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The Black-Scholes Equation and Formula

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Abstract

The purpose of this paper is to present fundamental arbitrage theory and its applications to pricing financial derivatives, in a way that does not require a previous knowledge of Stochastic Calculus or Measure theory. It starts off with an easy introduction to the financial market in Chapter 1 and tackles the tough tools, namely Stochastic calculus, in Chapter 2. We use this theory to derive the Black-Scholes equation and Black-Scholes formula.

Contents

Chapter 1

The Market

If you believe that the stock Eriksson B in six months would increase by 40 percent, what would you be willing to pay for a contract saying that in six months you can buy it for the same price as today?

In 1973 Fischer Black and Myron Scholes published the paper "The Pricing of Options and Corporate Liabilities" in the Journal of Political Economy, see [3]. It contains an equation which was going to become famous because it did something new. It describe how to create a risk free portfolio and also gave the explicit price for this portfolio. The real market is too complex to be modelled in its entirety so the models used were simplified, and thus did not give exact predictions on the real market.

Since then many papers have been published expanding the model. Often objects in the aforementioned model are extrapolated from the market and not always given a proper definition, for the reader unfamiliar with the market we will define the objects needed to develop an understanding of the model.

First when inspecting a financial market the most prominent objects are financial derivatives and assets. Financial derivatives is a broad categorisation of objects that are traded on a market and whose values are derived from some underlying assets.

Definition 1. *A* financial derivative *is an object on the market that can be bought or sold and is dependent on one or more assets.*

It follows from the definition that an asset is also a financial derivative although they are not usually classified as such. Some specific financial derivatives will be defined and used extensively later on. It is important to understand the assumptions going into developing the model, some of which we will introduce now.

Assumptions 1. *These assumptions explains how trading in the model works, since the model is an idealization of the real market.*

- *Short positions (negative financial derivatives) and fractional holdings are allowed.*
- *There is no cost for selling or buying assets or derivatives.*
- *The buying price is the same as the selling price.*
- *The market is liquid meaning it is always possible to buy and/or sell unlimited quantities on the market.*

Important assets are: A stock, it's an asset and modelled by a stochastic process.
A bond, is a risk-free asset modelled by a deterministic process with rate of return $r(t)$.

Note that bonds adds the possibility to borrow unlimited amounts by selling bonds short. A portfolio h is a collection of some stocks and bonds. A portfolio can be self-financing meaning there is no money put into the portfolio and all buying of new assets must be financed with the selling of other assets, this will be defined rigorous in Chapter 3. Other objects on the market, more often classified as financial derivatives are options, futures and forwards. In this paper we will only work with options.

Definition 2. *An option is a contract that gives the owner the right (but not the obligation) to buy or sell a derivative at a specified price called a striking price on or before a striking time T.*

Important options include European Put/Call options and American Put/Call options. These are the most common types of options. In this paper we will explore the European Call option.

Definition 3. *A European put/call option is an option to sell/buy a derivative at the exact time T.*

In contrary to a European option an American option can be exercised anytime between when the option is bought and time T. For a portfolio h we need something to track the value of that portfolio, we call that the value process $V(t, h)$. This makes it possible to define the most important assumption about the market, that the market is efficient, meaning it is free of arbitrage. This is a very basic condition, saying that it is not possible to make a risk-free profit.

Definition 4. *An* arbitrage possibility *on the market is a self-financed portfolio h such that:*

1. $V(0, h) = 0$.
2. $P(V(T, h) \geq 0) = 1$.

3. $P(V(T,h) > 0) > 0$.

The market is arbitrage free *if and only if there are no arbitrage possibilities.*

Now to the question posed in the beginning.

Example 1. *If you believe that the stock Eriksson B in six months would increase by 40 percent, what would you be willing to pay for a contract saying that in six months you can buy it for the same price as today?*

Today Eriksson B is say 80SEK.

Then this clearly is a European call option with strike price 80SEK and expiration date 6 months from now.

Chapter 2

Stochastic Calculus

This chapter will be devoted to the development of stochastic calculus. It is as the name suggests the calculus of stochastic processes and was first developed to determine rocket trajectories.
Recall that a stochastic process in continuous time is some function

$$X : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$$

where Ω is some probability space.

2.1 The Itô Integral

In this section we develop the stochastic integral and we begin by defining a special process that will be important. The presentation is based on [1].

