THE RATE OF MIXING FOR DIAGONAL FLOWS ON SPACES OF AFFINE LATTICES

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1. Introduction

Results on quantitative mixing play an important role in several recent developments in homogeneous dynamics; cf., e.g., [1, 4, 5, 6]. In the present paper our aim is to prove precise results on the rate of mixing on the spaces $\mathrm{ASL}(d,\mathbb{Z})\backslash\mathrm{ASL}(d,\mathbb{R})$. As far as we are aware, these cases have not been covered previously in the literature. We start with (and focus mainly on) the two-dimensional case; let G be the semidirect product $\mathrm{ASL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$, with multiplication

$$(M,\boldsymbol{v})\cdot(M',\boldsymbol{v}') = (MM', M\boldsymbol{v}'+\boldsymbol{v}).$$

Similarly, let Γ be the discrete subgroup $\mathrm{ASL}(2,\mathbb{Z})$ of G. We note that our central object of study, the homogeneous space $\Gamma\backslash G$, can be identified with the space of affine unimodular lattices in $\mathbb{R}^2$. The one-parameter subgroup of G consisting of elements $(\Phi'_t,\boldsymbol{0})$, where $\Phi'_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$, acts on $\Gamma\backslash G$ by right multiplication; given an element $x=\Gamma(M,\boldsymbol{v})\in\Gamma\backslash G$, we let

$$\Phi_t(x) = \Gamma(M,\boldsymbol{v})(\Phi'_t,\boldsymbol{0}) = \Gamma(M\Phi'_t,\boldsymbol{v}).$$

This transformation, together with a Haar measure μ on G (normalized so that the induced measure on $\Gamma\backslash G$ is a probability measure), gives rise to the measure-preserving dynamical system $(\Gamma\backslash G,\mu,\Phi_t)$. This dynamical system is mixing, that is to say; for all $f,g\in L^2(\Gamma\backslash G,\mu)$,

$$\lim_{t\to\infty}\int_{\Gamma\backslash G} f\,(g\circ\Phi_t)\,d\mu = \int_{\Gamma\backslash G} f\,d\mu\int_{\Gamma\backslash G} g\,d\mu.$$

By imposing certain regularity conditions on the functions f and g, we can obtain bounds on the rate of mixing. We let $S^{m,p}(\Gamma\backslash G)$ be the space of m times continuously differentiable functions on $\Gamma\backslash G$ such that the $L^p(\Gamma\backslash G)$ norms of all derivatives of order less or equal to m are finite. The Sobolev norm on this space is denoted $\|\cdot\|_{S^{m,p}(\Gamma\backslash G)}$ (we give a more precise definition of this space and norm in Section 2). We can now state our main results:

Theorem 1. *For $f,g\in S^{2,\infty}(\Gamma\backslash G)$ such that for all $M\in G'$*

$$\int_{[0,1]\times[0,1]} f(M,\boldsymbol{\xi})\,d\boldsymbol{\xi} = 0,$$

the flow Φ_t is mixing with exponential rate. More precisely, for $t\geq 0$,

$$\int_{\Gamma\backslash G} f(x)g(\Phi_t(x))\,d\mu(x) = O(e^{-\frac{t}{2}}\|f\|_{S^{2,\infty}(\Gamma\backslash G)}\|g\|_{S^{2,\infty}(\Gamma\backslash G)}),$$

where the implied constant is absolute.

By using results from [7] concerning the rate of mixing of diagonal flows on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$, we extend this to all f,g in $S^{2,\infty}(\Gamma\backslash G)$ (Corollaries 6 and 7). We then turn our attention to functions in $L^2(\Gamma\backslash G)$, and prove the following:

Theorem 2. *For $f,g\in S^{4,2}(\Gamma\backslash G)$, the flow Φ_t is mixing with exponential rate. More precisely, for $t\geq 1$,*

$$\int_{\Gamma\backslash G} f(x)g(\Phi_t(x))\,d\mu(x) = \int_{\Gamma\backslash G} f\,d\mu\int_{\Gamma\backslash G} g\,d\mu + O(te^{-\frac{t}{2}}\|f\|_{S^{4,2}(\Gamma\backslash G)}\|g\|_{S^{4,2}(\Gamma\backslash G)}),$$

where the implied constant is absolute.

After developing the necessary Fourier decomposition, these are proved in Sections 4 and 5. Finally, in Section 6, an analogue of Theorem 1 is proved for $\mathrm{ASL}(d,\mathbb{Z})\backslash \mathrm{ASL}(d,\mathbb{R})$, when $d \geq 3$.

2. Preliminaries

For notational convenience we denote $G' = \mathrm{SL}(2,\mathbb{R})$ and $\Gamma' = \mathrm{SL}(2,\mathbb{Z})$. We view $C^m(\Gamma\backslash G)$ as the space of m times continuously differentiable functions on G that are invariant under left multiplication by elements of Γ. For every element $X \in \mathfrak{g}$, $\mathfrak{g}$ being the Lie algebra of G ($\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^2$, with Lie bracket $[(M,\boldsymbol{v}),(N,\boldsymbol{w})] = (MN-NM, M\boldsymbol{w}-N\boldsymbol{v})$), the corresponding left-invariant differential operator is

$$[Xf](M) := \frac{\partial}{\partial t} f(M\exp(tX))\bigg|_{t=0}.$$

A basis for the Lie algebra is given by the elements

$$X_1 = \left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\boldsymbol{0}\right),\ X_2 = \left(\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\boldsymbol{0}\right),\ X_3 = \left(\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right),\boldsymbol{0}\right),\ X_4 = \left(0_2,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right),\ X_5 = \left(0_2,\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right).$$

For $f \in C^m(\Gamma\backslash G)$, we define

$$N_m f(M,\boldsymbol{v}) := \sum_{\mathrm{ord}(D)\leq m} |[Df](M,\boldsymbol{v})|,$$

where D is a monomial in $X_1,...,X_5$. A norm for f such that $f \in C^m(\Gamma\backslash G)$ and $N_m f \in L^p(\Gamma\backslash G)$ is given by

$$\|f\|_{S^{m,p}(\Gamma\backslash G)} := \|N_m f\|_{L^p(\Gamma\backslash G)}.$$

We denote the space of f such that $\|f\|_{S^{m,p}(\Gamma\backslash G)} < \infty$ as $S^{m,p}(\Gamma\backslash G)$. The space $L^p(\Gamma\backslash G)$ is identified with the space $L^p(\mathcal{F},\mu)$, where $\mathcal{F}$ is a suitable fundamental domain (a set containing at least one representative for every coset, and the set of repetitions having measure zero) for Γ in G. By choosing a fundamental domain, $\mathcal{F}'$, for $\Gamma'\backslash \mathrm{G}'$, the product space $\mathcal{F} = \mathcal{F}' \times \mathcal{J}$, $\mathcal{J}$ being a fundamental domain for the torus $\mathbb{T}^2 = \mathbb{Z}^2\backslash\mathbb{R}^2$, becomes a fundamental domain for $\Gamma\backslash G$. We note that the torus is the correct factor for the Cartesian product with $\mathcal{F}'$, since f being left-Γ invariant implies $f(M,\boldsymbol{v}) = f((I_2,\boldsymbol{m})(M,\boldsymbol{v})) = f(M,\boldsymbol{v}+\boldsymbol{m})$, $\forall M \in G'$, $\forall \boldsymbol{v} \in \mathbb{R}^2$, $\forall \boldsymbol{m} \in \mathbb{Z}^2$. Using Iwasawa decomposition, we can give an explicit description of one such $\mathcal{F}'$ and μ. This is done in the following way: for a given $M \in G'$, there is a unique decomposition

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\ x \in \mathbb{R},\ y \in \mathbb{R}^+,\ \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

For a given matrix $M \in G'$, we denote the variables $x,\ y,\ \theta$ given by the Iwasawa decomposition of M as $x(M),\ y(M),\ \theta(M)$. Conversely, for given $x \in \mathbb{R}$, $y \in \mathbb{R}^+, \theta \in \mathbb{R}/2\pi\mathbb{Z}$, we define

$$n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ a(y) := \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix},\ \kappa(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

A possible choice of a fundamental domain for $\Gamma'\backslash G'$ is

$$\mathcal{F}' = \{n(x)a(y)\kappa(\theta) : x^2+y^2 \geq 1,\ |x| \leq \frac{1}{2},\ \theta \in [0,\pi)\}.$$

The Haar measure on G is $d\mu = d\mu' d\boldsymbol{v}$, where $d\boldsymbol{v}$ is the Lebesgue measure on $\mathbb{R}^2$ and $d\mu'$ is the Haar measure on G'. The measure $d\mu'$ given in Iwasawa coordinates is then $\frac{3}{\pi^2}\frac{dx dy d\theta}{y^2}$.

3. Fourier Decomposition

The proofs of Theorems 1 and 2 use the Fourier decomposition of the given functions in their torus variables, and rely heavily on Lemmas 4 and 5 in [10]. The Fourier decomposition is as follows: for $f \in C^2(\Gamma\backslash G)$, let

$$\widehat{f}(M,\boldsymbol{k}) = \int_{\mathbb{T}^2} f(M,\boldsymbol{\xi}) e(-\boldsymbol{k}\cdot\boldsymbol{\xi})\, d\boldsymbol{\xi},$$

where $e(x) = e^{2\pi i x}$, $x \in \mathbb{R}$. Then

$$f(M,\boldsymbol{\xi}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \widehat{f}(M,\boldsymbol{k}) e(\boldsymbol{k}\cdot\boldsymbol{\xi}). \tag{1}$$

Since $f \in C^2(\Gamma\backslash G)$, for a fixed $M \in G'$ the corresponding function on the torus, $\boldsymbol{\xi} \mapsto f(M,\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \mathbb{T}^2$, is in $C^2(\mathbb{T}^2)$ and thus has an absolutely convergent Fourier series which is also uniformly convergent on compact subsets of G. Let $\widehat{\mathbb{Z}}^2$ denote the set of primitive lattice points in $\mathbb{Z}^2$, that is to say elements $\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right) \in \mathbb{Z}^2$ such that $\gcd(j,k) = 1$. This allows us to rewrite (1) as

$$f(M,\boldsymbol{\xi}) = \widehat{f}(M,\boldsymbol{0}) + \sum_{n=1}^{\infty} \sum_{\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right)\in\widehat{\mathbb{Z}}^2} \widehat{f}(M, n\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right)) e(n\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right)\cdot\boldsymbol{\xi}). \tag{2}$$

We now make use of the following lemma (Lemma 4 in [10]):

Lemma 3. $\forall\, T \in \Gamma'$, $\forall M \in G'$, $\forall \boldsymbol{k} \in \mathbb{Z}^2$, $\forall f \in C(\Gamma\backslash G)$,

$$\widehat{f}(TM,\boldsymbol{k}) = \widehat{f}(M, {}^{\mathrm{t}}T\boldsymbol{k}).$$

Proof.

