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Enflo, Karin
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Measuring Opportunity

Karin Enflo

Introduction

Is it possible to increase a person's freedom of choice by giving him an option that he would never choose? Would the Philosophy Department at Uppsala University increase freedom of choice for the students by offering a course that is so bad that no student would ever choose it (such as a course in the philosophy of skiing)? This kind of question has been debated in the literature on measures of freedom of choice. On one side of the debate there are thinkers, such as Van Hees and Wissenburg, who think that the preferences of the chooser are not relevant for assessing degrees of freedom of choice. On the other side there are thinkers, such as Sen, Pattanaik and Xu, who think the opposite.

Obviously, there is no theoretical problem with acknowledging that there may be several concepts of freedom of choice, where some are dependent on preferences and some are not. However, there are normative questions that are not solved by theoretical distinctions. Should one or both types of freedom of choice be considered a social good? Is one type of freedom of choice more important than the other?

The worry with a society endorsing only freedom of choice that is independent of preferences is that choices between bad options also will count as a social good. Some choices, such as a citizen's choice between imprisonment and execution, can hardly count as a social good. The worry with a society endorsing only freedom of choice that incorporates preferences is that freedom of choice may increase just by changing a citizen's preferences. A choice between imprisonment and execution may offer a citizen more freedom of choice than a choice between being a philosopher and a skier, if only the citizen could be manipulated to prefer the options of prison life and execution to the other two.

If freedom of choice should be regarded as a social good, perhaps the best candidate is to use a type of freedom of choice that depends on reasonable preferences rather than actual preferences. This way we can say that for a citizen who already has the option of living a normal life, the options of imprisonment and execution do not contribute to his or her freedom of choice, since no citizen could reasonably prefer these options to a normal life. The idea of using reasonable preferences as a basis for a measure of freedom of choice is Patattanaik and Xu's from 1998.

In this essay we shall discuss a person's freedom to choose whatever that person may reasonably prefer to choose. This type of freedom of choice may be regarded as a candidate for a social good, being a hybrid between freedom of choice and preference satisfaction utility. To separate this preference dependent freedom of choice from other types we may call it *opportunity*.

Here we shall not be concerned with the question whether opportunity should be considered a social good. Instead we shall be concerned with the more fundamental questions regarding how opportunity should be identified and measured. The first question is obviously more important, but it can hardly be answered without an answer to the other two. We shall especially focus on the question how opportunity should be measured. To answer this question we shall first discuss what conditions an opportunity measure should satisfy. It is proposed that an opportunity measure should satisfy at least four conditions. We shall then present a new class of measures which satisfy the four conditions, called the *Root measures*. Last we shall prove that these measures are the only ones that satisfy the conditions, given the extra assumption that the measures are analytic functions with non-zero partial derivatives.

Background model

Here we shall discuss opportunity as a candidate for a social good. We shall look at the opportunities of citizens in a particular society. What we are interested in are the opportunities offered by collectively available choice sets to citizens who are choosing options individually. Examples of such collectively available choice sets would be choice sets of universities to attend, TV-channels to watch and skiing tracks to use. These choice sets are available to three different groups of citizens, namely high school graduates, TV-viewers and skiers. In each of these groups the citizens choose options individually. A particular high school graduate, TV-viewer and skier may, for example, choose to study at Uppsala University, watch Sports-channels on TV and ski in Härjedalen. Another citizen, who belongs to the same groups, may choose to study at Stockholm University, watch History-channels on TV and ski in Värmland.

It is obvious that the citizens will have different preference orderings over the options. Here we shall also assume that most of these preference orderings are reasonable. It is reasonable to prefer to study at Uppsala over Stockholm, watch Sports-channels rather than History-channels and ski in Härjedalen over skiing in Värmland. It is also reasonable to have the opposite preferences. Some preference orderings are not reasonable, however. It is not reasonable to prefer skiing in Stockholm over skiing in Härjedalen, for example.

The assumption that there are several reasonable preference orderings is common in the discussion of freedom of choice and opportunity as social goods. It is used by, for example Pattanaik and Xu in 1998, Romero-Medina in 2001 and Peragine and Romero-Medina in 2006. There would not be much point in society offering its citizens freedom of choice over some type of option if there were just one reasonable preference ordering. In that case every reasonable person would choose the same option. If there were not several reasonable preference orderings over music, for example, it would suffice if society offered one radio channel, playing the same piece of music over and over.