Definition 5. *A stochastic process X is called a* Wiener process*, denoted W, if the following conditions hold.*

1. $W(0) = 0$.
2. *The process has independent increments, i.e. if $r < s \leq t < u$ then $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables. We will often use ΔW to denote such independent increments.*
3. *For $s < t$ the stochastic variable $W(t) - W(s)$ has the Normal distribution $N[0, t - s]$.*
4. *W has continuous trajectories.*

For the coming argument we will fix $\omega \in \Omega$ making

$$X : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$$

become

$$X : \mathbb{R} \times \{\omega\} \longrightarrow \mathbb{R}$$

with ω being unknown. To be able to continue we need a definition from measure theory. But since measure theory is outside the scoop of this paper it will be introduced in a heuristic way. For the reader well acquainted with measure theory the definition is the needed sigma algebra. This is needed so that parts of ω can be considered known, ω can be viewed as the infinite sequence of coin tosses and the following definition makes it possible to know a subsequence.

Definition 6. *The symbol F_t^X denotes "the information generated by X on the interval $[0,t]$". or alternatively "what has happened to X over the interval $[0,t]$". If, based upon observations of the trajectory $\{X(s); 0 \leq s \leq t\}$, it is possible to decide whether a given event A has occurred or not, we write this as*

$$A \in F_t^X$$

or say that "A is F_t^X-measurable". If the value of a given stochastic variable Z can be completely determined given observations of the trajectory $\{X(s); 0 \leq s \leq t\}$, then we also write

$$Z \in F_t^X.$$

If Y is a stochastic process such that we have

$$Y(t) \in F_t^X$$

for all $t \geq 0$ then we say that Y is adapted *to the* filtration $\{F_t^X\}_{t \geq 0}$.

To guarantee that a process can be integrated in a meaningful way we need to impose some restrictions.

Definition 7. *1. We say that the process g belongs to the class $\pounds^2[a,b]$ if the following conditions are satisfied:*

- $\int_a^b E[g^2(s)]ds < \infty$.
- *The process g is adapted to the F_t^W-filtration.*

2. We say that the process g belong to the class $\pounds^2$ if $g \in \pounds^2[0,t]$ for all $t > 0$.

An important observation, although not at all trivial is that for any process g

$$g \in \pounds^2[a,b] \Rightarrow g \in L^2[a,b] \qquad for\ any\ fixed \qquad \omega \in \Omega.$$

To prove this measure theory is needed (The Fubini Theorem) so it will not be done here. Now we begin defining the stochastic integral $\int_a^b g(s)dW(s)$ for $g \in \pounds^2[a,b]$. First suppose that g is simple, then there exist deterministic points in time $a = t_0 < t_1 < ... < t_n = b$ such that g is constant on each subinterval. This makes it natural to define the integral by

$$\int_a^b g(s)dW(s) = \sum_{k=0}^{n-1} g(t_k)[W(t_{k+1}) - W(t_k)].$$

For a general process $g \in \mathcal{L}^2[a,b]$ we approximate it with a sequence of simple processes $\{g_n\}_1^\infty$ such that $\int_a^b E[(g_n(s) - g(s))^2]ds \longrightarrow 0$ as $n \longrightarrow \infty$. Since $L^2[a,b]$ is complete (every Cauchy sequence converges) the integral $\int_a^b g_n(s)dW(s)$ converges to something in $L^2[a,b]$, and we let

$$\int_a^b g(s)dW(s) = \lim_{n\to\infty} \int_a^b g_n(s)dW(s).$$

There are some important properties of this integral.

Theorem 1. *Let g be a process such that g is adapted to the F_t^W-filtration and*

$$\int_a^b E\left[g^2(s)\right] ds < \infty.$$

Then the following holds

$$E\left[\int_a^b g(s)dW(s)\right] = 0,$$

and

$$E\left[\left(\int_a^b g(s)dW(s)\right)^2\right] = \int_a^b E\left[g^2(s)\right] ds.$$

Proof. The first part is pretty straight forward, giving

$$E\left[\int_a^b g(s)dW(s)\right] = E\left[\lim_{n\to\infty} \sum_{k=0}^{n-1} g_n(t_k)\left(W(t_{k+1}) - W(t_k)\right)\right] =$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} E\left[g_n(t_k)\right] E\left[W(t_{k+1}) - W(t_k)\right] = 0.$$

This holds since $g(t_k)$ is adapted to the filtration, meaning that it is only dependent on the behaviour of the wiener process on the interval $[0, t_k]$.

Since the Wiener increment is a forward increment dependent only on the interval $[t_k, t_{k+1}]$, the Wiener increment and $g(t_k)$ are independent. Last the expected values of the increments are 0.