$$\begin{aligned}\widehat{f}(TM,\boldsymbol{k}) = \int_{\mathbb{T}^2} f(TM,\boldsymbol{\xi}) e(-\boldsymbol{k}\cdot\boldsymbol{\xi})\, d\boldsymbol{\xi} &= \int_{\mathbb{T}^2} f(TM,T\boldsymbol{\xi}) e(-\boldsymbol{k}\cdot T\boldsymbol{\xi})\, d\boldsymbol{\xi} \\ &= \int_{\mathbb{T}^2} f(M,\boldsymbol{\xi}) e(-{}^{\mathrm{t}}T\boldsymbol{k}\cdot\boldsymbol{\xi})\, d\boldsymbol{\xi} = \widehat{f}(M, {}^{\mathrm{t}}T\boldsymbol{k}).\end{aligned}$$

The second equality is due to T being an area-preserving diffeomorphism of $\mathbb{T}^2$, the third is due to Γ-left invariance of f. □

Applying Lemma 3 to (2) now gives

$$f(M,\boldsymbol{\xi}) = \widehat{f}(M,\boldsymbol{0}) + \sum_{n=1}^{\infty} \sum_{\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right)\in\widehat{\mathbb{Z}}^2} \widehat{f}(\left(\begin{smallmatrix} * & * \\ j & k \end{smallmatrix}\right) M, \left(\begin{smallmatrix} 0 \\ n \end{smallmatrix}\right)) e(n\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right)\cdot\boldsymbol{\xi}), \tag{3}$$

where $\left(\begin{smallmatrix} * & * \\ j & k \end{smallmatrix}\right)$ is any matrix in Γ' with lower entries $\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right) \in \widehat{\mathbb{Z}}^2$. We define

$$\widetilde{f}_n(M) := \widehat{f}(M, \left(\begin{smallmatrix} 0 \\ n \end{smallmatrix}\right)). \tag{4}$$

The following lemma (Lemma 5 in [10]) provides a bound on the growth of the functions $\widetilde{f}_n$:

Lemma 4. *For any $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$ and $f \in S^{m,\infty}(\Gamma\backslash G)$, we have*

$$\left|\widetilde{f}_n\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)\right| \ll \frac{\|f\|_{S^{m,\infty}(\Gamma\backslash G)}}{n^m (c^2+d^2)^{\frac{m}{2}}},\ \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R}), \tag{5}$$

with the implied constant depending only on m.

Proof. We have $\exp(tX_4) = (I_2, \left(\begin{smallmatrix} t \\ 0 \end{smallmatrix}\right))$ and $\exp(tX_5) = (I_2, \left(\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right))$. Hence, parametrizing G by $((\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right))$ gives

$$[X_4 f](M,\boldsymbol{v}) = a\frac{\partial f}{\partial v_1}(M,\boldsymbol{v}) + c\frac{\partial f}{\partial v_2}(M,\boldsymbol{v}),$$

and

$$[X_5 f](M, \boldsymbol{v}) = b\frac{\partial f}{\partial v_1}(M, \boldsymbol{v}) + d\frac{\partial f}{\partial v_2}(M, \boldsymbol{v}).$$

Since

$$\widetilde{f}_n\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \int_{\mathbb{Z}\backslash\mathbb{R}}\int_{\mathbb{Z}\backslash\mathbb{R}} f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right))e(-n\xi_2)\, d\xi_2\, d\xi_1,$$

repeated integration by parts gives

$$(2\pi inc)^m \cdot \widetilde{f}_n\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \int_{\mathbb{Z}\backslash\mathbb{R}}\int_{\mathbb{Z}\backslash\mathbb{R}} [X_4^m f](\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right))e(-n\xi_2)\, d\xi_2\, d\xi_1 \tag{6}$$

and

$$(2\pi ind)^m \cdot \widetilde{f}_n\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \int_{\mathbb{Z}\backslash\mathbb{R}}\int_{\mathbb{Z}\backslash\mathbb{R}} [X_5^m f](\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right))e(-n\xi_2)\, d\xi_2\, d\xi_1. \tag{7}$$

Hence

$$\max(|c|^m, |d|^m) \cdot \left|\widetilde{f}_n\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)\right| \leq (2\pi n)^{-m}\|f\|_{S^{m,\infty}(\Gamma\backslash G)},$$

giving (5). □

4. Proof of Theorem 1

We now use the Fourier decomposition and the bound on the growth of the coefficients given by Lemma 4 to prove Theorem 1.

Proof of Theorem 1.

$$\int_{\Gamma\backslash G} f(x)g(\Phi_t(x))\, d\mu(x) = \int_{\mathcal{F}'}\int_{\mathbb{T}^2} f(M, \boldsymbol{\xi})g(M\Phi_t', \boldsymbol{\xi})\, d\boldsymbol{\xi}\, d\mu'(M).$$

Expanding both f and g in their Fourier decompositions gives

$$\int_{\mathcal{F}}\int_{\mathbb{T}^2}\Big(\widehat{f}(M, \boldsymbol{0}) + \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^2 \\ \boldsymbol{k}\neq\boldsymbol{0}}} \widehat{f}(M, \boldsymbol{k})e(\boldsymbol{k}\cdot\boldsymbol{\xi})\Big)\Big(\widehat{g}(M\Phi_t', \boldsymbol{0}) + \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^2 \\ \boldsymbol{m}\neq\boldsymbol{0}}} \widehat{g}(M\Phi_t', \boldsymbol{m})e(\boldsymbol{m}\cdot\boldsymbol{\xi})\Big)\, d\boldsymbol{\xi}\, d\mu'(M).$$

Since the two series are absolutely convergent, we may multiply out their product. Uniform convergence of the series on compact sets allows the exchange of order of summation and integration on the torus. The summands are then all zero, except for when $\boldsymbol{k} = -\boldsymbol{m}$, giving

$$\int_{\mathcal{F}'} \widehat{f}(M, \boldsymbol{0})\widehat{g}(M\Phi_t', \boldsymbol{0})\, d\mu'(M) + \int_{\mathcal{F}'}\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^2 \\ \boldsymbol{k}\neq 0}} \widehat{f}(M, \boldsymbol{k})\widehat{g}(M\Phi_t', -\boldsymbol{k})\, d\mu'(M). \tag{8}$$

By assumption, $\widehat{f}(M, \boldsymbol{0}) = 0$, so we need only concern ourselves with the rate of convergence to zero of

$$\int_{\mathcal{F}'}\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^2 \\ \boldsymbol{k}\neq 0}} \widehat{f}(M, \boldsymbol{k})\widehat{g}(M\Phi_t', -\boldsymbol{k})\, d\mu'(M). \tag{9}$$

Using Lemma 3 and (4), we can reparametrize the sum (9) similarly as we did when going from (1) to (2), obtaining:

$$\int_{\mathcal{F}'}\sum_{n=1}^{\infty}\sum_{\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right)\in\widehat{\mathbb{Z}}^2} \widetilde{f}_n(\left(\begin{smallmatrix} * & * \\ j & k \end{smallmatrix}\right) M)\widetilde{g}_n(-\left(\begin{smallmatrix} * & * \\ j & k \end{smallmatrix}\right) M\Phi_t')\, d\mu'(M). \tag{10}$$

Now, define the subgroup $\Gamma'_\infty := \{\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) : z \in \mathbb{Z}\}$. Each element $\left(\begin{smallmatrix} * & * \\ j & k \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right) \in \widehat{\mathbb{Z}}$ is the representative of a different coset in $\Gamma'_\infty\backslash\Gamma'$, and moreover every coset is represented (i.e. there is a bijection from $\widehat{\mathbb{Z}}^2$ to $\Gamma'_\infty\backslash\Gamma'$). This means that by taking the union of our fundamental domain $\mathcal{F}'$ multiplied from the left by elements $\left(\begin{smallmatrix} * & * \\ j & k \end{smallmatrix}\right)$, i.e. $\bigcup_{\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right)\in\widehat{\mathbb{Z}}^2}\left(\begin{smallmatrix} * & * \\ j & k \end{smallmatrix}\right)\mathcal{F}'$, we construct a fundamental domain for $\Gamma'_\infty\backslash G'$. We note that $\widetilde{f}$ and $\widetilde{g}$ are left Γ'_∞-invariant, and hence the function $M \mapsto \widetilde{f}_n(M)\widetilde{g}_n(-M\Phi'_t)$ is as well, so its integral is the same over any fundamental domain for $\Gamma'_\infty\backslash G'$. We choose to integrate over $\mathcal{I} = \{n(x)a(y)\kappa(\theta) : -\frac{1}{2} \leq x \leq \frac{1}{2},\ y > 0,\ \theta \in [0, 2\pi)\}$, a standard fundamental domain for $\Gamma'_\infty\backslash G'$. Hence, by the triangle inequality and Lebesgue's monotone convergence theorem, the absolute value of (10) is less than or equal to

$$\sum_{n=1}^{\infty} \sum_{\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right) \in \widehat{\mathbb{Z}}^2} \int_{\left(\begin{smallmatrix} * & * \\ j & k \end{smallmatrix}\right)\mathcal{F}'} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi'_t)|\, d\mu'(M) = \sum_{n=1}^{\infty} \int_{\mathcal{I}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi'_t)|\, d\mu'(M). \tag{11}$$

Lemma 4 is used to bound the contribution from a given n in (11) in the following manner: if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $c^2 + d^2 = \frac{1}{y(M)}$, hence denoting $y = y(M)$ and $\theta = \theta(M)$ gives

$$|\widetilde{f}_n(M)| \ll \min\left(\|f\|_{S^{0,\infty}(\Gamma\backslash G)}, \|f\|_{S^{2,\infty}(\Gamma\backslash G)} y n^{-2}\right) \ll \|f\|_{S^{2,\infty}(\Gamma\backslash G)} \min\left(1, yn^{-2}\right), \tag{12}$$

and

$$\begin{aligned} |\widetilde{g}_n(-M\Phi'_t)| &\ll \min\left(\|g\|_{S^{0,\infty}(\Gamma\backslash G)}, \|g\|_{S^{2,\infty}(\Gamma\backslash G)} y n^{-2}(e^t \sin^2\theta + e^{-t}\cos^2\theta)^{-1}\right) \\ &\ll \|g\|_{S^{2,\infty}(\Gamma\backslash G)} \min\left(1, yn^{-2}(e^t \sin^2\theta + e^{-t}\cos^2\theta)^{-1}\right). \end{aligned} \tag{13}$$

We note that the implied constants in these estimates are absolute. Now, using (12) and (13), we bound

$$\int_{\mathcal{I}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi'_t)|\, d\mu'(M)$$

by

$$\begin{aligned} \|f\|_{S^{2,\infty}(\Gamma\backslash G)} \|g\|_{S^{2,\infty}(\Gamma\backslash G)} \int_0^\infty \int_0^{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} &\min\left(1, yn^{-2}\right) \\ &\times \min\left(1, yn^{-2}(e^t \sin^2\theta + e^{-t}\cos^2\theta)^{-1}\right) \frac{dx\, d\theta\, dy}{y^2}. \end{aligned}$$