Assumptions

The word 'opportunity' is used both in an absolute and a comparative sense. In an absolute sense the word 'opportunity' is often used as a synonym to 'option', usually with positive connotations. The sentence '*P* has an opportunity to ski' just means that "skiing is an option for *P*", or perhaps that "skiing is a valuable option for *P*". In a comparative sense the expression 'more opportunity' is often used as a synonym to 'more or better options'. Here we shall stipulate that:

> A person *P* has *at least as much opportunity as* a person *P**, if and only if, *P* can choose whatever *P* may reasonably prefer to choose, to at least as high degree as *P** can choose whatever *P** may reasonably prefer to choose.

This definition of opportunity is quite vague, but how it may be interpreted shall be discussed later. First we shall make some technical assumptions. We shall assume that there is a finite *universal set X* that contains all the possible options x, y, z …. The set Z is the set of all non-empty subsets A, B … of X. There are several preference relations R_i over the options of X. They are assumed to be reflexive, transitive and complete. An ordered pair of a relation R_i and the set X is a preference ordering $Y_i = \langle X, R_i \rangle$. There is a set, Y, of all possible preference orderings. There is also a subset of Y that contains all reasonable preference orderings, called K.

We shall further assume that each preference ordering may be represented by real numbers through the use of utility functions. Let $\mathfrak{R}$ be the set of real numbers, and ≥ the numerical relation *at least as great as*. We shall assume that each preference ordering $\langle X, R_i \rangle$ can be represented by the relational system $\langle \mathfrak{R}, \geq \rangle$ through a utility function v_i that assigns numbers to options, in such a way that $v_i(x) \geq v_i(y)$ if, and only if, xR_iy. The utility functions are assumed to be ratio scale. For each preference ordering there are several utility functions that may be used since it is the case that if v is a measure of R_i, so is $g = \alpha v$ where $\alpha > 0$. A set that contains one utility function v_i for each preference ordering Y_i is called V_i. The set of all sets V_i is called V.

We shall assume that all the preference orderings are completely comparable. It is thus meaningful to say that a person P, with a preference ordering Y_1, values an option x at least as much as a different person P^*, with a preference ordering Y_2, values an option y. Since this is the case we shall only use the subsets of V that reflect this comparability. We may call these subsets W_i and the set of subsets W.

If we index each option x of X as $x_1, x_2 \dots x_n$ and index each utility function u_i representing a preference ordering Y_i as $v_1, v_2 \dots v_n$ then we can represent a value assignment in the form of a matrix. Below is an example of a value assignment for a set A containing three options evaluated by three utility functions:

	v_1	v_2	v_3
x_1	**9**	2	**9**
x_2	2	6	2
x_3	2	**9**	2

This matrix can be represented as an n-tuple or vector of real numbers, $\mathbf{u_A} = \langle v_1(x_1), v_2(x_1), \dots v_n(x_n)\rangle = \langle 9, 2, 9, 2, 6, 2, 2, 9, 2\rangle$. We shall mostly be interested in the maximal values of the options according to all reasonable preference orders. We then use the vector $\mathbf{Max\,v_A} = \langle \max u_1, \max u_2 \dots \max u_n\rangle$. In this case $\mathbf{Max\,v_A} = \langle 12, 12, 12\rangle$.

Last we shall assume that there is a binary relation on subsets of X, the relation: *offers at least as much opportunity as*. This relation is assumed to be reflexive, transitive and complete. A measure of opportunity is a function O on sets, such that $O(A) \geq O(B)$ if and only if *A offers as much opportunity as B*.

The Scale independence condition

To find an appropriate measure of opportunity it is useful to begin by formulating conditions that such a measure should satisfy. The first condition that we shall suggest is very general. We want a measure of opportunity to be scale independent, so that it does not matter which scale we use when measuring the values of the options. A measure may be scale independent in several different senses. In a weak sense of ‘scale independence’ it is only required that the ordering of sets in terms of opportunity is preserved when changing the scale by which the values of the options are measured. For this sense we may formulate the following condition:

> *The Weak scale independence condition*: For any measure of opportunity O, and all choice sets A, $B \in Z$ and their corresponding value vectors $\mathbf{v_A}$ and $\mathbf{v_B}$, it is the case that, if $O(A) \geq O(B)$ for some value assignment W_i and if all the values v of $\mathbf{v_A}$ and $\mathbf{v_B}$ are multiplied by some number $K > 0$, then it is still the case that $O(A) \geq O(B)$.