Next we have

$$E\left[\left(\int_a^b g(s)dW(s)\right)^2\right] = Var\left[\int_a^b g(s)dW(s)\right] =$$
$$= Var\left[\lim_{n\to\infty}\sum_{k=0}^{n-1} g_n(t_k)\left(W(t_{k+1}) - W(t_k)\right)\right].$$

It can be shown that these stochastic variables are uncorrelated giving

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} Var\left[g_n(t_k)\left(W(t_{k+1}) - W(t_k)\right)\right].$$

We have already argued that the increment and $g_n(t_k)$ are independent, giving us

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\Big(E[g_n(t_k)]^2 Var[W(t_{k+1}) - W(t_k)] +$$
$$+ E[W(t_{k+1}) - W(t_k)]^2 Var[g_n(t_k)] +$$
$$+ Var[W(t_{k+1}) - W(t_k)] Var[g_n(t_k)]\Big).$$

$Var[W(t_{k+1}) - W(t_k)] = t_{k+1} - t_k$ and so

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\left(E[g_n(t_k)]^2\left(t_{k+1} - t_k\right) + Var[g_n(t_k)]\left(t_{k+1} - t_k\right)\right).$$

Taking the limit we get, by the Riemann-Stiltjes Integral

$$\int_a^b E[g(t)]^2 + Var[g(t)]dt.$$

The identity $Var[X] = E[X^2] - E[X]^2$ completes the proof giving

$$\int_a^b E[g(t)]^2 + E[g^2(t)] - E[g(t)]^2 dt = \int_a^b E[g^2(t)]dt.$$

□

2.2 Itô's Lemma

Now we present one of the most important results in Stochastic calculus. It was first stated, boldly, by Itô in one of his papers as a lemma.
Let X be a stochastic process and suppose that there exist some $a \in R$ and $\mu(t), \sigma(t) \in \{F_t^X\}_{t\geq 0}$ such that the following hold for all $t \geq 0$

$$X(t) = a + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s).$$

Further assume that we have a C^2-function

$$f : R_+ \times R \longrightarrow R$$

and define a new process Z by

$$Z(t) = f(t, X(t)).$$

The arguments of the function has been omitted in favour for readability.

Theorem 2 (Itô's Lemma)**.** *Assume that the process X satisfies the integral equation above and Z is defined as above. Then Z satisfies the following integral equation*

$$Z(t) = Z(0) + \int_0^t \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X^2}\right) ds + \int_0^t \sigma\frac{\partial f}{\partial X}dW.$$

Proof. This proof is based on the sketches in [1] and [2] First we divide the interval $[0, t]$ into n equal subintervals such that $0 = t_0 < t_1 < ... < t_n = t$. Then

$$f(t, X(t)) - f(0, X(0)) = \sum_{k=0}^{n-1} \left(f(t_{k+1}, X(t_{k+1})) - f(t_k, X(t_k))\right).$$

Using Taylor's theorem we get

$$f(t_{k+1}, X(t_{k+1})) - f(t_k, X(t_k)) = \frac{\partial f(t_k, X(t_k))}{\partial t}\Delta t + \frac{\partial f(t_k, X(t_k))}{\partial X}\Delta X_k +$$
$$\frac{1}{2}\frac{\partial^2 f(t_k, X(t_k))}{\partial t^2}\Delta t^2 + \frac{\partial^2 f(t_k, X(t_k))}{\partial t\partial X}\Delta t\Delta X_k + \frac{1}{2}\frac{\partial^2 f(t_k, X(t_k))}{\partial X^2}(\Delta X_k)^2 + Q_k$$

where Q_k is the remainder and $\Delta t = \frac{t}{n} = t_{k+1} - t_k$. Also since μ and σ are adapted to the filtration there is a n big enough such that

$$\Delta X_k = X(t_{k+1}) - X(t_k) =$$

$$= \int_{t_k}^{t_{k+1}} \mu(s)ds + \int_{t_k}^{t_{k+1}} \sigma(s)dW(s) = \mu(t_k)\Delta t + \sigma(t_k)\Delta W_k.$$

This imply that for a big enough n

$$(\Delta X_k)^2 = \mu^2(t_k)\left(\Delta t\right)^2 + 2\mu(t_k)\sigma(t_k)\Delta t\Delta W_k + \sigma^2(t_k)\left(\Delta W_k\right)^2.$$