Substituting $y = n^2 v$ gives

$$\begin{aligned} &\ll \frac{\|f\|_{S^{2,\infty}(\Gamma\backslash G)} \|g\|_{S^{2,\infty}(\Gamma\backslash G)}}{n^2} \int_0^\infty \int_0^{2\pi} \min(1, v) \min\left(1, v(e^t \sin^2\theta + e^{-t}\cos^2\theta)^{-1}\right) \frac{d\theta\, dv}{v^2} \\ &\ll \frac{\|f\|_{S^{2,\infty}(\Gamma\backslash G)} \|g\|_{S^{2,\infty}(\Gamma\backslash G)}}{n^2} \int_0^\infty \min(1, v) \int_0^{\pi/2} \min\left(1, ve^{-t}\theta^{-2}\right) \frac{d\theta\, dv}{v^2} \\ &\ll \frac{\|f\|_{S^{2,\infty}(\Gamma\backslash G)} \|g\|_{S^{2,\infty}(\Gamma\backslash G)}}{n^2} \int_0^\infty \min(1, v) \left(\int_0^{\sqrt{ve^{-t}}} d\theta + \int_{\sqrt{ve^{-t}}}^\infty ve^{-t}\theta^{-2} d\theta \right) \frac{dv}{v^2} \\ &\ll \frac{\|f\|_{S^{2,\infty}(\Gamma\backslash G)} \|g\|_{S^{2,\infty}(\Gamma\backslash G)}}{n^2} e^{-\frac{t}{2}} \int_0^\infty \min(1, v) v^{\frac{1}{2}} \frac{dv}{v^2}. \end{aligned}$$

This gives

$$\int_{\mathcal{F}'} \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}^2 \\ \boldsymbol{k} \neq 0}} \widehat{f}(M, \boldsymbol{k}) \widehat{g}(M\Phi'_t, -\boldsymbol{k})\, d\mu'(M) = O\left(e^{-\frac{t}{2}} \|f\|_{S^{2,\infty}(\Gamma\backslash G)} \|g\|_{S^{2,\infty}(\Gamma\backslash G)}\right),$$

as required. □

Remark 1. In order to extend the result of Theorem 1 to all $f, g \in S^{2,\infty}(\Gamma\backslash G)$, we need to consider the term

$$\int_{\mathcal{F}'} \widehat{f}(M, \mathbf{0}) \widehat{g}(M\Phi'_t, \mathbf{0})\, d\mu'(M) \tag{14}$$

in (8). This is done by noting that for $f, g \in S^{m,2}(\Gamma\backslash G)$, $\widehat{f}(\cdot, \mathbf{0})$ and $\widehat{g}(\cdot, \mathbf{0})$ can be viewed as functions in $L^2(\Gamma'\backslash G')$, i.e. they are Γ'-left invariant and have finite $L^2(\mathcal{F}', \mu')$ norm. Clearly, the $L^2(\Gamma'\backslash G')$ norms are bounded by the $S^{m,2}(\Gamma\backslash G)$ norms of f and g respectively (and even by the $L^2(\Gamma\backslash G)$ norms). Recall also that $\mu'(\mathcal{F}')=1$, so for $f,g\in S^{m,\infty}(\Gamma\backslash G)$, the $S^{m,2}(\Gamma\backslash G)$ norms are bounded by the $S^{m,\infty}(\Gamma\backslash G)$ norms. The invariance condition is proved thus: let $\gamma\in\Gamma'$. Then

$$\begin{aligned}\widehat{f}(\gamma M,\mathbf{0}) &= \int_{\mathbb{T}^2} f(\gamma M,\boldsymbol{\xi})\,d\boldsymbol{\xi} = \int_{\mathbb{T}^2} f((\gamma,\mathbf{0})(M,\gamma^{-1}\boldsymbol{\xi}))\,d\boldsymbol{\xi}\\ &= \int_{\mathbb{T}^2} f(M,\gamma^{-1}\boldsymbol{\xi})\,d\boldsymbol{\xi} = \int_{\mathbb{T}^2} f(M,\boldsymbol{\xi})\,d\boldsymbol{\xi} = \widehat{f}(M,\mathbf{0}).\end{aligned}$$

The third equality holds due to Γ-left invariance of f and the fourth due to γ^{-1} being an area-preserving diffeomorphism of the torus. Naturally, the same also holds for $\widehat{g}(\cdot,\mathbf{0})$. By viewing $\widehat{f}(\cdot,\mathbf{0})$ and $\widehat{g}(\cdot,\mathbf{0})$ in this manner, we may use results regarding the rate of mixing on $\Gamma'\backslash G'$ to ascertain the rate at which (14) converges to

$$\begin{aligned}&\int_{\mathcal{F}'} \widehat{f}(M,\mathbf{0})\,d\mu'(M)\int_{\mathcal{F}'} \widehat{g}(M,\mathbf{0})\,d\mu'(M)\\ &= \int_{\mathcal{F}'}\int_{\mathbb{T}^2} f(M,\boldsymbol{\xi})\,d\boldsymbol{\xi}\,d\mu'(M)\int_{\mathcal{F}'}\int_{\mathbb{T}^2} g(M,\boldsymbol{\xi})\,d\boldsymbol{\xi}\,d\mu'(M)\\ &\qquad = \int_{\Gamma\backslash G} f\,d\mu\int_{\Gamma\backslash G} g\,d\mu.\end{aligned}$$

We denote the regular representation of G' on $L^2(\Gamma'\backslash G')$ by π, i.e. $\pi(M)g(\Gamma'N)=g(\Gamma'NM)$, and let $K(m)$ be the set of $L^2(\Gamma'\backslash G')$ functions g such that the map $\psi_g:\mathbb{R}\to L^2(\Gamma'\backslash G'),\theta\mapsto \pi(\kappa(\theta))g$ is m-times Fréchet differentiable.

Proposition 5. $f\in S^{m,2}(\Gamma\backslash G)\Rightarrow \widehat{f}(\cdot,\mathbf{0})\in K(m)$.

Proof. Assume $f\in S^{m,2}(\Gamma\backslash G)$. By differentiating through the integral, we have $\widehat{f}(\cdot,\mathbf{0})\in S^{m,2}(\Gamma'\backslash G')$ (we discuss this in greater detail in the beginning of Section 5). Now let g be an arbitrary function in $S^{1,2}(\Gamma'\backslash G')$. It suffices to show that $\theta\mapsto\pi(\kappa(\theta))[X'_\kappa g]$ is the Fréchet derivative of ψ_g, that is

$$\lim_{h\to 0}\frac{\|\psi_g(\theta+h)-\psi_g(\theta)-h\pi(\kappa(\theta))[X'_\kappa g]\|_{L^2(\Gamma'\backslash G')}}{h}=0,$$

where $X'_\kappa = X'_2 - X'_1$ (once again, we give precise definitions in Section 5). The representation is unitary, so

$$\frac{\|\pi(\kappa(\theta+h))g-\pi(\kappa(\theta))g-h\pi(\kappa(\theta))[X'_\kappa g]\|_{L^2(\Gamma'\backslash G')}}{h}=\frac{1}{h}\|\pi(\kappa(h))g-g-h[X'_\kappa g]\|_{L^2(\Gamma'\backslash G')}.$$

In the proof of Lemma 8, we note that $\frac{\partial}{\partial\varphi}g(M\kappa(\varphi))=[X'_\kappa g](M\kappa(\varphi))$, so by the fundamental theorem of calculus, the above expression equals

$$\frac{1}{h}\left(\int_{\mathcal{F}'}\left|g(M)+\int_0^h [X'_\kappa g](M\kappa(\varphi))\,d\varphi - g(M)-h[X'_\kappa g](M)\right|^2 d\mu'(M)\right)^{\frac{1}{2}}.$$

Minkowski's inequality for integrals gives

$$\begin{aligned}&\frac{1}{h}\left(\int_{\mathcal{F}'}\left|\int_0^h\Big([X'_\kappa g](M\kappa(\varphi))-[X'_\kappa g](M)\Big)\,d\varphi\right|^2 d\mu'(M)\right)^{\frac{1}{2}}\\ &\leq \frac{1}{h}\int_0^h\left(\int_{\mathcal{F}'}|[X'_\kappa g](M\kappa(\varphi))-[X'_\kappa g](M)|^2\,d\mu'(M)\right)^{\frac{1}{2}}d\varphi\\ &\leq \sup_{\varphi\in[0,h]}\|\pi(\kappa(\varphi))[X'_\kappa g]-[X'_\kappa g]\|_{L^2(\Gamma'\backslash G')}.\end{aligned}$$

This bound tends to zero as $h\to 0$, since G' acts continuously on $L^2(\Gamma'\backslash G')$; let us recall the standard proof of this fact: by density of $C_c(\Gamma'\backslash G')$ in $L^2(\Gamma'\backslash G')$, for a given $\epsilon>0$ there is a function $g_{\frac{\epsilon}{3}}\in C_c(\Gamma'\backslash G')$ such that $\|[X_\kappa g']-g_{\frac{\epsilon}{3}}\|_{L^2(\Gamma'\backslash G')}<\frac{\epsilon}{3}$. Since $g_{\frac{\epsilon}{3}}$ has compact support, it is uniformly continuous, and therefore there exists an $h_0 > 0$ such that for every φ with $|\varphi| < h_0$ we have $|g_{\frac{\epsilon}{3}}(M\kappa(\varphi)) - g(M)| < \frac{\epsilon}{3}$ $\forall M \in \mathcal{F}'$, and hence also $\|\pi(\kappa(\varphi))g_{\frac{\epsilon}{3}} - g_{\frac{\epsilon}{3}}\|_{L^2(\Gamma'\backslash G')} < \frac{\epsilon}{3}$. Now

$$\begin{aligned}\|\pi(\kappa(\varphi))[X'_\kappa g] - [X'_\kappa g]\|_{L^2(\Gamma'\backslash G')} &\leq \|\pi(\kappa(\varphi))[X'_\kappa g] - \pi(\kappa(\varphi))g_{\frac{\epsilon}{3}}\|_{L^2(\Gamma'\backslash G')}\\ &\quad + \|\pi(\kappa(\varphi))g_{\frac{\epsilon}{3}} - g_{\frac{\epsilon}{3}}\|_{L^2(\Gamma'\backslash G')}\\ &\quad + \|g_{\frac{\epsilon}{3}} - [X'_\kappa g]\|_{L^2(\Gamma'\backslash G')}\\ &< \epsilon.\end{aligned}$$

□

Now, Theorem 2 in [7] gives the following: for $f, g \in K(m)$ and $t \geq 1$,

$$\int_{\Gamma'\backslash G'} f\ \pi(\Phi_t)g\, d\mu' = \int_{\Gamma'\backslash G'} f\, d\mu' \int_{\Gamma'\backslash G'} g\, d\mu' + O\big((te^{-\frac{t}{2}})^{\alpha(m)}\|f\|_{S^{m,2}(\Gamma'\backslash G')}\|g\|_{S^{m,2}(\Gamma'\backslash G')}\big),$$

where $\alpha(2) = \frac{4}{5}$, and $\alpha(m) = 1$ for $m \geq 3$. We note that the explicit constants (that depend on f and g) are not given in the statement of this theorem, but by studying the proof (in particular, pages 286-288), we can extract the constants as stated here. This, together with Theorem 1, gives the following corollaries:

Corollary 6. *For $f, g \in S^{2,\infty}(\Gamma\backslash G), t \geq 1$,*

$$\int_{\Gamma\backslash G} f(x)g(\Phi_t(x))\, d\mu(x) = \int_{\Gamma\backslash G} f\, d\mu \int_{\Gamma\backslash G} g\, d\mu + O\big(t^{\frac{4}{5}}e^{-\frac{2t}{5}}\|f\|_{S^{2,\infty}(\Gamma\backslash G)}\|g\|_{S^{2,\infty}(\Gamma\backslash G)}\big),$$

where the implied constant is absolute.

