The Hilbert Transform

Axel Husin

Examensarbete i matematik, 15 hp
Handledare och examinator: Wolfgang Staubach

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THE HILBERT TRANSFORM

AXEL HUSIN

Abstract. We will investigate various aspects of the Hilbert transform, study some of its properties and also point out how it can be used in connection to the study of the norm convergence of Fourier series.

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1. Introduction

This text is supposed to introduce the reader to the Hilbert transform, reading this will make it easier to later understand more advanced texts on the subject. We will study the Hilbert transform on the real line and on the unit circle. However, by the Riemann mapping theorem it is possible to do similar constructions on more general domains. First we will remind the reader of some background material like the Lebesgue integral, Fourier series and the Fourier transform followed by a short study of harmonic functions that will motivate the definition of the Hilbert transform.

Then we define the Hilbert transform on the real line and study some of its main properties. Among the properties we will see how it relates to the Fourier transform and show that it is an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

To close this text we finally study the Hilbert transform on the unit circle and show how it can be used to prove norm convergence of Fourier series in $L^p(T)$ for $1 < p < \infty$.

2. Background material

2.1. Measure spaces. We will use the Lebesgue integral in this text. So we remind the reader about the definition of a measure space and the Lebesgue integral. However, this is a short introduction and is not intended to give the reader a full understanding of measure theory, for the full story and the details omitted here, see e.g. [6]. The French mathematician Henri Lebesgue (1875-1941) introduced his integral in 1904 and also laid the foundation of measure theory.

We start with the definition of a measure space. Let $X$ be a set and $\Sigma \subset P(X)$ a $\sigma$-algebra over $X$, that means that $\Sigma$ is non empty and closed under the complement and countable unions. The sets in $\Sigma$ are called measurable sets. Next we need to assign a measure to all the sets in $\Sigma$, the measure of the sets in $\Sigma$ is denoted by $\mu(E)$ (where $E \in \Sigma$). Formally we let $\mu : \Sigma \to \mathbb{R} \cup \{\infty\}$ be a function satisfying

$$
\mu(E) \geq 0 \quad \mu(\emptyset) = 0
$$

\begin{equation}
\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k),
\end{equation}

where the sets $E_k$ are disjoint. The triple $(X, \Sigma, \mu)$ is then called a measure space.

From the equations (2.1) we can derive some properties for the measure function, for example the following:

**Theorem 2.1.** Let $(X, \Sigma, \mu)$ be a measure space and let $A \subset B$ be measurable sets, then $\mu(B \setminus A) = \mu(B) - \mu(A)$ and $\mu(A) \leq \mu(B)$.

**Proof.** The set $B \setminus A$ is measurable and $A \cup (B \setminus A) = B$ is a disjoint union, so by (2.1) it yields that $\mu(A) + \mu(B \setminus A) = \mu(B)$ and hence $\mu(B \setminus A) = \mu(B) - \mu(A)$. And since $\mu(B \setminus A) \geq 0$ we get $\mu(A) \leq \mu(B)$. \qed

A very important example of a measure space which we will use throughout the text is the measure set $(\mathbb{R}, \sigma, \lambda)$ with $\lambda$ denoting the Lebesgue measure.
First define the outer Lebesgue measure $\lambda^*$ by

$$\lambda^*(E) = \inf \left\{ \sum_{k=1}^{\infty} (b_k - a_k), \ E \subset \bigcup_{k=1}^{\infty} [a_k, b_k] \right\}. $$

This makes $\lambda^*$ a translation invariant measure on $\mathbb{R}$ and one has $\lambda([a, b]) = b - a$, but $\lambda^*$ will by the axiom of choice not satisfy property 3 in (2.1), this will be shown in a moment. To remedy this situation we define $\sigma \subset P(\mathbb{R})$ by

$$\sigma = \{ E \mid \lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c) \text{ for every } A \subset \mathbb{R} \}$$

and we let $\lambda(E) = \lambda^*(E)$ for every $E \subset \sigma$. We skip the proof that $\sigma$ actually is a $\sigma$-algebra and that $\lambda$ satisfies the properties in equation (2.1).

In a similar way we can define the measure space $([0, 2\pi], \sigma, \lambda)$ which also will be important later on.

Now we answer the question why not every subset of $\mathbb{R}$ is measurable. The reason for this is that the outer Lebesgue measure $\lambda^*$ will not satisfy the properties in (2.1). Actually, it is impossible to find any translation invariant measure function $\mu$ satisfying (2.1) and $\mu([a, b]) = b - a$. Assume on the contrary that there is such a function $\mu$ and consider for example the set $\mathbb{R}/\mathbb{Q}$ of equivalence classes, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$ are in the same equivalence class if and only if $x - y \in \mathbb{Q}$. Each equivalence class $E$ has a representative in $[0, 1]$, since if we take any representative $r \in E$ we can add a rational number $q \in \mathbb{Q}$ such that $r + q \in [0, 1]$ and $r + q \in E$. Now use the axiom of choice to pick one representative in the interval $[0, 1]$ from every equivalence class, and collect them in a set $V$. The set $V$ we constructed is called a Vitali set. We note that

$$[0, 1] \subset \bigcup_{-1 \leq k \leq 1} V + k \subset [-1, 2]$$

and by taking measures of the sets we get

$$1 \leq \sum_{k \in \mathbb{Q}, -1 \leq k \leq 1} \mu(V + k) = \sum_{k \in \mathbb{Q}, -1 \leq k \leq 1} \mu(V) \leq 3,$$

which is a contradiction since the infinite sum must satisfy $\sum \mu(V) = 0$ or $\sum \mu(V) = \infty$. This shows that $\lambda^*$ does not satisfy property 3 in (2.1).

2.2. **Definition of the integral.** Let $(X, \Sigma, \mu)$ be a measure space. A function $f : X \to \mathbb{R}$ is called measurable if $f^{-1}(U)$ is measurable set for every open set $U \subset \mathbb{R}$. We state without proof some simple but important properties of measurable functions.

**Theorem 2.2.**

1. The characteristic function $\chi_E$ of a measurable set $E$ is measurable.
2. Simple functions $f = \sum_{k=1}^{n} c_k \chi_{E_k}$, where $E_k$ are measurable sets, are measurable.
3. Piecewise continuous functions are measurable.
4. If $f$ and $g$ are measurable, then $\lambda f$, $|f|^p$, $f + g$, $fg$, $f/g$, $\min(f, g)$ and $\max(f, g)$ are measurable, where $\lambda \in \mathbb{R}$ and $p > 0$ (the quotient part $f/g$ only holds if $g$ is nonzero).
5. If $\{f_k\}$ is sequence of measurable functions and $f_k \to f$ pointwise, then $f$ is measurable.
We define the integral in the following three steps.
1. Let \( f \) be a nonnegative simple function, \( f = \sum_{k=1}^{n} c_k \chi_{E_k} \) where \( c_k > 0 \) and \( E_k \) are measurable. Then we define
\[
\int_X f d\mu = \sum_{k=1}^{n} c_k \lambda(E_k).
\]
The third property in equation (2.1) ensures that this is a well defined.
2. Let \( f : X \to \mathbb{R} \) be a nonnegative measurable function, then we define
\[
\int_X f d\mu = \sup \left\{ \int_X g d\mu, \ 0 \leq g \leq f \text{ and } g \text{ is simple} \right\}.
\]
It can be shown that the definitions in step 1 and 2 coincide for the integral of a nonnegative simple function.
3. Let \( f : X \to \mathbb{R} \) be a measurable function where \( \int_X |f| d\mu < \infty \), then we call \( f \) integrable and define
\[
\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu,
\]
where \( f^+ = \max(f, 0) \) and \( f^- = -\min(f, 0) \).

Complex-valued functions can be dealt with by separating it into its real and imaginary part. We say that a function \( f : X \to \mathbb{C} \) is measurable if \( \text{Re}(f) \) and \( \text{Im}(f) \) are measurable, and we say that \( f \) is integrable if \( \text{Re}(f) \) and \( \text{Im}(f) \) are integrable, and in that case we define \( \int_X f d\mu = \int_X \text{Re}(f) d\mu + i \int_X \text{Im}(f) d\mu \).

2.3. **Properties of the integral.** Some of the most important properties of the Lebesgue integral are given in the following theorem which we state without proof.