Substituting these into the sum gives us

$$f(t, X(t)) - f(0, X(0)) = I_1 + I_2 + I_3 + \frac{1}{2}I_4 + \frac{1}{2}K_1 + K_2 + R$$

where

$$\begin{aligned}
I_1 &= \sum_{k=0}^{n-1} \frac{\partial f}{\partial t}(t_k, X(t_k))\Delta t \\
I_2 &= \sum_{k=0}^{n-1} \frac{\partial f}{\partial X}(t_k, X(t_k))\mu(t_k)\Delta t \\
I_3 &= \sum_{k=0}^{n-1} \frac{\partial f}{\partial X}(t_k, X(t_k))\sigma(t_k)\Delta W_k \\
I_4 &= \sum_{k=0}^{n-1} \frac{\partial^2 f}{\partial X^2}(t_k, X(t_k))\sigma^2(t_k)(\Delta W_k)^2 \\
K_1 &= \sum_{k=0}^{n-1} \left(\frac{\partial^2 f}{\partial X^2}(t_k, X(t_k))\mu^2(t_k) + \frac{1}{2}\frac{\partial^2 f(t_k, X(t_k))}{\partial t^2} + \frac{\partial^2 f(t_k, X(t_k))}{\partial t \partial X}\mu(t_k) \right) (\Delta t)^2 \\
K_2 &= \sum_{k=0}^{n-1} \left(\frac{\partial^2 f}{\partial t \partial X}(t_k, X(t_k))\sigma(t_k) + \frac{\partial^2 f}{\partial X^2}(t_k, X(t_k))\mu(t_k)\sigma(t_k) \right) \Delta t \Delta W_k \\
R &= \sum_{k=0}^{n-1} Q_k.
\end{aligned}$$

Now we have that

$$I_1 \longrightarrow \int_0^t \frac{\partial f}{\partial s}(s, X(s))ds \qquad I_2 \longrightarrow \int_0^t \frac{\partial f}{\partial X}(s, X(s))\mu(s)ds$$

$$I_3 \longrightarrow \int_0^t \frac{\partial f}{\partial x}(s, X(s))\sigma(s)dW(s)$$

by definition as $n \to \infty$.

$$I_4 \longrightarrow \int_0^t \frac{\partial^2 f}{\partial X^2}(s, X(s))\sigma^2(s)ds$$

is shown by naming $\frac{\partial^2 f}{\partial X^2}(t_k, X(t_k))\sigma^2(t_k) = a_k$ and evaluating

$$E\left[\left(\sum_{k=0}^{n-1} a_k(\Delta W_k)^2 - \sum_{k=0}^{n-1} a_k \Delta t\right)^2\right] = E\left[\left(\sum_{k=0}^{n-1} a_k\left((\Delta W_k)^2 - \Delta t\right)\right)^2\right] =$$

$$E\left[\sum_{k=0}^{n-1} a_k^2\left((\Delta W_k)^2 - \Delta t\right)^2 + 2\sum_{k\neq l} a_k a_l\left((\Delta W_k)^2 - \Delta t\right)\left((\Delta W_l)^2 - \Delta t\right)\right].$$

a_k, a_l, $(\Delta W_k)^2 - \Delta t$ and $(\Delta W_l)^2 - \Delta t$ are all independent, making the second sum disappear.

$$E\left[\sum_{k=0}^{n-1} a_k^2\left((\Delta W_k)^2 - \Delta t\right)^2\right] = \sum_{k=0}^{n-1} E\left[a_k^2\left((\Delta W_k)^4 - 2\Delta t(\Delta W_k)^2 + \Delta t^2\right)\right] =$$

$$\sum_{k=0}^{n-1} E\left[a_k^2\right]\left(3\Delta t^2 - 2\Delta t^2 + \Delta t^2\right) = 2\sum_{k=0}^{n-1} E\left[a_k^2\right]\Delta t^2 \to 0$$

by the coming argument. Proving that the sum converge in probability to the integral. For K_1 we call $\frac{\partial^2 f}{\partial X^2}(t_k, X(t_k))\mu^2(t_k) + \frac{1}{2}\frac{\partial^2 f(t_k, X(t_k))}{\partial t^2} + \frac{\partial^2 f(t_k, X(t_k))}{\partial t \partial X}\mu(t_k) = a(t_k)$ and use that $a(t)$ is bounded(everything in $a(t)$ can be approximated with bounded functions).