In a stronger sense of 'scale independence' it is also required that ratio relations between degrees of opportunity are preserved. For this sense we may formulate the following condition:

> *The Strong scale independence condition*: For any measure of opportunity O, and all choice sets A, $B \in Z$ and their corresponding value vectors $\mathbf{v_A}$ and $\mathbf{v_B}$, it is the case that for every number $K > 0$, there is a number $L = L(K) > 0$, where L is an increasing function of K, such that if all values in $\mathbf{v_A}$ are multiplied by K, then $O(\mathbf{v_A})$ is multiplied by L.

The *Strong scale independence condition* implies the *Weak scale independence condition*. Whether we should accept both conditions or only accept the weaker one depends on if we intend to measure opportunity on a ratio scale or just an ordinal scale. The *Strong scale independence condition* is necessary only for a ratio scale opportunity measure. It is not certain that the concept of opportunity allows for ratio scale comparisons. Nevertheless, we shall assume that it is and thus accept the *Strong scale independence condition* here.

The Dominance condition

The second condition that we shall suggest is uncontroversial. We can illustrate it with an example. Let us assume that there are just three reasonable preference orderings. As it happens these are the preference orderings of a philosopher, a skier and a philosophical skier. We are comparing the opportunities of two persons who are deciding where to live. Person P has the choice set A, while person P^* has the choice set B. Both sets contain three cities. Using the same function to represent values we have the following matrices:

Set *A*:

	P	*S*	*PS*
Uppsala	**8**	2	5
Umeå	5	**9**	**7**
Sveg	1	**9**	5

Set *B*:

	P	*S*	*PS*
Stockholm	**7**	**1**	**4**
Göteborg	5	**1**	3
Malmö	1	**1**	1

We may note that all the options of *A*, according to all reasonable preference orderings, are at least as good as all the options of *B*, and some are strictly better. In other words: *A dominates* over *B* in terms of value.

> *Definition*: For all sets A, $B \in Z$ and their corresponding value vectors $\mathbf{v_A}$ and $\mathbf{v_B}$, a set *A dominates* another set *B*, if and only if there is at least one bijection from $\mathbf{v_A}$ to $\mathbf{v_B}$ such that for every pair of values (v_{Ai}, v_{Bi}) it is the case that $v_{Ai} \geq v_{Bi}$ and there exists some pair of values (v_{Ai}, v_{Bi}) such that $v_{Ai} > v_{Bi}$.

It seems obvious that when a set *A* dominates another set *B* in terms of value the set *A* offers more opportunity than *B*. We can thus formulate a condition regarding dominance as follows:

> *The Dominance condition*: For any measure of opportunity O and all sets A, $B \in Z$, and their corresponding value vectors $\mathbf{v_A}$ and $\mathbf{v_B}$, if *A dominates B* in terms of all values then *A offers more opportunity than B* and thus $O(A) > O(B)$.

This condition is satisfied by all strictly increasing opportunity measures.

Two measures

Let us continue by looking at one proposal for a measure that satisfies both the *Strong scale independence condition* and the *Dominance condition*. It is a measure by Bossert from 1997. It is not a proposal for a measure of opportunity, but a proposal for a measure of overall wellbeing depending on freedom of choice. The suggestion is simply to sum the values of all the options in a set (Bossert 1997, pp. 101-102). Thus:

The Total sum measure: $O(A) = O(\mathbf{v_A}) = \sum_{i=1}^{n} v_i$, where O is a function from $\mathbf{R^n}$ to $\mathbf{R}$, A is a set and $\mathbf{v_A}$ is the corresponding vector of values v_i of the elements $x \in A$.