**Theorem 2.3.**

1. If \( f \) and \( g \) are integrable, then \( \int_X \lambda f d\mu = \lambda \int_X f d\mu \) and \( \int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu \).
2. If \( f \) is integrable and \( g \) is measurable with \( |g| \leq f \), then \( \int_X g d\mu \leq \int_X f d\mu \).
3. If \( f \) is integrable, then \( \int_X f d\mu \leq \int_X |f| d\mu \).
4. If \( f \) is integrable, then \( \int_X |f| d\mu = 0 \) if and only if \( f = 0 \) almost everywhere.

From now on, the statement \( f = 0 \) almost everywhere means that \( f \) is nonzero on a set of measure 0.

Now we present some theorems (without proofs), concerning the integral of the pointwise limit of a sequence of functions. These important theorems clarify some of the advantages of the Lebesgue integral as compared to the Riemann integral.

**Theorem 2.4 (Fatou’s lemma).** Let \( \{f_k\} \) be a sequence of nonnegative integrable functions converging pointwise to \( f \) and \( \int_X f_k d\mu \leq M \) for some \( M < \infty \). Then \( f \) is integrable and \( \int_X f d\mu \leq M \).

**Theorem 2.5 (Monotone convergence theorem (MCT)).** Let \( \{f_k\} \) be an increasing sequence of nonnegative measurable functions converging pointwise to \( f \). Then \( f \) is measurable and \( \int_X f d\mu = \lim_{k \to \infty} \int_X f_k d\mu \).
Theorem 2.6 (Dominated convergence theorem (DCT)). Let \{f_k\} be a sequence of integrable functions converging pointwise to \(f\) and \(|f_k| \leq g\) for some integrable function \(g\). Then \(f\) is integrable and \(\int_X f \, d\mu = \lim_{k \to \infty} \int_X f_k \, d\mu\).

In what follows we will denote by \(\int f(x) \, dx\), the integral of the function \(f\) with respect to the Lebesgue measure \(\lambda\) on \(\mathbb{R}\). We also state the following theorem, concerning interchange of order of integration in double integrals, only in the case of the Lebesgue measure.

Theorem 2.7 (Fubini-Tonelli). Let \(f(x,y)\) be function on \(\mathbb{R}^2\) such that either of the following integrals,
\[
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |f(x,y)| \, dx \right) \, dy
\]
or
\[
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |f(x,y)| \, dy \right) \, dx
\]
converge. Then one has
\[
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x,y) \, dx \right) \, dy = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x,y) \, dy \right) \, dx.
\]

2.4. \(L^p\) spaces. Since we will define the Hilbert transform on \(L^p\) spaces, this section covers some basic material concerning these spaces which will be used later on.

Let \(1 \leq p < \infty\) and \((X, \Sigma, \mu)\) denote a measure space, then we define \(L^p(X)\) as the space of complex-valued measurable functions \(f : X \to \mathbb{C}\) where \(\int_X |f|^p \, d\mu < \infty\). It can be shown that \(L^p(X)\) is a vector space by noting that if \(f \in L^p(X)\), then \(|\lambda f|^p = |\lambda|^p |f|^p\) so \(\lambda f\) is also in \(L^p(X)\), and if \(f, g \in L^p(X)\), then \(f + g \in L^p(X)\) by the Minkowski inequality 2.12. We also equip \(L^p(X)\) with the seminorm \(\|f\|_p = (\int_X |f|^p \, d\mu)^{1/p}\) where we can use the Minkowski inequality 2.12 to see that the triangle inequality holds.

The seminormed vector space \(L^\infty(X)\) is defined in a slightly different way. We define the space \(L^\infty(X)\) as the set of essentially bounded measurable functions, that is measurable functions bounded up to a set of measure zero. And we use the seminorm \(\|f\|_\infty = \inf\{C \geq 0 : |f(x)| \leq C\text{ for almost every }x\}\).

If we let \(N\) be the subspace of \(L^p(X)\) including all functions on \(X\) which are zero almost everywhere, we define the quotient space \(L^p(X) = L^p(X)/N\). This yields that \(f, g \in L^p(X)\) are equal to each other iff they are equal almost everywhere.

It can be shown that for \(1 \leq p \leq \infty\), \(L^p(X)\) is a Banach space, that is a complete normed vector space, this fact will be used many times throughout this text.

If \(X\) is a space of finite measure, then \(f \in L^p(X)\) implies \(f \in L^{p'}(X)\) for \(p' \leq p\), that is since \(|f|^{p'} < (|f| + 1)^{p'} \leq (|f| + 1)^p\) and since both \(f\) and the constant function 1 are in \(L^p(X)\), then \((|f| + 1)^p\) is integrable since \(L^p(X)\) is a vector space. However, if \(X\) does not have finite measure then the implication above does not hold. As an example we can take the function \(f(x) = 1/(1 + |x|)\), which is in \(L^2(\mathbb{R})\) but not in \(L^1(\mathbb{R})\).

Definition 2.8. One says that the sequence \(f_k \in L^p(X)\) converges in the norm or in the mean to a function \(f\), if, for every \(\epsilon > 0\), one has for large enough \(k\)
\[
\int_X |f_k - f|^p \, d\mu < \epsilon.
\]
Since $L^p(X)$ is a Banach space, it follows that $f$ will also belong to $L^p(X)$. In this connection we also have the following result.

**Lemma 2.9.** Let $\{f_k\}$ be a sequence of functions in $L^p(X)$ such that $f_k \to f$ pointwise and $f_k \to g$ in the $L^p$ norm. Then $f = g$ almost everywhere.

**Proof.** First we note that $|f_k - g|^p \to |f - g|^p$ pointwise. Since $f_k \to g$ in $L^p$, Fatou’s lemma yields that

$$\int_X |f - g|^p d\mu < \epsilon.$$ 

Since this is valid for every $\epsilon > 0$ we get

$$\int_X |f - g|^p d\mu = 0.$$ 

And by property 4 in Theorem 2.3 we get $|f - g|^p = 0$ almost everywhere and hence $f = g$ almost everywhere.

Some inequalities that will be important later on are the Hölder and the Minkowski inequalities.

**Theorem 2.10** (Hölder’s inequality). Let $f \in L^p(X)$ and $g \in L^q(X)$ where $1/p + 1/q = 1$ and $1 \leq p, q \leq \infty$, then

$$\int_X |fg| d\mu \leq ||f||_p ||g||_q.$$ 

**Proof.** If $p = \infty$ and $q = 1$, then $|fg| \leq ||f||_\infty |g|$ almost everywhere so $\int_X |fg| d\mu \leq ||f||_\infty \int_X |g| d\mu = ||f||_\infty ||g||_1$, similarly if $p = 1$ and $q = \infty$. Now assume $1 < p, q < \infty$. We start with the special case $||f||_p = ||g||_q = 1$. We can estimate the integrand $fg$ with Young’s inequality, which will be proved later:

$$|f(t)g(t)| \leq \frac{|f(t)|^p}{p} + \frac{|g(t)|^q}{q}.$$ 

Taking the integral of both sides gives

$$\int_X |fg| d\mu \leq \frac{||f||_p^p}{p} + \frac{||g||_q^q}{q} = 1 = ||f||_p ||g||_q.$$ 

Now let’s move on to the general case. If $||f||_p = 0$ or $||g||_q = 0$ the result is trivial since we have $fg = 0$ almost everywhere (by property 4 in Theorem 2.3) so the integral on the left hand side will vanish. Assume $||f||_p > 0$ and $||g||_q > 0$ and apply the version of Hölder’s inequality we just proved to the functions $f/||f||_p$ and $g/||g||_q$ to get

$$\int_X \frac{f}{||f||_p^p} \frac{g}{||g||_q^q} d\mu \leq 1.$$ 

Hölder’s inequality follows by multiplying (2.2) with $||f||_p ||g||_q$. 

In the proof above we used Young’s inequality, which says that for nonnegative $a$ and $b$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $1/p + 1/q = 1$ and $1 < p, q < \infty$. 

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One proof goes something like this. If \( b = 0 \) the inequality is trivial, so assume \( b > 0 \) and divide both sides by \( b^q \) to get
\[
\frac{a}{b^{q-1}} \leq \frac{a^p}{pb^q} + \frac{1}{q}.
\]
Now let \( t = a/b^q \), that gives \( 0 < t < \infty \) and \( t^p = a^p/b^{p(q-1)} = a^p/b^q \) by using the relation \( 1/p + 1/q = 1 \). The inequality can now equivalently be written as
\[
\frac{t^p}{p} - t + \frac{1}{q} \geq 0.
\]
We can verify the inequality by examining the differentiable function
\[
h(t) = t^{p/p} - t + 1/q.
\]
We have \( h(0) = 1/q \) and \( \lim_{t \to \infty} h(t) = \infty \) and \( h'(t) = t^{p-1} - 1 \), so \( h(1) = 0 \) is the global minimum of \( h \).

