $w \in \partial D_\delta$, $0 < t \leq \delta$.

In (ii) ϵ is the constant in Lemma 2.3.

Proof. Simply put

$$(2.6) \quad D_\delta = D \setminus \bigcup_{j=1}^k \overline{B_{cc}(w_j, \delta)}$$

where $w_j \in \partial D$ and $\{B_{cc}(w_j, \delta)\}_{j=1}^k$ is a finite covering of ∂D . Then D_δ satisfies (i). (ii) follows from Lemma 2.3. $\square$.

By Lemma 2.4 and Lemma 2.2 we can conclude that any bounded open set $D \subset \mathbb{R}^n$, $D \subset B_{cc}(0, R_0/2)$, can be approximated by a set D_δ as in Lemma 2.4 and that the continuous Dirichlet problem for the operator in (1.8) can be solved in D_δ . Given D and δ , we will in the bulk of the paper refer to D_δ as the

(2.7) regular approximating domain of D based on δ .

2.5. Convention concerning constants. Throughout the paper C will denote, unless otherwise stated, a positive constant ≥ 1 , not necessarily the same at each occurrence, depending at most on $p, n, \alpha, \beta, X, M$. In general, $C(a_1, \dots, a_m)$ denotes a positive constant ≥ 1 , which may depend at most on $p, n, \alpha, \beta, \mathbf{X}, M$ and $a_1, \dots, a_m$, and which is not necessarily the same at each occurrence. If $A \approx B$ then A/B is bounded from above and below by constants which, unless otherwise stated, depend at most on $p, n, \alpha, \beta, \mathbf{X}, M$.

3. BASIC ESTIMATES

Throughout the section we will, unless stated otherwise, assume (1.17) and we let Q denote the local homogeneous dimension (of $\mathbf{X}$) relative $B_{cc}(0, R_0)$.

Lemma 3.1. *There exists a constant $C = C(n, \mathbf{X}, Q)$, $1 \leq C < \infty$, and $\tilde{r} = \tilde{r}(n, \mathbf{X}, Q)$, $0 < \tilde{r} < R_0/2$, such that the following is true. Let $B_{cc}(x_0, r) \subset B_{cc}(0, R_0/2)$, $0 < r \leq \tilde{r}$. If $1 \leq q \leq Q$, let k satisfy $1 \leq k \leq Q/(Q-q)$, if $q \geq Q$ let k satisfy $1 \leq k < \infty$. Then,*

$$(3.1) \quad \left(\int_{B_{cc}(x_0, r)} |u|^{kq} dx \right)^{\frac{1}{kq}} \leq Cr \left(\int_{B_{cc}(x_0, r)} |\mathbf{X}u|^q dx \right)^{\frac{1}{q}},$$

whenever $u \in W_{\mathbf{X},0}^{1,q}(B_{cc}(x_0, r))$. Specifically, when $k = 1$ we have

$$(3.2) \quad \int_{B_{cc}(x_0, r)} |u|^q dx \leq Cr^q \int_{B_{cc}(x_0, r)} |\mathbf{X}u|^q dx,$$

for all $1 \leq q < \infty$.

Proof. For (3.1) see [CDG2] for references, (3.2) follows from (3.1) directly if $1 \leq q < Q$ and by an application of Hölder's inequality if $q \geq Q$. $\square$

Lemma 3.2. *Let $w \in \mathbb{R}^n$. Let u be a positive A -harmonic function in $B_{cc}(w, 2r)$. Then there exists $C = C(p, n, \alpha, \beta, \mathbf{X}, Q)$, $1 \leq C < \infty$, such that*

$$(i) \quad \sup_{B_{cc}(w, r)} u \leq C \inf_{B_{cc}(w, r)} u.$$

Furthermore, there exists $\sigma = \sigma(p, n, \alpha, \beta, \mathbf{X}, Q) \in (0, 1)$ such that if $x, y \in B_{cc}(w, r)$, then

$$(ii) \quad |u(x) - u(y)| \leq C \left(\frac{d(x, y)}{r} \right)^\sigma \sup_{B_{cc}(w, 2r)} u.$$

Proof. (i) is Theorem 2.3 in [D]. For the proof of (ii) we can use local Hölder continuity, Theorem 2.4 in [D], together with the classical iteration method described in Theorem 6.6 in [HKM]. See also [CDG1] for reference. $\square$