Corollary 7. *For $f, g \in S^{3,\infty}(\Gamma\backslash G), t \geq 1$,*

$$\int_{\Gamma\backslash G} f(x)g(\Phi_t(x))\, d\mu(x) = \int_{\Gamma\backslash G} f\, d\mu \int_{\Gamma\backslash G} g\, d\mu + O\big(te^{-\frac{t}{2}}\|f\|_{S^{3,\infty}(\Gamma\backslash G)}\|g\|_{S^{3,\infty}(\Gamma\backslash G)}\big),$$

where the implied constant is absolute.

5. Proof of Theorem 2

In order to prove Theorem 2, we make use of two bounds on the growth of $\widetilde{f}_n$ and $\widetilde{g}_n$. Both of these are based on Sobolev inequalities; the first (Lemma 8) on a one-dimensional inequality, the second (Lemma 9, which can be seen as a version of Lemma 4 for $S^{m,2}(\Gamma\backslash G)$ functions) on a three-dimensional inequality. These inequalities require the calculation of derivatives (on G') of the Fourier coefficients. This is done as follows: the Lie algebra of G', $\mathfrak{sl}(2,\mathbb{R})$, has the basis $X'_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $X'_2 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $X'_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and is embedded in $\mathfrak{g}$ by $X' \mapsto (X', \mathbf{0})$. Hence, for $f \in C^m(\Gamma\backslash G)$, D' a monomial in X'_1, X'_2, X'_3 of order not greater than m, we have

$$[D'\widetilde{f}_n]\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \int_{\mathbb{T}^2} [Df]\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right)\right)e(-n\xi_2)\, d\xi_1\, d\xi_2 = \widetilde{[Df]}_n\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right), \tag{15}$$

where D is the operator corresponding to the embedding of D' in $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. We can now state the two lemmas that will be of use when ascertaining the rate of mixing for $S^{4,2}(\Gamma\backslash G)$ functions.

Lemma 8. *For $f \in S^{1,2}(\Gamma\backslash G)$, $x \in \mathbb{R}$, $y \in \mathbb{R}^+$, $\theta \in [0, 2\pi)$, the following inequality holds*

$$|\widetilde{f}_n(n(x)a(y)\kappa(\theta))| \leq 2\sqrt{\int_0^{2\pi} F_n(n(x)a(y)\kappa(\varphi))\, d\varphi},$$

where $F_n := |\widetilde{f}_n|^2 + |\widetilde{[X_1 f]}_n|^2 + |\widetilde{[X_2 f]}_n|^2$.

Proof. The one-dimensional Sobolev inequality gives (by an abuse of notation, we let $\widetilde{f}_n(x,y,\theta) := \widetilde{f}_n(n(x)a(y)\kappa(\theta)))$

$$|\widetilde{f}_n(x,y,\theta)| \leq \sqrt{2\int_0^{2\pi} \left(|\widetilde{f}_n(x,y,\varphi)|^2 + |\frac{\partial}{\partial\varphi}\widetilde{f}_n(x,y,\varphi)|^2\right) d\varphi}. \tag{16}$$

Now,

$$\frac{\partial}{\partial\varphi}\widetilde{f}_n(x,y,\varphi) = \lim_{h\to 0}\frac{\widetilde{f}_n(n(x)a(y)\kappa(\varphi+h)) - \widetilde{f}_n(n(x)a(y)\kappa(\varphi))}{h},$$

and

$$\widetilde{f}_n(n(x)a(y)\kappa(\varphi+h)) = \widetilde{f}_n(n(x)a(y)\kappa(\varphi)\kappa(h)) = \widetilde{f}_n(n(x)a(y)\kappa(\varphi)\exp(h(X_2' - X_1'))).$$

Hence

$$\frac{\partial}{\partial\varphi}\widetilde{f}_n(x,y,\varphi) = [X_2'\widetilde{f}_n](x,y,\varphi) - [X_1'\widetilde{f}_n](x,y,\varphi).$$

By (15), we have $[X_1'\tilde{f}_n] = \widetilde{[X_1 f]}_n$ and $[X_2'\tilde{f}_n] = \widetilde{[X_2 f]}_n$. Hence

$$|\frac{\partial}{\partial\varphi}\widetilde{f}_n(x,y,\varphi)|^2 \leq 2|\widetilde{[X_1 f]}_n(x,y,\varphi)|^2 + 2|\widetilde{[X_2 f]}_n(x,y,\varphi)|^2. \tag{17}$$

Substituting (17) into (16) gives the desired inequality. □

Lemma 9. *For $f \in S^{m+2,2}(\Gamma\backslash G)$, $M \in G'$, we have*

$$|\widetilde{f}_n(M)| \ll \min\left(1, \frac{y(M)}{n^2}\right)^{\frac{m}{2}} \max\left(y(M), \frac{1}{y(M)}\right)^{\frac{1}{2}} \|f\|_{S^{m+2,2}(\Gamma\backslash G)}, \tag{18}$$

with the implied constant depending only on m.

Proof. Identities (6) and (7) give the following inequalities

$$|\widetilde{f}_n(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right))| \ll n^{-m}|c|^{-m}\left|\int_{\mathbb{T}^2}[X_4^m f](\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), \left(\begin{smallmatrix} \xi_1 \\ \xi_2\end{smallmatrix}\right))e(-n\xi_2)\, d\xi_1\, d\xi_2\right|, \tag{19}$$

$$|\widetilde{f}_n(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right))| \ll n^{-m}|d|^{-m}\left|\int_{\mathbb{T}^2}[X_5^m f](\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), \left(\begin{smallmatrix} \xi_1 \\ \xi_2\end{smallmatrix}\right))e(-n\xi_2)\, d\xi_1\, d\xi_2\right|. \tag{20}$$

Define the function $\widetilde{F}_{n,X_i^m}$ $(i = 4,5)$ on G' by

$$\widetilde{F}_{n,X_i^m}(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)) := \int_{\mathbb{T}^2}[X_i^m f](\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), \left(\begin{smallmatrix} \xi_1 \\ \xi_2\end{smallmatrix}\right))e(-n\xi_2)\, d\xi_1\, d\xi_2.$$

We note that $\widetilde{F}_{n,X_i^m}$ is not generally Γ'-left invariant (it is, however, Γ'_∞-left invariant). A bounded neighbourhood, $\mathcal{U}$, of the identity in G' is chosen, and we define the function h_M on $\mathcal{U}$; for $M \in G', U \in \mathcal{U}$, $h_M(U) := \widetilde{F}_{n,X_i^m}(MU)$. For $f \in S^{m+2,2}(\Gamma\backslash G)$, Sobolev's inequality (for $S^{2,2}(\Omega)$, Ω being a bounded domain in $\mathbb{R}^3$, after using the appropriate coordinate change) gives

$$|\widetilde{F}_{n,X_i^m}(M)| \ll \|h_M\|_{S^{2,2}(\mathcal{U})},$$

with the implied constant depending only on $\mathcal{U}$. Now

$$\|h_M\|_{S^{2,2}(\mathcal{U})} = \sum_{\mathrm{ord} D' \leq 2}\left(\int_{\mathcal{U}} |[D'\widetilde{F}_{n,X_i^m}](MU)|^2\, d\mu'(U)\right)^{\frac{1}{2}}.$$

Left invariance of the Haar measure gives

$$\|h_M\|_{S^{2,2}(\mathcal{U})} = \sum_{\mathrm{ord} D' \leq 2}\left(\int_{M\mathcal{U}} |[D'\widetilde{F}_{n,X_i^m}](N)|^2\, d\mu'(N)\right)^{\frac{1}{2}}.$$

By (15), we have

$$\|h_M\|_{S^{2,2}(\mathcal{U})} \leq \sum_{\mathrm{ord}D\leq 2} \left(\int_{M\mathcal{U}} \left| \int_{\mathbb{T}^2} [DX_i^m f](N, \boldsymbol{\xi}) e(-n\xi_2)\, d\boldsymbol{\xi} \right|^2 d\mu'(N) \right)^{\frac{1}{2}}.$$

Hence

$$\|h_M\|_{S^{2,2}(\mathcal{U})} \ll \left(\int_{M\mathcal{U}} \int_{\mathbb{T}^2} N_{m+2}^2 \, d\boldsymbol{\xi} d\mu' \right)^{\frac{1}{2}}.$$

For $M \in G'$ we denote the integral over the torus of N_{m+2}^2 by $I_{m+2}(M)$;

$$I_{m+2}(M) := \int_{\mathbb{T}^2} N_{m+2}^2(M, \boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

Noting that since I_{m+2} is Γ'-left invariant, the integral of I_{m+2} over $M\mathcal{U}$ is bounded by the integral of I_{m+2} over $\Gamma'\backslash G'$ multiplied by the maximal cardinality of the fibers of the projection of $M\mathcal{U}$ onto $\Gamma'\backslash G'$. More explicitly formulated; let π_M be the projection of $M\mathcal{U}$ onto $\Gamma'\backslash G'$, i.e.

$$\pi_M : M\mathcal{U} \to \Gamma'\backslash G', \quad N \mapsto \Gamma' N.$$

Define $p(M)$ by

$$p(M) := \max_{x\in\Gamma'\backslash G'} |\pi_M^{-1}(\{x\})|.$$

Then

$$\int_{M\mathcal{U}} I_{m+2}\, d\mu' \leq p(M) \int_{\Gamma'\backslash G'} I_{m+2}\, d\mu' \ll p(M)\|f\|^2_{S^{m+2,2}(\Gamma\backslash G)}.$$

As before, noting that $c^2 + d^2 = \frac{1}{y(M)}$, and making use of (19) and (20) gives a pointwise bound on $\widetilde{f}_n$:

$$|\widetilde{f}_n(M)| \ll \min\left(1, \frac{y(M)}{n^2}\right)^{\frac{m}{2}} p(M)^{\frac{1}{2}} \|f\|_{S^{m+2,2}(\Gamma\backslash G)}.$$

From Lemma 2.11 in [3], we acquire the following bound; $p(M) \ll \max(y(M), \frac{1}{y(M)})$, giving (18). □