Remark 2.11. The special case of \( p = q = 2 \) in Hölder’s inequality is called Cauchy-Schwarz inequality.

Theorem 2.12 (Minkowski’s inequality). Let \( f, g \in L^p(X) \) where \( 1 \leq p \leq \infty \), then
\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p.
\]

Proof. If \( p = 1 \) the result follows directly by taking the triangle inequality under the integral sign and if \( p = \infty \) we can again use the triangle inequality to get \( |f(x) + g(x)| \leq \|f\|_\infty + \|g\|_\infty \) almost everywhere, and hence \( \|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \). Now assume \( 1 < p < \infty \) and let \( q \) be defined by \( 1/p + 1/q = 1 \) and use Hölder’s inequality to get
\[
\|f + g\|_p^p = \int_X |f + g|^p d\mu = \int_X |f + g|^p |f + g|^{q-1} d\mu
\]
\[
\leq \int_X (|f| + |g|)(f + g)^{p-1} d\mu
\]
\[
= \int_X |f||f + g|^{p-1} d\mu + \int_X |g||f + g|^{p-1} d\mu
\]
\[
\leq (\|f\|_p + \|g\|_p) (\int_X |f + g|^{q(p-1)} d\mu)^{1/q}
\]
\[
= (\|f\|_p + \|g\|_p)\|f + g\|_p^{p-1}.
\]
Dividing both sides by \( \|f + g\|_p^{p-1} \) gives the Minkowski inequality. \( \square \)

2.5. Convolutions. The convolution is one of the most fundamental operations in mathematical analysis.

Definition 2.13. The convolution of the functions \( f \) and \( g \) is defined by
\[
f * g(x) = \int_{-\infty}^{\infty} f(t)g(x-t) dt.
\]
It can easily be seen that the convolution is commutative i.e. \( f * g(x) = g * f(x) \). This can be shown by a simple change of variables in the definition above.
One also has the following special case of the so called Young’s inequality.
Theorem 2.14. If \( f \in L^p(\mathbb{R}) \), \( 1 \leq p \leq \infty \) and \( g \in L^1(\mathbb{R}) \), then \( f \ast g(x) \) is in \( L^p(\mathbb{R}) \) and
\[
\|f \ast g\|_p \leq \|f\|_p \|g\|_1.
\]

Proof. The cases \( p = 1 \) and \( p = \infty \) are simple, so assume \( 1 < p < \infty \).
Let \( q \) be defined by \( 1/p + 1/q = 1 \). We get
\[
|f \ast g(x)| = \left| \int_{-\infty}^{\infty} f(t)g(x-t)dt \right|
\]
(2.3)
\[
\leq \int_{-\infty}^{\infty} |f(t)||g(x-t)|^{1/p}|g(x-t)|^{1/q}dt.
\]
By using Hölder’s inequality on (2.3) we get
\[
|f \ast g(x)| \leq \left( \int_{-\infty}^{\infty} |f(t)|^p|g(x-t)|dt \right)^{1/p} \left( \int_{-\infty}^{\infty} |g(x-t)|dt \right)^{1/q}.
\]
(2.4)
If we take \( L^p \)-norm of both sides of (2.4) and use Fubini’s theorem we get
\[
\|f \ast g\|_p \leq \|g\|_1^{1/q} \left( \int_{-\infty}^{\infty} |f(t)|^p \int_{-\infty}^{\infty} |g(x-t)|dxdt \right)^{1/p}
\]
\[
= \|g\|_1^{1/q} \|g\|_1^{1/p} \|f\|_p
\]
\[
= \|g\|_1 \|f\|_p.
\]
\[\square\]

2.6. Operator norm. An interesting property to study for linear operators is the operator norm. We will later show that the Hilbert transform is a bounded operator. The operator norm is defined as follows.

Definition 2.15. Let \( L : X \rightarrow Y \) be a linear operator between two normed vector spaces, then we define the operator norm of \( L \) by
\[
\|L\|_{op} = \sup \left\{ \frac{\|Lx\|}{\|x\|} : x \in X, x \neq 0 \right\}.
\]
If \( \|L\|_{op} < \infty \) we say that \( L \) is bounded.

3. Introduction to Fourier Analysis

3.1. Fourier series on \( L^p \). The Fourier series of a function is defined as follows.

Definition 3.1. Let \( f \in L^p(\mathbb{T}) \). Then the Fourier series of \( f \) is defined by
\[
\lim_{n \to \infty} S_n f(x) = \lim_{n \to \infty} \sum_{k=-n}^{n} \hat{f}(k)e^{ikx},
\]
where \( S_n f \) are called the partial sums of \( f \) and the Fourier coefficients \( \hat{f}(k) \) are defined by
\[
\hat{f}(k) = \frac{1}{2\pi} \int_{0}^{2\pi} f(t)e^{-ikt}dt.
\]
To simplify the notation we will set \( \sum_{-\infty}^{\infty} = \lim_{n \to \infty} \sum_{-n}^{n} \).

The Fourier coefficients will always exist since \( |f(t)e^{-ikt}| = |f(t)| \), where \( |f(t)| \) is in \( L^1 \) since it is in \( L^p \) and \( \mathbb{T} \) has finite measure. So the the main question is whether the Fourier series converges or not. That will be studied in the last section of this paper.

We define a trigonometric polynomial as a finite Fourier series of the form

\[
\sum_{k=-n}^{n} c_k e^{ikx}.
\]

The remarkable fact about the trigonometric polynomials is that all functions in \( L^p(\mathbb{T}) \), for \( 1 \leq p < \infty \), can be approximated by trigonometric polynomials.

**Theorem 3.2.** For \( 1 \leq p < \infty \), the set of trigonometric polynomials are a dense subset of \( L^p(\mathbb{T}) \), and if \( f \) is a trigonometric polynomial the Fourier series of \( f \) will converge to \( f \) in \( L^p(\mathbb{T}) \).

**Proof.** We can see that the set of trigonometric polynomials are dense in \( L^p(\mathbb{T}) \) by noting that the algebra of the trigonometric polynomials satisfy the condition of the Stone-Weierstrass theorem, so they are uniformly dense in \( C(\mathbb{T}) \), and \( C(\mathbb{T}) \) is in turn dense in \( L^p(\mathbb{T}) \) for \( 1 \leq p < \infty \) see e.g. [7]. This proves the first part of the theorem.

Now assume that if \( f \) is a trigonometric polynomial

\[
f(x) = \sum_{k=-n}^{n} c_k e^{ikx}.
\]

If we calculate the Fourier coefficients we get

\[
\hat{f}(k) = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \sum_{j=-n}^{n} c_j e^{ijt} \right) e^{-ikt} dt
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \left( \sum_{j=-n}^{n} c_j e^{i(j-k)t} \right) dt
\]

\[
= \frac{1}{2\pi} \sum_{j=-n}^{n} \int_{0}^{2\pi} c_j e^{i(j-k)t} dt = c_k,
\]

where we interpret \( c_k = 0 \) for \( |k| > n \). The Fourier series of \( f \) will clearly converge to \( f \) since all terms in the series with index larger than \( n \) will be zero, and therefore \( S_k f = f \) for \( k \geq n \).

### 3.2. Parsevals formula.

Among the \( L^p(\mathbb{T}) \) spaces the space \( L^2(\mathbb{T}) \) stands to be a Hilbert space. This means that one has an inner product in this space that defines the norm and the inner product enables us to measure angles and makes the geometry of the \( L^2(\mathbb{T}) \) spaces considerably easier than that of the other \( L^p(\mathbb{T}) \) spaces. As a result, it turns out that \( L^2(\mathbb{T}) \) is perfectly suited for the study of the Fourier series.

**Definition 3.3.** The inner product on \( L^2(\mathbb{T}) \) is defined by

\[
\langle f, g \rangle = \int_{0}^{2\pi} f(x) g(t) dt.
\]
One can easily see that this is well defined since $2|f(x)g(x)| \leq |f(x)|^2 + |g(x)|^2$. It is straight forward to show that (3.1) satisfies the conditions for being an inner product. Let $e_n(x) = e^{inx}$, $n \in \mathbb{Z}$. Then with the above inner product we have that $\langle e_n, e_m \rangle$ (where $n, m \in \mathbb{Z}$) is 0 if $n \neq m$ and is $2\pi$ if $n = m$, in other words the functions $e_n$ (where $n \in \mathbb{Z}$) is an orthogonal set with respect to the inner product (3.1). Therefore we can consider the Fourier coefficients $\hat{f}(n) = \langle f, e_n \rangle / \langle e_n, e_n \rangle$ as the coefficient $\hat{f}(n)$ in the projection $\hat{f}(n)e_n$ of $f$ onto the subspace generated by $e_n$.

