Numerical methods for glacial isostatic adjustment models

Juan Carlos Araujo-Cabarcas
Abstract

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Nordic countries experience post-glacial rebound, a movement where geographical contours slowly change elevation with respect to the mean sea level. The glacial isostatic adjustment (GIA) model aims to explain the phenomena, which combined with seismic data allows geoscientists to reconstruct elastic coefficients and viscosities of the Earth’s lithosphere and upper mantle.

The use of standard commercial codes are not adequate for GIA simulations and result in significant errors in the displacement field. This negative outcome suggests the development of GIA codes that include advection of pre-stress in the model. The problem set up consists on a solid 2D elastic layer under a flat Earth approximation, described by three different models suggested by current studies in geophysics. For space discretization the mixed finite element method (mFEM) is used and efficient preconditioners are built for the resulting algebraic system in saddle point form. A three level GMRES iterative solution strategy is proposed, based on Schur Complement preconditioners coupled with Multigrid techniques.

The implementation is presented as a ready-to-use toolbox that easily deals with problem parameters, geometries, compressible and fully incompressible materials and provides higher accuracy for the displacement field compared with the previously existent codes. It also can be easily extended to 3D geometries and allows the implementation of a viscoelastic mantle. The code is written in C++ using the deal.II library designed for FEM, permitting the use of readily-made software packages, such as Trilinos that are straightforwardly parallelizable.
“... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in... it is impossible to explain honestly the beauties of the laws of nature in a way that people can feel, without their having some deep understanding of mathematics and its methods...”

Richard Feynman
Glacial Isostatic Adjustment

Main stages in the GIA hypothesis.
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IX
1 Introduction

Nordic countries are currently experiencing a natural rebound of the ground, where geographical contours re-shape due to a changing elevation with respect to the sea level. Numerous studies claim that such phenomena can be attributed to what is called glacial isostatic adjustment (GIA) or more widely known as post glacial rebound; arguing that in periods of glaciation, large ice sheets grow over the Nordic/southern latitudes and compress the Earth in Isostacy or Isostatic equilibrium (horizontal pressure balance between Earth sub layers).

The balancing process induces material transport from regions of higher pressure, towards regions of lower pressure. On the other hand, in periods of warmer temperatures (interglacial), the covering ice melts in a relatively short time. Pressure is released from deeper layers allowing the ground to rebound. The sub-layers closest to the Earth surface are modeled as an elastic Lithosphere and a viscoelastic upper mantle with viscosity coefficients varying in the order of $10^{19} - 10^{22}$ Pa s. In movements with such high viscosities and long time scales, inertia does not play an important role and can be neglected.

There are several consequences of the compression and rebound of the Earth. For instance, the mean sea level varies depending on these movements. Knowledge on the evolution of these contours can help understanding human or animal migration trends. A clear example becomes apparent when the sea level is relatively low, natural bridges with zones of shallow waters appear where before only isolated islands existed, allowing natural migration.

Movements subject to typical viscosities in the mantle express themselves in relatively small displacements but require hundreds of years, which is very large compared to human life span, making it impossible to follow the process via measurements only. To tackle this difficulty, in recent geophysical research model inversion is currently being used, which is a method to fine-tune the physical parameters of the existing models, when large collections of data are available. In this way, by using gathered seismic data it is possible to investigate the physical properties of the Earth. Inverse GIA modeling leads to alternative ways to measure elastic coefficients and viscosities, helping to reconstruct the structure of the lithosphere and upper mantle, which has a significant importance in the ongoing scientific discussions.

The current mathematical model that is used to describe the visco-elastic movement, related to glacial rebound processes, reads as follows:

$$\nabla \cdot \sigma + \nabla (u \cdot \nabla p^{(0)}) - \rho_0 g \nabla \cdot u = 0. \quad (1.1)$$

Here $\sigma$ stands for the stress tensor, $u$ is the displacement vector, $p$ is the hydrostatic pressure and $\rho_0$ is the material density in relaxed state. The super-index 0 on the pressure denotes initial pressure-state. The gravitational acceleration vector is of the form $g = -g e_z$, where $e_z$ is a normal vector, pointing outwards from the Earth’s surface and $g$ is taken as a constant on the Earth’s surface, self-gravitation is left out of the main model in this thesis and saved for future analysis.
1.1 State of the art

The starting point of this work is the numerical study of (1.1) in [1], where only the elastic case was fully investigated. The code, developed in [1], was tested and compared to an implementation in ABAQUS, a commercial Finite Element (FEM) package, widely used in Geophysics for GIA simulations. The work in [2], shows that the use of such standard commercial codes for GIA simulations result in significant errors in the displacement field $\mathbf{u}$. This negative outcome lead to the implementation of the advection of pre-stress in [1] that was not possible with ABAQUS. The well-posedness of the GIA model has been studied in detail in [2].

The existing implementation in [1] takes into account two solid layers with different elastic properties delimited under a flat Earth as shown in Figure 2.2. The code uses as computational platform C++ within the deal.II library [3], designed for finite element methods (FEM), linked to a PETSc interface. The implementation in [1] could not be easily re-used at present time, as the code was tightly coupled to PETSc, which is not ideal for further revisions. Another obstacle encountered was the uneasy parameter interface implemented that made the code difficult to handle.

The GIA implementation in [1] uses modified Taylor Hood (MTH) elements $Q_1 \times \text{iso}Q_1$, consisting of two nested meshes, coarse and fine, obtained by one regular refinement of the coarse. The pressure degrees of freedom (dofs) are associated with the coarse, and the displacements with the fine mesh. Pair of function spaces used in simulations of elasticity and fluid dynamics, must satisfy the Babuska-Brezzi condition [4], and the MTH does.

1.2 Contributions of this thesis

A possible drawback of the MTH pair used in [1] is the less-accurate representation of the stress and strain tensors $\sigma, \varepsilon$. As both $\mathbf{u}$ and $p$ belong to the family of $C^{(1)}$ (one time differentiable) functions, then the stress

$$\sigma = \mu \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) + \lambda \nabla \cdot \mathbf{u}$$

is represented poorly in a $C^{(0)}$ family of step functions inside the quadrilaterals and undefined values at the quadrilaterals edges. To overcome this inconvenience, the classical $Q_2 \times Q_1$ pair that represent well both $\sigma$ and $\varepsilon$ is used in this work. This choice provides higher accuracy for the displacement field $\mathbf{u}$. At the same time, the matrices in the resulting linear algebraic system become slightly denser. The fact that the MTH pair is not available in deal.II is another motivation to use the readily available $Q_2 \times Q_1$ pair in the current study.

The present thesis project uses other available software packages, such as Trilinos [5], that has been proved to be a powerful high performance library\footnote{Many of the advantages of using Trilinos can be seen in [6].} for linear algebra calculations, extensively tested and easily parallelizable.

Despite the existence of a GIA code, after several attempts to reuse it, the conclusion was that the
existing implementation was not straightforward. A more flexible code that deals more easily with problem parameters, geometries, compressible and incompressible materials (including the fully incompressible case) and that could be updated to implement the $Q_2 \times Q_1$ pair of shape functions was needed to be used in current GIA studies at the Geocentrum, Uppsala University.

The difficulties mentioned above suggests further developing of GIA simulations and motivates the current thesis project. For this I have prepared a ready-to-use toolbox, to be used for studies in Geophysics, that can be easily extended to 3D geometries and that implements a viscoelastic mantle in the model.

It is the author’s opinion that the physics in post-glacial rebound phenomena and the research in numerical solution algorithms should be tightly coupled and considered together. The physical model should be taken into account in order to develop robust solution schemes that are supported by physical arguments. For this reason special attention has been given to the description of the GIA model from a physical point of view.

The research questions that this thesis address are several, but mainly related with the efficiency of the numerical solution method. Particularly, can appropriate preconditioners be chosen so that the iterative solution is numerically and computationally efficient? If so, the computational cost would be greatly diminished. The acknowledgment of this question imposes the use of high quality numerical techniques and for this purpose highly efficient Schur Complement preconditioners are proposed to use with Multigrid techniques.

1.3 Summary

The structure of this thesis is as follows. Section 2 presents a brief introduction to concepts in continuum mechanics, followed by theoretical fundamentals of the GIA model, such as derivations of the buoyancy and advection of pre-stress terms of the general model and a discussion about body forces and the deviatoric stress tensor. This section ends with a derivation of the viscoelastic Green's function withing the so-called Maxwell viscoelastic model. Section 3 is dedicated to the spatial discretization technique by using the mixed Finite Element method (mFEM) [4], necessary to avoid problems such as the Locking effect in elasticity. A short description of stable pairs in FE along with a discussion about continuity in the solution are given in this section.

Discrete GIA models, based on (1.1) lead to algebraic systems with matrices of saddle point form. In general, saddle point problems are a significant challenge for numerical solution methods. Therefore, a detailed discussion on how to overcome the numerical difficulties arising in the resulting system of equations is given in Section 4. Section 5 contains descriptions of the numerical experiments presented in this thesis. A final discussion about the performance, convergence of the numerical results is given in Section 6, along with comments on physical interpretation of the results.

Concepts concerning the linearity of the model and deformations due to ice load are discussed in detail by obtaining analytical solutions to the uniform load problem in Appendix A. In Appendix
B the discretization by using FEM of the stationary elastic problem is considered first as a base to understand the complete GIA model. The derivation of Korn’s inequality is included, to explain the choice of the particular preconditioners used in this thesis. The discretization of the viscoelastic problem is described in Appendix B.4, and a short description of multigrid methods and AMG is given in Appendix C.
2 The GIA model - physical processes and mathematical description

2.1 Theory of deformation

Dealing with bodies of dimensions constrained to the size of a point, in the mathematical sense, reduces considerably the relations that govern its physics. Namely, such artificial bodies can be built only when their constitutive material points are rigidly attached to each other. In other words, the distance $\Delta R_{ij} = |R_i - R_j| = C_{ij}$ separating two points in the body remains the same in time, hence $C_{ij} = \text{constant}$. Clearly, the rigid assumption neglects deformation, which is the main motivation of this work. For this, two frames of reference can be used in the deformation theory.

Lagrangian frames of reference are the most natural choice when working with a continuum material. The coordinates $R(r, t)$ follow the material along its movement and in such a frame, constant quantities will not vary in time as they are attached to the coordinate system. In general it is said that the frame of reference deforms with the material. Conservation of a property $\theta$ in a Lagrangian frame reads as

$$\frac{d\theta_L}{dt} = 0. \tag{2.1}$$

Alternatively, in the static Eulerian frame, $r$ will remain constant in time with an observer that measures the movement from this reference. For example, consider a fixed point and a physical property $\theta$, constant in the Lagrangian frame of reference that moves with the body. Say that the body travels with a velocity $v$ measured in the static grid. Then, the total derivative or material derivative of the measurable quantity $\theta$, will not remain constant in a fixed point of the Eulerian frame. Explicitly, its variation is written as

$$\frac{d\theta_E}{dt} = \frac{\partial \theta_E}{\partial t} + v \cdot \nabla \theta_E. \tag{2.2}$$

The traveling body might as well be subject to deformation. In such a case the conservation of the property $\theta_E$ reads

$$\frac{d\theta_E}{dt} = \frac{\partial \theta_E}{\partial t} + \nabla \cdot (\theta_E v). \tag{2.3}$$

At the initial time both frameworks coincide, but if there is movement they differ increasingly with time. This work utilizes as frame of reference an Eulerian grid. The stress tensor $\sigma$ is a mathematical representation of the forces in a given point $P(r)$, that can be of compressional or shear nature. Due to the stresses at a point $P$ the resulting force can be extracted from $\sigma(r)$ by the use of the Cauchy traction vector $\ell$, defined as

$$\ell = \sigma \hat{n}. \tag{2.4}$$

Here $\hat{n}$ is the outer unit normal vector to the given surface at $P$. This resulting force acting on $P$ located on an abstract surface, serves as the vector representation of the stresses in a given
surface that might coincide with the physical boundary surface of the body. The force balance or Newton’s second law in elasticity reads

$$\nabla \cdot \sigma + f = \rho \ddot{u},$$

(2.5)

where the body (volumetric) force $f$ represents external forces as the gravity or material responses to magnetic or electric fields, $\sigma$ is the stress tensor, $u$ the displacement field and $\ddot{u}$ the respective acceleration field.

All the terms arising from macroscopic internal forces are contained in the term $\nabla \cdot \sigma$ that accounts for the deformation forces in the body, such as boundary forces. Equation (2.5) has all the information needed to determine the elastic behavior of a material. However, in order to find unique solutions for (2.5), it is necessary to impose boundary conditions on $\partial \Omega$. The two types of boundary conditions, most often used in practice are

$$u = g(r) \text{ on } \Gamma_D \text{ (of Dirichlet type)},$$

$$\sigma \hat{n} = \ell(r) \text{ on } \Gamma_N \text{ (of Neumann type)},$$

(2.6)

where $\Gamma_N \subset \partial \Omega$, $\Gamma_D \subset \partial \Omega$ and $\Gamma_D \neq \emptyset$.

To express that a body is fixed along some part of its boundary (clamped), is natural to impose $u = 0$. On the other hand, if a part of the boundary is free, the corresponding boundary condition is $\sigma \hat{n} = 0$.

The last review is sufficient to model the macroscopic behavior of a continuum delimited by a finite body, provided that the initial conditions $u(t = t_0)$ and $\dot{u}(t = t_0)$ are known. At this point a constitutive law relating $u$ and $\sigma$ remains yet unknown and in the most general case a relationship of the form $\sigma = \sigma(\varepsilon, \dot{\varepsilon})$ is used, where $\varepsilon$ represents the dimensionless strain of the body due to deformation. In the following section more is discussed about the fundamentals of the linear theory of deformation.
Variations in a Hookean elastic body can be described by the Green’s deformation tensor

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_k \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right). \]  

(2.7)

This nonlinear relationship penalizes and constrains extreme deformations such as unphysical material overlapping that linear approaches may allow when the deformation is of the order of the body’s dimensions. As is well known, algorithms for solving nonlinear problems result in very elaborated and challenging solution schemes. Therefore, whenever possible, it is advisable to find suitable simplification that leads to linear models.

For the target problem in this work, see Figure 5.3, typical domains of interest are in the range of \( L \times W = 10^7 m \times 10^6 m \), and result in maximum GIA displacements of the order of \( 10^2 m \) in magnitude. Taking into account the typical characteristic length of the domain and scaling properly, it is clear that \( \Delta u / \Delta L \approx 10^2 / 10^7 = 10^{-5} \), which verifies that in the current setting second order strains are clearly negligible (\( 10^{-10} \)). Hence, a linear approximation of the deformation tensor is relevant.

In this order of ideas, for the given small displacements \( \frac{\partial u_i}{\partial x_j} \ll 1 \), higher order terms in the Green’s strain tensor (2.7) vanish, resulting in

\[ \varepsilon(u) = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right). \]  

(2.8)

Notice that the Jacobian of the transformation \( \nabla u \), denotes variations up to a first order approximation, of a body represented in a coordinate system. The Jacobian includes both, pure deformation (symmetric) and pure rotation (skew-symmetric). Hence, the resulting symmetric tensor \( \varepsilon(u) \) denotes solely pure deformation.

### 2.1.1 The extended Hooke’s law

Rheology is the study of the constitutive relationship describing the response \( \sigma \) due to a strain \( \varepsilon \) and other physical quantities as temperature, pressure and internal composition of a given material. The theory of deformation is broad and it can be as complex as desired. However, even though rheologies describing the mantle and other Earth’s sub-layers are in general of nonlinear nature, the displacements due to creep movement described in the GIA model tend to be small, continuous and smooth, making linear assumptions admissible. In such cases, the Hooke’s law can be written as

\[ \sigma = C \varepsilon \text{ (Elastic)}, \]  

(2.9)

where \( C \) is a fourth rank tensor describing particular directional responses of the material due to three-dimensional strains. Even though the relationship \( \sigma(\varepsilon) \) is linear, the complexity is still high, as the coefficients \( C_{ijkl} \) that couple directional responses with strains, in total 81 entries, allow for a complete linear combination of strains to represent complete stresses.
Examples varying from the most complex models such as the anisotropic\textsuperscript{2} transport in heterogeneous porous media to the simplest elastic deformation in rubber can be modeled by the extended Hooke law by choosing the right coefficients. As is known, the composition of the Earth is far from being isotropic and homogeneous. Nonetheless, the scope of this project is to model a simple two-layer Earth (lithosphere and mantle) both isotropic and homogeneous as a first approximation.

The setting of the model problem in Figure 2.2 is for two different materials \((i = 1, 2)\) with different physical properties \(\rho_i, E_i, \nu_i\) and \(\eta_i\) defined in the layers \(\Omega_1\) and \(\Omega_2\) in a flat Earth domain. In this thesis these material properties are constant piecewise functions in the domain. The simplest extreme of the last example is to represent the density as a \(C^{(0)}\) scalar function with constant densities and jumps between the different layers. Despite the discontinuity, higher order variations in the material density, due to compression for instance, can be accounted for through buoyant terms in the model. In each part of the domain the material is said to be homogeneous. Then, the Hooke’s tensor \(C\) can be simplified by symmetry arguments [7], and the stresses \(\sigma\) as functions of the displacements \(u\) become

\[
\sigma = 2\mu \varepsilon(u) + \lambda(\nabla \cdot u) I. \tag{2.10}
\]

The properties of the different materials are characterized by material parameters. The Lamé parameters \(\mu\) and \(\lambda\) are related to the more experimental parameters; Young’s modulus \(E\) and Poisson’s ratio \(\nu\) in the form

\[
\lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1 + \nu)}. \tag{2.11}
\]

Once equations (2.8) and (2.10) have been back-substituted in equation (2.5), the Navier-Lamé equation is obtained

\[
(\lambda + \mu) \nabla(\nabla \cdot u) + \mu \nabla^2 u + f = \rho \ddot{u} \quad \sigma = \sigma(\varepsilon). \tag{2.12}
\]

Equation (2.12) represents linear deformations in an elastic body, and lies within a Lagrangian framework, thus grids are constrained to move with the displacement field \(u\) at each time step. Notice, than here the stress tensor is assumed to depend only on the strains \(\sigma = \sigma(\varepsilon)\).