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a $\mathbf{X}$ -NTA-domain with constants M and r_0 . Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is a non-negative A -harmonic function in $\Omega \cap B_{cc}(w, 2r)$, continuous in $\bar{\Omega} \cap B_{cc}(w, 2r)$ and that $u = 0$ on $\partial\Omega \cap B_{cc}(w, 2r)$. Then there exist $C = C(p, n, \alpha, \beta, \mathbf{X}, Q, M)$, $1 \leq C < \infty$, $\sigma = \sigma(p, n, \alpha, \beta, \mathbf{X}, Q, M) \in (0, 1)$ such that if $x, y \in \Omega \cap B_{cc}(w, r)$, then*

$$|u(x) - u(y)| \leq C \left(\frac{d(x, y)}{r} \right)^\sigma \sup_{\Omega \cap B_{cc}(w, 2r)} u.$$

Proof. The lemma follows from Theorem 3.10 in [D] by standard arguments. $\square$

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a $\mathbf{X}$ -NTA-domain. Let $w \in \partial\Omega$, $0 < r < r_0$, and suppose that u is a non-negative A -harmonic function in $\Omega \cap B_{cc}(w, 2r)$, continuous in $\bar{\Omega} \cap B_{cc}(w, 2r)$ and that $u = 0$ on $\partial\Omega \cap B_{cc}(w, 2r)$. There exists $C = C(p, n, \alpha, \beta, \mathbf{X}, Q, M)$, $1 \leq C < \infty$, such that if $\tilde{r} = r/C$, then*

$$\sup_{\Omega \cap B_{cc}(w, \tilde{r})} u \leq C u(a_{\tilde{r}}(w)).$$

Proof. Using Lemma 3.2 and Lemma 3.3 the proof of the lemma follows along the lines of the classical argument in [CFMS]. $\square$

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\Lambda > 1$ be a constant. Then there exist a sequence of balls $\{B_i = B_{cc}(w_i, r_i)\}$, $w_i \in \Omega$, $r_i = d(w_i, \mathbb{R}^n \setminus \Omega)/(10\Lambda)$, and constants $C_1 = C_1(Q, \Lambda)$, $C_2 = C_2(Q, \Lambda) \geq \Lambda$, such that*

- (i) the balls $B_{cc}(w_i, r_i/\Lambda)$ are pairwise disjoint,
 - (ii) $\Omega = \cup_i B_i$, (iii) $B_{cc}(w_i, \Lambda r_i) \subset \Omega$,
 - (iv) $B_{cc}(w_i, C_2 r_i) \cap (R^n \setminus \Omega) \neq \emptyset$,
 - (v) a point in Ω is contained in at most C_1 balls B_i .
- (3.3)

Proof. This is Theorem 1.3 in [CW]. Since, in our setting, (Ω, d, dx) is a doubling metric space we see, following remark on p.67 in [CW], that (Ω, d, dx) is homogeneous space with constant depending on our local dimension Q . Note that we have, and this makes no difference in the proof, defined our radii with constants different to [CW]. $\square$

Lemma 3.6. *Let u be a non-negative A -subsolution to (1.8) in $B_{cc}(w, 4a^2 r) \subset B_{cc}(0, R_0/2)$, where $a = a(n, X, Q, R_0)$ is the constant in (2.4). Let $\hat{C} = 2a^2$. Then*

$$\int_{B_{cc}(w, r)} |\mathbf{X}u|^p dx \leq C \frac{|B_{cc}(w, \hat{C}r)|}{r^p} \left(\sup_{B_{cc}(w, \hat{C}r)} u \right)^p,$$

for some constant $C = C(p, n, \alpha, \beta, \mathbf{X}, Q)$, $1 \leq C < \infty$.

Proof. Let m denote the supremum of u on $B_{\mathbf{X}}(w, 2ar)$. Using Lemma 3.6 in [CGL] we let $\phi \in C_0^\infty(B_{\mathbf{X}}(w, 2ar))$ be such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B_{\mathbf{X}}(w, ar)$ and $|\mathbf{X}\phi| \leq Cr^{-1}$. Define $\psi = \phi^p(m+u)^p$. Then

$$\mathbf{X}\psi = p\phi^{p-1}(m+u)^p\mathbf{X}\phi + p\phi^p(m+u)^{p-1}\mathbf{X}u,$$

and using the notion of A -subsolution to (1.8) we see that

$$\int \langle A(x, \mathbf{X}u), \mathbf{X}\phi \rangle \phi^{p-1}(m+u)^p dx + \int \langle A(x, \mathbf{X}u), \mathbf{X}u \rangle \phi^p(m+u)^{p-1} dx \leq 0.$$