We may apply this measure to the previous example, set *A*:

	P	*S*	*PS*
Uppsala	**8**	2	5
Umeå	5	**9**	**7**
Sveg	1	**9**	5

If we want to rank the following subsets of *A*: *C* = (Uppsala, Umeå), *D* = (Uppsala, Sveg), *E* = (Umeå, Sveg) then *The Total sum measure* gives *O*(*C*) = 35, *O*(*D*) = 30, *O*(*E*) = 35, producing the ranking *C* and *E* over *D*.

This does not appear to be the correct ranking of the three sets in terms of opportunity. The ranking is based on values that are irrelevant for opportunity, being values of options that we may not reasonably choose. It is only the options that are at least as good as the others, according to some reasonable preference ordering, that we may reasonably choose. The other options and their values are irrelevant. Take set *E* as an example. A philosopher could not reasonably choose Sveg above Umeå, so the value of Sveg is irrelevant for the philosopher. A skier could reasonably choose either Umeå or Sveg, but not both, so one of their values is irrelevant for the skier. A philosophical skier could not reasonably choose Sveg above Umeå, so the value of Sveg is irrelevant for the philosophical skier. The relevant values of *E* are thus 5, 9 and 7.

If we accept this reasoning we should rather use a modified version of Bossert's measure:

The Maximal sum measure: $O(A) = O(\mathbf{Max\ v_A}) = \sum_{i=1}^{n} \max v_i$, where O is a function from $\mathbf{R^n}$ to $\mathbf{R}$, A is a set and $\mathbf{Max\ v_A}$ is the corresponding vector of maximal values v_i of the elements $x \in A$.

Applying this measure to the example we just had we get: *O*(*C*) = 8 + 9 + 7 = 24, *O*(*D*) = 8 + 9 + 5 = 22, *O*(*E*) = 5 + 9 + 5 = 19. The ranking now becomes *C* over *D* over *E*. This seems more correct, and not just for alphabetical reasons.

Since we have argued that only maximal values are relevant for opportunity we should not accept the *Dominance condition* as it stands, but a modified version:

> *The Maximal dominance condition*: For any measure of opportunity and for all sets A, $B \in K$, if A dominates B in terms of all maximal values then A offers more opportunity than B and thus $O(A) \geq O(B)$.

This condition is satisfied by the *Maximal sum measure*.

The Equality condition

Is the *Maximal sum measure* a good measure of opportunity? There is at least one relevant factor that the measure does not reflect: if opportunity is the ability to choose a valuable option, *whatever* we may reasonably prefer, then we should have more opportunity the less it matters which preference ordering we have. If there are just three reasonable preference orderings: the preferences of a philosopher, a skier, and a philosophical skier, we should have more opportunity the less it matters whether we are philosophers, skiers or philosophical skiers. For example:

Set *A*:

	P	*S*	*PS*
Köpenhamn	**5**	1	1
Oslo	**5**	1	1
Helsingfors	**5**	1	1

Set *B*:

	P	*S*	*PS*
Bergen	**5**	**5**	**5**
Helsingör	1	1	1
Mariehamn	1	1	1

In this example it seems as if set *B* offers more opportunity, since it offers better options regardless of whether a person is a philosopher, a skier or a philosophical skier. In more abstract terms: if a choice set has a total sum of 15 of maximal values for three reasonable preference orderings, we have more opportunity if the total sum is distributed according to $5 + 5 + 5$, than if the total sum is distributed according to $15 + 0 + 0$. We have more opportunity if the values are *equal* or *closer to being equal*.

This suggestion may seem reasonable. However, it is not perfectly clear. Obviously it is not problematic to recognize a perfectly equal distribution of a total sum, since it is the distribution where all individual values are equal. Neither is it problematic to recognize a completely unequal distribution of a total sum, since it is the distribution where the total sum equals one of the individual values. The problematic cases are the in-between cases. When is an unequal distribution closer to being equal than some other unequal distribution?

We shall not attempt to answer this question here. Instead we shall formulate two equality conditions for cases where the comparisons are obvious. This is the first one:

> *The Maximal equality condition*: For any measure of opportunity O and for all sets A, $B \in Z$ and their corresponding maximal value vector $\mathbf{Max\ v_A}$, and $\mathbf{Max\ v_B}$, such that the total sum of the values of $\mathbf{Max\ v_A}$, and $\mathbf{Max\ v_B}$ are equal, it is the case that if the individual values of $\mathbf{Max\ v_A}$ are equal and the individual values of $\mathbf{Max\ v_B}$ are not equal, then A offers more opportunity than B and thus $O(A) \geq O(B)$.