**Remark 3.4.** Some authors define the inner product with a factor $1/2\pi$, however we define it without to make the norm in $L^2(\mathbb{T})$ we defined earlier, coincide with the induced norm from the inner product $\|f\|_2 = \sqrt{\langle f, f \rangle}$.

**Lemma 3.5.** Assume that $f \in L^2(\mathbb{T})$, and that $S$ is any trigonometric polynomial of degree at most $n$ with $n \in \mathbb{Z}$. Then $\langle f - S_n f, S \rangle = 0$ and $\|S_n f - f\|_2 \leq \|S - f\|_2$.

**Proof.** $S(t)$ can be written as $S(t) = \sum_{-n}^{n} c_k e^{ikt}$. Then

$$\langle f - S_n f, S \rangle = \left\langle f - \sum_{k=-n}^{n} \frac{\langle f, e_k \rangle e_k}{2\pi}, \sum_{k=-n}^{n} c_k e_k \right\rangle$

(3.2)

$$= \sum_{k=-n}^{n} c_k \langle f, e_k \rangle - \sum_{k=-n}^{n} \frac{\langle f, e_k \rangle c_k \langle e_k, e_k \rangle}{2\pi} = 0.$$

This proves the first part.

By using (3.2) we get $\langle f - S_n f, S - S_n f \rangle = 0$, and hence by Pythagoras theorem we have

$$\|S - f\|_2^2 = \|S - S_n f\|_2^2 + \|S_n f - f\|_2^2 \geq \|S_n f - f\|_2^2.$$

The second part in the Lemma follows by taking the square root of both sides. □

Now we can prove the norm convergence of the Fourier expansion of $L^2(\mathbb{T})$ functions and also the Parseval identity.

**Theorem 3.6.** Let $f \in L^2(\mathbb{T})$, then $\lim_{n \to \infty} S_n f = f$ in the $L^2$ norm.

**Proof.** Let $\epsilon > 0$ be arbitrarily. Choose a trigonometric polynomial $P$ such that $\|f - P\|_2 < \epsilon$ and let $N$ be the degree of $P$. If $n \geq N$ we get by Lemma 3.5, $\|f - S_n f\|_2 \leq \|f - S_N f\|_2 \leq \|f - P\|_2 < \epsilon$, and the statement follows.

**Theorem 3.7** (Parseval’s identity). Let $f \in L^2(\mathbb{T})$, then the sequence $\{\hat{f}(k)\}$ is square summable and

$$2\pi \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 = \int_{0}^{2\pi} |f(t)|^2 dt.$$

**Proof.** By Lemma 3.5 $\langle f - S_n f, S_n f \rangle = 0$, so we can use Pythagoras theorem to get $\|f\|_2^2 - \|S_n f\|_2^2 = \|f - S_n f\|_2^2$. According to Theorem 3.6 $\|f - S_n f\|_2 \to 0$ so the result follows if we make the identifications

$$\|f\|_2^2 = \int_{0}^{2\pi} |f(t)|^2 dt.$$
and, by the orthogonal properties,
\[ \|S_n f\|_2^2 = \int_0^{2\pi} \left( \sum_{k=-n}^{n} \hat{f}(k) e^{ikt} \right) \left( \sum_{k=-n}^{n} \hat{f}(k) e^{-ikt} \right) dt = 2\pi \sum_{k=-n}^{n} |\hat{f}(k)|^2. \]

\[ \square \]

3.3. The Fourier transform on \( L^1 \). Another important linear operator which will be used in the study of the Hilbert transform is the Fourier transform. Let us review some of the basic properties of the Fourier transform.

Throughout this section, we will use the notation \( \tau_a \) for the translation operator defined by \( \tau_a f(x) = f(x-a) \), and \( R \) for the reflection operator \( Rf(x) = f(-x) \).

We start by defining the Fourier transform for functions in \( L^1(\mathbb{R}) \).

**Definition 3.8.** Let \( f \in L^1(\mathbb{R}) \). The Fourier transform of \( f \) is defined by
\[ \mathcal{F}f(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \]

This is well defined for all \( \omega \) since the integral is absolutely convergent.

**Example 3.9.** Calculate the Fourier transform of
\[ f_a(x) = \begin{cases} \frac{1}{2\pi} \left( 1 - \frac{|x|}{a} \right), & \text{if } |x| \leq a, \\ 0, & \text{if } |x| > a. \end{cases} \]

**Solution:**
\[ \hat{f}_a(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \]
\[ = \frac{1}{\pi} \int_0^a \left( 1 - \frac{t}{a} \right) \cos(\omega t) dt \]
\[ = -\frac{1}{\pi} \int_0^a \frac{1}{a\omega} \sin(\omega t) dt \]
\[ = \frac{1}{\pi a\omega^2} (-\cos(a\omega) + \cos(0)) \]
\[ = \frac{1}{\pi a\omega^2} (1 - \cos(a\omega)) \]
\[ = \frac{2 \sin^2(a\omega/2)}{\pi a\omega^2}. \]

**Remark 3.10.** Observe that \( \hat{f}_a \) in Example 3.9 is in \( L^1(\mathbb{R}) \) and by letting \( u = a\omega/2 \) and using a trick from the residue calculus we can calculate
\[ \int_{-\infty}^{\infty} \hat{f}_a(t) dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} du = 1. \]

By the dominated convergence theorem we get for all \( \delta > 0 \)
\[ \lim_{a \to \infty} \int_{|t| \geq \delta} \hat{f}_a(t) dt = 0. \]

We say that the sequence \( \{\hat{f}_a\} \) is an approximation of identity since it satisfies (3.3) and (3.4).
Theorem 3.11. If \( f \in L^1(\mathbb{R}) \) is differentiable and \( f' \in L^1(\mathbb{R}) \), then \( \hat{f}'(\omega) = i\omega \hat{f}(\omega) \).

Proof. Since \( f \in L^1(\mathbb{R}) \) we can choose sequences \( \{a_n\} \) and \( \{b_n\} \) such that \( \lim_{n \to \infty} a_n = -\infty \), \( \lim_{n \to \infty} b_n = \infty \) and \( \lim_{n \to \infty} |f(a_n)| = \lim_{n \to \infty} |f(b_n)| = 0 \). By partial integration we get

\[
\hat{f}'(\omega) = \lim_{n \to \infty} \int_{a_n}^{b_n} f'(t) e^{-i\omega t} dt = \lim_{n \to \infty} \left( [f(t) e^{-i\omega t}]^{b_n}_{a_n} - i\omega \int_{a_n}^{b_n} f(t) e^{-i\omega t} dt \right) = i\omega \hat{f}(\omega). \]

□

There is a simple formula for the Fourier transform of a convolution.

Theorem 3.12. Let \( f \in L^1(\mathbb{R}) \) and \( g \in L^1(\mathbb{R}) \), then

\[ \hat{f} \ast \hat{g}(\omega) = \hat{f}(\omega) \hat{g}(\omega). \]

Proof. We can use Fubini’s theorem since \( f \ast g \in L^1(\mathbb{R}) \).

\[
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(t) g(x-t) dt \right) e^{-i\omega x} dx = \int_{-\infty}^{\infty} f(t) \left( \int_{-\infty}^{\infty} g(x-t) e^{-i\omega x} dx \right) dt = \int_{-\infty}^{\infty} f(t) \left( \int_{-\infty}^{\infty} g(u) e^{-i\omega (u+t)} du \right) dt = \int_{-\infty}^{\infty} f(t) \hat{g}(\omega) e^{-i\omega t} dt = \hat{f}(\omega) \hat{g}(\omega). \]

□

3.4. The inversion theorem. We start first with the following lemma:

Lemma 3.13 (Hatmoving lemma). Let \( f, g \in L^1(\mathbb{R}) \), then

\[ \int_{-\infty}^{\infty} \hat{f}(x) g(x) dx = \int_{-\infty}^{\infty} f(x) \hat{g}(x) dx. \]

Proof.

\[
\int_{-\infty}^{\infty} \hat{f}(t) g(t) dt = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x) e^{-itx} dx \right] g(t) dt = \int_{-\infty}^{\infty} f(x) \left[ \int_{-\infty}^{\infty} g(t) e^{-itx} dt \right] dx = \int_{-\infty}^{\infty} f(x) \hat{g}(x) dx.
\]

The change of integration order is allowed by Fubini’s theorem since

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x) g(t) e^{-itx}| dx dt = \int_{-\infty}^{\infty} |f(x)| dx \int_{-\infty}^{\infty} |g(t)| dt < \infty. \]

□
The following lemma concerning density and continuity will be quite useful for us.