Using the inequality in the last display and the assumptions on A in Definition 1, we deduce that

$$(3.4) \quad \int |\mathbf{X}u|^p \phi^p(m+u)^{p-1} dx \leq C \int |\mathbf{X}u|^{p-1} |\mathbf{X}\phi| \phi^{p-1}(m+u)^p dx.$$

Hence

$$(3.5) \quad \int |\mathbf{X}u|^p \phi^p(m+u)^{p-1} dx \leq \frac{C}{r} \int |\mathbf{X}u|^{p-1} \phi^{p-1}(m+u)^p dx$$

and, using also the definition of m and ψ ,

$$(3.6) \quad \int |\mathbf{X}u|^p \psi dx \leq \frac{Cm^2}{r} \int |\mathbf{X}u|^{p-1} \psi^{(p-1)/p} dx.$$

Using the Hölder inequality and standard manipulations it follows from (3.6)

$$(3.7) \quad \int |\mathbf{X}u|^p \psi dx \leq \frac{Cm^{2p}}{r^p} |B_{\mathbf{X}}(w, 2ar)|.$$

Hence, by construction and using (2.4),

$$(3.8) \quad \begin{aligned} \int_{B_{cc}(w, r)} |\mathbf{X}u|^p dx &\leq \frac{Cm^p}{r^p} |B_{\mathbf{X}}(w, 2ar)| \\ &\leq C \frac{|B_{cc}(w, 2a^2r)|}{r^p} \left(\sup_{B_{cc}(w, 2a^2r)} u \right)^p, \end{aligned}$$

and the proof of the lemma is complete. $\square$

Lemma 3.7. *Let u be a non-negative A -subsolution to (1.8) in $B_{cc}(w, 2r)$ and assume $B_{cc}(w, 2r) \subset B_{cc}(0, R_0/2)$. Then,*

$$\int_{B_{cc}(w, s)} |\mathbf{X}u|^p dx \leq C \frac{|B_{cc}(w, s'')|}{(s'' - s)^p} \left(\sup_{B_{cc}(w, s'')} u \right)^p,$$

for some constant $C = C(p, n, \alpha, \beta, \mathbf{X}, Q)$, $1 \leq C < \infty$, whenever $r < s < s'' < 2r$.

Proof. Let $\hat{C}$ be as in the statement of Lemma 3.6 and let $\{B_{cc}(w_i, r_i)\}$ be a Whitney type covering of $B_{cc}(w, s'')$ in the sense of Lemma 3.5 with $\Lambda = 2\hat{C}$. Let Γ_s be the set of all balls in the set $\{B_{cc}(w_i, r_i)\}$ which intersects $B_{cc}(w, s)$. Then, using that $B_{cc}(w_i, C_2r_i) \cap (\mathbb{R}^n \setminus B_{cc}(w, s'')) = \emptyset$, C_2 the constant from Lemma 3.5, we can conclude that each ball $B_{cc}(w_i, r_i) \in \Gamma_s$ satisfies

$$(3.9) \quad s'' - s \leq (C_2 + 1)r_i.$$

Now, using Lemma 3.6 for each $B_{cc}(w_i, r_i) \in \Gamma_s$ we obtain

$$(3.10) \quad \begin{aligned} \int_{B_{cc}(w, s)} |\mathbf{X}u|^p dx &\leq \sum_{i: B_{cc}(w_i, r_i) \in \Gamma_s} \int_{B_{cc}(w_i, r_i)} |\mathbf{X}u|^p dx \\ &\leq C \sum_{i: B_{cc}(w_i, r_i) \in \Gamma_s} \int_{B_{cc}(w_i, r_i)} r_i^{-p} |B_{cc}(w_i, \hat{C}r_i)| \left(\sup_{B_{cc}(w_i, \hat{C}r_i)} u \right)^p \\ &\leq C (s'' - s)^{-p} |B_{cc}(w, s'')| \left(\sup_{B_{cc}(w, s'')} u \right)^p, \end{aligned}$$

and the proof is complete. $\square$

Lemma 3.8. *Let $O \subset \mathbb{R}^n$ be an open set. Let $u, \phi \in W_{\mathbf{X}}^{1,p}(O)$, let u be a solution to (1.8) in O and assume that $u - \phi \in W_{\mathbf{X},0}^{1,p}(O)$. Then there exists a constant $C = C(n, p, \alpha, \beta, \mathbf{X}, Q)$, $1 \leq C < \infty$*

$$(3.11) \quad \int_{\Omega} |\mathbf{X}u|^p dx \leq C \int_{\Omega} |\mathbf{X}\phi|^p dx.$$

Proof. See Lemma 2.7 in [D] for this quasi-minimizing property. $\square$

Theorem 3.1. *Let u be a non-negative A -superharmonic function in $B_{cc}(w, r) \subset B_{cc}(0, R_0/2)$, and let μ be the Riesz measure associated to u . Then*

$$\hat{C}^{-1} \mathbf{W}_{p,\mathbf{X}}^{\mu}(w, r/2) \leq u(w) \leq \hat{C} \inf_{B_{cc}(w, r/2)} u + \hat{C} \mathbf{W}_{p,\mathbf{X}}^{\mu}(w, r).$$

for some constant $\hat{C} = \hat{C}(p, n, \alpha, \beta, \mathbf{X}, Q)$, $1 \leq \hat{C} < \infty$.