Proof of Theorem 2. Let us first prove a slightly weaker version of Theorem 2, namely that for $f, g \in S^{5,2}(\Gamma\backslash G)$, we have

$$\int_{\Gamma\backslash G} f(x) g(\Phi_t(x))\, d\mu(x) = \int_{\Gamma\backslash G} f\, d\mu \int_{\Gamma\backslash G} g\, d\mu + O(te^{-\frac{t}{2}} \|f\|_{S^{5,2}(\Gamma\backslash G)} \|g\|_{S^{5,2}(\Gamma\backslash G)}). \tag{21}$$

We proceed as in the proof of Theorem 1; utilizing the Fourier decompositions of f and g, as well as the result from [7] as stated in Remark 1, we reduce the problem to bounding

$$\sum_{n=1}^{\infty} \int_{\mathcal{I}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')|\, d\mu'(M).$$

We now split the domain of integration in the following way; define the sets

$$\mathcal{A} := \{M \in \mathcal{I} \,:\, y(M) \leq 1\},$$

$$\mathcal{B} := \{M \in \mathcal{I} \,:\, y(-M\Phi_t') \leq 1\},$$

and

$$\mathcal{C} := \{M \in \mathcal{I} \,:\, y(M) > 1,\, y(-M\Phi_t') > 1\}.$$

Then

$$\int_{\mathcal{I}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')| d\mu'(M) \leq \int_{\mathcal{A}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')|\, d\mu'(M) + \int_{\mathcal{B}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')|\, d\mu'(M) + \int_{\mathcal{C}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')|\, d\mu'(M).$$

Substituting $N = -M\Phi_t'$ gives

$$\int_{\mathcal{B}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')|\, d\mu'(M) = \int_{\mathcal{A}} |\widetilde{g}_n(N)\widetilde{f}_n(-N\Phi_{-t}')|\, d\mu'(N).$$

We now have two integrals over $\mathcal{A}$ and one over $\mathcal{C}$ to consider. The integrals over $\mathcal{A}$ are dealt with first; from Lemma 9 we have the bounds

$$|\widetilde{f}_n(M)| \ll \min\left(1, \frac{y(M)}{n^2}\right)^{\frac{3}{2}} \max\left(y(M), \frac{1}{y(M)}\right)^{\frac{1}{2}} \|f\|_{S^{5,2}(\Gamma\backslash G)},$$

and

$$|\widetilde{g}_n(-M\Phi_t')| \ll \min\left(1, \frac{y(M)}{\alpha n^2}\right)^{\frac{3}{2}} \max\left(\frac{y(M)}{\alpha}, \frac{\alpha}{y(M)}\right)^{\frac{1}{2}} \|g\|_{S^{5,2}(\Gamma\backslash G)},$$

where

$$\alpha = e^t \sin^2(\theta(M)) + e^{-t}\cos^2(\theta(M)).$$

Hence

$$\int_{\mathcal{A}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')|\, d\mu'(M) \ll \|f\|_{S^{5,2}(\Gamma\backslash G)}\|g\|_{S^{5,2}(\Gamma\backslash G)} \int_0^1\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \min(1, n^{-2}y)^{\frac{3}{2}} \max(y, y^{-1})^{\frac{1}{2}} \times \min(1, n^{-2}y\alpha^{-1})^{\frac{3}{2}} \max(y\alpha^{-1}, \alpha y^{-1})^{\frac{1}{2}} \frac{dx\, d\theta\, dy}{y^2}. \tag{22}$$

Simplifying the right hand side of (22) gives

$$\begin{aligned}
&\ll \frac{\|f\|_{S^{5,2}(\Gamma\backslash G)}\|g\|_{S^{5,2}(\Gamma\backslash G)}}{n^3} \int_0^1\int_0^{2\pi} y^{-1}\min(1, y\alpha^{-1})^{\frac{3}{2}} \max(y\alpha^{-1}, \alpha y^{-1})^{\frac{1}{2}}\, d\theta\, dy \\
&\ll \frac{\|f\|_{S^{5,2}(\Gamma\backslash G)}\|g\|_{S^{5,2}(\Gamma\backslash G)}}{n^3} \int_0^1\int_0^{2\pi} \alpha^{-\frac{1}{2}} y^{-\frac{1}{2}} \min(1, y\alpha^{-1}) \max(1, \alpha y^{-1})\, d\theta\, dy \\
&= \frac{\|f\|_{S^{5,2}(\Gamma\backslash G)}\|g\|_{S^{5,2}(\Gamma\backslash G)}}{n^3} \int_0^1 y^{-\frac{1}{2}}\, dy \int_0^{2\pi} \alpha^{-\frac{1}{2}}\, d\theta.
\end{aligned}$$

In the same manner, we get the bound

$$\int_{\mathcal{A}} |\widetilde{g}_n(M)\widetilde{f}_n(-M\Phi_{-t}')|\, d\mu'(M) \ll \frac{\|f\|_{S^{5,2}(\Gamma\backslash G)}\|g\|_{S^{5,2}(\Gamma\backslash G)}}{n^3} \int_0^1 y^{-\frac{1}{2}}\, dy \int_0^{2\pi} \beta^{-\frac{1}{2}}\, d\theta,$$

where $\beta = e^t\cos^2\theta + e^{-t}\sin^2\theta$. Now,

$$\begin{aligned}
&\int_0^{2\pi} \beta^{-\frac{1}{2}}\, d\theta = \int_0^{2\pi} \alpha^{-\frac{1}{2}}\, d\theta \asymp \int_0^{\pi/2} \frac{1}{\max(\theta e^{\frac{t}{2}}, e^{-\frac{t}{2}}(\frac{\pi}{2}-\theta))}\, d\theta \\
&= e^{\frac{t}{2}} \int_0^{\frac{\pi}{2(1+e^t)}} \frac{1}{\frac{\pi}{2}-\theta}\, d\theta + e^{-\frac{t}{2}} \int_{\frac{\pi}{2(1+e^t)}}^{\frac{\pi}{2}} \theta^{-1}\, d\theta = e^{\frac{t}{2}}\log(e^{-t}+1) + e^{-\frac{t}{2}}\log(e^t+1) = O(te^{-\frac{t}{2}}).
\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \int_{\mathcal{A}\cup\mathcal{B}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')|\, d\mu'(M) = O(te^{-\frac{t}{2}}\|f\|_{S^{5,2}(\Gamma\backslash G)}\|g\|_{S^{5,2}(\Gamma\backslash G)}).$$

To bound the integral over $\mathcal{C}$, we use dyadic decomposition; for $j \in \mathbb{Z}_{\geq 0}$, let

$$\mathcal{C}_j := \{M \in \mathcal{I} : 2^j \leq y(M) \leq 2^{j+1}\},$$

and

$$\mathcal{C}'_{t,j} := \{M \in \mathcal{I} : 2^j \leq y(-M\Phi'_t) \leq 2^{j+1}\}.$$

Denoting the intersection as

$$\mathcal{C}_{t,j,k} := \mathcal{C}_j \cap \mathcal{C}'_{t,k}$$

gives

$$\sum_{n=1}^{\infty} \int_{\mathcal{C}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi'_t)| \, d\mu'(M) = \sum_{j,k\geq 0} \sum_{n=1}^{\infty} \int_{\mathcal{C}_{t,j,k}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi'_t)| \, d\mu'(M).$$

By the Cauchy-Schwarz inequality (and a change of variables),

$$\begin{aligned} &\sum_{n=1}^{\infty} \int_{\mathcal{C}_{t,j,k}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi'_t)| \, d\mu'(M) \\ &\quad \leq \sqrt{\sum_{n=1}^{\infty} \int_{\mathcal{C}_{t,j,k}} |\widetilde{f}_n(M)|^2 \, d\mu'(M)} \sqrt{\sum_{n=1}^{\infty} \int_{\mathcal{C}_{-t,k,j}} |\widetilde{g}_n(M)|^2 \, d\mu'(M)}. \end{aligned} \tag{23}$$

Lemma 8 gives

$$\sum_{n=1}^{\infty} \int_{\mathcal{C}_{t,j,k}} |\widetilde{f}_n(M)|^2 \, d\mu'(M) \ll \int_{\mathcal{C}_{t,j,k}} \int_0^{2\pi} \sum_{n=1}^{\infty} F_n(x,y,\varphi) \, d\varphi \, d\theta \, dx \, \frac{dy}{y^2},$$

and by Parseval's identity,

$$\begin{aligned} &\ll \int_{\mathcal{C}_{t,j,k}} \int_0^{2\pi} I_f(x,y,\varphi) \, d\varphi \, d\theta \, dx \, \frac{dy}{y^2} \\ &\qquad \ll \int_{2^j}^{2^{j+1}} |\{\theta \in [0,2\pi) : 2^k \leq y/\alpha \leq 2^{k+1}\}| \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\pi} I_f(x,y,\varphi) \, d\varphi \, dx \, \frac{dy}{y^2}, \end{aligned}$$

where

$$I_f := \int_{\mathbb{T}^2} (|f|^2 + |[X_1 f]|^2 + |[X_2 f]|^2) \, d\boldsymbol{\xi}.$$

Now, for $2^j \leq y \leq 2^{j+1}$,

$$\{\theta \in [0,2\pi) : 2^k \leq y/\alpha \leq 2^{k+1}\} \subseteq \{\theta \in [0,2\pi) : 2^{j-k-1} \leq \alpha \leq 2^{j-k+1}\},$$

giving

$$\ll |\{\theta \in [0,2\pi) : 2^{j-k-1} \leq \alpha \leq 2^{j-k+1}\}| \int_{2^j}^{2^{j+1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\pi} I_f(x,y,\varphi) \, d\varphi \, dx \, \frac{dy}{y^2}.$$

Since $e^{-t} \leq \alpha \leq e^t$, we have

$$|\{\theta \in [0,2\pi) : 2^{j-k-1} \leq \alpha \leq 2^{j-k+1}\}| = 0$$

if $e^{-t} > 2^{j-k+1}$ or $e^t < 2^{j-k-1}$, i.e. if $2^{j-k} \notin [\frac{1}{2e^t}, 2e^t]$. For $2^{j-k} \in [\frac{1}{2e^t}, 2e^t]$, we have

$$|\{\theta \in [0,2\pi) : 2^{j-k-1} \leq \alpha \leq 2^{j-k+1}\}| \ll |\{\theta \in [0,\tfrac{\pi}{2}) : 2^{j-k-1} \leq \alpha \leq 2^{j-k+1}\}|.$$

For $\theta \in [0,\frac{\pi}{2})$, we have $\frac{e^t\theta^2}{5} \leq \alpha$, so

$$|\{\theta \in [0,\tfrac{\pi}{2}) : 2^{j-k-1} \leq \alpha \leq 2^{j-k+1}\}| \leq |\{\theta \in [0,\tfrac{\pi}{2}) : \frac{e^t\theta^2}{5} \leq 2^{j-k+1}\}| \ll e^{-\frac{t}{2}} 2^{\frac{j-k}{2}}.$$