$$\begin{aligned}
|K_1| &\leq \sum_{k=0}^{n-1} |a(t_k)(\Delta t)^2| \\
&\leq \sup_{s\in[0,t]} |a(s)| \sum_{k=0}^{n-1} |(\Delta t)^2| \\
&= \sup_{s\in[0,t]} |a(s)||n\left(\frac{t}{n}\right)^2|
\end{aligned}$$

which goes to 0 as $n \to \infty$.
K_2 is similar to I_4 with $a_k = \frac{\partial^2 f}{\partial t \partial X}(t_k, X(t_k))\sigma(t_k) + \frac{\partial^2 f}{\partial X^2}(t_k, X(t_k))\mu(t_k)\sigma(t_k)$

$$
\begin{aligned}
&E\left[\left(\sum_{k=0}^{n-1} a_k \Delta t \Delta W_k\right)^2\right] = \\
&\sum_{k=0}^{n-1} E\left[(a_k)^2\right] E\left[\Delta W_k^2\right] \Delta t = \\
&\sum_{k=0}^{n-1} E\left[(a_k)^2\right] \Delta t^3 \to 0.
\end{aligned}
$$

Last but not least we have

$$
Q_k = o(|\Delta t|^2 + |\Delta X_k|^2) = o(|\Delta t|^2 + |\Delta t \Delta W_k| + |\Delta W_k|^2).
$$

And for any given ϵ there is a Δt small enough such that

$$
\sum_{k=0}^{n-1} |Q_k| \leq \sum_{k=0}^{n-1} \epsilon \left(|\Delta t|^2 + |\Delta t \Delta W_k| + |\Delta W_k^2|\right) = \sum_{k=0}^{n-1} \epsilon |\Delta t|^2 + \sum_{k=0}^{n-1} \epsilon |\Delta t \Delta W_k| + \sum_{k=0}^{n-1} \epsilon |\Delta W_k^2|.
$$

The first two sums we have shown goes to 0 faster than n goes to infinity and the last sum

$$
\sum_{k=0}^{n-1} \epsilon |\Delta W_k^2| \longrightarrow \int_0^t \epsilon dt = \epsilon t
$$

must hold for any ϵ. So

$$
R \longrightarrow 0 \quad as \quad n \longrightarrow \infty.
$$

The proof is thus completed.

□

2.3 Stochastic Differential Equations

Now we explore so called stochastic differential equations. Writing SDEs on the same form as ODEs and PDEs would require the formal time derivative of a Wiener process. However the Wiener process can be shown to be nowhere differentiable with probability 1, which poses a slight problem. To fix this we consider the integral form of the SDE namely

$$
X(t) = x_0 + \int_0^t \mu(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dW(s).
$$

The existence and uniqueness theorem for solutions to the SDE requires some inequalities that are outside the scope of this paper, so the proof will be omitted.

Theorem 3. *Suppose that there exist a constant K such that the following conditions are satisfied for all x, y and t*

$$||\mu(t,x) - \mu(t,y)|| \leq K||x-y||$$

$$||\sigma(t,x) - \sigma(t,y)|| \leq K||x-y||$$

$$||\mu(t,x)|| + ||\sigma(t,x)|| \leq K(1+||x||).$$

Then there exist a unique solution X to the SDE above, such that

1. *X is F_t^W-adapted.*

2. *X has continuous trajectories.*

3. *X is a Markov process.*

4. *There exist a constant C such that*

$$E[||X_t||^2] \leq Ce^{Ct}(1+||x_0||^2).$$

The fact that X is F_t^W-adapted implies that for every $\omega \in \Omega$, $W(t,\omega)$ is mapped to $X(t,\omega)$. This mapping is very complicated and makes solving SDEs explicitly almost impossible, but one can solve some simple examples.

Chapter 3

The Black-Scholes Model

Now we have the tools needed to develop the Black-Scholes model. The main reference is [1].

Definition 8. *Consider a financial market with vector price process $\hat{S}$ (this is the vector with all assets). A* contingent claim *with date of maturity (exercise date) T, also called a T-*claim*, is any stochastic variable $\chi \in F_T^{\hat{S}}$. A contingent claim χ is called a* simple claim *if it is of the form*

$$\chi = \Phi\left(\hat{S}(T)\right).$$

The function Φ is called the contract function*.*

From this point onwards we will only consider simple claims. On the market a claim is a contract that depends on Ŝ and the main problem is to determine a fair price on the claim (if it exist). $\Pi(t, \chi)$ is the price process of the claim χ. For $t = T$, χ is known and the price is exactly χ so $\Pi(T, \chi) = \chi$. For $t < T$, $\Pi(t, \chi)$ is unknown. Given N assets with values $Z_1(t), ..., Z_N(t)$ at time t a trading strategy is a N-dimensional stochastic process $(a_1(t), ..., a_N(t))$ that represent the allocations into the assets at time t. Then at time t we have

$$V(t,h) = \sum_{n=1}^{N} a_n(t) Z_n(t).$$

Now we can give the self-financing criteria in a formal statement.