\textsuperscript{2}Anisotropic materials can be found in Earth’s mantle, olivine is a common crystal on the mantle that presents a particular shape that exerts different directional responses to compressional stresses.
Fluids can be treated in a similar way by considering the fact that the coalescence forces that keep a material “glued” is much weaker in fluids than in solids. Hence, strains per se are not the main cause of stresses. This argument comes intuitively, as the stretching of a viscous fluid does not produce internal forces. However, the rate of strain is observed to cause stresses coming from viscous nature, and the relationship \( \sigma = \sigma(\dot{\varepsilon}) \) is used instead. By following the same steps as in elasticity but working with stresses due to a viscous response in an Eulerian framework, the velocity field \( \mathbf{v} \) becomes the target instead of \( \mathbf{u} \). Nonetheless the same linear relationship holds

\[
\sigma = C \dot{\varepsilon} \quad \text{(Viscous)}.
\]  

\[ \text{(2.13)} \]

### 2.1.2 Incompressible materials

Materials with \( \nu = \frac{1}{2} \) are referred to as (volumewise) incompressible. From (2.11) it is evident that \( \nu = \frac{1}{2} \) entails that \( \lambda \) becomes undefined. In order to handle purely incompressible materials, in fluids the hydrostatic pressure is often suppressed from the stress tensor by using the deviatoric stress tensor \( \sigma_s \)

\[
\sigma = \sigma_s - p \mathbf{I}.
\]

(2.14)

Here, \( p \) accounts for the compressional (negative sign) hydrostatic stresses. The divergence of this term reads as

\[
\nabla \cdot \sigma = \nabla \cdot \sigma_s - \nabla p.
\]

(2.15)

Applying the same relations (2.10), (2.8) and (2.11) to the deviatoric stress tensor results in

\[
\sigma_s = 2\mu \dot{\varepsilon} + \lambda (\nabla \cdot \mathbf{v}) \mathbf{I},
\]

(2.16)

\[
\dot{\varepsilon} = \frac{1}{2} \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right),
\]

(2.17)

\[
\lambda = \frac{\eta \nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{\eta}{2(1 + \nu)},
\]

(2.18)

where \( \eta \) is the viscosity of the fluid. The case \( \nu = \frac{1}{2} \) in (2.18) produces an undefined value \( \lambda \), to handle it more knowledge about the physical system is required. The extra information can be supplied by the continuity equation in Eulerian frames of reference, namely,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.
\]

(2.19)

Notice that for perfect incompressible materials, the density should remain constant, which means,

\[
\nabla \cdot \mathbf{v} = 0.
\]

(2.20)

Relation (2.20) is widely used in fluid mechanics to represent incompressible fluids in an Eulerian framework. The incompressible Navier-Stokes equation can be derived by using the condition (2.20) in (2.16) and substituting the resulting equation in (2.5). In the process the relations (2.17), (2.15) and \( \dot{\mathbf{v}} = \ddot{\mathbf{u}} \) are used and the incompressible Navier-Stokes equation reads

\[
\mu \nabla^2 \mathbf{v} - \nabla p + \mathbf{f} = \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right), \quad \sigma = \sigma(\dot{\varepsilon}).
\]

(2.21)
Notice that the term in brackets represents the total derivative in time in an Eulerian frame and \( \mu \) stands for the dynamic viscosity of the fluid. The main difference between equations (2.12) and (2.21) is the term \( \nabla p \) and the material derivative. The subtraction of the hydrostatic pressure presented in this section can be extended similarly to incompressible elastic materials, such as rubber, defining a scaled pressure that is derived in Section 2.2.3.

2.2 Description of the GIA model

The generally accepted mathematical model, used to describe glacial isostatic adjustment is given by the following equation

\[
\nabla \cdot \sigma + \nabla (u \cdot \nabla p_0) - \rho_0 g \nabla \cdot u = 0.
\]  

(2.22)

This section describes the role of each of the terms in (2.22) and the connection to the general theory of elasticity as outlined in Section 2. Crucial approximations are shown in detail and the scope of such simplifications is justified whenever possible.

2.2.1 The body force \( f \)

It turns out that due to the geological time-scale, certain simplifications and approximations in the underlying equations are relevant.

The removal of inertial terms from equation (2.5), \( (\rho \ddot{u} \approx 0) \) is a common practice in Geophysics when the problem of interest describes the movement of the inner layers of the Earth such as the mantle. The resulting movement is known as creep and it is attributed to the slow acceleration field \( \ddot{u} \approx 1 \text{ mm/yr/Ma} \approx 10^{-17} \text{ m/s}^2 \), where mm/yr/Ma means millimeter/year/Mega year. Despite that the theory of gravity on homogeneous spheres predicts a variable gravity field, in this work \( g \) is chosen constant. Notice that the effect of gravity inside of an homogeneous sphere can be correctly represented as being produced only by a point at the center of the sphere, carrying a mass \( M(r) = 4\pi \rho r^3 / 3 \). This concentrated point gathers the effects of inner layers below the evaluation point, but the gravity due to layers above the evaluation point is known to be null by symmetry arguments. The resulting relation is a linear variation of \( g(r) \) with depth inside spherical bodies [9].

\[
g = -G \frac{M(r)}{r^2} e_r = -\frac{4\pi \rho G}{3} r e_r.
\]

The opposite argument claims that the Earth is inhomogeneous and its variable density accounts for a more complex gravity field. Measurements from seismic data along with inversions on the PREM model [8], reveal that a constant gravity is a fairly good approximation\(^3\) due to the variation in density in superficial layers, that contrasts the effect of variable gravity starting from approximately 3000 km depth until the Earth’s surface (Figure 2.3). If greater depths are reached, this approximation is no longer valid, see Figure 2.3. However, even with a variable gravity field, it is always possible to split the initial problem into two separate problems due to the

\(^3\)For an extension of Earth’s inner gravity field refer to Chapters 3 in [8] and 5 in [9].
Figure 2.3: Radial variations of internal density (left), gravity (right: thick curve) and pressure (right: thin curve) according to Earth model PREM (data source: Dziewonski, 1989). Taken with permission from [8].

This transformation allows to resolve only for deformations produced by the boundary forces in linearly the problem. The motivation with the splitting is to isolate the effects of hydrostatic displacements and solve for GIA displacements only. To illustrate the last idea, consider the relation

$$\nabla \cdot \sigma - \rho_r g e_z = 0,$$  \hspace{1cm} (2.23)

that can be factored as

$$\nabla \cdot \sigma - \rho_r g e_z = \nabla \cdot \sigma - \nabla (\rho_r g z) = \nabla \cdot (\sigma - \rho_r g z I),$$

with $I$ the identity operator. Equation (2.24) represents the transformed force balance

$$\nabla \cdot \sigma' = 0.$$  \hspace{1cm} (2.24)

This transformation allows to resolve only for deformations produced by the boundary forces in the deviatoric stress tensor $\sigma'$. Furthermore, the linearity of the model in use ensures the possibility to retrieve the solution to the original problem as the superposition of the two separate problems. In reality the hydrostatic pressure has already displaced the Earth to its current deformation state and hence, displacements cannot be measured. Such isolated hydrostatic pressure $p_h = \rho_r g$ is depicted with a thin line in Figure 2.3.

Further calculations in Appendix (A.5) show that GIA displacements are of much lower magnitude than those of hydrostatic nature making the splitting necessary.
2.2.2 Linear Buoyancy and advection of pre-stress

The phenomena of variable density accounts for buoyancy of the material. In this study the external body force $\mathbf{f}$ is of gravitational nature. As a first approximation, it is relevant to set $\rho \approx \rho_0 + \Delta \rho$ and assume that the variation in density, up to a first order, is proportional to the divergence of the displacement vector. To see why this is the case, consider a material enclosed by a volume $d\Omega$ which is subject to a compressive/expansive force. Then, an increase/decrease of the material density results in sinking/ floating, due to the hydrostatic pressure. The last analysis reveals that this proportionality\(^4\) is of the form

$$\Delta \rho \propto -\nabla \cdot \mathbf{u},$$

where a positive value of $\nabla \cdot \mathbf{u}$ is understood as expansion. It is easy to notice that the proportionality constant should have dimensions of density. For this purpose $\rho_0$ is used, denoting the density of the material in a relaxed state. Then it is possible to set an intuitive relationship

$$\Delta \rho = -\rho_0 \nabla \cdot \mathbf{u}.$$ 

Hence, the body force can be written as $\mathbf{f} = \rho \mathbf{g} \approx \rho_0 \mathbf{g} + \Delta \rho \mathbf{g}$ and finally

$$\mathbf{f} = \rho_0 \mathbf{g} - (\nabla \cdot \mathbf{u}) \rho_0 \mathbf{g}. \quad (2.25)$$

Relation (2.25) derived intuitively, can be seen as a first order truncation in the density\(^5\) where higher order variations have been neglected. The continuity equation (2.19) for the Eulerian framework can be written as

$$\frac{\partial}{\partial t} (\rho_0 + \Delta \rho) + \nabla \cdot [(\rho_0 + \Delta \rho) \mathbf{v}] = 0,$$

by using $\rho \approx \rho_0 + \Delta \rho$, and truncating until $\rho_0$ in the right side leads to

$$\frac{\partial \Delta \rho}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}) = 0.$$

Notice that due to the linear approximation the displacement $\mathbf{u}$ must be small in magnitude compared to the length scale in the problem. In the framework considered here, the velocity $\mathbf{v}$ is even smaller\(^6\) (of the order of $9 \text{ mm/yr}$), therefore the term $\Delta \rho \mathbf{v}$ is of second order and hence neglected. Then integration in time reveals

$$\Delta \rho = -\rho_0 \nabla \cdot \mathbf{u}, \quad (2.26)$$

\(^4\)The notation $A \propto B$ reads “$A$ is proportional to $B$”.

\(^5\)The calculations showed here are based on the linearization of the equations done in [10], second chapter. However, the arguments have been slightly shortened as readers coming from numerical analysis are more familiarized with truncation and approximations.

\(^6\)For a comprehensive geophysical discussion of the GIA observations see [2].
which leads to the same result obtained in (2.25). The truncation argument is equivalent to stating that variations of density are due to compression and expansion only, which is the intuitive argument.

In the force balance (2.24), the inclusion of the buoyant term reads

\[ \nabla \cdot \sigma(e, \dot{e}) + b = 0, \quad b = -\rho_0 g \nabla \cdot u. \]  

(2.27)

This equation holds true for all \( t \) in a Lagrangian frame. In the viscoelastic case, \( \sigma \) depends on time in (2.27) even if there are no time derivatives. The evolution in time is given by a memory integral term introduced in Section B.4. The resulting adjustment/relaxation movement is often denoted as the creep motion, and creep adjustment is the fundamental idea behind isostacy.

In Eulerian frameworks, the frame of reference where the body is represented remains still while the material is displacing. This fact presents a computational advantage of working in Eulerian frameworks, where there is no need to modify the initial mesh. For this reason, Equation (2.27) is transformed into a Eulerian frame by introducing the so-call advection of pre-stress.

Physical quantities measured in a Lagrangian reference coincide with an Eulerian until there is movement. Consider the infinitesimal period of time \( t_0 < t < t_0 + \delta t \), where both representations are equivalent but differ increasingly as \( t \) grows. The following analysis is valid only in the limit \( \delta t \to 0 \). Next, introduce the deviatoric stress tensor

\[ \sigma = \sigma_s - p_0 I, \]

where \( p_0 \) is referred to as the initial hydrostatic pre-stress that in the GIA case is due to gravitational forces. In the Lagrangian framework \( p_0 \) does not change in time\(^7\) over \( \delta t \)

\[ \frac{d(p_0)_L}{dt} \bigg|_{\delta t} = 0. \]

While there is no motion, then \( (p_0)_E = (p_0)_L \). When motion is initiated the final expression projecting into an Eulerian framework is obtained by using the material derivative (2.3)

\[ \frac{\partial (p_0)_E}{\partial t} = -\mathbf{v} \cdot \nabla (p_0)_E. \]

Taking into account that \( p_0 \) does not change in time \([10]\), the initial pressure is advected only. Integrating over the initial \( \delta t \) follows

\[ p_0(t_0 + \delta t) - p_0(t_0) = -\mathbf{u} \cdot \nabla p_0. \]

The term on the right-hand side shows that the initial stress is advected in an Eulerian framework. The lack of time derivatives due to the removal of the inertial terms may lead to the erroneous idea that the GIA problem has the same representation in both frameworks, making it necessary

\(^7\)Refer to Cathles interpretation in [10], Ch. 1 for a detailed explanation.
to calculate explicitly the advection of pre-stress. In practice, when building the Eulerian force balance it is found

$$\nabla \cdot \sigma = \nabla \cdot (\sigma_s - p_0(t_0 + \delta t) I)$$

which after back-substitution, the term $p_0(t_0)$ can be then inserted back into the original stress tensor. Then the complete transformation reads

$$\nabla \cdot \sigma + \nabla(u \cdot \nabla p_0) - \rho_0 g \nabla \cdot u = 0.$$  \hspace{1cm} (2.28)

Equation (2.28) is broadly accepted in GIA studies and it is also the mathematical model used in this thesis. The solutions presented in the rest of the thesis are based on the following (sub)models on different layers $\Omega_i$:

(M0) $\nabla \cdot \sigma = 0$.

(M1) $\nabla \cdot \sigma + \rho_i \nabla(u \cdot g) = 0$.

(M2) $\nabla \cdot \sigma + \rho_i \nabla(u \cdot g) - \rho_i g (\nabla \cdot u) = 0$.

Clearly (M0) deals with boundary forces on a perfectly elastic body described in a Lagrangian frame, (M1) projects (M0) into an Eulerian coordinate system and (M2) introduces first order density variations with the so-called buoyant term.

### 2.2.3 The scaled pressure

Section 2.1.2 shows that in order to avoid singularities for $\lambda$, the incompressibility condition $\nu = \frac{1}{2}$ enforces the usage of the hydrostatic pressure $p$. The pressure defined in that way, resolves the problem for constant values of the Lamé parameters. In general, the elastic parameters are functions of the spatial coordinates $\mu = \mu(r)$, $\lambda = \lambda(r)$ and the way to overcome undefined values of $\lambda$ is to work with a scaled pressure, related to the displacements as follows

$$p = \frac{\lambda}{\mu} \nabla \cdot u.$$  \hspace{1cm} (2.29)

The use of the auxiliary variable $p$ as in (2.29) avoids possible lost of convergence rate in FE solutions due to the Locking effect observed in cases with extreme deformation conditions [11].

Substituting (2.10) in (2.28) using $\lambda \nabla \cdot u = \mu p$ leads to the following relations

$$\mu \nabla \cdot \varepsilon + \nabla(\mu p) + a' = 0, \quad a'(u) = \rho_i \nabla(u \cdot g) - \rho_i g (\nabla \cdot u).$$

The system of equations to solve becomes

$$\begin{align*}
\mu \nabla \cdot \varepsilon(u) + \nabla(\mu p) + a'(u) &= 0, \\
\nabla \cdot u - \frac{\mu}{\lambda} p &= 0. \hspace{1cm} (2.30)
\end{align*}$$

This system of equations is equivalent to the Navier equation (2.12) without inertial terms, allowing the treatment of fully incompressible materials as well as nearly incompressible materials in the limit $\nu \to \frac{1}{2}, \mu/\lambda \to 0$. 
2.3 Complementary models

Even though the current project works with material properties $\rho, \mu, \lambda$ and $\eta$ that are piecewise discontinuous $C^{(0)}$ functions that remain constant in each subdomain (as depicted in Figure 2.2), it is possible to define smoother and more complete models with density functions of the form

$$\rho = \rho(\rho_0, \rho_1, \ldots, \rho_n; \phi; \mathbf{r}, t).$$

This family of problems is known as multiphase problems. There are several models used to simulate numerically multiphase problems [6]. As an example, in the so-called diffuse interface model, each phase is represented by a scalar function $\phi(\mathbf{r}, t)$. The material properties $\rho_k, \mu_k, \lambda_k, \eta_k$ are then attached to each pure phase $\phi(r_k, t)$ and in this way transported within the domain.

One of the mathematical tools within the diffuse interface model is the Cahn-Hilliard model

$$\frac{\partial \phi}{\partial t} = \nabla^2 \left( -\frac{\xi^2}{2} \nabla^2 \phi - \phi^3 \right), \quad (2.31)$$

where $\xi$ represents the natural thickness of the juncture between two pure phases (solid or melted rock for example), but it can be extended to more than two phases as well. Notice that in the multiphase configuration the material properties as density are advected with time. Working with (2.31) can be useful to simulate the lithosphere and mantle as a two phase flow system that keeps track of the movement of its physical parameters.

Solution algorithms for multiphase problems include nonlinear solvers and the time discretization is carefully chosen as problems arising from multiphase flows result in stiff PDE’s. A similar successful solution scheme involving three phase flows (air, water and low concentration of surfactants) is given in [12].