**Lemma 3.14.**

(a) Let $1 \leq p < \infty$. Then the space $C_0$ of continuous functions with compact support on $\mathbb{R}$ is dense in $L^p(\mathbb{R})$.

(b) Let $1 \leq p < \infty$. Then $\lim \|r_h f - f\|_p = 0$.

**Proof.** (a) follows from Theorem 1.3 in [8] and (b) follows from Proposition 1.4 in [8]. □

**Remark 3.15.** One can in fact show that the space $C^k_0$, $k = 0, 1, 2, \ldots, \infty$ of $k$ times continuously differentiable functions with compact support on $\mathbb{R}$ is dense in $L^p(\mathbb{R})$, $1 \leq p < \infty$, see e.g. [8].

**Theorem 3.16.** Assume $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then $\mathcal{F}^{-1} \mathcal{F} f = f$ almost everywhere. The inverse Fourier transform $\mathcal{F}^{-1}$ is defined by

$$\mathcal{F}^{-1} f(x) = \frac{1}{2\pi} R\mathcal{F}(x) = \frac{1}{2\pi} \hat{f}(-x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega)e^{ix\omega}d\omega.$$  

**Proof.** Let

$$k_a(x) = \begin{cases} \frac{1}{2\pi} \left(1 - \frac{|x|}{a}\right), & \text{if } |x| \leq a, \\ 0, & \text{if } |x| > a. \end{cases}$$  

Then, by Example 3.9,

$$\hat{k}_a(t) = \frac{2\sin^2(at/2)}{\pi at^2}.$$  

Let $f \in L^1(\mathbb{R})$. By Lemma 3.13 and the fact that $\hat{f(x-t)}(\omega) = \hat{f}(-\omega)e^{-ix\omega}$ (where the Fourier transform on the left hand side is taken with respect to the variable $t$) we get

$$\int_{-\infty}^{\infty} f(x-t)\hat{k}_a(t)dt = \int_{-\infty}^{\infty} \hat{f}(-t)e^{-itx}k_a(t)dt = \int_{-\infty}^{\infty} \hat{f}(t)e^{itx}k_a(t)dt.$$  

By the dominant convergence theorem the right hand side tends as $a \to \infty$ pointwise to $\mathcal{F}^{-1} \mathcal{F} f(x)$, and by Fubini’s theorem the left hand side tends to $f$ in $L^1$ since

$$\|f * \hat{k}_a - f\|_1 \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x-t) - f(x)|\hat{k}_a(t)dx.$$  

Moreover $\|\tau_{2n/a} f - f\|_1 \leq 2\|f\|_1$ and $\|\tau_{2n/a} f - f\|_1 \to 0$ (Lemma 3.14 part (b)), so we can use the dominated convergence theorem to see that $\|f * \hat{k}_a - f\|_1 \to 0$. This implies, by Lemma 2.9 that $\mathcal{F}^{-1} \mathcal{F} f = f$ almost everywhere. □

As a consequence of this we have the important Plancherel theorem.
**Theorem 3.17.** Let \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \), then \( \|\hat{f}\|_2^2 = 2\pi\|f\|_2^2 \), or in integral notation
\[
\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |f(t)|^2 dt.
\]

**Proof.** The proof of this uses a similar technique as the proof of Theorem 3.16 so we start with defining the functions \( k_a \) as in equation (3.5). Let \( g = f * \hat{Rf} \), then by Theorem 3.12, \( \hat{g} = \hat{f} \hat{Rf} = |\hat{f}|^2 \). We can also see that \( g \) is uniformly continuous since
\[
|g(x) - g(y)| = \left| \int_{-\infty}^{\infty} f(t)(f(t-x) - f(t-y))dt \right|
\leq \int_{-\infty}^{\infty} |f(t)||f(t-x) - f(t-y)|dt
\leq \|f\|_2 \|\tau_x f - \tau_y f\|_2
= \|f\|_2 \|\tau_{x-y} f - f\|_2,
\]
where we have used Cauchy-Schwarz inequality. Now letting \( |x - y| \to 0 \), Lemma 3.14 part (b) yields that \( |g(x) - g(y)| \to 0 \).

By Lemma 3.13 we get
\[
\int_{-\infty}^{\infty} g(t)\hat{k}_a(t)dt = \int_{-\infty}^{\infty} |\hat{f}(t)|^2 \hat{k}_a(t)dt.
\]

The right hand side tends as \( a \to \infty \) to \( \frac{1}{2\pi} \|\hat{f}\|_2^2 \) by the monotone convergence theorem. Moreover the left hand side tends to \( g(0) = \|f\|_2^2 \). That is since \( g \) is continuous at 0 so that for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( |g(t) - g(0)| < \epsilon/2 \) for all \( |t| < \delta \), and we get for big enough \( a \) that
\[
\left| \int_{-\infty}^{\infty} g(t)\hat{k}_a(t)dt - g(0) \right| \leq \int_{|t| \leq \delta} |g(t) - g(0)|\hat{k}_a(t)dt
\leq \int_{|t| \leq \delta} |g(t)|\hat{k}_a(t)dt + \int_{|t| \geq \delta} 2|g| \max(\hat{k}(a))dt
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

This works for all \( \epsilon > 0 \) so we must have \( \|\hat{f}\|_2^2 = 2\pi\|f\|_2^2 \).

\[\square\]

### 3.5. The Fourier transform on \( L^2 \)

Now we define the Fourier transform for functions in \( L^2(\mathbb{R}) \). Since the integral in definition 3.8 is not absolutely convergent for all functions in \( L^2(\mathbb{R}) \) we need to make the definition in terms of sequences.

**Definition 3.18.** Let \( f \in L^2(\mathbb{R}) \). Then since \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) is dense in \( L^2(\mathbb{R}) \) there is a sequence of functions \( f_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) converging to \( f \) in \( L^2 \)-sense. The Fourier transform of \( f \) is defined by
\[
(3.7) \quad \hat{f} = \lim_{n \to \infty} \hat{f}_n,
\]
where the limit is to be taken in \( L^2 \).
Since \( \{f_n\} \) is a Cauchy sequence one can show that \( \{\widehat{f}_n\} \) is also a Cauchy sequence by
\[
\|\widehat{f}_n - \widehat{f}_m\|_2^2 = \|f_n - f_m\|_2^2 = 2\pi \|f_n - f_m\|_2^2,
\]
so the limit in (3.7) will always exist. Now we show that the limit (3.7) is unique. Assume both \( \{f_n\} \) and \( \{g_n\} \) are sequences of \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) functions converging to \( f \) in \( L^2 \) and set \( \widehat{f} = \lim_{n \to \infty} \widehat{f}_n \) and \( \widehat{g} = \lim_{n \to \infty} \widehat{g}_n \), then
\[
\|\widehat{f} - \widehat{g}\|_2 \leq \|\widehat{f} - \widehat{f}_n\|_2 + \|\widehat{f}_n - \widehat{g}_n\|_2 + \|\widehat{g}_n - \widehat{g}\|_2
\]
\[= \|\widehat{f} - \widehat{f}_n\|_2 + \sqrt{2\pi} \|f_n - g_n\|_2 + \|\widehat{g}_n - \widehat{g}\|_2
\]
\[\leq \|\widehat{f} - \widehat{f}_n\|_2 + \sqrt{2\pi} \|f_n - f\|_2 + \sqrt{2\pi} \|f - g_n\|_2 + \|\widehat{g}_n - \widehat{g}\|_2,
\]
where all the terms go to zero, so \( \widehat{f} = \widehat{g} \).

**Remark 3.19.** For functions in \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) this new definition coincides with definition 3.8 since we can get convergence to such functions by a constant sequence.

Now it is a simple task to extend the inversion theorem, Plancherel theorem and the convolution theorem to \( L^2(\mathbb{R}) \).