Definition 9. *A portfolio is called* self-financing *if the value process $V(t,h)$ satisfies the condition*

$$V(t,h) = \sum_{n=1}^{N} \int_0^t a_n(s) dZ_n(t).$$

Lemma 1. *The rate of interest in an arbitrage free model must be unique.*

Proof. We denote the rate of interest r and suppose there exist some risk free derivative with return c. If $c < r$ then arbitrage can be achieved by selling the derivative short and buy bonds. If $c > r$ then arbitrage can be achieved by selling bonds short and buy the derivative. So for the model to be arbitrage free we must have $c = r$.

□

We need some more assumptions that we could not include earlier since they are a bit technical.

Assumptions 2. *We assume that*

1. *The derivative instrument in question can be bought and sold on a market.*
2. *The market is free of arbitrage.*
3. *The price process for the derivative asset is of the form*

$$\Pi(t, \chi) = F(t, S(t))$$

where F is some smooth function.

The last assumption, specific to the Black-Scholes model, is that the market consist of two assets satisfying

$$B(t) = \int_0^t r \cdot B(s)ds$$

$$S(t) = \int_0^t S(s)\mu(s, S(s))ds + \int_0^t S(s)\sigma(s, S(s))dW(s).$$

Where $B(t)$ is a riskfree asset with deterministic constant rate of interest r. By the assumption we have $\Pi(t, \Phi(S(T))) = F(t, S(t))$, so by the Itô formula we get

$$\Pi(t, \Phi(S(T))) = F(0, S(0)) + \int_0^t \left(\frac{\partial F}{\partial s} + S\mu \frac{\partial F}{\partial S} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 F}{\partial^2 S} \right) ds + \\ + \int_0^t S\sigma \frac{\partial F}{\partial S} dW.$$

Let us consider a self-financing portfolio consisting of the underlying stock and bond. To get the right proportions we use the trading strategy $(\alpha(t), \beta(t))$ to form the replicating portfolio $F(t, S(t)) = \alpha(t)S(t)+\beta(t)B(t)$. The self financing assumption implies that

$$F(t, S(t)) = F(0, S(0)) + \int_0^t \alpha(s)dS(s) + \int_0^t \beta(s)dB(s) =$$

$$= F(0, S(0)) + \int_0^t \alpha(s)\mu(s)S(s)ds + \int_0^t \alpha(s)\sigma(s)S(s)dW(s) + \int_0^t \beta(s)rB(s)ds.$$

Since $\Pi(t, \Phi(S(T))) = F(t, S(t))$ we get

$$\int_0^t \left(\frac{\partial F}{\partial s} + S\mu\frac{\partial F}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 F}{\partial^2 S}\right) ds + \int_0^t S\sigma\frac{\partial F}{\partial S}dW =$$

$$= \int_0^t \alpha(s)\mu(s)S(s)ds + \int_0^t \alpha(s)\sigma(s)S(s)dW(s) + \int_0^t \beta(s)rB(s)dt.$$

Collecting integrals we get

$$\int_0^t \left(\frac{\partial F}{\partial s} + S\mu\frac{\partial F}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 F}{\partial S^2} - \alpha\mu S - \beta rB\right) ds +$$

$$+ \int_0^t \left(S\sigma\frac{\partial F}{\partial S} - S\sigma\alpha\right) dW = 0.$$

Now we see that if we let $\alpha = \frac{\partial F}{\partial S}$ the Stochastic integral becomes 0 and we have a risk-free portfolio. Then we get

$$\int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2} + \beta rB\right) dt = 0.$$

It is evident that this choice of portfolio also removes the μ process from the integral equation. This is the ground breaking discovery made by Black and Scholes. Since μ is almost impossible to estimate, the discovery that it is not needed to price certain portfolios was the birth of financial mathematics.