The multiphase description is useful when dealing with large displacements due to potentials that tend to separate the phases. In the two phase case, take for example the movement of plates in the lithosphere, where $\rho_0$ denotes the density of solid rock and $\rho_1$ of melted rock. Then, if $\phi(\mathbf{r}) \in [-1, 1]$ and $\phi(\mathbf{r}) = 1$ denotes solid rock, $\phi(\mathbf{r}) = -1$ melted rock and the states in between can be represented as a linear combination of the pure phases; the two-phase problem can be set as

$$\rho(\rho_0, \rho_1; \mathbf{r}, t) = \frac{\rho_0}{2} (\phi(\mathbf{r}, t) + 1) + \frac{\rho_1}{2} (1 - \phi(\mathbf{r}, t)).$$

The change of phase is justified by the so-called Free energy term that defines phase interactions. In Geophysical studies the free energy is related to the magma production or mass transfer from solid to liquid, between the upper mantle and lithosphere [13] that can be inserted as an inhomogeneous contribution in two separated continuity equations for $\rho_0(\phi(\mathbf{r}) + 1)$ and $\rho_1(1 - \phi(\mathbf{r}))$.

The final configuration can be introduced in the force balance and the energy equation to define a closed set of equations denoted as the McKenzie equations for magma migration. For a comprehensive work in this field, refer to [10], [13], [2], [14] and [6].
2.4 Modeling of viscoelastic materials

Materials that exhibit internal stresses due to strains and rate of strains, $\sigma = \sigma(\epsilon, \dot{\epsilon})$, are said to be of viscoelastic nature. This section starts with a brief description of the model used to represent viscoelastic material behavior in the GIA models.

There exist several methods to model viscoelastic behavior. Among the most often used are the Maxwell and Kelvin models, which account for the most important features in viscoelastic response. Often, combining those two models leads to complete and accurate simulations of viscoelastic behavior. For an extensive analysis of viscoelastic models, see [7].

In the problem considered here, the movement is fairly slow and smooth and therefore the Maxwell model turns out to be rather suitable. It incorporates creep motion, that is the base of isostacy that plays an important role in GIA models.

In the Maxwell model, a Hookean spring is connected in serial with a linear damper, see Figure 2.4. The resulting strain is the sum of each individual contribution, elastic ($e$) and viscous ($v$)

$$\epsilon = \epsilon_e + \epsilon_v. \quad (2.32)$$

Taking derivatives in (2.32)

$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_v,$$

then differentiating $\sigma(t)$, related with $\epsilon$ via the Hooke’s law as $\sigma = E\epsilon_e$, and using the viscous response $\sigma = \eta \dot{\epsilon}_v$, in last equation yields

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta},$$

$$\dot{\sigma} + \alpha \sigma = E\dot{\epsilon}, \quad \alpha = \frac{E}{\eta}. \quad (2.33)$$
Here \( \alpha \) represents the inverse of the typical relaxation time of the material.

Interestingly enough, the fundamental solution for the viscoelastic problem can be obtained in a closed term. The derivation of the Green’s function is presented based on the analysis in [15] Section 3.10, done for the wave equation.

Let \( H(t_0) \) be the Heaviside step function, defined as

\[
H(t_0) = \begin{cases} 
0, & t < t_0 \\
1, & t \geq t_0
\end{cases}
\]  

(2.34)

Similarly, the impulse function can be defined as

\[
I(t_0, t_1) = H(t_0) - H(t_1)
\]

(2.35)

Definition (2.35) can be used to represent any function as a linear superposition of discrete jumps of size \( f(t_{k+1}) - f(t_k) \). In the limit \( \tau = t_1 - t_0 \rightarrow 0 \) the resulting function becomes continuous.

Constructing Green’s functions is a technique to solve inhomogeneous linear ordinary differential equations (ODE) when particular initial conditions are known. It consists of finding a solution of the ODE, given an arbitrary forcing function, by superposition of discrete impulse contributions. The forcing term is represented as a discrete step function of finite length \( \tau \) (Figure 2.5, right) and the total response as a superposition of individual contributions.

To gain some knowledge of equation (2.33), it is desired to evaluate at first a constant strain rate \( \dot{\varepsilon}_0 \), take then \( E \dot{\varepsilon} = E \dot{\varepsilon}_0 = a \) with \( a \) a constant forcing term

\[
\dot{\sigma} + \alpha \sigma = a.
\]

(2.36)

This equation is linear and its solution is the sum of the homogeneous solution and its particular solution

\[
\sigma(t) = \sigma_c + \sigma_p.
\]

Using separation of variables, the solution \( \sigma_c \) of the homogeneous equation \( \dot{\sigma} + \alpha \sigma = 0 \), ie.,

\[
\frac{d\sigma}{dt} = -\alpha \sigma; \quad \text{becomes} \quad \sigma_c(t) = Ae^{-\alpha t}.
\]

A natural assumption regarding the particular solution is similar to the forcing term. Thus, assume that \( \sigma_p = c \) and replace it in (2.36) to retrieve

\[
0 + \alpha c = a \rightarrow \sigma_p = \frac{a}{\alpha}.
\]

The solution sought so far is

\[
\sigma(t) = Ae^{-\alpha t} + \frac{a}{\alpha}.
\]

The trivial initial condition \( \sigma(t = 0) = 0 \) is used for simplicity, which after rearranging terms
leads to
\[ \sigma(t) = \frac{a}{\alpha}(1 - e^{-\alpha t}). \]  
(2.37)

If the forcing term acts only until time \( \tau \), by using the linearity of the solution, it is possible to represent the solution at time \( t \geq \tau \) as the linear superposition \( \sigma(t) = \sigma(t) - \sigma(t - \tau) \)
\[ \sigma(t) = \frac{a}{\alpha}(1 - e^{-\alpha t}) - \frac{a}{\alpha}(1 - e^{-\alpha (t-\tau)}). \]

Figure 2.5 (left) shows the viscoelastic response for such a constant impulse function. Consider next the limit case when \( \tau \to 0 \). Using a first order expansion shows that \( e^{\alpha \tau} \approx 1 + \alpha \tau \) and hence, the behavior can be represented as
\[ \sigma(t) = \frac{a}{\alpha}e^{-\alpha t}(e^{\alpha \tau} - 1), \]  
(2.38)
\[ \sigma(t) = a\tau e^{-\alpha \tau}. \]  
(2.39)

If the strain rate is represented as a composed (echelon) step function (Figure 2.5, right) the solution can be expressed as the superposition of each individual response \( E\dot{\varepsilon}_i \) as follows
\[ \sigma(t) = \sum_i E\dot{\varepsilon}_i \tau e^{-\alpha(t-\tau)}. \]  
(2.40)

Finally, the continuous solution can be obtained by
(i) taking the limit \( \tau \to 0 \)
\[ \sigma(t) = \int_{-\infty}^{t} E\dot{\varepsilon}(t')e^{-\alpha(t-t')} dt'. \]  
(2.41)

(ii) Integration by parts,
\[ \sigma(t) = [Ee^{-\alpha(t-t')}\varepsilon(t')]_{-\infty}^{t} - \int_{-\infty}^{t} E\alpha \varepsilon(t')e^{-\alpha(t-t')} dt', \]  
(2.42)
and assuming \( \varepsilon(t < 0) = 0 \). Then, the final viscoelastic response is found as

\[
\sigma(t) = E \varepsilon(t) - \int_0^t E \alpha \varepsilon(t') e^{-\alpha(t-t')} \, dt',
\]

or written in its original terms

\[
\sigma(t) = \sigma_e(t) - \int_0^t \frac{\partial \chi(t, t')}{\partial t'} \sigma_e(t') \, dt', \quad \chi(t, t') = e^{-\alpha(t-t')}.
\]  

The kernel function \( \chi(t, t') \) is referred in the literature as the Green’s function of the inhomogeneous ODE. Note that most of the rheologies in geodynamics are essentially nonlinear. Here the derivation is based on the linearity of equation (2.33), and the approximation is restricted to the linear theory while the earth’s behavior is of nonlinear nature.

### 2.5 The scaled GIA model

The complete GIA model reads as follows:

\[
\nabla \cdot \sigma + \nabla (u \cdot \nabla p^{(0)}) - (\nabla \cdot u) \rho_0 g + \rho_0 g = 0,
\]

where \( \sigma \) stands for the stress tensor, \( u \) is the displacement vector, \( p \) is the hydrostatic pressure and \( \rho_0 \) is the material density in relaxed state. The super-index \((0)\) on the pressure means initial conditions or pre-stress that is advected in an Eulerian frame of reference. Below the dimensionless form of (2.47) is given by using typical values that characterize the physical quantities involved in the model. To this end, consider the more general force balance

\[
\nabla \cdot \mathbf{\sigma}(\varepsilon; \mathbf{r}) + \nabla (\mathbf{u} \cdot \mathbf{b}) - (\nabla \cdot \mathbf{u}) \mathbf{c} + \mathbf{f}(\mathbf{r}) = 0.
\]

Let \( S, L \) and \( U \) be typical values for stresses, length and displacements. Consider then the dimensionless scaled values denoted by ‘\( \sim \)’:

\[
\nabla = \frac{1}{L} \mathbf{\tilde{\nabla}}; \quad \bar{\sigma} = \frac{\sigma}{S}; \quad \bar{r} = \frac{r}{L}; \quad \mathbf{\tilde{u}} = \frac{u}{U},
\]

and replacement of the scaling terms (2.46) in (2.45) leads to

\[
\frac{1}{L} \mathbf{\tilde{\nabla}} \cdot [S \bar{\sigma}(\varepsilon; \mathbf{r})] + \frac{U}{L} \mathbf{\tilde{\nabla}} (\mathbf{\tilde{u}} \cdot \mathbf{b}) - \frac{U}{L} (\nabla \cdot \mathbf{\tilde{u}}) \mathbf{c} + \mathbf{f}(L \bar{r}) = 0, \quad \text{in } \Omega.
\]

Notice, that \( \varepsilon \) is proportional to \( \nabla \mathbf{u} \), then a proper scaling \( \mathbf{\tilde{e}} = (U/L)\varepsilon \) yields

\[
\frac{US}{L^2} \mathbf{\tilde{\nabla}} \cdot [\bar{\sigma}(\mathbf{\tilde{e}}; \mathbf{\tilde{r}})] + \frac{U}{L} \mathbf{\tilde{\nabla}} (\mathbf{\tilde{u}} \cdot \mathbf{b}) - \frac{U}{L} (\nabla \cdot \mathbf{\tilde{u}}) \mathbf{c} + \mathbf{f}(L \bar{r}) = 0,
\]

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that, after normalizing, becomes
\[
\tilde{\nabla} \cdot [\tilde{\sigma}(\tilde{\varepsilon}; \tilde{r})] + L S \tilde{\nabla} (\tilde{u} \cdot b) - L S (\tilde{\nabla} \cdot \tilde{u}) c + \frac{L^2}{US} \mathbf{f}(L \tilde{r}) = 0, \quad \text{in } \tilde{\Omega}. \tag{2.47}
\]

Now use the constitutive equation (2.48) and the linear strain relationship (2.8)
\[
\tilde{\sigma} = 2\mu \tilde{\varepsilon}(\tilde{u}) I; \quad \tilde{\varepsilon}(\tilde{u}) = \frac{1}{2} \left( \tilde{\nabla} \tilde{u} + (\tilde{\nabla} \tilde{u})^T \right). \tag{2.48}
\]

Proceeding similarly as in Section 2.1.1 the Lamé coefficients yields
\[
\left( \frac{\lambda}{S} + \frac{\mu}{S} \right) \tilde{\nabla} (\tilde{\nabla} \cdot \tilde{u}) + \frac{\mu}{S} \tilde{\nabla}^2 \tilde{u} + \frac{L}{S} \tilde{\nabla} (\tilde{u} \cdot b) - \frac{L}{S} (\tilde{\nabla} \cdot \tilde{u}) c + \frac{L^2}{US} \mathbf{f} = 0,
\]
or using the dimensionless scaled pressure \( \tilde{p} \)
\[
\frac{\mu}{S} \tilde{\nabla}^2 \tilde{u} + \tilde{\nabla} \left( \frac{\mu}{S} \tilde{\nabla} \tilde{p} \right) + \frac{L}{S} \tilde{\nabla} (\tilde{u} \cdot b) - \frac{L}{S} (\tilde{\nabla} \cdot \tilde{u}) c + \frac{L^2}{US} \mathbf{f} = 0. \tag{2.49}
\]

Equation (2.49) is referred as the scaled elastic GIA model. For the scaling of the viscoelastic contributions, it is possible to take the scaling
\[
\tilde{t} = \frac{t}{T}; \quad \tilde{\tau} = \frac{\tau}{T},
\]
where \( T \) is a typical time. Defining the time-dependent viscoelastic coefficients \( \mu, \lambda \) and \( \alpha_0 \) as
\[
\lambda(r, t, \tau) = \lambda_e(r) e^{-\alpha_0(t-\tau)}; \quad \mu(r, t, \tau) = \mu_e(r) e^{-\alpha_0(t-\tau)}; \quad \alpha_0 = \frac{\eta(r)}{\mu_e(r)},
\]
where \( \eta \) is the dynamic viscosity and \( \alpha_0 \) is the inverse of the Maxwell time. Thus, the scaling leads to
\[
\lambda = \lambda_e e^{-\alpha_0 T(t-\tilde{t})}; \quad \mu = \mu_e e^{-\alpha_0 T(t-\tilde{t})},
\]
where
\[
\tilde{\alpha} = \alpha_0 T; \quad \tilde{\mu_e} = \frac{\mu_e}{S}; \quad \tilde{\lambda_e} = \frac{\lambda_e}{S}.
\]
are dimensionless quantities. On the other hand, the load vector has to be scaled with the stress tensor and its implicit inner dependencies \( \tilde{\varepsilon}, \tilde{\nabla} \tilde{u} \) as follows
\[
\tilde{\ell} = \frac{L}{US} \ell.
\]
If the ‘\( \sim \)’ signs are dropped, the final scaled GIA model reads
\[
\begin{align*}
\nabla \cdot \sigma + \frac{L}{S} \left( \nabla (u \cdot b) - (\nabla \cdot u) c \right) + \frac{L^2}{US} \mathbf{f} & = 0, \quad (2.50) \\
\sigma(t) & = \sigma_e(t) - \int_0^t \frac{\partial \chi(t, t')}{\partial t'} \sigma_e(t') dt'; \quad \chi(t, t') = e^{-\alpha(t-t')}.
\end{align*}
\]
A simple check on the order of magnitude in the different terms in (2.50) highlights the dominant terms of the model. With typical values $L = 10^7 \, m$, $S = 10^{11} \, Pa$ and $U = 1 \, m$, and $b = c = -\rho_i g_0 \mathbf{e}_z \approx 10^4 \, Pa/m$, replace in the first equation in (2.50) to obtain

$$\nabla \cdot \sigma + 10^{-4} \left( 10^4 \nabla (u \cdot \mathbf{e}_z) - 10^4 (\nabla \cdot u) \mathbf{e}_z \right) + 10^3 f = 0,$$

$$\nabla \cdot \sigma - \left( \nabla (u \cdot \mathbf{e}_z) - (\nabla \cdot u) \mathbf{e}_z \right) + 10^3 f = 0.$$  

Claiming that the maximum stress value $\sigma$ lies in the load $\ell = -\rho_i g_0 h \mathbf{e}_z L/US \approx -10^3 \mathbf{e}_z$, the last approximated model reads

$$10^3 (\nabla^2 u - \nabla p) + \nabla (u \cdot \mathbf{e}_z) - (\nabla \cdot u) \mathbf{e}_z + 10^3 f = 0. \quad (2.51)$$

Equation (2.51) shows that the stresses $\sigma$ coming from the first term and the body force $f$ are the dominant terms. If the deviatoric stress tensor is used as in Section 2.2.1, then $f = 0$. The first term in (2.51) is three orders of magnitude greater than the remaining terms $\nabla (u \cdot \mathbf{e}_z) - (\nabla \cdot u) \mathbf{e}_z$ coming from the advection of pre-stress and buoyant contributions. This result is used in Section 4, and motivates the usage of the dominant anisotropic Laplace operator $\nabla u : \nabla u$ to build efficient numerical solution strategies.
3 Discretization of the continuous model

3.1 Space discretization using the finite element method

As already discussed, to enable an adequate handling of fully incompressible materials in the model, a stabilized form of the elasticity equation is used, which includes the scaled hydrostatic pressure. The equations used in sequel are as follows

\[ \nabla \cdot \sigma + \nabla (u \cdot \rho g) - \rho g \nabla \cdot u = 0, \]
\[ \mu \nabla \cdot u - \frac{\mu^2}{\lambda} p = 0, \]  

(3.1)

where the product of the second equation in (2.30) with \( \mu \) has been taken in order to scale the \( \nabla \cdot u \) term. Equation (3.1) is equipped with either of the following constitutive relations:

\[ \sigma = 2\mu \varepsilon(u) + \mu p I \]  

(3.2)

for the elastic case and

\[ \sigma(t) = 2\mu \varepsilon(u(t)) + \mu p(t) - \int_0^t \alpha \chi(t, t') (2\mu \varepsilon(u(t')) + \mu p(t')) \, dt', \quad \chi(t, t') = e^{-\alpha(t-t')}, \]  

(3.3)

for the visco-elastic case.

Since (3.1) is a regularized system of equations for the unknown displacements and pressure, the mixed finite element formulation is used. Consider first elastic materials only, then the problem reads:

Find \( u \in V_g = \{ \phi \in H^1(\Omega) : \phi_{\Gamma_D} = g_D \}, p \in Q = L^2(\Omega) \) so that

\[ \int_\Omega (2\mu \varepsilon(v) : \varepsilon(u) + v \cdot (\nabla (u \cdot b) - b \nabla \cdot u)) \, d\Omega + \int_\Omega \mu \nabla \cdot v \, p \, d\Omega = \int_\Omega v \cdot f \, d\Omega + \int_\Gamma v \cdot t \, d\Gamma, \]

\[ \int_\Omega \mu \nabla \cdot u \, q \, d\Omega - \int_\Omega \frac{\mu^2}{\lambda} \, p \, q \, d\Omega = 0, \]  

(3.4)

for all test functions \( v \in V_0 = \{ \phi \in H^1(\Omega)^d : \phi_{\Gamma_D} = 0 \}, q \in Q \). The variational form (3.4) is often referred as the weak\(^8\) form of the elastic GIA model (3.1).