**Theorem 3.20.** Let \( f \in L^2(\mathbb{R}) \), then
\[
\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |f(t)|^2 dt.
\]

**Proof.** Let \( \{f_n\} \) be a sequence of functions in \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) that converges to \( f \). Then by the continuity of the \( L^2 \)-norms (see [8])
\[
\|\hat{f}\|_2 = \lim_{n \to \infty} \|\hat{f}_n\|_2 = \sqrt{2\pi} \lim_{n \to \infty} \|f_n\|_2 = \sqrt{2\pi} \|f\|_2.
\]

**Theorem 3.21.** Assume \( f \in L^2(\mathbb{R}) \), then \( \mathcal{F}^{-1}\mathcal{F} f = f \) almost everywhere.

**Proof.** Let \( \{f_n\} \) be a sequence of \( C_0^2(\mathbb{R}) \) functions such that \( f_n \to f \). The existence of such a sequence follows from Remark 3.15. Then, by Theorem 3.11, \( \hat{f}_n(\omega) = -\omega^2 \hat{f}_n(\omega) \) so \( |\hat{f}_n(\omega)| \leq \|\hat{f}_n\|_\infty/\omega^2 \) and, hence, \( \hat{f}_n \in L^1(\mathbb{R}) \). We get
\[
\|\frac{1}{2\pi} R \hat{f} - f\|_2 \leq \|\frac{1}{2\pi} R \hat{f} - \frac{1}{2\pi} R \hat{f}_n\|_2 + \|f_n - f\|_2 = \|f - f_n\|_2 + \|f_n - f\|_2,
\]
where the right hand side tends to zero.

**Remark 3.22.** Since the Fourier transform commutes with reflection we also have
\[
\mathcal{F}\mathcal{F}^{-1} f = \mathcal{F} \frac{1}{2\pi} R \mathcal{F} f = \mathcal{F} \frac{1}{2\pi} \mathcal{F} \mathcal{F} f = \mathcal{F}^{-1}\mathcal{F} f = f.
\]

**Theorem 3.23.** Let \( f \in L^2(\mathbb{R}) \) and \( g \in L^1(\mathbb{R}) \), then
\[
\mathcal{F}[f * g] = \mathcal{F} f \mathcal{F} g.
\]

**Proof.** Let \( f_n \) be a sequence of \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) converging to \( f \) in \( L^2 \) sense. Then
\[
\|\mathcal{F}[f * g] - \mathcal{F} f \mathcal{F} g\|_2
\]
\[\leq \|\mathcal{F}[f * g] - \mathcal{F}[f_n * g]\|_2 + \|\mathcal{F}[f_n * g] - \mathcal{F} f_n \mathcal{F} g - \mathcal{F} f \mathcal{F} g\|_2
\]
\[= \sqrt{2\pi} \|f_n * g\|_2 + 0 + \|\mathcal{F} f_n - f\|_2 \mathcal{F} g\|_2
\]
\[\leq \sqrt{2\pi} \|f - f_n\|_2 \|g\|_1 + \sqrt{2\pi} \|f_n - f\|_2 \|\mathcal{F} g\|_\infty,
\]
where the right hand side tends to zero.
4. Relation to harmonic functions

4.1. Cauchy type integrals. In this section, we shall start by reviewing some facts from complex analysis and the theory of harmonic functions. To this end, we need to briefly discuss Cauchy type integrals, that is functions of the form

\( \Phi(z) = \frac{1}{2\pi i} \int_\gamma \frac{\phi(\zeta)}{\zeta - z} d\zeta, \)

where \( \gamma \) is a smooth curve and \( \phi \) is a complex valued function defined on \( \gamma \).

Let \( \gamma \) be a closed curve and \( D \) the domain inside \( \gamma \), and let \( \phi \) be an holomorphic function on some open set containing \( D \cup \gamma \). Then by the Cauchy integral formula and Cauchy integral theorem \( \Phi(z) = \phi(z) \) if \( z \in D \) and \( \Phi(z) = 0 \) if \( z \notin D \cup \gamma \). It is interesting to note that even if \( \gamma \) is not necessarily closed or if \( \phi \) is just continuous on \( \gamma \), then the function \( \Phi \) behaves nicely by the following theorem.

**Theorem 4.1.** Let \( \gamma \) be a finite smooth curve and \( \phi \) be a continuous function on \( \gamma \). Then the function \( \Phi \) in (4.1) is a holomorphic function on \( \mathbb{C} \setminus \gamma \) and

\[ \Phi'(z) = \frac{1}{2\pi i} \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^2} d\zeta. \]

**Proof.** Consider the following difference for small enough \(|h|\):

\[
\left| \frac{\Phi(z + h) - \Phi(z)}{h} \right| \leq \frac{1}{2\pi} \left| \int_\gamma \frac{\phi(\zeta)}{h(\zeta - h)(\zeta - z)^2} d\zeta \right|
\]

where \( m \) is the maximum of \(|\phi|\) on \( \gamma \) and \( d \) is half of the distance from \( z \) to \( \gamma \) and \( l \) is the length of \( \gamma \). The expression clearly tends to zero as \( h \to 0 \). \( \Box \)

It is interesting to investigate what happens with (4.1) if \( z \) approaches a point \( w \) on \( \gamma \) which is not an endpoint. It turns out that in some situations there will exist a limit, but the limit might be different depending on if we approach the curve from the left or from the right. If we approach the curve from the left we will denote the limit by \( \Phi^+(w) \), and if the curve is approached from the right we denote the limit by \( \Phi^-(w) \). The values \( \Phi^+(w) \) and \( \Phi^-(w) \) are given by **Sokhotzki-Plemelj jump theorem**. Before we state **Sokhotzki-Plemelj jump theorem** we fix our notation for a disk of radius \( R \) and define the principal value of an integral.

**Definition 4.2.** Let \( x_0, y_0 \in \mathbb{R} \) and define \( \mathbb{D}_R(x_0, y_0) \) by

\[ \mathbb{D}_R(x_0, y_0) = \{ (x, y) : |(x - x_0, y - y_0)| < R \}. \]
Definition 4.3. Let $\gamma$ be a smooth curve and $\phi$ be a complex-valued function defined on $\gamma$ with a singularity at the point $z$. Then we define the principal value integral by

$$\text{PV} \int_\gamma \phi(\zeta) d\zeta = \lim_{\epsilon \to 0} \int_{\gamma_\epsilon} \phi(\zeta) d\zeta,$$

where $\gamma_\epsilon$ is the part of $\gamma$ outside of $D_\epsilon(z)$.

Theorem 4.4 (Sokhotzki-Plemelj jump theorem). Let $\gamma$ be a finite smooth curve and $\phi$ be a complex-valued function defined on $\gamma$ and satisfying the Hölder condition

$$|\phi(x) - \phi(y)| \leq C|x - y|^\alpha,$$

for some $0 < \alpha \leq 1$ and $C > 0$. Let

$$\Phi(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\phi(\zeta)}{\zeta - z} d\zeta$$

and let $w$ be a point on $\gamma$ which is not an endpoint. Then both sides of the following equality will exist and be equal.

$$\Phi^+(w) = \frac{1}{2\pi i} \text{PV} \int_{\gamma} \frac{\phi(\zeta)}{\zeta - w} d\zeta \pm \frac{\phi(w)}{2}.$$

For the proof of Sokhotzki-Plemelj jump theorem we refer the reader to [5].

4.2. Introduction to harmonic functions. A harmonic function $f(x, y)$ is a function satisfying the Laplace equation $f_{xx} + f_{yy} = 0$. The Dirichlet problem in an open domain $\Omega$ is the problem of finding a function $f(x, y)$ which is harmonic in $\Omega$ and continuous in $\overline{\Omega}$ (the closure of $\Omega$) and takes specific values on $\partial \Omega$ (the boundary of $\Omega$). The following theorem shows that if the boundary of $\Omega$ is a piecewise smooth closed curve and if a solution to Dirichlet’s problem exists, then the solution is unique.

Theorem 4.5. Let $\Omega$ be an open domain where the boundary is a piecewise smooth closed curve. If $f(x, y)$ and $g(x, y)$ are harmonic in $\Omega$, continuous on $\overline{\Omega}$ and $f(x, y) = g(x, y)$ on $\partial \Omega$, then $f(x, y) = g(x, y)$ in $\overline{\Omega}$.