We note that we get the same bound when replacing t with $-t$, as α is replaced by $\beta = e^{-t}\sin^2\theta + e^t\cos^2\theta$, and

$$\begin{aligned}|\{\theta \in [0,2\pi) \,:\, 2^{j-k-1} \leq \beta \leq 2^{j-k+1}\}| &= |\{\theta \in [\pi/2, 5\pi/2) \,:\, 2^{j-k-1} \leq \beta \leq 2^{j-k+1}\}| \\ &= |\{\theta \in [0,2\pi) \,:\, 2^{j-k-1} \leq \alpha \leq 2^{j-k+1}\}|.\end{aligned}$$

Returning to (23), we now have

$$\sum_{n=1}^{\infty} \int_{\mathcal{C}_{t,j,k}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')| \, d\mu'(M) = 0,$$

if $2^{j-k} \notin [\frac{1}{2e^t}, 2e^t]$, and

$$\begin{aligned}&\sum_{n=1}^{\infty} \int_{\mathcal{C}_{t,j,k}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')| d\mu'(M) \\ &\quad \ll \sqrt{e^{-\frac{t}{2}} 2^{\frac{j-k}{2}} \int_{2^j}^{2^{j+1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\pi} I_f(x,y,\varphi)\, d\varphi\, dx\, \frac{dy}{y^2}} \\ &\quad\quad \times \sqrt{e^{-\frac{t}{2}} 2^{\frac{k-j}{2}} \int_{2^k}^{2^{k+1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{2\pi} I_g(x,y,\varphi)\, d\varphi\, dx\, \frac{dy}{y^2}} \\ &\quad\quad = e^{-\frac{t}{2}} \sqrt{\int_{\mathcal{C}_j} I_f(M)\, d\mu'(M)} \sqrt{\int_{\mathcal{C}_k} I_g(M)\, d\mu'(M)}\end{aligned}$$

otherwise. We can now bound the sum of the integrals over $\mathcal{C}$ by

$$\begin{aligned}&\sum_{n=1}^{\infty} \int_{\mathcal{C}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')| \, d\mu'(M) \\ &\quad \ll e^{-\frac{t}{2}} \sum_{\substack{j,k\geq 0 \\ |j-k|\leq 1+\frac{t}{\log 2}}} \sqrt{\int_{\mathcal{C}_j} I_f(M)\, d\mu'(M)} \sqrt{\int_{\mathcal{C}_k} I_g(M)\, d\mu'(M)} \\ &\quad \ll e^{-\frac{t}{2}} \sum_{\substack{j,k\geq 0 \\ |j-k|\leq 3t}} \sqrt{\int_{\mathcal{C}_j} I_f(M)\, d\mu'(M)} \sqrt{\int_{\mathcal{C}_k} I_g(M)\, d\mu'(M)} \\ &\quad \ll e^{-\frac{t}{2}} \sum_{d\in[-3t,3t]\cap\mathbb{Z}} \sum_{j=\max(0,d)}^{\infty} \sqrt{\int_{\mathcal{C}_j} I_f(M)\, d\mu'(M)} \sqrt{\int_{\mathcal{C}_{j-d}} I_g(M)\, d\mu'(M)}.\end{aligned}$$

Once again, the Cauchy-Schwarz inequality (and taking the smallest possible lower limit of summation) gives

$$\begin{aligned}&\ll e^{-\frac{t}{2}} \sum_{d\in[-3t,3t]\cap\mathbb{Z}} \sqrt{\sum_{j=0}^{\infty} \int_{\mathcal{C}_j} I_f(M)\, d\mu'(M)} \sqrt{\sum_{j=0}^{\infty} \int_{\mathcal{C}_j} I_g(M)\, d\mu'(M)} \\ &\ll te^{-\frac{t}{2}} \sqrt{\int_{\mathcal{I}\setminus\mathcal{A}} I_f(M)\, d\mu'(M)} \sqrt{\int_{\mathcal{I}\setminus\mathcal{A}} I_g(M)\, d\mu'(M)}.\end{aligned} \tag{24}$$

By the definitions of I_f and I_g, as well as noting that $\mathcal{I}\setminus\mathcal{A} \subset \mathcal{F}' \cup \mathcal{F}'\kappa(\pi)$, we conclude that the integrals in (24) are bounded by $\|f\|^2_{S^{1,2}(\Gamma\backslash G)}$ and $\|g\|^2_{S^{1,2}(\Gamma\backslash G)}$, respectively, giving

$$\sum_{n=1}^{\infty} \int_{\mathcal{C}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi_t')| \, d\mu'(M) = O(te^{-\frac{t}{2}} \|f\|_{S^{1,2}(\Gamma\backslash G)} \|g\|_{S^{1,2}(\Gamma\backslash G)}),$$

completing the proof of (21).

We will next describe how to improve the above argument to get the desired result (that is, for $S^{4,2}(\Gamma\backslash G)$). The bound on $p(M)$ used in Lemma 9 is very much a worst-case estimate, and on average greatly overestimates the value of $p(M)$ for M such that $y(M) < 1$. Instead, we may bound p by the so-called *invariant height function*, $\mathcal{Y}_{\Gamma'}$. This is done thus: let $\mathbb{H}$ be the upper-half plane

$$\mathbb{H} := \{x + iy \ : \ y > 0,\, x, y \in \mathbb{R}\},$$

together with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

We let G' act on $\mathbb{H}$ by fractional linear transformations, i.e. for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G'$, $z \in \mathbb{H}$, let

$$M \cdot z = \frac{az + b}{cz + d}.$$

We note that for $M = n(x)a(y)\kappa(\theta)$, $M \cdot i = x + iy$. In Lemma 9 we can choose $\mathcal{U}$ to be the set

$$\mathcal{U} := \{M \in G' \ : \ \mathsf{d}(M \cdot i, i) < 1\},$$

where d is the distance induced by the given metric. This means that

$$p(M) = \sup_{U \in \mathcal{U}} \#\{\gamma \in \Gamma' \ : \ \mathsf{d}(\gamma M U \cdot i, M \cdot i) < 1\}.$$

We see that p is Γ'-left invariant, so we need only consider p on $\mathcal{F}'$ ($\mathcal{F}'$ as chosen in Section 2). Let $\mathcal{F}'' \subset \mathbb{H}$ be the set $\{x(M) + iy(M) \ : \ M \in \mathcal{F}'\}$. Since $\mathcal{F}'$ is a fundamental domain for $\Gamma'\backslash G'$, and $\Gamma'\backslash G'/\operatorname{SO}(2) \cong \Gamma'\backslash\mathbb{H}$ (see, e.g., [2] or [3]), we conclude that $p(M)$ is less than or equal to the number of Γ' translates of $\mathcal{F}''$ that a (hyperbolic) unit ball around $x(M)+iy(M)$ intersects. For all $M \in \mathcal{F}'$ such that $y(M)$ is less or equal to 3, $p(M)$ is uniformly bounded, as the set of all $w \in \mathbb{H}$ which have distance not greater than one to $\{z \in \mathcal{F}'' \ : \ \operatorname{Im} z \leq 3\}$ is compact and hence intersects only a finite number of Γ'-translates of $\mathcal{F}''$. For $y(M)$ larger than 3, the only translates of $\mathcal{F}''$ that a unit ball around $x(M) + iy(M)$ can intersect are translates of the type $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right) \mathcal{F}''$, $k \in \mathbb{Z}$, i.e. integer translations of $\mathcal{F}''$ along the x-axis. Hence, for $M \in \mathcal{F}'$, $p(M)$ is asymptotically bounded by the Euclidean width along the x axis of a hyperbolic unit ball around $x(M) + iy(M)$. For a point $z \in \mathbb{H}$, this width is $O(\operatorname{Im} z)$, hence for all $M \in G'$,

$$p(M) \ll \sup_{\gamma \in \Gamma'} y(\gamma M) = \sup_{\gamma \in \Gamma'} \operatorname{Im}(\gamma M \cdot i).$$

For $z \in \mathbb{H}$, we define

$$\mathcal{Y}_{\Gamma'}(z) := \sup_{\gamma \in \Gamma'} \operatorname{Im}(\gamma \cdot z).$$

By changing the estimate for $p(M)$ used in Lemma 9, now we obtain the bounds

$$|\widetilde{f}_n(M)| \ll \min\left(1, \frac{y(M)}{n^2}\right) \mathcal{Y}_{\Gamma'}(M \cdot i)^{\frac{1}{2}} \|f\|_{S^{4,2}(\Gamma\backslash G)},$$

$$|\widetilde{g}_n(-M\Phi'_t)| \ll \min\left(1, \frac{y(M)}{\alpha n^2}\right) \max\left(\frac{y(M)}{\alpha}, \frac{\alpha}{y(M)}\right)^{\frac{1}{2}} \|g\|_{S^{4,2}(\Gamma\backslash G)}.$$

We need only consider the integral over $\mathcal{A}$, which now reads

$$\begin{aligned}
&\int_{\mathcal{A}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi'_t)| \, d\mu'(M) \\
&\quad \ll \|f\|_{S^{4,2}(\Gamma\backslash G)} \|g\|_{S^{4,2}(\Gamma\backslash G)} \int_0^1 \int_0^{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \min(1, n^{-2}y) \mathcal{Y}_{\Gamma'}(x + iy)^{\frac{1}{2}} \\
&\qquad\qquad \times \min(1, n^{-2}y\alpha^{-1}) \max(y\alpha^{-1}, \alpha y^{-1})^{\frac{1}{2}} \, \frac{dx\, d\theta\, dy}{y^2}.
\end{aligned}$$

By Proposition 2.2 in [9],

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{Y}_{\Gamma'}(x+iy)^{\frac{1}{2}}\,dx \leq C,$$

for all $0<y<1$ and some positive constant C. This gives

$$\frac{\|f\|_{S^{4,2}(\Gamma\backslash G)}\|g\|_{S^{4,2}(\Gamma\backslash G)}}{n^2}\int_0^1\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} y^{-1}\min(1,y\alpha^{-1})\max(y\alpha^{-1},\alpha y^{-1})^{\frac{1}{2}}\mathcal{Y}_{\Gamma'}(x+iy)^{\frac{1}{2}}\,dx\,d\theta\,dy$$

$$\ll \frac{\|f\|_{S^{4,2}(\Gamma\backslash G)}\|g\|_{S^{4,2}(\Gamma\backslash G)}}{n^2}\int_0^1\int_0^{2\pi}\alpha^{-\frac{1}{2}}y^{-\frac{1}{2}}\min(1,y\alpha^{-1})\max(1,\alpha y^{-1})\,d\theta\,dy$$

$$\ll \frac{\|f\|_{S^{4,2}(\Gamma\backslash G)}\|g\|_{S^{4,2}(\Gamma\backslash G)}}{n^2}\int_0^{2\pi}\alpha^{-\frac{1}{2}}\,d\theta \ll \frac{\|f\|_{S^{4,2}(\Gamma\backslash G)}\|g\|_{S^{4,2}(\Gamma\backslash G)}}{n^2}te^{-\frac{t}{2}}.$$