Considering finite dimensional subspaces \( v, q \) belonging to the stable\(^9\) Taylor-Hood pair \( Q2 \times Q1 \) leads to the corresponding discrete form of (3.4):

\[
\begin{bmatrix}
A & B^T \\
B & -C
\end{bmatrix}
\begin{bmatrix}
u_h \\
p_h
\end{bmatrix}
=
\begin{bmatrix}
f \\
0
\end{bmatrix},
\]  

(3.5)

\(^8\)The detailed procedure to reach the weak form (3.4) can be followed in Appendix B.

\(^9\)Appendix B.2 gives a short description of stability in FE spaces.
with bilinear forms:

\[ K_{ij} = \int_\Omega 2\mu \varepsilon(v_i) : \varepsilon(v_j) \, d\Omega, \]

\[ K'_{ij} = \int_\Omega v_i \cdot \{ \nabla(v_j \cdot b) - b \nabla \cdot v_j \} \, d\Omega. \]

\[ B_{ij} = \int_\Omega \mu \nabla \cdot v_j q_i \, d\Omega, \]

\[ C_{ij} = \int_\Omega \frac{\mu^2}{\lambda} q_i q_j \, d\Omega, \]

\[ f_i = \int_\Omega v_i \cdot f \, d\Omega + \int_\Gamma v_i \cdot \ell \, d\Gamma, \]

where \( b = \rho \, g \). Notice that as discussed in Section 2.2.3, the Lamé coefficients can be varying in space. This can be seen as a domain composed by different materials (discontinuous) and hence independent elastic properties for each one. It is often needed in Geophysics to study the coupling and interaction of two different layers, such as the lithosphere and upper mantle.

As it was shown in Section 2.5, the additional terms included (advection of pre-stress and buoyancy) are not numerically dominant in the resulting matrix \( A = K + K' \). The only mayor difference is that \( A \), being non-symmetric imposes the need to use non-symmetric iterative solvers. For this purpose GMRES is used in this thesis.

Korn’s inequality (Section B.1.1) ensures that the matrix \( K \) is positive definite. This matrix can be understood as a multidimensional stiffness matrix with extra terms that couple weakly the \( u \) components. The most important property is that \( K \) is positive definite and dominant in the resulting GIA system of equations. This fact is exploited in the construction of the GIA preconditioners.

The discrete vectors \( u_h \) and \( p_h \) coming from the FE discretization should not be confused with the physical variables \( u \) and \( p \). From this point the subscript \( h \) is dropped and \( u \) and \( p \) are understood as FE solution vectors.

### 3.2 Saddle point systems

A linear system \( A \mathbf{x} = \mathbf{b} \) on two-by-two Block-form

\[
\begin{bmatrix}
A & B_1^T \\
B_2 & -C
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
\]

(3.7)

is of saddle point nature\(^{10}\) if it satisfies one or more of the following conditions:

\(^{10}\)For an extensive treatment of saddle point problems refer to [16].
• C1) $A$ is symmetric.
• C2) The symmetric part of $A$, $K = \frac{1}{2}(A + A^T)$ is positive definite.
• C3) $B_1 = B_2 = B$.
• C4) $C$ is symmetric ($C = C^T$) and positive definite.
• C5) $C = 0$.

In the present case the system (3.5) arising from the GIA discretization complies with C2 as $A$ is dominated by the elasticity Laplace operator as seen in Section 2.5, which is positive definite from Korn’s inequality. It satisfies clearly C3 and C4 is also fulfilled as $C$ is symmetric and positive definite since it arises from the pressure mass matrix. Hence, the resulting system (3.5) for the elastic GIA, is a saddle point problem and requires special numerical solution methods.
4 Numerical solution methods for the arising linear systems

As is well known, the two approaches to solve systems of linear algebraic equations are to use

i) direct methods, or

ii) iterative methods.

Direct solution methods are often much preferred by applied scientists due to their general applicability to various types of linear systems and due to their robustness with respect to problem and discretization parameters. It is also known that direct methods pose in general high demands for computer resources, both in terms of computation and memory requirements. The direct solution techniques for sparse matrices, that are available through numerical linear algebra packages, show high efficiency and good (close to linear) scalability for increasing problem sizes, in particular, for two-dimensional problems. Still, the computational cost and the memory demands become a bottleneck (prohibitive), for large enough problems, specially for three-dimensional applications with, for instance, 500 000 degrees of freedom (dofs).

The alternative class of methods are the iterative solvers, and in particular, Krylov subspace methods. The iterative methods (for sparse matrices, as in the considered case) are known to have an optimal computational cost per iteration. Here, ‘optimal’ means linearly proportional to the number of dofs. At the same time, these methods, when applied straightforwardly, may suffer from slow convergence and even may not converge in certain situations.

Slow convergence or lack of convergence is observed for linear systems with matrices that are ill-conditioned. To quantify well- or ill-conditioning, the condition number of the matrix (say, $\mathcal{A}$), is defined as

$$\kappa(\mathcal{A}) = \|\mathcal{A}\| \cdot \|\mathcal{A}^{-1}\|.$$  \hspace{1cm} (4.1)

For symmetric and positive definite matrices, the so-called ‘spectral condition number’ is often used,

$$\kappa(\mathcal{A}) = \frac{\max|\lambda(\mathcal{A})|}{\min|\lambda(\mathcal{A})|},$$  \hspace{1cm} (4.2)

where $\lambda(\mathcal{A})$ are the eigenvalues of $\mathcal{A}$. The larger the condition number is, the more ill-conditioned the matrix becomes.

A matrix can be ill-conditioned due to various factors. One such factor, that is often encountered, is that

$$\kappa(\mathcal{A}) \approx O(h^{-\alpha}), \quad \alpha > 1, \quad \text{even} \quad \alpha \geq 2,$$

where $h$ is the characteristic mesh size. Thus, when $h$ is decreased twice in order to improve the quality of the discretization, the condition number increases four times. Further, $\kappa(\mathcal{A})$ may be large due to problem parameters, in particular when these approach critical values, such as $\nu \to \frac{1}{2}$, or due to bad scaling.
4.1 The notion of preconditioning

The condition number of a matrix is always larger or equal to one ($\kappa(A) \geq 1$), and it is equal to one for the identity matrix. Therefore, the best condition number is close to one, and ideally is independent of the various parameters, originating from the model or introduced after discretization.

Consider the solution of a system

$$A x = b.$$ 

The classical remedy to improve the conditioning of $A$ is by using a suitable preconditioner. The preconditioner is most often another matrix, $P$, such that the so-called ‘preconditioned matrix’ $P^{-1}A$ is well-conditioned. When using a preconditioner, the system to solve is

$$P^{-1}A x = P^{-1}b,$$

which is equivalent to the original one, and has the same solution.

The general requirements for $P$ to be an efficient preconditioner for $A$ are:

i) $\kappa(P^{-1}A) < \kappa(A)$, and hopefully, $\kappa(P^{-1}A) \ll \kappa(A)$.

ii) The overall preconditioned solution scheme must offer a gain in performance with respect to the original system.

iii) The construction of $P$ must be cheap and solving a system with it (occurring once during each iteration) must be much easier than solving systems with $A$.

iv) Since it is aimed to solve very large problems, requiring parallel computations is natural. Thus, solutions of systems with $P$ must be well-parallelizable.

4.2 Types of preconditioners

The preconditioning matrix can be chosen in different ways.

a) Left preconditioning

In this case $A x = b$ is replaced by $P^{-1}A x = P^{-1}b$.

b) Right preconditioning

It is also possible to apply $P$ from the right side of $A$, namely

$$\text{solve } \tilde{A}y = b; \quad \tilde{A} = AP^{-1}$$

$$\text{set } x = P^{-1}y$$
c) Symmetrized preconditioners

If \( P \) is symmetric and positive definite, then \( P^{-1/2} \) exists and is also symmetric and positive definite. In certain cases the symmetricity can be preserved by combining a) and b) as follows:

\[
\text{solve } \tilde{A}y = P^{-1/2}b; \quad \tilde{A} = P^{-1/2}AP^{-1/2}
\]

set \( x = P^{-1/2}y \)

\[\text{d) Approximate inverse preconditioners}\]

In some cases it turns out to be possible to construct directly an approximation of \( A^{-1} \). Then, no solutions of systems with the preconditioner are required but only multiplications with it, i.e.

\[
\text{solve } P\tilde{A}x = Pb; \quad \text{where } P \approx A^{-1}
\]

This approach is very attractive when the construction of a sparse approximation of \( A^{-1} \) is available. Note, that if \( A \) is sparse, then \( A^{-1} \) is in general dense. This approach has led to a large class of preconditioning techniques, referred to as SPAI (from ‘sparse approximate inverses’).

e) Implicitly defined preconditioners

Preconditioning by no means requires that \( P \) has to be explicitly available. In some advanced preconditioning techniques, \( P \) is defined via a procedure, that corresponds to a solution of a system, where the matrix is only implicitly defined.

Examples of such methods are Multigrid (MG), Algebraic Multigrid (AMG), Algebraic Multilevel Iteration (AMLI) methods and Domain Decomposition (DD) methods.

In such cases, the implicitly preconditioned system is denoted as

\[
[P^{-1}]A\tilde{x} = [P^{-1}]b. \tag{4.3}
\]

To illustrate an implicitly defined preconditioner, a brief description of MG followed by a simple test are presented in Appendix C.

4.3 Preconditioners for systems of two-by-two block form

As the system matrix in (3.5) is of two-by-two block form, it is natural to be solved or preconditioned using its algebraic structure.

4.3.1 Reduction to Schur complement form

To illustrate the Schur complement based solution techniques consider the system

\[
\begin{bmatrix}
A & B^T \\
B & -C
\end{bmatrix}
\begin{bmatrix}
u \\
p
\end{bmatrix}
= \begin{bmatrix}
f \\
g
\end{bmatrix}
\text{ or } A\mathbf{u} + B^T\mathbf{p} = \mathbf{f}, \quad B\mathbf{u} - C\mathbf{p} = \mathbf{g}. \tag{4.4}
\]
Assume that $A^{-1}$ exists, solve for $u$ in the first equation and substitute it in the second equation,

$$-Sp = g - BA^{-1}f; \quad S = C + BA^{-1}B^T. \quad (4.5)$$

The matrix $S$ is known in the literature as the Schur complement of the system matrix $A$. After solving (4.5) for $p$, the displacements $u$ are found by solving

$$Au = f - B^Tp. \quad (4.6)$$

A mayor drawback of the Schur complement reduction is that there is the need to solve systems with $S$ requiring to compute $A^{-1}$ explicitly, being clearly unfeasible unless $A$ is diagonal, which is not the case for the considered problems. In order to obtain the action of $S$ on a vector, the remaining option is to use an inner solution method for $A$. Note, that in this case the inner system has to be solved to machine accuracy in order to accurately apply the action of $S$. Thus, even if it is possible to solve efficiently systems with $A$, the inner solver will be expensive. In addition, since $S$ is still a large system, and should be solved iteratively, it is needed to precondition it. Therefore, the reduction to Schur complement is not recommended.

4.4 Existing solution algorithms

4.4.1 Uzawa-like methods

An alternative way of solving (4.4) is to iterate on a relaxation scheme

$$Au_{k+1} = f - B^T p_k,$$
$$p_{k+1} = p_k + \omega(Bu_{k+1} - C p_k), \quad (4.7)$$

known as the extended Uzawa iteration in [17], [14] and [16]. Here $\omega$ is a parameter that can be determined so that an optimal convergence rate can be achieved. This approach can be extended further to the Arrow-Hurwicz iteration among other of similar nature.

$$u_{k+1} = u_k + \alpha Q^{-1}_A(f - Au_k - B^T p_k),$$
$$p_{k+1} = p_k + \omega Q^{-1}_B(Bu_{k+1} - C p_k), \quad (4.8)$$

where $\alpha$ is a parameter that plays a similar role as $\omega$ but for the displacements. The choice of preconditioners $Q^{-1}_A, Q^{-1}_B$ is broad in the literature. Despite the great efforts to improve Uzawa-like schemes, the inconvenient low convergence rate of these methods motivate the need for more robust and effective schemes.

Contemporary alternative methods have been also proposed for saddle point problems, for instance the Augmented Lagrangian (AL) method. Research based on the application of AL methods on slightly modified equations for the dynamics of the mantle is found in [18]. Proponents firmly argue that preconditioning with AL result in a fully size-independent solution scheme. For a comprehensive extension on AL, please refer to [19] and [18].
4.4.2 Block-factorization preconditioning methods

Consider the general matrix $\mathcal{A}$ in two-by-two form,

$$
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}.
$$

The matrix $\mathcal{A}$ possesses an exact block-factorization

$$
\mathcal{A} = \begin{bmatrix}
I_1 & 0 \\
A_{21}A_{11}^{-1} & I_2
\end{bmatrix} \begin{bmatrix}
A_{11} & 0 \\
0 & S_{\mathcal{A}}
\end{bmatrix} \begin{bmatrix}
I_1 & A_{11}^{-1}A_{12} \\
0 & I_2
\end{bmatrix} = L_{\mathcal{A}} D_{\mathcal{A}} U_{\mathcal{A}},
$$

where $S_{\mathcal{A}} = A_{22} - A_{21}A_{11}^{-1}A_{12}$. Based on (4.9), various approximations of $\mathcal{A}$ can be derived and used as preconditioners.

The literature on preconditioners for matrices of two-by-two block form is vast [20, 21, 16, 22, 23]. Based on the available theory and the fact that $\mathcal{A}$ in the current application is non-symmetric of saddle point form, this thesis is confined to a preconditioner of lower-triangular form, known to be potentially very efficient from [1].

4.4.3 Proposed preconditioner

The system matrix $\mathcal{A}$ in (3.5) can be factored exactly as follows

$$
\begin{bmatrix}
A & B^T \\
B & -C
\end{bmatrix} = \begin{bmatrix}
A & 0 \\
B & -S
\end{bmatrix} \begin{bmatrix}
I_1 & A^{-1}B^T \\
0 & I_2
\end{bmatrix};
$$

$S = C + BA^{-1}B^T$. (4.10a)

The second matrix in the factorization is approximated by the identity. If neglected a good candidate for a preconditioner is

$$
\mathcal{P} = \begin{bmatrix}
\hat{A} & 0 \\
\hat{B} & -\hat{S}
\end{bmatrix},
$$

where $\hat{A}$ and $\hat{S}$ are some approximations of $A$ and the negative Schur complement of $\mathcal{A}$, $S$.

The preconditioner $\mathcal{P}$ can be expected to be a good preconditioner for $\mathcal{A}$, provided that $\hat{A}$ and $\hat{S}$ are good approximations of $A$ and $S$ respectively. To illustrate the latter, consider

$$
\mathcal{P}_0 = \begin{bmatrix}
A & 0 \\
B & -S
\end{bmatrix},
$$

known as the ‘ideal preconditioner’ for saddle point matrices [21]. To see the effect of $\mathcal{P}_0$ as a preconditioner for $\mathcal{A}$, consider the generalized eigenvalue problem

$$
\mathcal{A}\mathbf{x} = \lambda \mathcal{P}_0 \mathbf{x}.
$$
If \( P_0 \) is a good preconditioner for \( A \), the eigenvalues \( \lambda \) are expected to be close to 1 (clustered around one). Note that \( P_0 \) is strongly non-symmetric, thus \( \lambda \) are in general complex numbers. To simplify the derivations, instead of \( A \mathbf{x} = \lambda P_0 \mathbf{x} \), consider \( (A - P_0) \mathbf{x} = (\lambda - 1)P_0 \mathbf{x} \) and let \( \mu = (\lambda - 1) \).

Clearly, when \( \mu = 0 \) or \( \mu \approx 0 \), then \( \lambda = 1 \) or \( \lambda \approx 1 \), what is wanted. Computations show that

\[
A - P_0 = \begin{bmatrix} 0 & B^T \\ 0 & -C + S \end{bmatrix} \quad \text{and} \quad P_0^{-1} = \begin{bmatrix} A^{-1} & 0 \\ S^{-1}BA^{-1} & -S^{-1} \end{bmatrix},
\]

then

\[
P_0^{-1}(A - P_0) = \begin{bmatrix} A^{-1} & 0 \\ S^{-1}BA^{-1} & -S^{-1} \end{bmatrix} \begin{bmatrix} 0 & B^T \\ 0 & -C + S \end{bmatrix} = \begin{bmatrix} 0 & A^{-1}B^T \\ 0 & S^{-1}BA^{-1}B^T + S^{-1}C - S^{-1}S \end{bmatrix} = \begin{bmatrix} 0 & A^{-1}B^T \\ 0 & S^{-1}(BA^{-1}B^T + C - S) \end{bmatrix} = \begin{bmatrix} 0 & A^{-1}B^T \\ 0 & 0 \end{bmatrix}.
\]

From (4.11) is clearly apparent that for the ideal preconditioner \( \mu = 0 \), thus \( \lambda = 1 \). Therefore, an iterative method to solve \( A \), preconditioned by \( P_0 \) will converge for at most two iterations. In practice, it is unfeasible to use the exact blocks \( A \) and \( S \), and they need to be approximated.

Remark: The theory of preconditioned iterative methods for nonsymmetric problems reveals, that clustering of eigenvalues in one or few clusters is not sufficient to guarantee fast convergence. For a complete analysis, some information about the eigenvectors is required as well (see for instance [24]). Such information is, in general, very difficult to obtain. In this study it is considered enough control to monitor the behavior of the spectrum of the preconditioned matrix.