Proof. This is essentially Theorem 5.1 in [TW]. Indeed, given $w \in \mathbb{R}^n$, $r > 0$, let

$$(3.12) \quad \tilde{\mathbf{W}}_{p,\mathbf{X}}^{\mu}(w, r) = \int_0^r \left(\frac{\mu(B_{cc}(w, t))}{\text{Cap}(B_{cc}(w, t/2), B_{cc}(w, t))} \right)^{1/(p-1)} \frac{dt}{t},$$

be the Wolff potential associated to μ and introduced in display (5.9) in [TW]. Here $\text{Cap}(B_{cc}(w, t/2), B_{cc}(w, t))$ is the subelliptic capacity, defined in [TW], of the condenser $(B_{cc}(w, t/2), B_{cc}(w, t))$. However, using $B_{cc}(w, r) \subset B_{cc}(0, R_0/2)$ and (1.17) it follows that $\text{Cap}(B_{cc}(w, t/2), B_{cc}(w, t))$ is comparable to $t^{-p}|B_{cc}(w, t)|$ and hence our notion of Wolff potential is equivalent to the notion of Wolff potential considered in [TW]. $\square$

4. PROOF OF THEOREM 1.3

Let p , $1 < p < \infty$, be fixed and assume that $A = (A_1, \dots, A_m) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is as in Definition 1 for some α, β . Let u be a non-negative A -subharmonic function in $\mathbb{R}^n$ and let μ be the Riesz measure associated to u . To start the proof of Theorem 1.3 we introduce

$$(4.1) \quad M(w, r) = \sup_{x \in B_{cc}(w, r)} u(x).$$

Lemma 4.1. *Let u be a non-negative A -subharmonic function in $\mathbb{R}^n$ and let μ be the Riesz measure associated to u . Then there exists $\theta = \theta(p, n, \alpha, \beta, \mathbf{X}, Q)$, $0 < \theta < 1$, such that if $M(w, tr) \leq \theta M(w, r)$, for some $0 < t \leq 1$ and $r > 0$, then*

$$M(w, r) \leq C \left(r^p \frac{\mu(B_{cc}(w, 2r))}{|B_{cc}(w, 2r)|} \right)^{1/(p-1)},$$

where $C = C(t, p, n, \alpha, \beta, \mathbf{X}, Q)$, $1 \leq C < \infty$.

Proof. We will prove the lemma along the lines of the proof of Lemma 3.1 in [KZ] and to this end we divide the proof into the cases $p \geq 2$ and $1 < p < 2$.

The case $p \geq 2$. Let $\delta = r/10^6$, $D := B_{cc}(w, \frac{3}{2}r + \delta)$, and let D_{δ} regular approximating domain of D based on δ , see (2.7). Note that $B_{cc}(w, \frac{3}{2}r) \subset D_{\delta}$ and let h be A -harmonic in D_{δ} with boundary values defined by u , i.e.,

$$\begin{cases} L_{A,p}h = 0 & \text{in } D_{\delta} \\ h = u & \text{in } W_{\mathbf{X},0}^{1,p}(D_{\delta}). \end{cases}$$

By the comparison principle we have $0 \leq u \leq h$ on D_δ and using the Harnack inequality we see that

$$(4.2) \quad \inf_{B_{cc}(w,r)} h \geq C^{-1} \sup_{B_{cc}(w,r)} h \geq C^{-1} \sup_{B_{cc}(w,r)} u = C^{-1} M(w, r).$$

Thus we can conclude that if we let $\theta = C^{-1}/2$, then

$$(4.3) \quad \begin{aligned} h(x) - u(x) &\geq \inf_{B_{cc}(w,r)} h - \sup_{B_{cc}(w,tr)} u \geq C^{-1} M(w, r) - M(w, tr) \\ &\geq (C^{-1} - \theta) M(w, r) \geq \theta M(w, r), \end{aligned}$$

for all $x \in B_{cc}(w, tr)$. Let

$$\phi = \begin{cases} \min(h(x) - u(x), \theta M(w, r)) & x \in D_\delta, \\ 0 & \text{elsewhere.} \end{cases}$$