□

6. Mixing in Higher Dimensions

We now let $G=\mathrm{ASL}(d,\mathbb{R})=\mathrm{SL}(d,\mathbb{R})\ltimes\mathbb{R}^d$, $\Gamma=\mathrm{ASL}(d,\mathbb{Z})=\mathrm{SL}(d,\mathbb{Z})\ltimes\mathbb{Z}^d$. As before, multiplication is

$$(M_1,\boldsymbol{v}_1)\cdot(M_2,\boldsymbol{v}_2)=(M_1M_2,M_1\boldsymbol{v}_2+\boldsymbol{v}_1),$$

and we also reuse the notation $\mathrm{G}'=\mathrm{SL}(d,\mathbb{R})$ and $\Gamma'=\mathrm{SL}(d,\mathbb{Z})$. As in the two-dimensional case, we can view $\Gamma\backslash G$ as the space of affine unimodular d-dimensional lattices. Let X_i denote the basis elements of the Lie algebra $\mathfrak{g}=\mathfrak{sl}(d,\mathbb{R})\oplus\mathbb{R}^d$ corresponding to the $\mathbb{R}^d$ coordinates, i.e. $X_i=(0,{}^{\mathrm{t}}(0,...,1,...,0))$. The left-invariant differential operator corresponding to X_i is then

$$[X_if](M,\boldsymbol{v})=\frac{\partial}{\partial t}f((M,\boldsymbol{v})\exp(tX_i))\bigg|_{t=0}=\sum_{j=1}^{d}M_{ji}\frac{\partial}{\partial v_j}f(M,\boldsymbol{v}), \tag{25}$$

where M and $\boldsymbol{v}$ are parametrized by $\{M_{ji}\}_{1\leq j,i\leq d}$ and $\{v_j\}_{1\leq j\leq d}$ respectively. We have the Iwasawa decomposition for G': $G'=NAK$, where $K=\mathrm{SO}(d)$, N is the subgroup of matrices

$$n_d(x)=\begin{pmatrix}1 & x_{12} & \cdots & x_{1d}\\ & \ddots & \ddots & \vdots\\ & & \ddots & x_{d-1,d}\\ & & & 1\end{pmatrix},\qquad x_{jk}\in\mathbb{R},$$

and A is the subgroup consisting of

$$a_d(y)=\begin{pmatrix}y_1 & & \\ & \ddots & \\ & & y_d\end{pmatrix},\qquad y_j\in\mathbb{R}^+.$$

Since A is a subgroup of G', $y_d=\prod_{i=1}^{d-1}y_i^{-1}$. The Haar measure on G' in these coordinates is, with $M=n(x)a(y)k$ ([11], Section 2.1),

$$d\mu'(M)=\frac{2^{d-1}\pi^{d(d+1)/4}}{\prod_{j=1}^{d}\Gamma(\frac{j}{2})\prod_{j=2}^{d}\zeta(j)}\rho(y)dn(x)da(y)dk,$$

where dk is the normalized Haar measure on $\mathrm{SO}(d)$, $dn(x)=\prod_{1\leq j<k\leq d}dx_{jk}$ and $da(y)=\prod_{j=1}^{d-1}y_j^{-1}dy_j$. The function $\rho(y)$ is

$$\rho(y)=\prod_{j=1}^{d}y_j^{2j-d-1}=\prod_{j=1}^{d-1}y_j^{2(j-d)}.$$

We now state our result concerning the rate of mixing in higher dimensions:

Theorem 10. *Let*

$$\Phi'_t = \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_d t} \end{pmatrix},$$

where $\sum_{i=1}^d \lambda_i = 0$. *Given* $\Gamma\backslash G \ni x = \Gamma(M, \boldsymbol{v})$, *we define*

$$\Phi_t(x) := \Gamma(M, \boldsymbol{v}) \cdot (\Phi'_t, \mathbf{0}_d).$$

For $f, g \in S^{d,\infty}(\Gamma\backslash G)$ *such that* $\int_{\mathbb{T}^d} f \, d\boldsymbol{\xi} = 0$ *and* $t \geq 0$,

$$\int_{\Gamma\backslash G} f(x) g(\Phi_t(x)) \, d\mu(x) = O(e^{-\lambda_{\max} t} \|f\|_{S^{d,\infty}(\Gamma\backslash G)} \|g\|_{S^{d,\infty}(\Gamma\backslash G)}),$$

where $\lambda_{\max} = \max(\lambda_1, \dots, \lambda_d)$.

The proof of Thereom 10 once again relies on bounds of Fourier coefficients; for a function $f \in S^{m,\infty}(\Gamma\backslash G)$, $m \geq 2$, we have the Fourier decomposition with respect to the torus variable

$$f(M, \boldsymbol{v}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \widehat{f}(M, \boldsymbol{k}) e({}^{\mathrm{t}}\boldsymbol{k}\boldsymbol{v}), \tag{26}$$

where

$$\widehat{f}(M, \boldsymbol{k}) = \int_{\mathbb{T}^d} f(M, \boldsymbol{\xi}) e(-{}^{\mathrm{t}}\boldsymbol{k}\boldsymbol{\xi}) \, d\boldsymbol{\xi}.$$

As in the two-dimensional case, we note the following property of $\widehat{f}$: for $T \in \Gamma'$, $M \in G'$, $\boldsymbol{k} \in \mathbb{Z}^d$, we have (with a proof which is completely analogous to the proof of Lemma 3)

$$\widehat{f}(TM, \boldsymbol{k}) = \widehat{f}(M, {}^{\mathrm{t}}T\boldsymbol{k}).$$

This allows the reformulation of (26);

$$f(M, \boldsymbol{v}) = \widehat{f}(M, \mathbf{0_d}) + \sum_{n=1}^{\infty} \sum_{\boldsymbol{k} \in \widehat{\mathbb{Z}}^d} \widehat{f}\left(\left(\begin{smallmatrix} * \\ {}^{\mathrm{t}}\boldsymbol{k} \end{smallmatrix} \right) M, \left(\begin{smallmatrix} \mathbf{0}_{d-1} \\ n \end{smallmatrix} \right)\right) e(n\, {}^{\mathrm{t}}\boldsymbol{k}\boldsymbol{v}),$$

where $\widehat{\mathbb{Z}}^d$ is the set of d-dimensional primitive integer vectors, and $\left(\begin{smallmatrix} * \\ {}^{\mathrm{t}}\boldsymbol{k} \end{smallmatrix} \right)$ is any matrix in Γ' with dth row vector equal to ${}^{\mathrm{t}}\boldsymbol{k}$ (such a matrix always exists; cf. e.g., Theorem 31 in [8]). Let

$$\widetilde{f}_n(M) := \widehat{f}\left(M, \left(\begin{smallmatrix} \mathbf{0}_{d-1} \\ n \end{smallmatrix} \right)\right).$$

This gives the following generalization of Lemma 4:

Lemma 11. *For any* $f \in S^{m,\infty}(\Gamma\backslash G)$,

$$|\widetilde{f}_n(M)| \ll \frac{\|f\|_{S^{m,\infty}(\Gamma\backslash G)}}{n^m \|M_{d,\cdot}\|^m}, \tag{27}$$

where $\|M_{d,\cdot}\| = \left(\sum_{j=1}^d M_{d,j}^2\right)^{\frac{1}{2}}$, *and the implied constant depends only on* d *and* m.

Proof. Since

$$\widetilde{f}_n(M) = \int_{\mathbb{T}^d} f(M, \boldsymbol{\xi}) e(-n\xi_d) \, d\boldsymbol{\xi},$$

(25) and repeated integration by parts gives

$$(2\pi i n M_{d,j})^m \cdot \widetilde{f}_n(M) = \int_{\mathbb{T}^d} [X_j^m f](M, \boldsymbol{\xi}) e(-n\xi_d) \, d\boldsymbol{\xi}.$$

Hence

$$\max_{1 \leq j \leq d} (|M_{d,j}|^m) |\widetilde{f}_n(M)| \ll n^{-m} \|f\|_{S^{m,\infty}(\Gamma\backslash G)},$$

giving (27). ∎

We note that $\widetilde{f}_n$ is Γ'_H-left invariant, where

$$H = \{T \in G' \,:\, {}^{\mathrm{t}}T\boldsymbol{e}_d = \boldsymbol{e}_d\} = \left\{ \begin{pmatrix} A & \boldsymbol{w} \\ {}^{\mathrm{t}}\boldsymbol{0}_{d-1} & 1 \end{pmatrix} \,:\, A \in \mathrm{SL}(d-1,\mathbb{R})\,,\, \boldsymbol{w} \in \mathbb{R}^{d-1} \right\},$$

and $\Gamma'_H := \Gamma' \cap H$. To integrate over $\Gamma'_H \backslash G'$, we need a fundamental domain for the action of Γ'_H: let

$$\mathcal{F} := \left\{ n_d(x)a_d(y)k_d \,:\, x_{ji} \in (-\tfrac{1}{2}, \tfrac{1}{2}],\, 0 < y_{j+1} \le \frac{2}{\sqrt{3}} y_j \ (j = 1, ..., d-2),\, k_d \in K \right\}.$$

Proposition 12. *$\mathcal{F}$ contains a fundamental domain for $\Gamma'_H \backslash G'$.*

Proof. We need to show that for a given $M = n_d(x)a_d(y)k_d$, there is a $\gamma \in \Gamma'_H$, and an $M' \in \mathcal{F}$ such that $\gamma M' = M$. Let

$$S = \begin{pmatrix} y_1 & y_2 x_{1,2} & \cdots & y_{d-1}x_{1,d-1} \\ & y_2 & \cdots & y_{d-1}x_{2,d-1} \\ & & \ddots & \vdots \\ & & & y_{d-1} \end{pmatrix} \in \mathrm{GL}(d-1,\mathbb{R}),\ \boldsymbol{x} = \begin{pmatrix} x_{1,d} \\ \vdots \\ x_{d-1,d} \end{pmatrix} \in \mathbb{R}^{d-1}.$$

This allows M to be written in block matrix form:

$$M = \begin{pmatrix} S & y_d \boldsymbol{x} \\ {}^{\mathrm{t}}\boldsymbol{0}_{d-1} & y_d \end{pmatrix} k_d.$$

Since $y_d^{\frac{1}{d-1}} S \in \mathrm{SL}(d-1,\mathbb{R})$, there is an $A \in \mathrm{SL}(d-1,\mathbb{Z})$, and $T = n_{d-1}(u)a_{d-1}(v)k'_{d-1}$ such that $AT = y_d^{\frac{1}{d-1}} S$ and $T \in \mathcal{S}_{d-1}$, where $\mathcal{S}_{d-1}$ is the Siegel set (see e.g. [11], Section 2.1)

$$\mathcal{S}_{d-1} := \Big\{ n_{d-1}(u)a_{d-1}(v)k'_{d-1} \,:\, u_{ji} \in (-\tfrac{1}{2}, \tfrac{1}{2}],\\ 0 < v_{j+1} \le \frac{2}{\sqrt{3}} v_j \ (j = 1, ..., d-2),\, k'_{d-1} \in \mathrm{SO}(d-1) \Big\}.$$