From the strategy equation $F(t) = \beta(t)B(t) + \alpha(t)S(t)$ we get that $B(t) = \frac{F(t)-\alpha(t)S(t)}{\beta(t)}$. Substituting this into the equation give us

$$\int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF\right) ds = 0.$$

Dropping the integral and adding the initial condition $F(T,S) = \Phi(S)$ give us the famous Black-Scholes Equation.

$$\begin{aligned}&\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF = 0\\ &F(T,S) = \Phi(S).\end{aligned}$$

3.1 Feynman-Kač

In this section we attempt to solve the PDE starting with the special case $r = 0$. Then the equation looks like

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2(t,x)x^2\frac{\partial^2 F}{\partial x^2}(t,x) = 0$$

$$F(T,x) = \Phi(x).$$

We use a stochastic representation formula to find a solution to the equation. First we assume that there exist a solution F to the equation. Then we fix a point in time t and a point in space x. Define the stochastic process X on the time interval $[t,T]$ as the solution to the SDE

$$X(s) = x + \int_t^s \sigma(\tau, X(\tau))X(\tau)dW(\tau).$$

Applying the Itô formula to the process $F(s, X(s))$ gives us

$$\begin{aligned}&F(T,X(T)) = F(t,X(t)) +\\ &+\int_t^T \left(\frac{\partial F}{\partial s}(s,X(s)) + \frac{1}{2}\sigma^2(s,X(s))X^2(s)\frac{\partial^2 F}{\partial X^2}(s,X(s))\right) ds +\\ &+\int_t^T \sigma(s,X(s))X(s)\frac{\partial F}{\partial X}(s,X(s))dW(s).\end{aligned}$$

The time integral vanishes since F satisfies the PDE, and if the process $\sigma(s,X(s))X(s)\frac{\partial F}{\partial X}(s,X(s))$ is adapted to the F_t^W-filtration the stochastic integral will disappear under the expected value since $F(t,x)$ can be considered constant (we fixed t and x) and $E_{t,x}\left[F(T,X(T))\right] = E_{t,x}\left[\Phi(X(T))\right]$, we get that

$$F(t,x) = E_{t,x}\left[\Phi(X(T))\right].$$

Let $r \in \mathbb{R}$. Then we have the equation

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2(t,x)x^2\frac{\partial^2 F}{\partial x^2}(t,x) + rx\frac{\partial F}{\partial x}(t,x) - rF(t,x) = 0$$

$$F(T,x) = \Phi(x).$$

Consider the function $G(t,x) = e^{-rt}F(t,x)$. Multiply the equation with e^{-rt} and use $G(t,x)$ to rewrite it as

$$\frac{\partial G}{\partial t}(t,x) + \frac{1}{2}\sigma^2(t,x)x^2\frac{\partial^2 G}{\partial x^2}(t,x) + r\frac{\partial G}{\partial x}(t,x) = 0$$

$$G(T,x) = \Phi(x)e^{rt}.$$

This is similar to what we had before. However there is another term in the equation. This problem can be fixed by changing the SDE previously defined to be

$$X(s) = x + \int_t^s \sigma(\tau, X(\tau))X(\tau)dW(\tau).$$

For $r \in \mathbb{R}$ we instead define $X(s)$ as

$$X(s) = x + \int_t^s rX(\tau)d\tau + \int_t^s \sigma(\tau, X(\tau))X(\tau)dW(\tau).$$

This will make sure that the time integral in the Itô formula disappears. Also if the process $\sigma(s,X(s))X(s)\frac{\partial G}{\partial X}(s,X(s))$ is adapted to the F_t^W-filtration the stochastic integral will disappear under the expected value.

We get

$$G(t,x) = E_{t,x}\left[\Phi(X(T))e^{rT}\right].$$

Changing back to $F(t,x)$ and moving some exponents around we get the Feynman-Kač stochastic representation formula

$$F(t,x) = e^{-r(T-t)}E_{t,x}\left[\Phi(X(T))\right].$$

This is an explicit pricing formula for certain derivatives, it certainly has its limitations but is used extensively by traders to evaluate a fair price on, for example, European Call Options. The Feynman-Kač stochastic representation formula can be evaluated numerically with for example a Monte-Carlo method. Another way to numerically evaluate the function would be directly from the PDE with possibly a finite difference method.