We then obtain that $\phi \geq 0$, $\phi \in W_{\mathbf{X},0}^{1,p}(D_\delta)$ and

$$\phi = \theta M(w, r) \quad \text{on } B_{cc}(w, tr).$$

Let Γ be the set of points where $\mathbf{X}\phi \neq 0$ in the weak sense. Then, using (1.11) we see that

$$(4.4) \quad \begin{aligned} \int_{B_{cc}(w,2r)} |\mathbf{X}\phi|^p dx &\leq \int_{\Gamma} (|\mathbf{X}u| + |\mathbf{X}h|)^{p-2} |\mathbf{X}u - \mathbf{X}h|^2 dx \\ &\leq C \int_{B_{cc}(w,2r)} (A(x, \mathbf{X}h) - A(x, \mathbf{X}u)) \cdot \mathbf{X}\phi dx \\ &= -C \int_{B_{cc}(w,2r)} A(x, \mathbf{X}u) \cdot \mathbf{X}\phi dx = C \int_{B_{cc}(w,2r)} \phi d\mu \\ &\leq C\theta M(w, r) \mu(B_{cc}(w, 2r)). \end{aligned}$$

Furthermore, using Lemma 3.1 and the definition of ϕ , we see that

$$(4.5) \quad \begin{aligned} (\theta M(w, r))^p |B_{cc}(w, tr)| &\leq \int_{B_{cc}(w,2r)} \phi^p dx \\ &\leq Cr^p \int_{B_{cc}(w,2r)} |\mathbf{X}\phi|^p dx. \end{aligned}$$

Combining (4.4), (4.5) and (1.16) we can therefore conclude that

$$M(w, r) \leq C \left(r^p \frac{\mu(B_{cc}(w, 2r))}{|B_{cc}(w, 2r)|} \right)^{1/(p-1)}$$

and hence the proof is complete in the case $p \geq 2$.

The case $1 < p < 2$. Let $s_j = 2r(1 - 2^{-j})$, $j = 1, 2, \dots$, then $r \leq s_j < \bar{s}_j < s_{j+1} \leq 2r$, where $\bar{s}_j = (s_j + s_{j+1})/2$. Let in this case $\delta_j = (s_{j+1} - s_j)/10^6$, $D := B_{cc}(w, \bar{s}_j)$, and let D_{δ_j} regular approximating domain of D based on δ , see (2.7). Note that

$$B_{cc}(w, s_j) \Subset D_{\delta_j} \subset B_{cc}(w, \bar{s}_j)$$

and let $h = h(\cdot, \bar{s}_j)$ be the A -harmonic function in D_{δ_j} such that $h - u \in W_{0,\mathbf{X}}^{1,p}(D_{\delta_j})$. Let $x_1, x_2 \in \bar{B}_{cc}(w, s_j)$ s.t. $\sup_{B_{cc}(w, s_j)} h = h(x_1)$ and $\inf_{B_{cc}(w, s_j)} h = h(x_2)$, then we can form a Harnack chain connecting x_1, x_2 inside D_{δ_j} s.t.

$$(4.6) \quad \inf_{B_{cc}(w, s_j)} h \geq c_j \sup_{B_{cc}(w, s_j)} h \geq c_j M(w, s_j),$$

where $c_j = C^{-1} \left(\frac{\bar{s}_j - s_j}{\bar{s}_j} \right)^\gamma$, where $\gamma = \gamma(p, n, \alpha, \beta, \mathbf{X}, Q)$, $\gamma > 0$. Assume

$$(4.7) \quad c_j M(w, s_j) \geq 2\theta M(w, r),$$

for some $\theta \in (0, 1)$ to be chosen. Then, using (4.6) and (4.7) we see that

$$(4.8) \quad \begin{aligned} h(x, \bar{s}_j) - u(x) &\geq \inf_{B_{cc}(w, s_j)} h - \sup_{B_{cc}(w, tr)} u \geq c_j M(w, s_j) - \theta M(w, r) \\ &\geq \frac{c_j}{2} M(w, s_j), \end{aligned}$$

whenever $x \in B_{\mathbf{X}}(w, tr)$. It is enough to show that (4.7) implies the following