Now, ${}^{\mathrm{t}}\boldsymbol{u} = {}^{\mathrm{t}}(u_{1,d}, ..., u_{d-1,d}) \in (-\frac{1}{2}, \frac{1}{2}]^{d-1}$ and $\boldsymbol{w} \in \mathbb{Z}^{d-1}$ are chosen so that

$$\boldsymbol{x} = A\boldsymbol{u} + \boldsymbol{w}.$$

Then

$$\begin{pmatrix} A & \boldsymbol{w} \\ {}^{\mathrm{t}}\boldsymbol{0}_{d-1} & 1 \end{pmatrix} \begin{pmatrix} n_{d-1}(u) & \boldsymbol{u} \\ {}^{\mathrm{t}}\boldsymbol{0}_{d-1} & 1 \end{pmatrix} \begin{pmatrix} y_d^{-\frac{1}{d-1}} a_{d-1}(v) & \boldsymbol{0}_{d-1} \\ {}^{\mathrm{t}}\boldsymbol{0}_{d-1} & y_d \end{pmatrix} \begin{pmatrix} k'_{d-1} & \boldsymbol{0}_{d-1} \\ {}^{\mathrm{t}}\boldsymbol{0}_{d-1} & 1 \end{pmatrix} k_d = M.$$

□

Remark 2. By a similar discussion, using in particular the fact that $\mathcal{S}_{d-1}$ is contained in a finite union of fundamental domains for $\mathrm{SL}(d-1,\mathbb{Z})\backslash \mathrm{SL}(d-1,\mathbb{R})$, it follows that $\mathcal{F}$ is optimal in the sense that $\mathcal{F}$ is contained in a finite union of fundamental domains for $\Gamma'_H \backslash G'$.

Proof of Theorem 10. Assume $f, g \in S^{m,\infty}(\Gamma\backslash G)$, $m \ge 2$. As in the proof of Theorem 1, using the Fourier expansion of both functions we get

$$\left| \int_{\Gamma\backslash G} f(x)g(\Phi_t(x))\, d\mu(x) \right| \le \sum_{n=1}^{\infty} \int_{\Gamma'_H \backslash G'} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi'_t)|\, d\mu'(M).$$

By Lemma 11 and Proposition 12, we have

$$\int_{\Gamma'_H\backslash G'} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi'_t)|\, d\mu'(M)$$
$$\ll \|f\|_{S^{m,\infty}(\Gamma\backslash G)}\|g\|_{S^{m,\infty}(\Gamma\backslash G)} \int_0^\infty \int_0^{\frac{2}{\sqrt{3}}y_1}\cdots\int_0^{\frac{2}{\sqrt{3}}y_{d-2}} \min\left(1, \frac{y_d^{-1}}{n}\right)^m$$
$$\times \int_K \min\left(1, \frac{y_d^{-1}}{n\sqrt{\sum_{i=1}^d k_{d,i}^2 e^{2\lambda_i t}}}\right)^m dk \prod_{j=1}^{d-1} y_j^{2(j-d)-1}\, dy_{d-1}...dy_1.$$

For $j = 1, ..., d-1$, let $v_j = y_j n^{-\frac{1}{d-1}}$, and $v_d = \prod_{j=1}^{d-1} v_j^{-1}$. This substitution gives

$$\ll n^{-d}\|f\|_{S^{m,\infty}(\Gamma\backslash G)}\|g\|_{S^{m,\infty}(\Gamma\backslash G)} \int_0^\infty \int_0^{\frac{2}{\sqrt{3}}v_1}\cdots\int_0^{\frac{2}{\sqrt{3}}v_{d-2}} \min\left(1, v_d^{-1}\right)^m$$
$$\times \int_K \min\left(1, \frac{v_d^{-1}}{\sqrt{\sum_{i=1}^d k_{d,i}^2 e^{2\lambda_i t}}}\right)^m dk \prod_{j=1}^{d-1} v_j^{2(j-d)-1}\, dv_{d-1}...dv_1. \tag{28}$$

Note that the push-forward of the Haar measure under the map $K \to \mathrm{S}^{d-1}$, $k \mapsto (k_{d,1}, \ldots, k_{d,d})$ is a K-invariant measure on S^{d-1}; hence it is a constant multiple of the volume measure $dV_{\mathrm{S}^{d-1}}$ on S^{d-1}. It follows that

$$\int_K \min\left(1, \frac{v_d^{-1}}{\sqrt{\sum_{i=1}^d k_{d,i}^2 e^{2\lambda_i t}}}\right)^m dk$$
$$\ll \int_{\mathrm{S}^{d-1}} \min\left(1, \frac{v_d^{-1}}{\sqrt{\sum_{i=1}^d x_{d,i}^2 e^{2\lambda_i t}}}\right)^m dV_{\mathrm{S}^{d-1}}(\boldsymbol{x}).$$

Passing to spherical coordinates gives

$$\ll \int_0^\pi \min\left(1, \frac{v_d^{-1}}{\sqrt{e^{2\lambda_{\max}t}\cos^2\phi}}\right)^m \sin^{d-2}\phi\, d\phi \ll \int_0^\pi \min\left(1, \frac{v_d^{-1}}{\sqrt{e^{2\lambda_{\max}t}\sin^2\phi}}\right)^m d\phi.$$

We use the same approximations as before, giving (recall $m \geq 2$)

$$\ll \int_0^1 \min\left(1, \frac{v_d^{-1}}{\phi\, e^{\lambda_{\max}t}}\right)^m d\phi \leq \int_0^{e^{-\lambda_{\max}t}v_d^{-1}} d\phi + \int_{e^{-\lambda_{\max}t}v_d^{-1}}^\infty v_d^{-m}e^{-\lambda_{\max}tm}\phi^{-m}d\phi = O(e^{-\lambda_{\max}t}v_d^{-1}).$$

Using this bound in (28), together with $v_d^{-1} = v_1v_2...v_{d-1}$, gives

$$\int_{\mathcal{F}} |\widetilde{f}_n(M)\widetilde{g}_n(-M\Phi'_t)|\, d\mu'(M)$$
$$\ll e^{-\lambda_{\max}t}n^{-d}\|f\|_{S^{m,\infty}(\Gamma\backslash G)}\|g\|_{S^{m,\infty}(\Gamma\backslash G)} \int_0^\infty \int_0^{\frac{2}{\sqrt{3}}v_1}\cdots\int_0^{\frac{2}{\sqrt{3}}v_{d-2}} \min\left(1, v_1v_2...v_{d-1}\right)^m$$
$$\times \prod_{j=1}^{d-1} v_j^{2(j-d)}\, dv_{d-1}...dv_1.$$

In order to finish the proof, we require that the integral

$$\int_0^\infty \int_0^{\frac{2}{\sqrt{3}}v_1}\cdots\int_0^{\frac{2}{\sqrt{3}}v_{d-2}} \min\left(1, v_1v_2...v_{d-1}\right)^m \prod_{j=1}^{d-1} v_j^{2(j-d)}\, dv_{d-1}...dv_1 \tag{29}$$

is finite. We now make the substitution $e^{u_j} = v_j$, giving

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\alpha+u_1}\cdots\int_{-\infty}^{\alpha+u_{d-2}} \exp\left(m\min\left(0,\sum_{j=1}^{d-1}u_j\right)+\sum_{j=1}^{d-1}(2(j-d)+1)u_j\right)du_{d-1}...du_1, \tag{30}$$

where $\alpha = \log\frac{2}{\sqrt{3}}$. We now make another change of variables, letting $w_j = \sum_{i=1}^{j} u_i$. The bounds of integration in (30) give

$$\begin{aligned} &-\infty < w_1 < \infty \\ &-\infty < w_2 - w_1 \le \alpha + w_1 \\ &-\infty < w_3 - w_2 \le \alpha + (w_2 - w_1) \\ &\qquad\vdots \\ &-\infty < w_{d-1} - w_{d-2} \le \alpha + (w_{d-2} - w_{d-3}). \end{aligned} \tag{31}$$

We simplify this system of linear inequalities by repeated row operations. This implies, for $2 \le j \le d-1$,

$$-\infty < w_j \le \frac{j}{2}\alpha + \frac{j}{j-1}w_{j-1}. \tag{32}$$

We now reverse the system, giving

$$\begin{aligned} -\infty <& w_{d-1} < \infty \\ \tfrac{d-2}{d-1}w_{d-1} - \tfrac{d-2}{2}\alpha \le& w_{d-2} < \infty \\ \tfrac{d-3}{d-2}w_{d-2} - \tfrac{d-3}{2}\alpha \le& w_{d-3} < \infty \\ &\vdots \\ \tfrac{w_2}{2} - \tfrac{\alpha}{2} \le& w_1 < \infty. \end{aligned}$$

This gives the integral

$$\int_{-\infty}^{\infty}\exp(m\min(0,w_{d-1})-w_{d-1})\int_{\frac{d-2}{d-1}w_{d-1}-\frac{d-2}{2}\alpha}^{\infty}\exp(-2w_{d-2})\cdots\int_{\frac{w_2}{2}-\frac{\alpha}{2}}^{\infty}\exp(-2w_1)\,dw_1...dw_{d-1}.$$

Repeated evaluation of the innermost integral gives, for $j = 2, 3, \dots, d-2$:

$$\ll \int_{-\infty}^{\infty}\exp(m\min(0,w_{d-1})-w_{d-1})\cdots\int_{\frac{j}{j+1}w_{j+1}-\frac{j}{2}\alpha}^{\infty}\exp(-2\lambda_j w_j)\,dw_j...dw_{d-1},$$

where $\lambda_1 = 1$, and $\lambda_{j+1} = 1 + \frac{j}{j+1}\lambda_j$. By induction, $\lambda_j = \frac{j+1}{2}$, so by using the above bound for $j = d-2$ and once more evaluating the inner integral, we get

$$\ll \int_{-\infty}^{\infty}\exp(m\min(0,w_{d-1})-(d-1)\,w_{d-1})\,dw_{d-1},$$

which is finite for $m \ge d$. □

Remark 3. As in the two-dimensional case, for general $f, g \in S^{m,p}(\Gamma\backslash G)$, $\widehat{f}(\cdot,\mathbf{0})$ and $\widehat{g}(\cdot,\mathbf{0})$ are functions in $S^{m,p}(\Gamma'\backslash G')$. Exponential rate of mixing for $\mathrm{SL}(d,\mathbb{Z})\backslash\mathrm{SL}(d,\mathbb{R})$ is known from Oh [6] (see also e.g. Kleinbock and Margulis [4, 5]). Thus it seems that we should be able to obtain a precise exponential mixing result also without the assumption $\int_{\mathbb{T}^d} f\,d\boldsymbol{\xi} = 0$ (analogously to the way the corresponding results for the case $d = 2$ were obtained in Remark 1), however we have not yet worked this out.

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