3.2 Black-Scholes formula

The SDE

$$X(s) = x + \int_t^s rX(\tau)d\tau + \int_t^s \sigma(\tau, X(\tau))X(\tau)dW(\tau)$$

is very special and can be explicitly solved. Consider the function $f(x) = \ln(x)$ and the SDE above, then by the Itô formula we get

$$\ln(X(t)) = \ln(x) + \int_t^s r - \frac{1}{2}\sigma^2(\tau, X(\tau))d\tau + \int_t^s \sigma(\tau, X(\tau))dW(\tau)$$

$$X(t) = x \cdot \exp\left(\int_t^s r - \frac{1}{2}\sigma^2(\tau, X(\tau))d\tau + \int_t^s \sigma(\tau, X(\tau))dW(\tau)\right).$$

When σ is constant this yields the nice expression

$$X(t) = x \cdot \exp\left(\left(r - \frac{1}{2}\sigma^2\right)(s-t) + \sigma(W(s) - W(t))\right)$$

$Y = \left(r - \frac{1}{2}\sigma^2\right)(s-t) + \sigma(W(s) - W(t))$ can be viewed as a normally distributed stochastic variable.

The Feynman-Kač formula for such processes can be writen as

$$F(t, s) = e^{-r(T-t)} \int_{-\infty}^{\infty} \Phi(xe^y)f(y)dy$$

where f is the density function for the stochastic variable Y.

For an arbitrary contract function Φ this integral can not be solved explicitly, so we will solve it for a European call option where

$$\Phi(x) = \max[x - K, 0],$$

K is the striking price of the contract. Normalizing Y and using the standarized normal variable Z the integral becomes

$$\int_{-\infty}^{\infty} \max\left[xe^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K, 0\right]\phi(z)dz$$

where ϕ is the density of the $N[0,1]$ distribution.

The integrand is 0 when $z < z_0$

$$z_0 = \frac{\ln\left(\frac{K}{x}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

$$\int_{z_0}^{\infty} \left(x e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K \right) \phi(z)dz$$
$$= \int_{z_0}^{\infty} x e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} \phi(z)dz - \int_{z_0}^{\infty} K\phi(z)dz.$$

The second integral is obviously $K \cdot P(Z \geq z_0) = K \cdot P(Z \leq -z_0)$, the symmetry in the normal distribution is responsible for the change of inequality. In the first integral we write out the density function and complete the square.

$$\int_{z_0}^{\infty} x e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} \phi(z)dz$$
$$= \frac{x e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)}}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{\sigma\sqrt{T-t}z - \frac{z^2}{2}} dz$$
$$= \frac{x e^{r(T-t)}}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{-\frac{1}{2}\left(z-\sigma\sqrt{T-t}\right)^2} dz.$$

We recognize the integral as the density of the $N[\sigma\sqrt{T-t}, 1]$ distribution.

$$\frac{x e^{r(T-t)}}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{-\frac{1}{2}\left(z-\sigma\sqrt{T-t}\right)^2} dz$$
$$= x e^{r(T-t)} \cdot P(Z - \sigma\sqrt{T-t} \geq z_0) = x e^{r(T-t)} \cdot P(Z \leq -z_0 + \sigma\sqrt{T-t}).$$

That completes the famous Black-Scholes formula.

Theorem 4. *The price of a European call option with strike price K and time of maturity T is given by $\Pi(t) = F(t, S(t))$, where*

$$F(t,s) = sP(Z \leq -z_0 + \sigma\sqrt{T-t}) - e^{-r(T-t)}P(Z \leq -z_0).$$

Here Z is $N(0,1)$ distributed and $z_0 = \frac{\ln\left(\frac{K}{s}\right) - \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$.

Example 2. *We notice from the derivation that we do not have everything needed in Example 1 to calculate the correct price for the contract.*

The interest rate is easy enough to check, let us say it is 1 *percent. The volatility on the other hand can be hard to estimate, we estimate it to be* $\sigma = 0.4$.

Now we have everything we need to compute the value of the contract from Example 1 we start by calculating z_0 *for*

$$\begin{aligned} \sigma &= 0.4 \\ K &= 80SEK \\ s &= 80SEK \\ r &= 0.01 \\ T - t &= 0.5 \end{aligned}$$

give us $z_0 = 0.1237$. *Plugging everything into*

$$F(t,x) = xP(Z \leq -z_0 + \sigma\sqrt{T-t}) - e^{-r(T-t)}P(Z \leq -z_0)$$

and we have

$$F(0,80) = 80P(Z \leq -0.1237 + 0.4\sqrt{0.5}) - e^{-0.01(0.5)}P(Z \leq -0.1237) \approx 9.1755SEK.$$

So according to the model you should be willing to pay about $9SEK$ *for this option.*

Note that the expected increase in the price of the stock does not affect the price of the option. Please compare the proof of the Black-Scholes theorem where the drift μ, as we saw, disappears.

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