### 4.5 Approximations for the pivot block \( A \)

Since \( A \) is a large block, the best choice would be not to approximate it by another matrix but to solve it by an inner iterative method instead. In this way a stopping criteria, offers the possibility to control the quality of the resulting approximation for \( A \). For a stopping criterion close to machine accuracy, the final effect is \( \hat{A} = A \), which is very expensive. Therefore, a good preconditioner is needed. As it is suggested in [11] for the standard formulation of the elasticity problem (Appendix B.1) the block-diagonal part \( \hat{K} \) of the stiffness matrix \( K \) for the Lamé-Navier equation results in an optimal preconditioner for \( K \). The explicit form of the matrices is

\[
K_{ij} = \int_{\Omega} 2\mu \varepsilon(v_i) : \varepsilon(v_j) \, d\Omega,
\]

\[
\hat{K}_{ij} = \int_{\Omega} \mu \nabla v_i : \nabla v_j \, d\Omega.
\]
Namely, if
\[
K = \begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix},
\]
then \( \hat{K} = \begin{bmatrix}
K_{11} & 0 & 0 \\
0 & K_{22} & 0 \\
0 & 0 & K_{33}
\end{bmatrix} \).

Also in [11], a sharper bound for the spectral condition number \( \kappa(\hat{K}^{-1}K) \) is found as
\[
\kappa(\hat{K}^{-1}K) \leq 2 \frac{1 + \nu}{1 - 2\nu},
\]
where \( \nu \) is the Poisson’s ratio. It is clearly apparent that the condition number deteriorates considerably when \( \nu \to \frac{1}{2} \).

The problem presented in this thesis differs from the Lamé-Navier equations in two respects:

i) it is formulated as a mixed problem, i.e. even if no convective terms are included, the pivot block differs from that in the standard formulation;

ii) the system matrix, including the pivot block is nonsymmetric, due to the inclusion of the pre-stress and the buoyancy terms.

Nevertheless, the block \( A \) is solved by an inner iterative method, and its block-diagonal part \( \hat{A} \) is used as a preconditioner. To measure the quality of \( \hat{A} \) consider the eigenvalue problem
\[
A\mathbf{u} = \lambda\hat{A}\mathbf{u},
\]
where in the case \( A = \hat{A}, \lambda = 1 \).

Figure 4.1 plots \( \lambda \) for M0, M1 given in Section 2.2.2 and refinments to the initial mesh \( k = 3, 4 \). For M0 \( \lambda \) is real, and for M1 the imaginary part is small compared with the real part, meaning that the problem is not dominated by advection. It is clearly seen in Figure 4.1 that Re(\( \lambda \)) is clustered around one, which confirms the good quality of the preconditioner. The clustering of Re(\( \lambda \)), and hence the condition number, does not grow with the refinments \( k \) of the mesh, making \( \hat{A} \) an ideal preconditioner. Finally, the efficiency of this approach is confirmed by all numerical experiments in [25] and [1].

4.6 Element-by-element Schur complement approximation

The approximation \( \hat{S} \) is obtained in this work by using the element-by-element (EBE) technique. EBE is only applicable in a FEM setting, and is defined as follows.

The element system matrix \( \mathcal{A}_e \) has also a two-by-two block form
\[
\mathcal{A}_e = \begin{bmatrix}
A_e & B_e^T \\
B_e & -C_e
\end{bmatrix}.
\]
Figure 4.1: Comparison of eigenvalues $\lambda$ from the generalized problem $Au = \lambda \hat{A}u$. The index $i$ represents min-max reordering.
where the subscript $e$ refers to the element index. Then

$$\hat{S} = \sum_{\text{elements}} \left( C_e + B_e (A_e + h^2 I_e)^{-1} B_e^T \right).$$

Here $h$ stands for the characteristic mesh size, and its use is to ensure that the element matrices become non-singular, as local matrices away from the boundaries are essentially singular.

The EBE approximation of the Schur complement has very compelling properties\textsuperscript{11} for nonsymmetric matrices in two-by-two block form and is a very attractive preconditioner for being cheap to calculate. The spectral proximity with the exact $S$ is what makes $\hat{S}$ of real value. To illustrate this fact Figure 4.2 shows the eigenvalues of a small case (4 refinements) of Problem 1 with the configuration in Table 1. Figure 4.2 (top), shows the eigenvalues for $S$, $\hat{S}$ and $C$. The eigenvalues of $\hat{S}$ fit closer and better represents detailed features of $S$ compared with $C$. A deep comparison of eigenvalues coming from the generalized problems $S \mathbf{p} = \lambda \hat{S} \mathbf{p}$ and $S \mathbf{p} = \lambda C \mathbf{p}$ in Figure 4.2 (bottom) suggest the best fit of $\hat{S}$ as the eigenvalues $\lambda(\hat{S})$ are clustered around one whereas $\lambda(C)$ are more dispersed.

\textsuperscript{11} More on the special properties of $\hat{S}$ can be found in [26]
4.7 Solution scheme

Below, the solution procedure used to solve a system with the matrix $\mathcal{A}$ as given in (3.5) is summarized.

Systems of the form $\mathcal{A}x = b$ are solved once in the elastic case and for each time increment in the viscoelastic case. The outer solution method is preconditioned GMRES [24]. Within each outer iteration, a system with the preconditioner is solved. The complete algorithm to solve with $\mathcal{P}$ is as follows

$$\mathcal{P} = \begin{bmatrix} \hat{\mathcal{A}} & 0 \\ B & -\hat{\mathcal{S}} \end{bmatrix},$$

where $\hat{\mathcal{S}}$ is the EBE approximation of the exact Schur complement $\mathcal{S}$.

The solution with $\mathcal{P}$ decomposes into the following steps

$$\begin{bmatrix} \hat{\mathcal{A}} & 0 \\ B & -\hat{\mathcal{S}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \quad (4.14)$$

Step 1: Solve $A y_1 = f_1$.

The solution is done by an inner preconditioned GMRES method, where the preconditioner is the block-diagonal matrix $\hat{\mathcal{A}}$. The inner solver for $A$ is referred to as the ‘level-1 inner solver for $A$’.

Each diagonal block of $A$ is solved by an AMG-preconditioner. The latter block solver is referred to as the ‘level-2 inner solver for $A$’.

Step 2: Compute $z = f_2 - B y_1$.

Step 3: Solve $\hat{\mathcal{S}} y_2 = -z$

Systems with $\hat{\mathcal{S}}$ are also solved by an AMG-preconditioned GMRES.

Note, that although the systems that are solved are linear, the solution procedure is highly non-linear due to the inner solvers involved, and, thus, the preconditioner is variable.
Table 1: Physical parameters used in the uniform load, Footing and Extended Footing problem.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_T$</td>
<td>4000 km</td>
</tr>
<tr>
<td>$I_s$</td>
<td>1000 km</td>
</tr>
<tr>
<td>$h$</td>
<td>2.17 km</td>
</tr>
<tr>
<td>$U$</td>
<td>1 m</td>
</tr>
<tr>
<td>$L$</td>
<td>10' m</td>
</tr>
<tr>
<td>$\rho_i$</td>
<td>981 kg/m$^3$</td>
</tr>
<tr>
<td>$\rho_r$</td>
<td>3300 kg/m$^3$</td>
</tr>
<tr>
<td>$g_0$</td>
<td>9.8 m/s$^2$</td>
</tr>
<tr>
<td>$E$</td>
<td>4 x 10$^{11}$ Pa</td>
</tr>
<tr>
<td>$S$</td>
<td>10$^{11}$ Pa</td>
</tr>
</tbody>
</table>

5 Numerical experiments

In this section the numerical experiments set to test the final code are presented together with a description of the domain, model parameters and stopping criteria used for the outer and the inner solvers. All the experiments are presented for the following three cases:

M0) $\nabla \cdot \sigma = 0$.

M1) $\nabla \cdot \sigma + \rho_r \nabla(u \cdot g) = 0$.

M2) $\nabla \cdot \sigma + \rho_r \nabla(u \cdot g) - \rho_r g (\nabla \cdot u) = 0$.

The two dimensional displacement field is denoted by $u = [u, v]^T$ and the imposed boundary conditions applied on all experiments read as follows:

\[
\begin{align*}
    u &= 0 \quad \text{on } \Gamma_D \text{ (no slip)}, \\
    u &= 0, \quad \frac{\partial v}{\partial y} = 0 \quad \text{on } \Gamma_V \text{ (vertical slip)}, \\
    \sigma \hat{n} &= \rho_i g h \quad \text{on } \Gamma_L \text{ (load)}, \\
    \frac{\partial u}{\partial \hat{n}} &= 0 \quad \text{on } \Gamma_N \text{ (homogeneous Neumann)},
\end{align*}
\]

The imposed boundary conditions can be described as $\Gamma_V$ is set to vertical slip, which stands for homogeneous Dirichlet (clamped) on the displacements in the x-direction but homogeneous Neumann on the vertical displacements. $\Gamma_D$ is set to no slip, meaning no displacements at all. Finally, $\Gamma_L$ is subject to a load force $\ell$ of magnitude $\rho_i g_0 h$, imposing a vertical compressive stress exerted over the domain by a block of ice of thickness $h$ with “ice density” $\rho_i$.

A constant value of the gravity field $g_0$ is used, and the inner Earth will have a constant density $\rho_r$ (rock). A square initial mesh is used, that is then refined in a regular way. The space discretization is done by stable Taylor-Hood $Q2 \times Q1$ elements. Each problem’s domain starts with an initial number of squares and from there the meshes are refined $k$ consecutive times.

The common model parameters for the problem are given in Table 1 and $g = -g_0 e_y$, with $e_y$ a vertical vector pointing upwards.

The system tolerance controlling the relative $L^2$-error in the global scheme is set to $TOL(\mathcal{A}) = 10^{-7}$ for all the experiments. The tolerances of the inner solvers for computing the effect of $\hat{S}$
and $\hat{A}$ have to be chosen carefully as a compromise between accuracy and time applies. Namely, a small $TOL(\hat{A})$ reproduces a more accurate effect of $A^{-1}$ in the preconditioning algorithm (4.7) reducing both, the total system iterations and the iterations for $\hat{A}$, but at the same time each iteration becomes more expensive to compute. On the other hand, a more permissive value of $TOL(\hat{A})$ results in less iterations for $\hat{A}$ but poor approximation of the inverse $A^{-1}$, demanding a higher number of system iterations. For a suitable choice several runs revealed that the optimal values $TOL(\hat{A}) = 0.001$, $TOL(\hat{S}) = 0.1$ in relative $L^2$-error minimize the execution time. The serial code for all the experiments was run on the Kalkyl Cluster with Quad-core Intel Xeon 5520 (Nehalem 2.26 GHz, 8MB cache) processors.

The numerical experiments presented in this thesis are enumerated as Problem 1, 2 and 3:

\section{5.1 Benchmark: Uniform load problem}

This section compares numerical experiments with the available analytical solutions derived in Appendix (A), that serve as a comparison point to fine-tune the computational algorithm.

A convenient benchmark for linear elasticity models is the “uniform load” test 1, depicted in Figure 5.1: a homogeneous elastic body, occupying a rectangular (in 2D) domain is subjected to uniform vertical load (on $\Gamma_L$).

\textbf{Problem 1} The uniform load, sketched in Figure 5.1, is the benchmark used to verify convergence of the discretization. The problem mesh starts with one quadrilateral (cell) on refinement $k = 0$ and squared dimensions of sides $L_x = L_y = 4\,000$ km. For measuring the convergence of the method at a particular refinement $k$, the difference of the analytical solution $u$ given in Appendix A and the numerical solution $u_h$ measured in $L^2$ norm is calculated as

$$||e_h||^2 = \int_{\Omega} e_h \cdot e_h \, d\Omega; \quad e = u - u_h,$$

(5.2)
<table>
<thead>
<tr>
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<th>$h$</th>
<th>cells</th>
<th>dofs($u$, $p$)</th>
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<td>(18+4)</td>
</tr>
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<td>4</td>
<td>(50+9)</td>
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<tr>
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<td>0.1</td>
<td>16</td>
<td>(162+25)</td>
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<tr>
<td>3</td>
<td>0.05</td>
<td>64</td>
<td>(578+81)</td>
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<tr>
<td>4</td>
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<td>256</td>
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<tr>
<td>5</td>
<td>0.0125</td>
<td>1024</td>
<td>(8450+1089)</td>
</tr>
<tr>
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<td>4096</td>
<td>(33282+4225)</td>
</tr>
<tr>
<td>7</td>
<td>0.003125</td>
<td>16384</td>
<td>(132098+16641)</td>
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<tr>
<td>8</td>
<td>0.0015625</td>
<td>65536</td>
<td>(526338+66049)</td>
</tr>
</tbody>
</table>

Table 2: Computational parameters in the uniform load (problem 1) experiment.

Figure 5.2: Results for Problem 1, model M1 for number of refinements $k = 4$, $\nu = 0.2, 0.3, 0.4, 0.5$, and physical parameters shown in table (1).

which is tested for each refinement $k$.

Table 2 shows the correlation between refinements $k$, mesh size $h$, number of cells and dofs for $u$ and $p$.

5.2 Footing problem

The major difference with Problem 1 is that the upper segment is partly subjected to a constant forcing load on the region $I_x$ and homogeneous Neumann for the rest. The physical values for Problem 2 are shown in Table 1 and the sketch of the problem is shown in Figure 5.3.

**Problem 2** Footing problem, sketched in Figure 5.3 and dimensions $L_x = 10\ 000\ km$. The problem mesh starts with a mesh of 10 divisions on $L_x$ and 4 divisions on $L_y$.  

![Graph 1](image1.png)

(a) $u(y)$ at a middle point $x = x_m$.

![Graph 2](image2.png)

(b) $v(y)$ at a middle point $x = x_m$. 

35
<table>
<thead>
<tr>
<th>$k$</th>
<th>$| e_k |$</th>
<th>iter</th>
<th>$| e_k |$</th>
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<th>$| e_k |$</th>
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</tr>
</thead>
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<td>8(15,3)</td>
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<td>11(15,4)</td>
<td>0.000340491</td>
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<td>9(27,5)</td>
</tr>
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<td>8</td>
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<td>10(63,6)</td>
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</table>

Table 3: Numerical experiments for Problem 1 with $TOL(\mathcal{A}) = 10^{-7}$, $TOL(\hat{\mathcal{A}}) = 10^{-3}$ and $TOL(\hat{\mathcal{S}}) = 0.1$ for testing the different models $M_0$, $M_1$ and $M_2$ over different number of $k$ refinements and Poisson ratios $\nu$. Deviations in $L^2$ norm and iterations in the form $\mathcal{A}(\hat{\mathcal{A}}, \hat{\mathcal{S}})$ are shown.
Table 4: Results for Problem 2, where $t_a$ stands for ‘assembly time’ and $t_s$ ‘solver time’ in seconds.

In order to evaluate the numerical solutions of the problem depicted in was run in two similar experiments with same physical parameters in Table (1) but different domains, problem 2 with $L_x = 10\,000\, km$ and problem 3 $L_x = 25\,000\, km$ respectively.

**Problem 3** Extended Footing, with same sketch as the Footing in Figure 5.3, but $L_x = 25\,000\, km$. 

```
<p>| | | | | |</p>
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<td>2</td>
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<td>$t_s$</td>
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<td>$t_s$</td>
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<td>1.31</td>
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(a) Results for Poisson’s ratios $\nu = 0.2, 0.3$. (b) Results for Poisson’s ratios $\nu = 0.4, 0.5$. 
```
Figure 5.3: Configuration for the footing problem.

Table 5: Computational configuration of Problem 2.

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<th>dofs</th>
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<tr>
<td>5</td>
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<td>(329474+41409)</td>
</tr>
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</table>
The initial mesh is obtained by dividing $L_x$ in 25 intervals and $L_y$ in four intervals.

Figure 5.4: Results for Problem 2, model M1.
Figure 5.5: Results for Problem 3, model M1.
Figure 5.6: Comparison between models in Problem 3 for different Poisson ratios $\nu$. 
Figure 5.7: Problem 3, $v = 0.4$: Color representation for the M2 model with shortened domain to fit the page. Right) magnitude $||\mathbf{u}||$ color map and exaggerated (100:1) Lagrangian representation, middle) component $u$ color map with vector field $\mathbf{u}$ in arrows and left) component $v$ color map.
6 Discussion

6.1 Results for the benchmark: Problem 1

Several global conclusions can be drawn from the results in Problem 1

i) The fact that the optimal \( TOL(\hat{S}) \) is relatively high reflects the excellent quality of the preconditioner \( \hat{S} \), that keeps the iterations for \( \hat{S} \) low at a cheap cost.

ii) Figure 5.2a shows the remaining numerical error of the scheme, as analytically \( u(y) = 0 \) and the numerical test reproduces the horizontal displacements of the order of the numerical error \( (10^{-4}m) \). The analytical solutions of Problem 1, model M1 is given in Appendix A.2, equation (A.13).

iii) The exponential behavior of \( v(y) \) is well reproduced in Figure 5.2b. In the incompressible case \( \nu = 0.5 \), equation (A.13) predicts \( v(y) = 0 \), which is fulfilled in the numerical solution.

iv) Table 3 shows the deviations \( ||e_k|| \) in \( L^2 \) norm calculated by using equations (A.10), (A.13) and (A.17), where \( k \) represents the number of initial mesh refinements. The major conclusion is that the numerical scheme is consistent as convergence is reached when the number of refinements is increased. This can be seen as the deviations for each case (M0, M1, M2) diminishes with the refinements \( k \).

v) The number of iterations increased with higher Poisson ratios \( \nu \) in each case, being the incompressible case \( (\nu = 0.5) \) the most expensive problem.

vi) The number of outer iterations shows that the preconditioned scheme achieved the imposed performance requirements. The number of iterations is nearly mesh-independent, giving an acceptable scalability for the problem. There is a slight growth of the outer iterations when \( \nu = 0.5 \).

vii) Error estimates for mixed problems in FE are of the form

\[
||e_k|| \leq C_1 ||u - u_h|| + C_2 ||p - p_h||
\]

and unfortunately, it is not straight forward to compare convergence rates. For this reason, results in Table 3 do not suggest fixed convergence rates. However, the method shows convergence when comparing results in pairs of refined meshes.