The other obvious comparison between two sets is the one below. It describes the case where the only difference between two sets, A and B, is in terms of an equal sum of two values being distributed differently between the two individual values.

> *The Limited equality condition*: For any measure of opportunity and for all sets A, $B \in Z$ and their corresponding vectors of values $\mathbf{Max\ v_A}$, $\mathbf{Max\ v_B}$ such that all values are equal in $\mathbf{Max\ v_A}$ and $\mathbf{Max\ v_B}$ except for $v_i \in \mathbf{Max\ v_A}$ and $v_k \in \mathbf{Max\ v_B}$ and $v_j \in \mathbf{Max\ v_A}$ and $v_l \in \mathbf{Max\ v_B}$ and $v_i + v_j = v_k + v_l$, if $| v_i - v_j | < | v_k - v_l |$ then A offers more opportunity than B and thus $O(A) \geq O(B)$.

As it turns out, the *Limited equality condition* implies the *Maximal equality condition* so it is sufficient to use the second condition. We may rename it the *Equality condition*. This condition is not fulfilled by Bossert's measure, so we must find another.

The Symmetry condition

Before looking at alternative measures we should look at another consideration that may be important. Let us assume that two sets A and B have the same total sum of maximal values, 15. The maximal values of both sets are maximally equal. Both sets have taken three preference orderings into account. The distribution is thus 5 + 5 + 5 in both cases. However, there is a difference: in the set A there is one option, a, that accounts for all the values of 5, while in the set B, it is all three options.

Set *A*:

	P	*S*	*PS*
Bergen	**5**	**5**	**5**
Visby	1	1	1
Helsingör	1	1	1

Set *B*:

	P	*S*	*PS*
Lund	**5**	1	1
Brunflo	1	**5**	1
Täby	1	1	**5**

Any reasonable person who would choose from *A* would choose Bergen. But a reasonable person who would choose from *B* could choose Lund, Brunflo or Täby, depending on if the reasonable person has the preferences of a philosopher, a skier or a philosophical skier. Could this mean that the opportunity of *A* is higher than the opportunity of *B*? There seem to be reasons both for and against this view.

If we just look at the stipulative definition of 'opportunity' there is nothing that suggests that *A* would offer more opportunity than *B*. A person *P*'s opportunity is regarded as *P*'s ability to choose whatever *P* may reasonably prefer to choose. It is not regarded as *P*'s ability to choose a single option that *P* may reasonably prefer to choose, whatever *P* may reasonably prefer to choose. We thus cannot use the definition to decide whether *A* offers more opportunity than *B*. Instead we must ask: which type of opportunity is a better candidate for a social good: the type that says that *A* offers as much opportunity as *B*, or the type that says that *A* offers more opportunity than *B*?

Now, there are certainly some cases when it seems to be better for *P* to choose from *A*. One case would be when *P* is uncertain of his own preferences, but knows what could be reasonable preferences. Choosing from *A* would not involve any uncertainty regarding what to choose beforehand and no risk of regret afterwards. However, this line of reasoning presupposes that there is a difference between what is reasonable in general and what is reasonable for *P*. If all reasonable preference orderings are reasonable also for *P* then there is no reason for *P* to be uncertain or regretful when choosing from *B*. Another case when it seems to be better for *P* to choose from *A* would be when *P* has different preferences at different times. Choosing from *A* would offer *P* an option that he could consider best at several different times (evolving from a skier to a philosophical skier to a philosopher, for example).

In the case where *P* is certain about his preferences there certainly would be no advantage for *P* in choosing from *A*. If *P* had cared about other preference orderings than his own, then *P* would have a different preference ordering. If the philosopher would have cared for the preferences of a skier he would have adopted the preferences of a philosophical skier, for example.

Perhaps there are better arguments for the view that A offers more opportunity than B. It does not seem to be any overwhelming reason to say that A offers more opportunity than B. Therefore we shall accept the following condition:

> *The Symmetry condition*: For any measure of opportunity and all sets $A, B \in Z$ and their corresponding vectors of values **Max v$_\mathbf{A}$** and **Max v$_\mathbf{B}$**, if there is at least one bijection from **Max v$_\mathbf{A}$** to **Max v$_\mathbf{B}$** such that for every pair of values (v_{Ai}, v_{Bi}) it is the case that $v_{Ai} = v_{Bi}$ then $O(A) = O(B)$.