$$(4.9) \quad \begin{aligned} M(w, s_j)^{p/2} &\leq C \bar{s}_j^p c_j^{-p/2} (s_{j+1} - \bar{s}_j)^{-p(2-p)/2} \\ &\quad \times M(w, s_{j+1})^{p(2-p)/2} \left(\frac{\mu(B_{cc}(w, 2r))}{|B_{cc}(w, 2r)|} \right)^{p/2}, \end{aligned}$$

for every $j = 1, 2, \dots$. To see why (4.9) is enough let us as in [KZ] introduce the functions

$$(4.10) \quad \begin{aligned} \Psi(\tau) &= M(w, \tau)^{p/2} \left(r^p \frac{\mu(B_{cc}(w, 2r))}{|B_{cc}(w, 2r)|} \right)^{\frac{-p}{2(p-1)}}, \\ k_j &= C s^p c_j^{-p/2} (s_{j+1} - \bar{s}_j)^{-p(2-p)/2} r^{-p^2/2}. \end{aligned}$$

Using this notation we see that (4.9) can be expressed as

$$(4.11) \quad \Psi(s_j) \leq k_j (\Psi(s_{j+1}))^{2-p}, \quad j = 1, 2, \dots$$

We now end up in two cases, either (4.7) holds for all $j = 1, 2, \dots$, this gives using (4.11) for each j , that $M(w, r) = 0$, since $M(w, 2r) < \infty$, in which case the Lemma is true. The other case is that (4.7) fails for some $j \geq 1$. Choosing $\theta = (2C5^\gamma)^{-1}$ gives that (4.7) holds for $j = 1$. Let now j_0 be the smallest $j > 1$ such that (4.7) fails. Since (4.11) holds for all $j = 1, \dots, j_0 - 1$ we can run the final iteration argument on p.190 in [KZ], and get that there exists a constant $C = C(t, p, n, \alpha, \beta, \mathbf{X}, Q) > 0$ such that

$$\Psi(r) \leq C < \infty,$$

which is just another way of writing the Lemma. To prove that (4.7) implies (4.9) we let

$$\phi = \begin{cases} \min(h(x, \bar{s}_j) - u(x), \frac{1}{2} c_j M(w, s_j)) & x \in D_{\delta_j} \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\phi \in W_{\mathbf{X},0}^{1,p}(D_{\delta_j})$ and $\phi \geq 0$. Let Γ be the set of points where $\mathbf{X}\phi \neq 0$ in the weak sense and let

$$(4.12) \quad \begin{aligned} I_1 &:= \left(\int_{\Gamma} (|\mathbf{X}h| + |\mathbf{X}u|)^{p-2} |\mathbf{X}h - \mathbf{X}u|^2 dx \right), \\ I_2 &:= \left(\int_{D_{\delta_j}} (|\mathbf{X}h| + |\mathbf{X}u|)^p dx \right). \end{aligned}$$

Then, using the Hölder inequality we find that

$$(4.13) \quad I := \int_{D_{\delta_j}} |\mathbf{X}\phi|^p dx \leq I_1^{p/2} I_2^{(2-p)/2}.$$

Now, repeating the argument in (4.4) we deduce that

$$(4.14) \quad I_1 \leq C c_j M(w, s_j) \mu(D_{\delta_j}) \leq C c_j M(w, s_j) \mu(B_{cc}(w, 2r)).$$

Using quasi minimizing property of A -harmonic functions, see Lemma 3.8, we see that

$$(4.15) \quad I_2 = \int_{D_{\delta_j}} (|\mathbf{X}h| + |\mathbf{X}u|)^p dx \leq C \int_{D_{\delta_j}} (|\mathbf{X}u|)^p dx \leq C \int_{B_{cc}(w, \bar{s}_j)} (|\mathbf{X}u|)^p dx.$$

Hence, applying Lemma 3.7 we deduce that

$$(4.16) \quad I_2 \leq \frac{C}{(s_{j+1} - \bar{s}_j)^p} |B_{cc}(w, s_{j+1})| (M(w, s_{j+1}))^p.$$

Putting together the estimates in (4.14) and (4.16) we can conclude that

$$(4.17) \quad \begin{aligned} I &\leq C [c_j M(w, s_j) \mu(B_{cc}(w, 2r))]^{p/2} \\ &\times \left[\frac{1}{(s_{j+1} - \bar{s}_j)^p} |B_{cc}(w, s_{j+1})| (M(w, s_{j+1}))^p \right]^{(2-p)/2}. \end{aligned}$$

Again, using Lemma 3.1 and the definition of ϕ , we see that

$$(4.18) \quad \begin{aligned} (c_j M(w, s_j))^p |B_{cc}(w, tr)| &\leq \int_{B_{cc}(w, \bar{s}_j)} \phi^p dx \\ &\leq C \bar{s}_j^p \int_{B_{cc}(w, \bar{s}_j)} |\mathbf{X}\phi|^p dx. \end{aligned}$$