6.2 Unphysical boundary conditions in Problem 2

In Figure 5.4 variations for the horizontal \( u(x_s) \) and vertical \( v(x_s) \) displacements are shown for evaluation at the upper surface \( (x_s, 0) \), in order to compare the responses with different Poisson
ratios $\nu$. The first conclusion is the tight and smooth proximity of graphs with $\nu = 0.49$ and $\nu = 0.5$ with no discontinuities or jumps in the numerical solution.

The length of the used domain is short on the right boundary $x = L_x = 10\,000\,km$ enforcing the displacements to get a wrong shape. The displacements are expected to decay smoothly to zero, which is not seen in this experiment that is first obtained in [1]. Furthermore, it is noticed in the point $u(x_s = L_x)$, that the graph shows $\frac{\partial u}{\partial x_s} = 0$ with a positive $u \neq 0$ at the right end. In the same case, the vertical $v(x_s = L_x)$ shows a negative slope $\frac{\partial u}{\partial x_s} < 0$ tending to a linear decay $\frac{\partial^2 u}{\partial x_s^2} = 0$.

In contrast to Problem 2, Figure 5.5 (Problem 3) shows a natural relaxation towards zero, for both $u(x_s)$ and $v(x_s)$ at the surface in the limit $x_s \to \infty$. As a consequence, after the forebulge $v(x_s)$ is never negative contrary to what results in Problem 2 suggested. It is easy to see that the unphysical boundary condition in $x_s$ showed in Figure 5.4, introduces an artificial force that alters the horizontal location of the forebulge. In the vertical displacements the maximum and minimum remained fairly similar, but the shape of the plots changed dramatically.

These unphysical features of the numerical solution are consequence of imposing boundary conditions before the curve gets relaxed. The straightforward solution is to enlarge the domain, so that the physical requirements are satisfied. The proposed extended domain in Problem 3 is set to $L_x = 25\,000\,km$. Notice that the extended domain on a flat Earth shows a correct physical decay towards relaxation ($u, v = 0$) away from the load, an segment of arc of $25\,000\,km$ in a sphere with radius $R = 6\,400\,km$, represents an angle $\theta \approx 224^\circ$, which falls out from a good approximation. The last problem suggests to implement an spherical domain to test the GIA model with fewer approximations.

### 6.3 Results for Problem 3 and model comparison

Several conclusions about the numerical experiments can be drawn by considering Figures 5.5, 5.6 and Table 4:

i) It is clearly apparent from Figure 5.5 that in the limit $\nu \to 0$ the minimum in both horizontal and vertical displacements increases in depth, but the forebulge in the vertical highest point decreases from a sharp curve when $\nu = 0.5$, until being completely flattened ($\nu = 0.2$). As expected, the last analysis suggests that higher values of $\nu$ result in forces opposing the load compression due to the increment in shear stresses.

ii) It is notorious that M1 and M2 give very close solutions in all the plots in Figure 5.6 and main discrepancies between them are due to higher order approximation as the buoyant term in M2. However, they differ greatly from M0 with more pronounced variations in the M2 model, mainly in the horizontal displacements that have been somehow left aside in former GIA studies. For instance, the maximum in $u(x_s)$ is displaced about 800-1000 km to the right, which is a large value compared with the dimensions of the problem. Hence,
it can be said that the inclusion of such higher order terms are justified only when \( u(x) \) is of great value in the current research.

iii) From the vertical displacements with \( \nu = 0.2 \), it is clear that the lack of buoyancy in the M1 model lead to an overestimated advection of pre-stress term, that is being corrected at some point by the buoyant term included in M2.

iv) In the incompressible case M2 and M1 overlap, the only difference between these models is the buoyant term containing \( \nabla \cdot \mathbf{u} \), which is clearly satisfied (\( \nabla \cdot \mathbf{u} = 0 \) when \( \nu = 0.5 \)). This fact can be used as a physical test to corroborate the good behavior of the solutions.

v) From Table 4 it is seen in almost all cases (exception \( \nu = 0.4 \)) that model M1 lessen the outer iterations at the expense of increasing the inner iterations for \( \hat{A} \), which resulted in more computational time, which can be understood that the introduction of the buoyant term (M2) relaxes the system in a way that the computational time is lowered.
Future work

Aiming at better representing the response of the earth to ice loads, it is necessary to account for an elastic lithosphere and viscoelastic mantle with different material properties that are allowed to advect with time. For this purpose a two phase problem can be included where a pure phase accounts for a particular material property.

It is widely known that the Earth is a dynamic system called Geoid. Redistribution of land heights and sea-level due to GIA movements result in considerable changes in the gravitational field which varies with this redistribution. A notorious improvement in the model would be to introduce the gravitational potential and solve for the Poisson equation at every iteration.

Simple calculations for the used 2D domain ($10000\ km \times 4000\ km$) show that a flat Earth approximation is not a good choice as $L_y$ is of the order of the Earth radius. In order to set up a more realistic model, a generalization to a spherical domain must be included in the list of improvements.

As argued in Section 2.1.1 GIA displacements are small and sufficiently smooth for considering a linear approach. Nonetheless, when considering the influence of temperature in the boundary near the mantle Non-linear rheologies should arise. A final possibility is to study a GIA model that accounts for nonlinear terms.
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[^12]: www.latex-project.org
[^13]: www.vim.org
[^14]: http://www.texample.net/tikz/
[^15]: http://www.vtk.org
[^16]: http://www.paraview.org
[^17]: http://comunidad.udistrital.edu.co/grupofisinfor/
[^18]: http://www.colfuturo.org
7 Introducción, estado del arte y contribución de la tesis

Actualmente los países nórdicos experimentan un movimiento de los suelos, en donde los contornos geográficos se modifican en elevación con respecto al nivel del mar. Numerosos estudios atribuyen tal fenómeno a lo que se conoce como ajuste post glacial (AP); argumentando que en periodos de glaciación crecieron extensas capas de hielo sobre latitudes nortes/sureñas comprimiendo al planeta en equilibrio isostático (Presión de equilibrio horizontal entre las capas internas de la tierra).

El proceso de balance induce el transporte de material desde regiones de alta presión hacia regiones de baja presión. Sin embargo, en periodos de temperaturas cálidas (interglaciares), las capas de hielo se derriten en periodos relativamente cortos permitiendo que la presión sea liberada desde capas mas profundas resultando en un levantamiento de los suelos. Las capas cercanas a la superficie de la tierra se modelan como una litosfera elástica y un manto superficial visco elástico con coeficientes de viscosidad que varían en el orden de los $10^{19} - 10^{22} \text{ Pa} \cdot \text{s}$. En movimientos sujetos a tales viscosidades y escalas de tiempo tan extensas la inercia no juega un papel decisivo y puede ser descartada.

Las consecuencias debidas al movimiento post glacial de la tierra son varias; por ejemplo el nivel promedio del mar varía dependiendo de él. El conocimiento de la evolución de dichos contornos es un factor decisivo en la comprensión de las tendencias migratorias de animales e incluso la raza humana. Un ejemplo claro de ello son las conexiones o puentes naturales que se forman entre zonas aisladas o islas cuando el nivel del mar es bajo, que claramente permite la migración de las especies.

Movimientos característicos de dichas viscosidades en el manto se manifiestan en pequeños desplazamientos pero requieren de cientos de años para ser significativamente notorios. Estos tiempos son mayores que el promedio de vida humana imposibilitando, así un seguimiento adecuado por medio de mediciones. La inversión en geodinámica es un mecanismo de reconstrucción de las características primordiales del suelo que se basa en la utilización de la vasta colección de datos sísmicos junto con modelos teóricos para predecir y ajustar los valores de los parámetros del modelo en uso. De esta manera podemos inferir las propiedades y características fundamentales de las capas del planeta cuando se dispone de una base de datos sísmicos. La inversión del modelo de AP conlleva a la creación de vías alternativas de la medición de los coeficientes de elasticidad en el manto superficial, tema que ha sido motivo de extensa investigación y ha generado discusiones de gran importancia en la comunidad científica.

A continuación se presenta el modelo usado en este trabajo para describir los movimientos relativos al ajuste post glacial:

$$\nabla \cdot \sigma + \nabla (u \cdot \nabla p^{(0)}) - \rho_0 g \nabla \cdot u = 0$$

Aquí, $\sigma$ representa el tensor de esfuerzos, $u$ el campo vectorial de desplazamiento, $p$ la presión hidrostática y $\rho_0$ la densidad del material en estado de relajación. El superíndice 0 en la presión indica un estado inicial. El campo vectorial de gravitación será tomado de la forma $g = -ge_z$, donde $g$ es la aceleración de la gravedad.
donde $e_z$ es el versor saliente desde la superficie de la tierra. El modelamiento de auto gravitación será dejado aparte para futuros estudios.

### 7.1 Estado del arte

El punto de partida del presente trabajo es el estudio numérico del modelo (7.1) según [1], donde solo el caso elástico fue estudiado en detalle. El código fuente desarrollado en dicho estudio fue ejecutado y comparado con el software comercial ABAQUS, que es una aplicación reconocida por su amplio uso en investigación geodinámica, en particular AP, por medio del método de elementos finitos (MEF). El trabajo presentado en [2] muestra que el uso del mencionado software para simular AP repercute en errores significativos en el campo de desplazamiento $u$. Este resultado negativo llevó a la necesidad de la inclusión del término de advección del preesfuerzo en [1] siendo imposible de implementar con ABAQUS. La correcta formulación del modelo de AP ha sido ampliamente estudiada en [2].

La implementación inicial que me fue entregada [1] toma en cuenta el modelamiento de las dos capas mas superficiales de la tierra como capas sólidas, cuyas propiedades elásticas se delimitan dentro de un planeta plano como el ilustrado en 2.2. Dicho código fuente fue escrito en C++ usando el paquete de librerías diseñado para MEF, conocido como deal.II [3] vinculado a la interfaz de computación en paralelo PETSc. Esta primera implementación no pudo ser reusada con suficiente facilidad y dedicar tiempo al mantenimiento de dicho código requería el ajuste a las nuevas herramientas provenientes de nuevas versiones de software existentes en deal.II y PETSc. Adicionalmente el código producido en [1] no fue diseñado en miras del fácil manejo de los parámetros internos del aplicativo, lo que implicaba un procedimiento entorpecedor cada vez que se deseaba modificar el experimento.

La configuración específica dada en [1] consiste en el uso de elementos tipo Taylor Hood modificado en par $Q_1 \times isoQ_1$, que acopla dos mallas, fina y gruesa, como resultado de un refinamiento de la gruesa. Los grados de libertad de $p$ se asocian a la malla gruesa y los de $u$ en la fina. Los espacios de funciones en uso, como es requerido, satisfacen la condición de Babuska-Brezzi enunciada en [4].

### 7.2 Contribución de la tesis

El posible inconveniente de usar elementos tipo Taylor Hood modificado es la baja calidad en la representación de los tensores de esfuerzo y deformación $\sigma, \varepsilon$ que son las entidades más importantes a resolver en problemas de elasticidad y fluidos. Esto debido a que las funciones usadas en estos elementos pertenecen a la familia de funciones $C^{(1)}$ diferenciables. De esta manera el tensor de esfuerzos

$$\sigma = \mu \left( \nabla u + (\nabla u)^T \right) + \lambda \nabla \cdot u$$

es pobremente representado en $C^{(0)}$ donde la función no tiene representación en los ejes del elemento. Estas funciones pueden entenderse como funciones discontinuas a saltos y constantes
dentro de los cuadriláteros. La solución de este inconveniente es la implementación en esta tesis del espacio de funciones $Q2 \times Q1$ que representan adecuadamente tanto $\sigma$ como a $\varepsilon$. Esta elección conlleva a resultados más acertados para los desplazamientos $u$, pagando el precio de que las matrices resultantes del sistema algebraico son mayormente densas. Las funciones utilizadas en [1] no están disponibles en deal.II motivando así la actualización del espacio de funciones a uno más común en el presente trabajo.

Además de las mejoras mencionadas anteriormente, el código resultante de esta tesis implementa herramientas de tecnología de punta como Trilinos [5], que es reconocida como líder en computación de alto rendimiento en cálculos de álgebra lineal, cuyos códigos admiten fácil implementación de computación en paralelo.

A pesar de la existencia del código de AP inicial, luego de numerosos intentos de ejecución, se determinó que el presente trabajo de grado se enfocara al mejoramiento y actualización del aplicativo en campos como precisión del método, implementación de la interfaz de parámetros de fácil uso, geometrías generalizables, implementación de materiales compresibles e incompresibles y que pudiera ser actualizado con facilidad para implementar tecnologías ya existentes.

Al momento de la presentación de este trabajo de maestría se ha diseñado un aplicativo fácil de usar en estudios de geofísica que permite la extensión a geometrías en 3D y a estudios de viscoelasticidad con una preparación mínima. El modelo físico aquí tratado fue recopilado de varias fuentes e incluido en el presente material en un lenguaje sencillo como contribución propia, ya que hasta el momento había estado desvinculado como se puede apreciar en [1]. Como licenciado en física traté de mantener siempre una relación directa entre la física involucrada en el ajuste postglacial y el estudio de los algoritmos de resolución numéricos, ya que en mi opinión personal ambos campos deben retroalimentarse constantemente. Con esta motivación se le ha prestado atención especial a la descripción y derivación de los principios involucrados en el ajuste postglacial desde una perspectiva fenomenológica.

Las preguntas de investigación sugeridas en este trabajo abarcan temas desde el campo de la eficiencia y rendimiento del método numérico. Específicamente se desea conocer si se pueden diseñar precondicionadores de tal manera que la solución numérica exhiba cálculos con eficiencia extrema. De ser positiva la respuesta, podría construirse dicho algoritmo de tal manera que el costo de construcción sea reducido al mínimo? Aseverar cualquier conclusión en el tema presupone el uso de tecnología y métodos de estudio numérico de punta con lo que se cuenta con precondicionadores de alto rendimiento y técnicas de multigrilla.

\[19\) Interesados en conocer especificaciones pueden consultar [6].
References


Appendix

A Analytical solutions for the uniform load problem

In this section analytical solutions are sought in terms of the parameters involved, therefore no numerical values are needed, but some orders of magnitude are mentioned. The procedure starts by computing explicitly the strain tensor $\varepsilon(\mathbf{u})$, the stress tensor $\sigma(\mathbf{u}) = 2\mu \varepsilon(\mathbf{u}) + \lambda (\nabla \cdot \mathbf{u}) I$, $\nabla p(0)$ and $\nabla (\mathbf{u} \cdot \nabla p(0))$ as follows

$$\varepsilon(\mathbf{u}) = \frac{1}{2} \begin{bmatrix} 2 \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2 \frac{\partial v}{\partial y} \end{bmatrix}; \quad \nabla \cdot \mathbf{u} = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right).$$

$$\sigma(\mathbf{u}) = \mu \begin{bmatrix} 2 \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2 \frac{\partial v}{\partial y} \end{bmatrix} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\nabla p(0) = \begin{bmatrix} 0 \\ -\rho_i g \end{bmatrix}; \quad \nabla (\mathbf{u} \cdot \nabla p(0)) = -\rho_i g \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix}.$$  \hspace{1cm} (A.1)

The $\nabla \cdot \sigma$ reads

$$\nabla \cdot (2\mu \varepsilon(\mathbf{u}) + \lambda (\nabla \cdot \mathbf{u}) I) = \mu \begin{bmatrix} 2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial^2 v}{\partial y^2} \end{bmatrix} + \lambda \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \end{bmatrix}.$$  

The traction force $\sigma \mathbf{n} = \ell$, on $\Gamma_L$ is expressed as

$$\sigma \mathbf{n} = \begin{bmatrix} (2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y} \end{bmatrix} n_x + \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) n_y = \begin{bmatrix} 0 \\ -\rho_i g h_i \end{bmatrix}. \hspace{1cm} (A.1)$$

Due to the symmetry of the problem, the load contributes to deformation only in the vertical direction. Hence, derivatives with respect to $x$ vanish within $\Omega$. As a consequence, the lateral
boundary conditions \( u = 0 \) are immediately satisfied. The latter argument suggests to use the ansatz

\[
\begin{align*}
u(x, y) = 0; \quad v(x, y) = v(y) \quad \forall \, x, y \in \Omega.
\end{align*}
\] (A.2)

On \( \Gamma_L \) the outer normal vector \( \hat{n} = [0, 1]^T \) and accounting for the horizontal symmetry of the problem \( u(x, y) = 0 \), (A.1) reduces to

\[
\left. \frac{dv}{dy} \right|_{y=0} = -\frac{\rho_i g h_i}{2\mu + \lambda}.
\] (A.3)

Valuable complementary information, needed in other to evaluate the numerical test (5.1), dwells in the scaled pressure defined in (2.29). The main reason for its inclusion is to broaden the model considering all possible combinations of \( \mu \) and \( \nu \), admitting incompressibility as well. For this purpose consider

\[
\tilde{p} = \frac{\lambda}{\mu} \nabla \cdot \mathbf{u} = \frac{\lambda}{\mu} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),
\] (A.4)

and in the uniform load context (\( \partial u/\partial x = 0 \)) it reduces to

\[
\tilde{p} = \frac{\lambda}{\mu} \frac{\partial v}{\partial y}.
\] (A.5)