The Symmetry condition says that it does not matter for opportunity how the values are distributed among individual options. Having a single option that is at least as good as the others according to all reasonable preference orderings do not affect opportunity more than having different options that are at least as good as the other according to different reasonable preference orderings.

The Root measures

There is a class of measures that satisfy the *Strong scale independence condition*, the *Maximal dominance condition*, the *Equality condition* and the *Symmetry condition*. The measures are root functions of the following form:

> *The Root measures*: $O(A) = O(\mathbf{Max\ v_A}) = \alpha \sum_{i=1}^{n} (\max v_i)^r$, where $0 < r < 1$, $\alpha > 0$, where O is a function from $\mathbf{R^n}$ to $\mathbf{R}$, A is a set and **Max v$_\mathbf{A}$** is the corresponding vector of maximal values v_i of the elements $x \in A$.

The *Root measures* satisfy the *Strong scale independence condition* because if all values in **Max v$_\mathbf{A}$** are multiplied by some number K then $O(A)$ is multiplied by K^r. They satisfy the *Dominance condition* because they are strictly increasing functions and they satisfy the *Equality condition* because they are strictly concave functions. Last, they satisfy the *Symmetry condition* because they are symmetric functions on the maximal values of **Max v$_\mathbf{A}$**.

The problem with the *Root measures* is that they are not one measure but a class of measures. Different choices of r may thus result in different rankings of sets in terms of opportunity. One solution to the problem would be to try to find an additional condition that restricts the value of r further. Another solution would be to choose the square root function as measure of opportunity, since this is a familiar function. We would then get the following measure of opportunity:

The Square root measures: $O(A) = O(\mathbf{Max\ v_A}) = \alpha \sum_{i=1}^{n} \sqrt{\max v_i}$, $\alpha > 0$, where O is a function from $\mathbf{R^n}$ to $\mathbf{R}$, A is a set and $\mathbf{Max\ v_A}$ is the corresponding vector of maximal values v_i of the elements $x \in A$.

We may apply this measure to the previously given example of cities:

Set *A*:

	P	*S*	*PS*
Uppsala	**8**	2	5
Umeå	5	**9**	**7**
Sveg	1	**9**	5

Set *B*:

	P	*S*	*PS*
Stockholm	**7**	**1**	**4**
Göteborg	5	**1**	3
Malmö	1	**1**	1

We get that $O(A) = \sqrt{8} + \sqrt{9} + \sqrt{7} \approx 8.5$ and $O(B) = \sqrt{7} + \sqrt{1} + \sqrt{4} \approx 4.4$, ranking *A* over *B* in terms of opportunity.

The *Root measures* are not just a class of measures that happen to satisfy the four conditions. It is possible to prove that the *Root measures* are the only measures that satisfy the four conditions, given the extra assumption that a measure of opportunity should be an analytic function with non-zero partial derivatives. If we think that the four conditions are reasonable for a measure of opportunity, there are thus four reasons to adopt the *Root measures* as measures of opportunity.[1]

[1] Thanks to all who have given valuable comments on earlier versions of this paper, especially to Constanze Binder, Per Enflo and Martin van Hees, but also to Per Algander, Thomas Anderberg, Jason Alexander, Jim Blevins, Luc Bovens, Johan Gustafsson, Bengt Hansson, Victor Moberger, Erik Olsson, Prasanta Pattanaik, Martin Peterson, Karl Pettersson, Wlodek Rabinowicz, Howard Sobel, Maria Svedberg, Folke Tersman, Torbjörn Tännsjö and Jan Österberg. Thanks also to Erik Carlson, who did not get the opportunity to comment on this version.

Appendix

A. Proof that the Root measures satisfy the 4 conditions

In this section we prove the following theorem:

> *The Realization theorem*: the *Root measures* satisfy the four conditions of *strong scale-independence*, *maximal dominance*, *equality* and *symmetry*.