Proof. Let $h(x, y) = f(x, y) - g(x, y)$. Then $h$ is harmonic in $\Omega$ and is equal to 0 on $\partial \Omega$. Let $F = (-h h_y, h h_x)$. Since a harmonic function is actually smooth, Greens theorem tells us that

$$\oint_{\partial \Omega} F \cdot d\mathbf{r} = \iint_{\Omega} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy$$

$$= \iint_{\Omega} \left( \left( \frac{\partial h}{\partial x} \right)^2 + \left( \frac{\partial h}{\partial y} \right)^2 \right) dxdy + \iint_{\Omega} \left( h \frac{\partial^2 h}{\partial x^2} + h \frac{\partial^2 h}{\partial y^2} \right) dxdy.$$

The integral on the left hand side vanishes since $F$ is zero on $\partial \Omega$, and the right most integral vanishes since $h$ is harmonic. So we can conclude that

$$\iint_{\Omega} \left( \left( \frac{\partial h}{\partial x} \right)^2 + \left( \frac{\partial h}{\partial y} \right)^2 \right) dxdy = 0$$
and therefore that \( h_x \) and \( h_y \) are constant 0 on \( \Omega \) which means that \( h(x, y) = C \) (constant) on \( \Omega \). But \( h \) is continuous on \( \overline{\Omega} \) and \( h(x, y) = 0 \) on \( \partial \Omega \), so the constant \( C \) must be zero. This means that \( f(x, y) = g(x, y) \) on \( \overline{\Omega} \). \( \square \)

4.3. A first glance at the Hilbert transform. From a historical point of view, the Hilbert transform originated in the work of David Hilbert on integral equations and boundary value problems in 1905.

The Hilbert transform of a continuous function \( f \) defined on the boundary of a simply connected open domain \( \Omega \) can be thought of as the values on the boundary of \( \Omega \), of the harmonic conjugate to the solution of the Dirichlet problem with boundary value \( f \). By the existence of solution of the Dirichlet’s problem (see e.g. [1]), the solution will exist. And since we are working in a simply connected domain the harmonic conjugate to the solution will always exists, but will only be unique up to a constant. To make the Hilbert transform unique we have to choose how to set this constant. However we will not define the Hilbert transform in this way since we would like to be able to transform functions for which a solution to Dirichlet’s problem doesn’t exist.

In this section we will find formulas for the Hilbert transform on the unit circle and the real line.

4.4. Poisson formula for the unit disk. First we solve the Dirichlet problem on a disk of radius \( R \) centered at the origin, and to make it easier for us we assume that there exist a solution which is harmonic in the closed disk with radius \( R \) centered at the origin (this means that we assume the existence of a harmonic function in an open domain slightly larger then the disk of radius \( R \) centered at the origin). In the actual theorem we don’t assume the existence of the solution and we only require the solution to be continuous on the closed disk of radius \( R \) (not harmonic), but this variant of the theorem is harder to prove. For the proof of the full theorem the interested reader is refereed to [1].

**Theorem 4.6.** Let \( f(x, y) \) be real and harmonic on the closed disk \( \overline{D}_R(0, 0) \). Then the values of \( f \) in \( D_R(0, 0) \) are given by

\[
(4.4) \quad f(re^{i\theta}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{R^2 - 2Rr \cos(\theta - t) + r^2} dt.
\]

**Proof.** Since \( f \) is harmonic on the simply connected closed disk \( \overline{D}_R(0, 0) \), \( f \) is the real part of an holomorphic function \( h(z) = f(z) + ig(z) \), with \( g(0) = 0 \), this makes \( h \) unique. By the Cauchy integral formula and Cauchy integral theorem we get

\[
(4.5) \quad h(z) = \frac{1}{2\pi i} \oint_C \frac{h(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i} \oint_C \frac{h(\zeta)d\zeta}{\zeta - z^*},
\]

where \( C \) is the positively oriented circle \(|\zeta| = R\) and \( z^* = R^2/\zeta \) is the reflection of \( z \) in \( C \). If we set \( z = re^{i\theta} \) in (4.5) we get
\[ h(re^{i\theta}) = \frac{1}{2\pi i} \oint_C \frac{h(\zeta) d\zeta}{\zeta - z} - \frac{1}{2\pi i} \oint_C \frac{h(\zeta) d\zeta}{\zeta - z^*} \]
\[ = \frac{1}{2\pi i} \oint_C \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - z^*} \right) h(\zeta) d\zeta \]
\[ = \frac{1}{2\pi i} \oint_C \left( \frac{1}{\zeta - z} + \frac{\zeta}{\zeta(\zeta - z)} \right) h(\zeta) d\zeta \]
\[ = \frac{1}{2\pi i} \oint_C \left( \frac{|\zeta|^2 - |z|^2}{|\zeta||\zeta - z|^2} \right) h(\zeta) d\zeta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{R^2 - r^2}{|Re^{it} - re^{i\theta}|^2} \right) h(Re^{it}) dt \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - t) + r^2} \right) h(Re^{it}) dt. \]

Taking the real part of (4.6) gives (4.4). \( \square \)

Now we continue our studies on the unit circle (letting \( R = 1 \)) to see what happens with the harmonic conjugate to \( f(x, y) \) when we let \((x, y)\) approach the unit circle. To do this we first write (4.4) in complex form. Observe that if \( \zeta = e^{it} \) and \( z = re^{i\theta} \), then we have

\[
\text{Re} \left( \frac{\zeta - z}{\zeta + z} \right) = \text{Re} \left( \frac{e^{-it}(e^{it} + re^{i\theta})}{e^{-it}(e^{it} - re^{i\theta})} \right)
\]
\[ = \text{Re} \left( \frac{1 + r \cos(\theta - t) + ir \sin(\theta - t)}{1 - r \cos(\theta - t) - ir \sin(\theta - t)} \right)
\[ = \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2}.
\]

Now we can write (4.4) with \( R = 1 \) as

\[
\text{Re} \frac{1}{2\pi i} \oint_C \frac{\zeta + z f(\zeta)}{\zeta} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} f(e^{it}) dt = f(z),
\]

where \( C \) is the positively oriented circle \(|\zeta| = 1\). Let us define \( \tilde{h} \) as

\[ \tilde{h}(z) = \frac{1}{2\pi i} \oint_C \frac{\zeta + z f(\zeta)}{\zeta} d\zeta = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{z}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta(\zeta - z)} d\zeta. \]

Then \( \tilde{h} \) is an holomorphic function on \(|z| < 1\) (by Theorem 4.1) with real part \( \text{Re} \tilde{h} = f \) and imaginary part \( \text{Im} \tilde{h} = g \) (since \( \text{Im} \tilde{h}(0) = g(0) = 0 \)), so we get \( \tilde{h}(z) = h(z) \) for all \(|z| < 1\). If we let \( z \) approach the point \( w \in C \) on the unit circle in (4.7), then we get from Sokhotzki-Plemelj jump Theorem 4.3 (observe that in order to apply Sokhotzki-Plemelj we need that \( f \) satisfies the Hölder condition (4.2) on \( C \), so from now on we assume
that).

\[ h(w) = f(w) + ig(w) = \frac{f(w)}{2} + \frac{f(w)}{2} + \frac{1}{2\pi i} \text{PV} \oint_{\gamma} \frac{\zeta + w f(\zeta)}{\zeta - w} d\zeta \]

\[ = f(w) + \frac{1}{2\pi} \text{PV} \int_{0}^{2\pi} f(e^{it}) e^{i(t/2 - \theta/2)} (e^{i\theta} + e^{it}) dt \]

\[ = f(w) + \frac{1}{2\pi} \text{PV} \int_{0}^{2\pi} f(e^{it}) e^{i(\theta - t/2) + e^{-i(\theta - t)/2}} dt \]

\[ = f(w) + \frac{i}{2\pi} \text{PV} \int_{0}^{2\pi} f(e^{it}) \cot \left( \frac{\theta - t}{2} \right) dt. \]

Since the real part of \( h \) is \( f \), we get

\[ g(w) = \text{Im} \, h(w) = \frac{1}{2\pi} \text{PV} \int_{0}^{2\pi} f(e^{it}) \cot \left( \frac{\theta - t}{2} \right) dt. \]

This formula is what we call the Hilbert transform on the unit circle.

Let us formulate this as a theorem.

**Theorem 4.7.** Let \( f(x, y) \) be real and harmonic on the closed unit disk and let \( f \) satisfy the Hölder condition (4.2) on the unit circle. Then the values on the unit circle of the harmonic conjugate \( g \) to \( f \) with \( g(0) = 0 \) are given by

\[ g(e^{i\theta}) = \frac{1}{2\pi} \text{PV} \int_{0}^{2\pi} f(e^{it}) \cot \left( \frac{\theta - t}{2} \right) dt. \]

4.5. **Poisson formula for the upper half plane.** Now we will develop similar formulas for the solution of the Dirichlet’s problem in the upper half plane. In this case the solution is not unique, we can for example always add the function \( f(x, y) = y \) to the solution to get a new one. However, if we require that the solution is bounded then we get a unique solution, see [1].