Estimates (4.17) and (4.18) imply that (4.9) holds. $\square$

4.1. Proof of Theorem 1.3. We first prove the inequality on the left hand side. Let u be as in Theorem 1.3 and let μ be its Riesz measure. Recall (4.1) and let $v = M(w, r) - u$. Then v is a non-negative A -superharmonic function in $B_{cc}(w, r)$ and

$$(4.19) \quad L_{A,p} v = -\mu.$$

The inequality on the left hand side in Theorem 3.1 gives

$$(4.20) \quad \hat{C}^{-1} \mathbf{W}_{p,\mathbf{X}}^\mu(w, \frac{1}{2}r) \leq M(w, r) - u(w),$$

with $\hat{C} \geq 1$ from Theorem 3.1, this completes the proof of the inequality on the left hand side in Theorem 1.3. To prove the inequality on the right hand side in Theorem 1.3 we let, as in [KZ], $\gamma = (2\hat{C} - 1)/(2\hat{C}) \in (0, 1)$. Let k be the positive integer such that $\gamma^k < \theta < \gamma^{k-1}$, where θ is as in the statement of Lemma 4.1, and let $t = 2^{-k}$. Suppose that there exists j , $1 \leq j \leq k$, such that

$$(4.21) \quad M(w, 2^{-j}r) \geq \gamma M(w, 2^{-j+1}r).$$

Then, since

$$M(w, 2^{-j+1}r) - u$$

is a non-negative A -superharmonic function in $B_{cc}(w, 2^{-j+1}r)$, we can apply Theorem 3.1 and conclude that

$$(4.22) \quad \begin{aligned} M(w, 2^{-j+1}r) - u(w) &\leq \hat{C}(1 - \gamma)M(w, 2^{-j+1}r) + \hat{C} \mathbf{W}_{p,\mathbf{X}}^\mu(w, r) \\ &= \frac{1}{2}M(w, 2^{-j+1}r) + \hat{C} \mathbf{W}_{p,\mathbf{X}}^\mu(w, r). \end{aligned}$$

In particular,

$$(4.23) \quad M(w, tr) \leq M(w, 2^{-j+1}r) \leq 2u(w) + C \mathbf{W}_{p,\mathbf{X}}^\mu(w, r).$$

If on the contrary, there exists no j , $1 \leq j \leq k$, such that (4.21) holds, then

$$(4.24) \quad M(w, 2^{-j}r) < \gamma M(w, 2^{-j+1}r),$$

for all $j = 1, 2, \dots, k$, and hence

$$(4.25) \quad M(w, tr) = M(w, 2^{-k}r) < \gamma^k M(w, r) \leq \theta M(w, r).$$

In this case we can now use Lemma 4.1 to obtain that

$$M(w, tr) \leq M(w, r) \leq C \left(r^p \frac{\mu(B_{cc}(w, 2r))}{|B_{cc}(w, 2r)|} \right)^{1/(p-1)} \leq C \mathbf{W}_{p,\mathbf{X}}^\mu(w, 4r).$$

Since either (4.23) or (4.24) is true we obtain the conclusion of Theorem 1.3. $\square$

5. PROOF OF THEOREM 1.4

To prove the existence of the boundary measure μ , we only need to show that our extended function is in fact an A -subsolution, since then the existence of the measure μ that satisfies (1.20) follows from (1.14). First let $\hat{u}$ be the following

$$\hat{u}(x) = \begin{cases} u(x) & x \in \bar{\Omega} \cap B_{cc}(w, 2r) \\ 0 & x \in B_{cc}(w, 2r) \setminus \Omega. \end{cases}$$

To prove that $\hat{u}$ is a A -subsolution inside $B_{cc}(w, 2r)$, we will use the test-function $\psi \in C_0^\infty(B_{cc}(w, 2r))$, $\psi \geq 0$, and as in [LNy, Lemma 2.2] let $\tilde{\psi}_{\epsilon, \eta} = [(\eta + \max[\hat{u} - \epsilon, 0])^\epsilon - \eta^\epsilon]\psi$, where $\eta, \epsilon > 0$ are small. From the fact that u is a solution in $\Omega \cap B_{cc}(w, 2r)$ and that $\tilde{\psi}_{\epsilon, \eta} \in W_{\mathbf{X}, 0}^{1, p}(\Omega \cap B_{cc}(w, 2r) \cap \{\hat{u} \geq \epsilon\})$, $\tilde{\psi} \geq 0$, we have

$$(5.1) \quad \int_{B_{cc}(w, 2r)} A(x, \mathbf{X}\hat{u}) \cdot \mathbf{X}\tilde{\psi}_{\epsilon, \eta} dx = 0.$$