1. Proof that the Root measures satisfy the Strong scale independence condition

> *The Strong scale independence condition*: For any measure of opportunity O, and all choice sets A, $B \in Z$ and their corresponding value vectors $\mathbf{v_A}$ and $\mathbf{v_B}$, it is the case that for every number $K > 0$, there is a number $L = L(K) > 0$, where L is an increasing function of K, such that if all values in $\mathbf{v_A}$ are multiplied by K, then $O(\mathbf{v_A})$ is multiplied by L.

Proof: The Root measures trivially satisfy the *Strong scale independence condition* because if all the values in **Max** $\mathbf{v_A}$ are multiplied by K, then $O(A)$ is multiplied by K^r.

2. Proof that the Root measures satisfy the Maximal dominance condition

> *The Maximal dominance condition*: For any measure of opportunity and for all sets A, $B \in K$, if A dominates B in terms of all maximal values then A offers more opportunity than B and thus $O(A) \geq O(B)$.

Proof: This is trivial since the function is strictly increasing.

3. Proof that the Root measures satisfy the Equality condition

> *The Equality condition*: For any measure of opportunity and for all sets A, $B \in Z$ and their corresponding vectors of values **Max** $\mathbf{v_A}$, **Max** $\mathbf{v_B}$ such that all values are equal in **Max** $\mathbf{v_A}$ and **Max** $\mathbf{v_B}$ except for $v_i \in$ **Max** $\mathbf{v_A}$ and $v_k \in$ **Max** $\mathbf{v_B}$ and $v_j \in$ **Max** $\mathbf{v_A}$ and $v_l \in$ **Max** $\mathbf{v_B}$ and $v_i + v_j = v_k + v_l$, if $| v_i - v_j | < | v_k - v_l |$ then A offers more opportunity than B and thus $O(A) \geq O(B)$.

Proof: By hypothesis it is the case that $v_i + v_j = v_k + v_l$ and
$| v_i - v_j | < | v_k - v_l |$, thus $v_i - v_k = v_l - v_j$.

Suppose that $v_i < v_j$ and that $v_k < v_l$. In this case, since $v_i + v_j = v_k + v_l$, we have $v_k < v_i < v_j < v_l$.

We then apply the *Mean Value Theorem*. Assume that the function f is continuous on the closed interval $[v_k, v_i]$ and differentiable on the open interval (v_k, v_i). Then there exists some θ_1 in (v_k, v_i) such that

$$f'(\theta_1) = \frac{f(v_i) - f(v_k)}{v_i - v_k}.$$

The function f is also continuous on the closed interval $[v_j, v_l]$ and differentiable on the open interval (v_j, v_l).

So there also exists some θ_2 in (v_j, v_l) such that $f'(\theta_2) = \dfrac{f(v_l) - f(v_j)}{v_l - v_j}$.

We know that $v_i - v_k = v_l - v_j$. We also know that $\theta_1 < \theta_2$ since they are located in different intervals. This means that $f'(\theta_1) > f'(\theta_2)$ since the function f is concave and the derivative is monotonically decreasing.

So $f(v_i) - f(v_k) = f'(\theta_1)(v_i - v_k) = f'(\theta_1)(v_l - v_j) > f'(\theta_2)(v_l - v_j) = f(v_l) - f(v_j)$.

So $f(v_i) - f(v_k) > f(v_l) - f(v_j)$ and therefore $f(v_i) + f(v_j) > f(v_k) + f(v_l)$. Thus $O(A) \geq O(B)$.

4. Proof that the Root measures satisfy the Symmetry condition

> *The Symmetry condition*: For any measure of opportunity and all sets A, $B \in Z$ and their corresponding vectors of values **Max v**

Proof: The Root measures are symmetric functions of the values and trivially satisfy the *Symmetry condition*.

B. Proof of the uniqueness of the Root measures

In this section we will prove the following theorem:

> *The Characterization theorem*: if a measure O is an analytic function with non-zero partial derivatives of some function on the values of options, and if O satisfies the four conditions of *scale-independence, maximal dominance*, *equality* and *symmetry*, then O is a *Root measure* with $0 < r < 1$.