Finally, in order to remove the singularity \( \lambda \to \nu = \frac{1}{2} \) in the solution, transform and obtain after arrangements

\[
\frac{1}{2\mu + \lambda} = \frac{1}{\lambda \left( \frac{2\mu}{\lambda} + 1 \right)}.
\] (A.6)

### A.1 Model M0

Consider the problem

\[
\nabla \cdot \sigma = 0,
\]

subjected to boundary conditions (5.1). After some manipulations the final equations take the form

\[
(2\mu + \lambda) \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + (\mu + \lambda) \frac{\partial^2 v}{\partial x \partial y} = 0,
\] (A.7)

\[
(2\mu + \lambda) \frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x^2} + (\mu + \lambda) \frac{\partial^2 u}{\partial x \partial y} = 0.
\]

Applying the symmetry condition (A.2) to (A.7) results in

\[
(2\mu + \lambda) \frac{d^2 v}{dy^2} = 0; \quad \frac{dv}{dy} = A,
\] (A.8)

what gives a solution of the form

\[
v(y) = Ay + B,
\] (A.9)
with $A$ and $B$ constants to be determined. To determine these constants, use (A.3) and $v(y_b) = 0$ to obtain

$$u = 0; \quad v(y) = \frac{\rho_i h_i}{2\mu + \lambda} (y_b - y); \quad \tilde{p} = -\frac{\rho_i h_i}{\mu \left(\frac{2\mu}{\delta} + 1\right)}.$$  \hspace{1cm} (A.10)

### A.2 Model M1 (inclusion of the advection of pre-stress)

Consider the problem

$$\nabla \cdot \sigma + \nabla (u \cdot \nabla \rho^{(0)}) = 0,$$  \hspace{1cm} (A.11)

where the advection of pre-stress term is included. After some rearrangements the final equations take the form

$$(2\mu + \lambda) \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + (\mu + \lambda) \frac{\partial^2 v}{\partial x \partial y} - \rho_i g \frac{\partial v}{\partial x} = 0,$$  \hspace{1cm} (A.12)

$$(2\mu + \lambda) \frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x^2} + (\mu + \lambda) \frac{\partial^2 u}{\partial x \partial y} - \rho_i g \frac{\partial v}{\partial y} = 0,$$

that results in the reduced balance equation (A.12)

$$(2\mu + \lambda) \frac{d^2 v}{dy^2} - \rho_i g \frac{dv}{dy} = 0.$$

Define the auxiliary variable $\phi = dv/dy$ and rearrange

$$\frac{d\phi}{\phi} = \frac{\rho_i g}{2\mu + \lambda} dy.$$  

Integrate over the vertical domain to obtain

$$\ln \phi = \frac{\rho_i g}{2\mu + \lambda} y + \ln \phi_0,$$  

$$\phi = \phi_0 e^{\frac{\rho_i g}{2\mu + \lambda} y}.$$  

At this point it is necessary to impose the boundary condition (A.3) yielding $\phi_0 = -\frac{\rho_i h_i}{2\mu + \lambda}$, and integrating this last expression

$$v(y) = -\frac{\rho_i h_i}{\rho_r} e^{\frac{\rho_i g}{2\mu + \lambda} y} + v_0.$$  

Imposing homogeneous Dirichlet b.c on the bottom ($y = y_b$)

$$v_0 = \frac{\rho_i h_i}{\rho_r} e^{\frac{\rho_i g}{2\mu + \lambda} y_b},$$

gives the desired solution

$$u = 0; \quad v(y) = \frac{\rho_i h_i}{\rho_r} \left( e^{\rho_i g \delta y_b} - e^{\rho_i g \delta y} \right); \quad \tilde{p} = -\frac{\rho_i h_i}{\mu \left(\frac{2\mu}{\delta} + 1\right)} e^{\rho_i g \delta y}; \quad \delta = \frac{1}{2\mu + \lambda}.$$  \hspace{1cm} (A.13)
Table 6: Typical values of the physical parameters.

<p>| | | | |</p>
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<td>$U$</td>
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<td>10$^7$ s</td>
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A.3 Model M0 + buoyancy

In order to analyse the effect of the buoyancy term only, start by considering the equation

$$\nabla \cdot \sigma - \rho_r g (\nabla \cdot u) = 0.$$  \hspace{1cm} (A.14)

After some rearrangements the final equation takes the componentwise form

$$(2\mu + \lambda) \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + (\mu + \lambda) \frac{\partial^2 v}{\partial x \partial y} = 0,$$

$$(2\mu + \lambda) \frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x^2} + (\mu + \lambda) \frac{\partial^2 u}{\partial x \partial y} - \rho_r g \frac{\partial v}{\partial x} = 0.$$  \hspace{1cm} (A.15)

Following the same steps as in Section (A.2), the solution reads

$$u = 0; \quad v(y) = \frac{\rho_i h_i}{\rho_r} \left( e^{-\rho_r g \delta y} - e^{-\rho_r g \delta y_b} \right); \quad p = -\frac{\rho_r g h_i}{\mu \left( \frac{2\mu}{\lambda} + 1 \right)} e^{-\rho_r g \delta y}; \quad \delta = \frac{1}{2\mu + \lambda}.$$  \hspace{1cm} (A.16)

A.4 Model M2 (inclusion of buoyancy)

Finally, this section deals with the problem in its full complexity

$$\nabla \cdot \sigma + \nabla (u \cdot \nabla p^{(0)}) - \rho_r g \nabla \cdot u = 0,$$

where

$$-\rho_r g \nabla \cdot u = \rho_r g \left[ \begin{array}{c} 0 \\ \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \end{array} \right].$$

The componentwise balance equation takes now the form

$$(2\mu + \lambda) \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + (\mu + \lambda) \frac{\partial^2 v}{\partial x \partial y} - \rho_r g \frac{\partial v}{\partial x} = 0,$$

$$(2\mu + \lambda) \frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x^2} + (\mu + \lambda) \frac{\partial^2 u}{\partial x \partial y} - \rho_r g \frac{\partial v}{\partial y} + \rho_r g \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0.$$  \hspace{1cm} (A.16)
Applying the ansatz (A.2), (A.16) reduces to

$$(2\mu + \lambda)\frac{\partial^2 v}{\partial y^2} = 0,$$

which can be solved in a similar way as in the simple case $M.0$. The solution reads

$$u = 0; \quad v(y) = \frac{\rho_i gh_i}{2\mu + \lambda}(y_b - y); \quad \bar{p} = -\frac{\rho_i gh_i}{\mu(2\frac{\mu}{\lambda} + 1)}. \quad (A.17)$$

### A.5 Test case: complete body force

To illustrate why excluding the material weight $\rho_0 g$ is necessary, consider the uniform load problem defined in Section (5.1) and solve for the problem (2.23). The balance equation, after some explicit calculations given in (A), reads

$$(2\mu + \lambda)\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + (\mu + \lambda)\frac{\partial^2 v}{\partial x\partial y} = 0,$$

$$(2\mu + \lambda)\frac{\partial^2 v}{\partial y^2} + \mu\frac{\partial^2 v}{\partial x^2} + (\mu + \lambda)\frac{\partial^2 u}{\partial x\partial y} - \rho_r g = 0,$$

and with the ansatz (A.2), it reduces to

$$(2\mu + \lambda)\frac{d^2 v}{dy^2} - \rho_r g = 0, \quad \rightarrow \frac{d^2 v}{dy^2} = w; \quad w = \frac{\rho_r g}{(2\mu + \lambda)}. \quad (A.19)$$

Integration leads to obtain

$$\frac{dv}{dy} = wy + A; \quad v(y) = \frac{w}{2}y^2 + Ay + B. \quad (A.20)$$

Allow to displace the vertical axis so that $y_t$ corresponds to the upper boundary and $y = 0$ the bottom surface instead. Evaluate the lower boundary condition as $v(0) = 0$ to retrieve the value $B = 0$. Continue by evaluating the upper boundary condition given in (A.1) at $y = y_t$ to obtain

$$A = -\frac{g}{2\mu + \lambda}(\rho_i h_i - \rho_i y_t).$$

Plugging all the terms, the solution reads

$$v(y_t) = \frac{g}{2\mu + \lambda}(\rho_r y + \rho_r y_t - \rho_i h_i) y_t \quad (A.21)$$

and evaluating on the surface (maximum displacement) it is found

$$v(y_t) = \frac{g}{2\mu + \lambda}(2\rho_r y_t - \rho_i h_i) y_t. \quad (A.22)$$
From this expression it is easily seen how the load seems not to contribute enough for the phenomena and the difference $2\rho_r y_i - \rho_i h_i \approx 2\rho_r y_i$ hides the effect of any GIA displacement, as $y_i(4000\ km) \gg h(2.17\ km)$. With some typical order of magnitude of the problem parameters $\mu \approx \lambda \approx 10^{11}\ Pa$, $\rho_r \approx \rho_i \approx 10^3\ kg/m^3$, $y_i \approx 10^6\ m$, $h \approx 10^3\ m$ and $g \approx 10\ m/s^2$ from (A.22) results in

$$O(y_i) \approx \frac{10^1}{10^{11}} \left( 10^3 \cdot 10^6 \right)^{10^6} = 10^5\ m.$$  

Thus, the outcome of the inclusion of the term $f = \rho_r g$ shows vertical displacements of the order $10^5\ m$ that are definitely higher than the displacements of the order of $10^2\ m$ predicted by the elastic GIA model. This is the main reason to decompose into the non-hydrostatic stress tensor $\sigma'$.  

### B FEM discretization in space of vector problems

FEM is specially advantageous when working with discontinuous problems, as it approaches the weak form (lower spatial derivatives) of the PDE rather than the strong representation (full spatial derivatives). It is possible to define a mesh such as the jumps or spatial discontinuities in the material are left in the boundaries of the element $\partial \Omega_k$ without harming the convergence of the final scheme. Discretizing in space can be relatively easy, as FEM is very flexible with geometries compared with other techniques (Finite Differences or Spectral Methods).

Scalar problems, like the Poisson’s equation $-\nabla^2 c(r) = f(r)$, can be approximated with functions $\phi \in H_1$ with the Galerkin projection as follows

$$c(r) = \sum_{k=1}^{N} c_k \phi_k(r). \quad (B.1)$$

The problem is then reduced to finding the coefficients $c_k$ by choosing a method to discretize the PDE. Namely, the finite elements methods or spectral methods are of common use for this purpose.

When problems are of vector nature, $\dim(\mathbb{R}^d) = d$, the Galerkin projection becomes slightly more elaborated. Here, it would be needed to solve for each component of the solution vector as a scalar problem. For this, consider the set of vector test functions in the 3D case ($d = 3$)

$$\Phi_k^1 = (\phi_k, 0, 0)^T; \quad \Phi_k^2 = (0, \phi_k, 0)^T; \quad \Phi_k^3 = (0, 0, \phi_k)^T.$$  

In last equation, each test function $\phi_k$ in the scalar problem generates $d$ new vectorial test functions. Hence, if there is a set of $N$ different scalar test functions $\phi_k$, the span of functions in the vector problem would be $d \times N$ in total; one for each component $k$ of each test function $\phi_j$. After this, introduce the FE vector test functions $v_j$

$$v_j(r) = \Phi_k^n(r); \quad n = \text{int} \left( \frac{N - 1 + j}{N} \right); \quad k = \text{int} \left( \frac{d - 1 + j}{d} \right); \quad j = 1, 2 \cdots d \times N; \quad k = 1, 2 \cdots N; \quad n = 1, 2 \cdots d. \quad (B.2)$$
Hence, the Galerkin projection over a vector problem can be arranged as

\[ u(r) = \sum_{j=1}^{dN} u_j v_j(r) \quad v \in H_1^d. \]

In this way each component \((d = 3)\) of the displacement vector \(u = (u, v, w)^T\) is treated as an independent but coupled scalar problem. The coefficients \(u_j\) are gathered as components of a vector \(u_h\) with dimension \(\text{dim}(u_h) = dN\), and \(\text{int}[\alpha]\) is the integer part of \(\alpha\).

In order to set the spatial discretization, let \(\phi(r)\) be a scalar test function with the special property \(\{\phi : |
abla\phi| + |\phi| < \infty\}\), meaning that \(\phi(r)\) is at least one time differentiable with respect to each spatial coordinate \((\phi \in C^1)\). The resulting function space is called Hilbert space \(H_1\), and solutions represented as a linear combination of functions in this space have the property of being well behaved solutions. For discretizing vector fields, the extended basis functions take the form in the three-dimensional case \((d = 3)\)

\[ \Phi^1_k = (\phi_k, 0, 0)^T; \quad \Phi^2_k = (0, \phi_k, 0)^T; \quad \Phi^3_k = (0, 0, \phi_k)^T. \]  

(B.3)

From this point, the displacements \(u\) are projected into the Hilbert space \(H_1^3\), such that \(u \in H_1^3\) by using the Galerkin projection

\[ u(r) = \sum_{j=1}^{3N} u_j v_j(r). \]  

(B.4)

The basis vectors \(v_j\) are defined as in equation B.2.

**B.1 The stationary elastic problem**

A stationary state can be found by deforming a body in an adiabatic process such that the body reaches an inflection point of the mechanic energy with \(\dot{u} = 0\). At this point the body experiences \(\ddot{u} = 0\) and remains steady in the deformation state.

The stationary problem of (2.5) can be written as

\[-\nabla \cdot \sigma = f.\]

Introduce an arbitrary vector test function \(v\) such that \(v \in H_1^d\), where \(d = \text{dim}(u)\). Use the left product with \(v\) and integrate over \(\Omega\)

\[-\int_{\Omega} v \cdot (\nabla \cdot \sigma) \, d\Omega = \int_{\Omega} v \cdot f \, d\Omega.\]

At this stage, it is necessary to introduce the double contraction of tensors or Frobenius tensor product. Let \(A\) and \(B\) be 2nd rank tensors of dimension \(d\). The double contraction is defined\(^{20}\) as

\[ A : B = \sum_{i,j=1}^{d} A_{ij} B_{ij}. \]  

(B.5)

\(^{20}\)Notice that \(A : B\) is a scalar, completely different from the matrix product \(AB\) resulting in a matrix.
Using the vector identity \( \nabla \cdot (v \cdot \sigma) = \nabla v : \sigma + v \cdot (\nabla \cdot \sigma) \) yields to
\[
- \int_\Omega \nabla \cdot (v \cdot \sigma) \, d\Omega + \int_\Omega \nabla v : \sigma \, d\Omega = \int_\Omega v \cdot f \, d\Omega.
\]

Use the Gauss theorem
\[
\int_\Omega \nabla \cdot (v \cdot \sigma) \, d\Omega = \int_\Gamma v \cdot (\sigma \cdot \hat{n}) \, d\Gamma \tag{B.6}
\]
to obtain the weak form
\[
\int_\Omega \nabla v : \sigma \, d\Omega = \int_\Omega v \cdot f \, d\Omega + \int_\Gamma v \cdot \ell \, d\Gamma, \tag{B.7}
\]
where \( \ell \) is the traction vector defined in (A.3).