Carrying out the differentiation in (5.1) we get

$$\begin{aligned} & \int_{B_{cc}(w, 2r)} A(x, \mathbf{X}\hat{u}) \cdot (\mathbf{X}\psi[(\eta + \max[\hat{u} - \epsilon, 0])^\epsilon - \eta^\epsilon] \\ & \quad + \psi\epsilon\mathbf{X}\hat{u}(\eta + \max[\hat{u} - \epsilon, 0])^{\epsilon-1}) dx = 0. \end{aligned}$$

Using (1.11) we see that for any $\eta, \epsilon > 0$ the following holds

$$\int_{B_{cc}(w, 2r) \cap \{\hat{u} \geq \epsilon\}} [(\eta + \max[\hat{u} - \epsilon, 0])^\epsilon - \eta^\epsilon] A(x, \mathbf{X}\hat{u}) \cdot \mathbf{X}\psi dx \leq 0.$$

Using dominated convergence we get, by first letting $\eta \rightarrow 0$ and then letting $\epsilon \rightarrow 0$, that

$$(5.2) \quad \int_{B_{cc}(w, 2r)} A(x, \mathbf{X}\hat{u}) \cdot \mathbf{X}\psi dx \leq 0,$$

which tells us that $\hat{u}$ is an A -subsolution in $B_{cc}(w, 2r)$.

To start the proof of (1.21) we first note, using Lemma 3.4, and Harnack's inequality, that there exists a constant $C = C(p, n, \alpha, \beta, \mathbf{X}, Q, M)$, $1 \leq C < \infty$, such that if $\tilde{r} = r/C$, then

$$(5.3) \quad u(a_{\tilde{r}}(w)) \leq \max_{\Omega \cap B_{cc}(w, 2\tilde{r})} u \leq C u(a_{\tilde{r}}(w)).$$

To prove inequality on the left hand side in (1.21) we let $\phi \in C_0^\infty(B_{cc}(w, 2\tilde{r}))$ be such that, $0 \leq \phi \leq 1$, $\phi = 1$ on $B_{cc}(w, \tilde{r})$, and $|\mathbf{X}\phi| \leq Cr^{-1}$. Then, using Hölder's inequality, Lemma 3.4, Lemma 3.7, Harnack's inequality and the doubling of metric balls, see (1.15), we see that

$$\begin{aligned} \mu(B_{cc}(w, \tilde{r})) & \leq \int \phi d\mu \leq \int |\mathbf{X}\phi| |A(x, \mathbf{X}u)| dx \leq Cr^{-1} \int_{\Omega \cap B_{cc}(w, 2\tilde{r})} |\mathbf{X}u|^{p-1} dx \\ & \leq Cr^{-1} |\Omega \cap B_{cc}(w, 2\tilde{r})|^{1/p} \left(\int_{\Omega \cap B_{cc}(w, 2\tilde{r})} |\mathbf{X}u|^p dx \right)^{(p-1)/p} \\ (5.4) \quad & \leq Cr^{-p} |B_{cc}(w, 2\tilde{r})| (u(a_{\tilde{r}}(w)))^{p-1}. \end{aligned}$$

This proves the inequality on the left hand side in (1.21). To prove the inequality on the right hand side we can use Lemma 4.1. Recall (4.1) and recall that Lemma 4.1 states that if

$$(5.5) \quad M(w, t\tilde{r}) \leq \theta M(w, \tilde{r}), \text{ for some } 0 < t < 1 \text{ and } \theta \text{ given by the lemma,}$$

then

$$(5.6) \quad M(w, \tilde{r}/4) \leq C \left(\tilde{r}^p \frac{\mu(B_{cc}(w, \tilde{r}/2))}{|B_{cc}(w, \tilde{r}/2)|} \right)^{1/(p-1)}.$$

Once (5.6) is established the inequality on the right hand side in (1.21) follows from a rearrangement of terms, (5.3) and Harnack's inequality. Hence we only have to verify (5.5). To prove (5.5), we see that Lemma 3.3 gives

$$u(x) \leq Ct^\sigma M(w, r),$$

whenever $x \in B_{cc}(w, t\tilde{r})$, $0 < t < 1/2$ and where σ is from Lemma 3.3. In particular, choosing $t = (\theta/C)^{1/\sigma}$ completes the proof of (5.5). $\square$

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