Proof: Let the max values of A be written as a vector $\mathbf{v_A} = \langle v_1, v_2, \dots, v_m \rangle$ with m = the number of reasonable value orders. We get $O(A) = O(f(v_1), f(v_2), \dots, f(v_m))$. Since O is symmetric and differentiable as a function of $f(v_1), f(v_2), \dots, f(v_m)$, the expansion of a differentiable symmetric function via elementary symmetric functions gives:

$$O(A) = O(\mathbf{v_A}) = C(n_1)\left(\sum_{i=1}^{m}\big(f(v_i)\big)\right) + C(n_2)\left(\sum_{i=1, i\neq j}^{m}\ \sum_{j=1}^{m}\big(f(v_i)f(v_j)\big)\right) +$$

$$C(n_3)\left(\sum_{i=1}^{m}\big(f(v_i)\big)^2\right) + \text{higher order terms in the } f(v_i)\,f(v_j)\text{'s.} \dots \text{(I)}$$

(See Glaeser 1963, p. 205-206).

Since O has a non-vanishing derivative at 0 as a function of $f(v_1), f(v_2), \dots, f(v_m)$, we must have $C(n_1)$ different from 0.

Now consider $A = A_j$ with $v_1 = v_2 = \dots = v_j = v$ and $v_{j+1} = v_{j+2} = \dots = v_m = 0$, $1 \le j \le m$. Putting that into (I) gives:

$O(A_j) = g_j(v) = j\ C(n_1)\,f(v) + \{C(n_2)J + C(n_3)\ j\ \}(\,f(v))^2$ + higher order terms with $J = j(j-1)$ = the number of $(v_i)(v_j)$'s different from 0. …. (II)

Multiplying v by a number K will multiply $g_j(v)$ by a number $L = L(K)$. It is a well-known mathematical fact that this gives $g_j(v) = \alpha(j)v^r$. Since $f(v) \to 0$ as $v \to 0$, the first degree term in $f(v)$ on the right hand side of (II) will dominate as $v \to 0$ and so from (II) we get that $\dfrac{\alpha(j)v^r}{jC(n_1)f(v)} \to 1$ as $v \to 0$. Since this holds for all j, $1 \le j \le m$, we have that $\dfrac{\alpha(j)}{j}$ is constant, say $\dfrac{\alpha(j)}{j} = \alpha$ and $\dfrac{g_j(v)}{j} = g(v) = \alpha v^r$.

By dividing (II) by j this gives:

$$\frac{g_j(v)}{j} = \alpha v^r = C(n_1)\,f(v) + \{C(n_2)\frac{J}{j} + C(n_3)\ \}(\,f(v))^2 + \text{higher order terms}$$
… (III)

Since this holds for all j, $1 \leq j \leq m$ and $\frac{J}{j} = j - 1$ we must have $C(n_2) = 0$ in (III) and so $C(n_2) = 0$ in (I). Thus putting $v = v_i$ in (III) and putting that back into (I) we get $O(\mathbf{v_A}) = \alpha \sum_{i=1}^{m} (v_i)^r$. The *Equality condition* gives that this function is strictly concave and the *Maximal dominance condition* gives that this function is strictly increasing so $0 < r < 1$, $\alpha > 0$. Thus we have: $O(\mathbf{v_A}) = \alpha \sum_{i=1}^{n} (v_i)^r$, where $0 < r < 1$, $\alpha > 0$.

References

Bossert, Walter. 'Opportunity sets and individual well-being.' *Social Choice and Welfare* 14, 1997, pp. 97-112

Glaeser, G. 'Fonctions composés différentiables.' *Annals of Mathematics* vol. 77, 1963, pp. 193-209

Pattanaik, Prasanta K. and Xu, Yongsheng. 'On Preference and Freedom.' *Theory and Decision* Vol. 44, 1998, pp. 173-198.

Peragine, Vito and Romero-Medina, Antonio. 'On Preference, Freedom and Diversity.' *Social Choice and Welfare* 27, 2006, pp. 29-40.

Romero-Medina, Antonio. 'More on Preference and Freedom.' *Social Choice and Welfare* 18, 2001, pp. 179-191.

Sen, Amartya. 'Welfare, Preference and Freedom.' *Journal of Econometrics* 50, 1991, pp. 15-29.

Van Hees Martin and Wissenburg, Marcel. 'Freedom and Opportunity.' *Political studies* XLVII, 1999, pp. 67-82.