Next, use (2.10) in the first term of (B.7), along with the fact that \( \nabla v : I = \text{tr}(\nabla v) = \nabla \cdot v \)
\[
\int_\Omega \nabla v : \sigma \, d\Omega = 2 \int_\Omega \mu \nabla v : \varepsilon(u) \, d\Omega + \int_\Omega \lambda (\nabla \cdot v)(\nabla \cdot u) \, d\Omega.
\]

Finally, it is easy to derive a symmetric weak form by using the properties of the double contraction \( A : B^T = A^T : B \) and \( A^T : B^T = A : B \). Start by using these properties to represent \( \nabla v : \varepsilon(u) \)
\[
\varepsilon(v) : \varepsilon(u) = \frac{1}{2} \left( \nabla v + (\nabla v)^T \right) : \frac{1}{2} \left( \nabla u + (\nabla u)^T \right)
\]
\[
= \frac{1}{4} \nabla v : \nabla u + \frac{1}{4} \nabla v : (\nabla u)^T + \frac{1}{4} (\nabla v)^T : \nabla u + \frac{1}{4} (\nabla v)^T : (\nabla u)^T
\]
\[
= \frac{1}{2} \nabla v : \nabla u + \frac{1}{2} \nabla v : (\nabla u)^T
\]
\[
= \nabla v : \varepsilon(u).
\]

Introduce the weak form
\[
\int_\Omega 2\mu \varepsilon(v) : \varepsilon(u) \, d\Omega + \int_\Omega \lambda (\nabla \cdot v)(\nabla \cdot u) \, d\Omega = \int_\Omega v \cdot f \, d\Omega + \int_\Gamma v \cdot \ell \, d\Gamma. \tag{B.8}
\]

### B.1.1 Korn’s inequality

The bilinear operator
\[
a(u, v) = \int_\Omega 2\mu \varepsilon(v) : \varepsilon(u) \, d\Omega + \int_\Omega \lambda (\nabla \cdot v)(\nabla \cdot u) \, d\Omega,
\]
can be shown to be bounded independently of the boundary conditions. For a simple case consider homogeneous Dirichlet boundary conditions \( u = 0 \) on \( \Gamma \), meaning that the test functions
must also vanish \( \mathbf{v} = 0 \) on the boundaries. Following the procedure given in [27], take first

\[
\int_{\Omega} \mathbf{\varepsilon}(\mathbf{v}) : \mathbf{\varepsilon}(\mathbf{v}) \, d\Omega = \int_{\Omega} \sum_{i,j=1}^{d} \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \, d\Omega
\]

\[
= \frac{1}{4} \int_{\Omega} \sum_{i,j=1}^{d} \left[ \left( \frac{\partial v_i}{\partial x_j} \right)^2 + 2 \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} + \left( \frac{\partial v_j}{\partial x_i} \right)^2 \right] \, d\Omega
\]

\[
= \frac{1}{2} \| \nabla \mathbf{v} \|^2 + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{d} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, d\Omega.
\]

To deal with the last integral take the identities

\[
\frac{\partial}{\partial x_j} \left( v_i \frac{\partial v_j}{\partial x_i} \right) = \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} + v_i \frac{\partial}{\partial x_j} \left( \frac{\partial v_j}{\partial x_i} \right) = \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} + v_i \frac{\partial^2 v_j}{\partial x_j \partial x_i},
\]

apply the Gauss theorem twice and use the fact that \( \mathbf{v} \) vanishes on \( \Gamma \). The procedure is as follows

\[
\int_{\Omega} \sum_{i,j=1}^{d} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, d\Omega = \sum_{i,j=1}^{d} \left[ \int_{\Omega} \frac{\partial}{\partial x_j} \left( v_i \frac{\partial v_j}{\partial x_i} \right) \, d\Omega - \int_{\Omega} v_i \frac{\partial^2 v_j}{\partial x_j \partial x_i} \, d\Omega \right]
\]

\[
= \sum_{i,j=1}^{d} \left[ \int_{\Gamma} v_i \frac{\partial v_j}{\partial x_i} n_j \, d\Gamma - \int_{\Omega} v_i \frac{\partial^2 v_j}{\partial x_j \partial x_i} \, d\Omega \right]
\]

\[
= \sum_{i,j=1}^{d} \left[ - \int_{\Gamma} \frac{\partial}{\partial x_i} \left( v_i \frac{\partial v_j}{\partial x_j} \right) \, d\Gamma + \int_{\Omega} \frac{\partial v_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} \, d\Omega \right]
\]

\[
= \sum_{i,j=1}^{d} \left[ - \int_{\Gamma} v_i \frac{\partial v_j}{\partial x_j} n_i \, d\Gamma + \int_{\Omega} \frac{\partial v_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} \, d\Omega \right]
\]

\[
= \int_{\Omega} \left( \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i} \right) \left( \sum_{j=1}^{d} \frac{\partial v_j}{\partial x_j} \right) \, d\Omega
\]

\[
= \int_{\Omega} (\nabla \cdot \mathbf{v})^2 \, d\Omega \geq 0.
\]

As a conclusion

\[
\int_{\Omega} \mathbf{\varepsilon}(\mathbf{v}) : \mathbf{\varepsilon}(\mathbf{v}) \, d\Omega = \frac{1}{2} \| \nabla \mathbf{v} \|^2 + \| \nabla \cdot \mathbf{v} \|^2 \geq 0.
\]
which reflects the coercivity of the bilinear form
\[ a(u, u) = 2\mu \|\varepsilon(u)\|^2 + \lambda \|\nabla \cdot u\|^2 \geq 2\mu \|\varepsilon(u)\|^2 \geq C \|\nabla u\|^2 \geq C \|v\|^2, \]
where in the last part was used the Poincaré inequality
\[ \nabla u : \nabla u = \|\nabla u\|^2 \geq C \|v\|. \]
is used with a positive constant $C$. For Neumann boundary conditions a similar procedure can be applied but using at some point the trace inequality defined in $\Gamma$. For a detailed procedure refer to [27].

### B.2 The LBB condition

It is known that in mixed formulations for problems in elasticity and fluid mechanics, the choice of function spaces is not trivial. It has been proven necessary to choose a stable pair such that the dimensions of the pair must satisfy $N \neq M$ for low order shape functions. The resulting $2 \times 2$ block matrix of the form of equation (3.5), is a saddle point matrix. As it has been pointed out by Gilbert Strang in his MIT lectures\(^\text{21}\), solving saddle point problems may result in the checkerboard pattern, which is a vector filled with $\gamma(1, -1, 1, -1, 1 \cdots, 1)$ with $\gamma$ a constant depending on the grid size. Clearly, a checkerboard solution is not a physical solution nor a stable solution. The apparent reason, is that unfortunately the resulting vector belongs to the null space of $B$, this is $B^T p = 0$.

To understand the concept behind saddle point systems, state that for every $p$, there must exist a vector $u$ such that
\[ uB^T p \geq \beta \sqrt{u^T A u} \sqrt{p^T p}, \]
for a fixed $\beta > 0$. (B.9)

Condition (B.9) fails if there exists a nonzero $p$ such that $B^T p = 0$, and the system matrix $A$ becomes rank-deficient to solve for both $u$ and $p$. The choice of an appropriate pair is not trivial and the set of stable basis functions needs to fulfill (B.9), known as the inf-sup or Ladyshenskaya-Babuska-Brezzi condition [4]. The pair used in this work is the $Q2 \times Q1$ Taylor-Hood basis pair, consisting of (quadratic/bilinear) basis functions for the (displacements $u$/pressure $p$), designed for quadrilaterals elements in 2D, which is the set of basis functions used in this thesis, or bricks in 3D.

There exist different arguments justifying the need to use different basis functions for the physical variables $u$ and $p$. For a starting view, the evaluation of $\nabla \cdot u$ measures the discontinuity of a vector field, if the measurement of such discontinuity lives in the one point (FEM nodes) where exists discontinuity, the measurement does not contain enough information about the physical problem. The problem might be discretized using staggered grids, evaluating $u$ and $p$ on different points in a way that the information needed for calculating the divergence at a particular point involves different sources.

A straight consequence that involves the choice of an adequate function space, becomes apparent for the space discretization of the stress field $\sigma$ in (2.10), which involves spatial derivatives of $u$. For smooth and well behaved solutions, $\sigma$ must be at least $C^{(1)}$ continuous, hence $u$ must be represented at least with a $C^{(2)}$ family of functions.

The modified Taylor-Hood family $Q1 \times Q^{'}1$ (bilinear in $u$ and $p$)\textsuperscript{22} shape functions pair was implemented in [1]. The different meshes for $u$ and $p$ satisfies condition (B.9) with $C^{(1)}$ functions. However, satisfying the LBB condition is not enough guarantee for obtaining good solutions in every problem, the argument given above demands at least $C^{(2)}$ continuous piece-wise shape functions for the $u$ field in the particular GIA problem.

### B.3 The mixed Finite Element Method

To avoid the Locking effect [11] observed in elasticity problems, the physical scalar variable, $p$ will be projected into the FE space as

$$p(r) = \sum_{j}^{M} p_j q_j(r) \quad q \in H^1_1, \quad (B.10)$$

For the mixed FE discretization, insert the scaled pressure definition $\lambda \nabla \cdot u = \mu p$ in (B.8) to reach

$$\int_{\Omega} 2\mu \varepsilon(v) : \varepsilon(u) \, d\Omega + \int_{\Omega} \mu \nabla \cdot v \, p \, d\Omega = \int_{\Omega} v \cdot f \, d\Omega + \int_{\Gamma} v \cdot \ell \, d\Gamma.$$  

For completeness, take the definition (2.29) times $q$, integrate over $\Omega$ and rearrange

$$\int_{\Omega} \nabla \cdot u \, q \, d\Omega - \int_{\Omega} \frac{\mu}{\lambda} \, p \, q \, d\Omega = 0.$$  

Take the product with $\mu$ to get symmetry between the equations and obtain the weak form

$$\int_{\Omega} 2\mu \varepsilon(v) : \varepsilon(u) \, d\Omega + \int_{\Omega} \mu \nabla \cdot v \, p \, d\Omega = \int_{\Omega} v \cdot f \, d\Omega + \int_{\Gamma} v \cdot \ell \, d\Gamma,$$

$$\int_{\Omega} \mu \nabla \cdot u \, q \, d\Omega - \int_{\Omega} \frac{\mu^2}{\lambda} \, p \, q \, d\Omega = 0.$$  

the FE representation is now composed by a vector $x$, and test function space $\Phi$

$$x = \begin{pmatrix} u_h \\ p_h \end{pmatrix}, \quad \Phi = \begin{pmatrix} v \\ q \end{pmatrix}, \quad \dim(x) = dN + M.$$  

$N$ and $M$ represent the span of different scalar test functions available for $u$, $v$ and $p$, $q$ respectively.

\textsuperscript{22}The prime in $Q^{'}1$ corresponds to a coarser mesh where $p$ lives.
B.4 FEM discretization of the viscoelastic GIA model

The general model can be written as

\[-\nabla \cdot \sigma - \nabla (b \cdot u) + (\nabla \cdot u)c = f,\]

(B.11)

with the definitions \( b = \nabla p^{(0)} \), \( c = \rho \mathbf{g} \) and \( \sigma \) given by the viscoelastic form (2.43)

\[\sigma(t) = \sigma_E(t) - \int_0^t \frac{\partial \chi(t, t')}{\partial t'} \sigma_E(t') \, dt' \quad \chi(t, t') = e^{-\alpha(t-t')}.
\]

To proceed with the discretization, take the left product with a vector test function \( v \) and integrate over \( \Omega \).

\[-\int_\Omega v \cdot (\nabla \cdot \sigma) \, d\Omega - \int_\Omega v \cdot \nabla (b \cdot u) \, d\Omega + \int_\Omega (\nabla \cdot u) v \cdot c \, d\Omega = \int_\Omega v \cdot f \, d\Omega.
\]

As it has been previously used, the elastic contribution \( \sigma_E \) is expressed by using (2.10). Apply the same vector identities and divergence theorem for the term \( v \cdot \nabla \cdot \sigma_E \) that was used for the stationary problem and obtain the boundary contributions. Once this has been done, it is easy to obtain the weak form

\[
\begin{align*}
a(u(t), v) - \int_0^t \tilde{a}(u(t), v; t, \tau) \, d\tau &+ b(v, p(t)) - \int_0^t \tilde{b}(v, p(t); t, \tau) \, d\tau = \tilde{g}(v; t, \tau), \\
b(u(t), q) - \int_0^t \tilde{b}(u(t), q; t, \tau) \, d\tau &- c(p(t), q) + \int_0^t \tilde{c}(p(t), q; t, \tau) \, d\tau = 0,
\end{align*}
\]

(B.12)

bilinear forms are defined as follows:

\[
\begin{align*}
a(u, v) &= \int_\Omega 2\mu_E \varepsilon(u) : \varepsilon(v) \, d\Omega + \int_\Omega -\nabla (b \cdot u) \cdot v + (\nabla \cdot u)c \cdot v \, d\Omega, \\
\tilde{a}(u(t), v; t, \tau) &= \int_\Omega 2\alpha_0 e^{-\alpha_0(t-t')} \mu_E \varepsilon(u(t')) : \varepsilon(v) \, d\Omega, \\
b(u(t), q) &= \int_\Omega \mu_E p(t) \nabla \cdot v \, d\Omega, \\
\tilde{b}(u(t), q; t, \tau) &= \int_\Omega \alpha_0 e^{-\alpha_0(t-t')} \mu_E p(t) \nabla \cdot v \, d\Omega, \\
c(p(t), q) &= \int_\Omega \frac{\mu_E^2}{\lambda_E} p(t) q \, d\Omega, \\
\tilde{c}(p(t), q; t, \tau) &= \int_\Omega \alpha_0 e^{-\alpha_0(t-t')} \frac{\mu_E^2}{\lambda_E} p(t) q \, d\Omega, \\
\tilde{g}(v; t) &= \int_\Omega v \cdot f \, d\Omega + \int_\Gamma v \cdot \ell \, d\Gamma - \int_0^t \int_\Gamma \alpha_0 e^{-\alpha_0(t-t')} v \cdot \ell \, d\Gamma \, d\tau.
\end{align*}
\]
The following step at this point is to discretize in time, such that \( t_0 = 0, t_1 = t_0 + \Delta t_n, \ldots, t_n = T \), where \( \Delta t_n = t_n - t_{n-1} \) is the time step, and \( u^n, p^n \) the resulting displacement and pressure at each time step. The resulting integral terms can be split as follows

\[
I^n_0 = \int_0^{t_n} \tilde{a}(u(\tau), v; t_n, \tau) \, d\tau = \int_0^{t_{n-1}} \tilde{a}(u(\tau), v; t_n, \tau) \, d\tau + \int_{t_{n-1}}^{t_n} \tilde{a}(u(\tau), v; t_n, \tau) \, d\tau
\]

\[
= \tilde{I}^{n-1}_0 + I^n_{n-1},
\]

and similarly in all terms with ‘\( \sim \)’ in (B.12). The use of the trapezoidal rule

\[
\int_{t_{n-1}}^{t_n} f(t) \, dt \approx \frac{\Delta t_n}{2} (f(t_n) + f(t_{n-1}))
\]

results in a system of the form\(^{23}\)

\[
(\mathcal{A} - \frac{\Delta t_n}{2} \mathcal{A}_0) \begin{pmatrix} u^n \\ p^n \end{pmatrix} = \begin{pmatrix} r^{(1)} \\ r^{(2)} \end{pmatrix},
\]

where

\[
\mathcal{A} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \quad \text{and} \quad \mathcal{A}_0 = \begin{bmatrix} A_0 & B_0^T \\ B_0 & -C_0 \end{bmatrix},
\]

with matrices \( A, B \) and \( C \) varying in time. The system (B.14) is of the same form as system (3.7) and its numerical solution implies a solution scheme similar to the given in Section 4.7 for each time step.

\section*{C Multigrid algorithms}

Iterative solvers as Krylov space solvers, present a good compromise between performance and problem-size provided that the system is preconditioned appropriately. However, iterative methods seem to work efficiently over high frequencies of the error but take a long time to smooth the low components. The essential idea with multigrid (MG) algorithms\(^{24}\) is to define a set of refined grids and use them to switch to coarsened versions of the problem reducing in this way the error by specific frequencies. As a fact, iterating on coarser grids is cheaper than iterating further on finer grids (Figure C.1), but the greatest advantage is that the convergence rate is enhanced when projecting the error to coarser grids.

After only a small number of iterations over the finer grid the high-frequencies of the error will decay significantly, but the convergence will slow down leaving a smoother remaining lower-frequency error. Then the error is projected on coarser grids and iterated for faster decay of some

\(^{23}\)For a complete derivation refer to [1].

\(^{24}\)For an introduction to the multigrid algorithms approach refer to sec. 4.6.2 on [28].
specific components of the error. When the error is reduced sufficiently in all the spectrum, the solution is interpolated back to the original grid.

In the MG, the coarse-grid variables rely strongly on the geometry of the fine levels and it is necessary to develop refined grids along with prolongation, restriction and smoothing operators to transform the solution between grids. This is often a problem and a difficult task to achieve, if not impossible on complex geometries. A more robust and flexible algorithm can be thought of as a black box scheme where the geometrical properties of the problem are left aside, omitting the construction of artificial grids. The algebraic multigrid (AMG), deals with the mentioned features, building the refined grids by relying solely on information in the system matrix (algebraic), identifying coupled variables by comparing the relative size of the matrix coefficients (threshold). The AMG is ideal to work with FEM frameworks as the degrees of freedom often present strong coupling patterns.

**AMG test**

In this section a simple experiment is set up to demonstrate the advantages of using AMG as a preconditioner. Consider the Poisson problem in a square domain $x, y \in [-1, 1]$

$$-\nabla^2 u = 1, \quad u \in \Omega,$$

$$u = 0, \quad u \in \partial \Omega,$$

(C.1)

![Figure C.1: Grids refined consecutively used in multigrid methods.](image)
with standard FEM discretization, take the product with a set of test functions, integrate and proceed
\[- \int_{\Omega} v \nabla^2 u \, d\Omega = \int_{\Omega} v \cdot 1 \, d\Omega,\]
\[\int_{\Omega} \nabla v \cdot \nabla u \, d\Omega - \int_{\partial \Omega} v \nabla u \cdot \hat{n} \, d\Gamma = \int_{\Omega} v \, d\Omega,\]
\[\int_{\Omega} \nabla v \cdot \nabla u \, d\Omega = \int_{\Omega} v \, d\Omega.\]

The resulting system of equations after applying Dirichlet boundary conditions is of the form \( Ku = b \) with \( K \), the reduced stiffness matrix after boundary conditions and \( b \) the load vector.

The FEM basis functions are the quadratic Lagrange over rectangular quadrilaterals (Q2). For the solution of the linear system of equations three distinct methods are used. Namely, the Conjugate Gradient (CG) method preconditioning with AMG, CG without preconditioning and finally a direct UMFPACK solver. All tested for a number of refinements \( k \) in the global squared mesh.

<table>
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<th>CG (s)</th>
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<td>380</td>
<td>0.767</td>
</tr>
<tr>
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<td>1.68</td>
<td>13</td>
<td>3.62</td>
<td>751</td>
<td>3.98</td>
</tr>
<tr>
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<td>6.65</td>
<td>14</td>
<td>25.2</td>
<td>1446</td>
<td>24.5</td>
</tr>
</tbody>
</table>

Table 7: Statistics on the solution of the Poisson system in 2D with relative tolerance \( TOL = 10^{-12} \) with the use of CG preconditioned with AMG, without preconditioning and UMFPACK.

Several conclusions can be drawn from Table 7 showing the results of the numerical test. First, the impossibility to solve any problem by direct methods, the times thrown by UMFPACK grow considerably fast with the mesh refinement. The scalability of the AMG as it remains fairly constant around 12 iterations for any refinement. The use of AMG in the solution algorithm introduces extra complexity in the algorithm, which is not justified for small problems. Even though the iterations remained low, each iteration became more costly than those using only CG without preconditioner, this is clear for refinements lower than 9. For larger number of refinements preconditioning becomes a necessity.

The problem (C.1) serves as a simple example to help understanding the tools used in the solution scheme but in general more complex problems can get worse condition numbers and the urge of preconditioning might appear with less number of refinements as in the GIA case.