We start by proving a formula for the bounded solutions of Dirichlet’s problem in the upper half plane. The technique used here is very similar to that in Section 4.4. Again we assumes the existence of a solution in the closed upper half plane.

**Theorem 4.8.** Let \( f(x, y) \) be the real part of a bounded holomorphic function on the closed upper half plane \( y \geq 0 \). Then the values of \( f \) where \( y > 0 \) are given by

\[ f(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t, 0)}{(t - x)^2 + y^2} dt. \]

**Proof.** Let \( h(z) = f(z) + ig(z) \) be a bounded holomorphic function with real part \( f \). By the Cauchy integral formula and Cauchy integral theorem we get, for large enough \( R \),

\[ h(z) = \frac{1}{2\pi i} \int_{-R}^{R} \left( \frac{1}{t - z} - \frac{1}{t - \zeta} \right) h(t, 0) dt + \frac{1}{2\pi i} \int_{C_R} \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - \zeta} \right) h(\zeta) d\zeta, \]

where \( C_R \) is the positively oriented half circle \( |\zeta| = R, y \geq 0 \). If we set \( z = x + iy \) and simplify we get

\[ h(x, y) = \frac{y}{\pi} \int_{-R}^{R} \frac{h(t, 0)}{(t - x)^2 + y^2} dt + \frac{y}{\pi} \int_{C_R} \frac{h(\zeta)}{(\zeta - z)(\zeta - \zeta)} d\zeta. \]
If now \( \|h\|_{\infty} \leq M \), then one has
\[
\left| \int_{C_R} \frac{h(\zeta)}{(\zeta - z)(\zeta - \bar{z})} d\zeta \right| \leq \int_{C_R} \frac{|h(\zeta)|}{(|\zeta| - |z|)^2} |d\zeta| = \frac{MR\pi}{R^2(1 - \frac{z}{R})^2},
\]
which tends to 0 as \( R \to \infty \). Hence, as \( R \to \infty \) it follows from (4.9) that
\[
(4.10) \quad h(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{h(t, 0)}{(t - x)^2 + y^2} dt.
\]
The proof completes by taking the real part of (4.10).

To see what happens to the harmonic conjugate as \( y \to 0 \), we write (4.8) in complex form by noting that if \( \zeta \) is real we have
\[
\text{Im} \left( \frac{1}{\zeta - z} \right) = \text{Im} \left( \frac{1}{\zeta - x - iy} \right) = \frac{y}{(\zeta - x)^2 + y^2}.
\]
And that gives us
\[
\text{Re} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(t, 0)}{t - z} dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(t - x)^2 + y^2} f(t, 0) dt = f(x, y).
\]
We define
\[
(4.11) \quad \tilde{h}(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(t, 0)}{t - z} dt.
\]
Then \( \tilde{h} \) is a holomorphic function in \( y > 0 \) (by Theorem 4.1) with \( \text{Re} \tilde{h} = f \) and \( \lim_{y \to \infty} \tilde{h}(x, y) = 0 \), so if we choose \( g \) to have the property \( \lim_{y \to \infty} g(x, y) = 0 \) we get \( \tilde{h}(x, y) = h(x, y) \) for all \( y > 0 \). If we let \( z \) approach the point \( s \in \mathbb{R} \) on the real line in (4.11), assuming \( f \) satisfies the H"{o}lder condition (4.2) on \( \mathbb{R} \), a modification of the argument above yields
\[
(4.12) \quad h(s) = f(s) + \frac{1}{\pi i} \text{PV} \int_{-\infty}^{\infty} \frac{f(t, 0)}{t - s} dt.
\]
For the details see [4]. Since the real part of (4.12) is \( f(s) \), we can prove the following theorem by taking the imaginary part of (4.12).

**Theorem 4.9.** Let \( f(x, y) \) be the real part of a bounded holomorphic function on the closed upper half plane and let \( f \) satisfy the H"{o}lder condition (4.2) on the real line. Then there exists a harmonic conjugate \( g \) to \( f \) with \( \lim_{y \to \infty} g(x, y) = 0 \) and the values of \( g \) on the real line are given by
\[
g(s, 0) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(t, 0)}{s - t} dt.
\]
5. The Hilbert transform on the real line

5.1. Definition. The work we done so far with harmonic functions makes the following definitions of the Hilbert transform on the real line natural.

Definition 5.1. The Hilbert transform on the real line is defined pointwise by

\[ Hf(x) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} \, dt. \]

Similarly we can also define the Hilbert transform on the unit circle.

Definition 5.2. The Hilbert transform on \( \mathbb{T} = [0, 2\pi] \) is defined pointwise by

\[ Hf(x) = \frac{1}{2\pi} \text{PV} \int_{0}^{2\pi} f(x) \cot \left( \frac{x-t}{2} \right) \, dt. \]

Remark 5.3. In the definitions, we are defining the Hilbert transform as a pointwise limit as \( \epsilon \to 0 \) in the principal value. But in many applications where \( f \in L^p \) for \( 1 < p < \infty \) it is useful to take the limit in \( L^p \) sense. It can be shown that these definitions are equivalent, see e.g. [3].

5.2. Properties of the Hilbert transform. In this section we will take a close look at some of the basic properties of the Hilbert transform on the real line. It follows directly from the definition of the Hilbert transform that the associated operator is linear.

Another slightly less obvious property is that the Hilbert transform commutes with translations and positive dilations. Let \( \tau_a \) be the translation operator defined by \( \tau_a f(x) = f(x-a) \), and let \( S_a \) for \( a > 0 \) be the dilation operator \( S_a f(x) = f(ax) \). We can get \( H \tau_a f = \tau_a H f \) and \( H S_a f = S_a H f \) by a simple change of variables:

\[
H \tau_a f(x) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(t-a)}{x-t} \, dt = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(u)}{x-a-u} \, du = \tau_a H f(x),
\]

\[
H S_a f(x) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(at)}{x-t} \, dt = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(u)}{x-u} \, du = S_a H f(x).
\]

If we now let \( R \) be the reflection operator \( Rf(x) = f(-x) \), the we can by another chance of variables get \( HRf = -RHf \).

\[
HRf(x) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(-t)}{x-t} \, dt = -\frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(u)}{x-u} \, du = -RHf(x).
\]

There is no simple formula for the Hilbert transform of a product of two functions. However, we will discuss the special cases of the Hilbert transform of \( xf(x) \) and \( f(x)/x \).

If we want to consider the Hilbert transform of \( x^n f(x) \) for \( n \in \mathbb{Z} \) we can iterate the following formulas.

Theorem 5.4. Let \( f \) be integrable. Then

\[
H(xf(x)) = xHf(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \, dt.
\]

If we instead require that \( f/x \) is integrable, then

\[
H \left( \frac{f(x)}{x} \right) = \frac{Hf(x) - Hf(0)}{x}.
\]
Proof.

\[ H(xf(x)) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{tf(t)}{x-t} \, dt \]
\[ = \frac{x}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{f(t)}{x-t} \, dt - \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{(x-t)f(t)}{x-t} \, dt \]
\[ = xHf(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \, dt \]

and by using (5.1) on the function \( f(x)/x \) we get

\[ Hf(x) = xH\left(\frac{f(x)}{x}\right) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t} \, dt \]
\[ = xH\left(\frac{f(x)}{x}\right) + Hf(0). \]

(5.2) follows by moving \( Hf(0) \) to the other side and dividing by \( x \). \( \Box \)

The Hilbert transform of a function \( f \) can be seen as the convolution of \( f \) with \( k(x) = (\pi x)^{-1} \), where one has to remember to calculate the integral as a principal value. To make it formal we let

\[ k_\epsilon(x) = \begin{cases} (\pi x)^{-1} & \text{if } |x| \geq \epsilon, \\ 0 & \text{if } |x| < \epsilon, \end{cases} \]

and get

\[ Hf(x) = \lim_{\epsilon \to 0} H_\epsilon f(x) = \lim_{\epsilon \to 0} f * k_\epsilon(x), \]

where \( H_\epsilon f = f * k_\epsilon \) is called the truncated Hilbert transform.

5.3. Some Hilbert transforms. Let us calculate the Hilbert transform of some basic functions.

The Hilbert transform of a constant \( c \) is given by

\[ H(c) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{c}{x-t} \, dt \]
\[ = \frac{c}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{1}{u} \, du = 0. \]

We can calculate the Hilbert transform of \( \sin x \) by

\[ H(\sin x) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{\sin t}{x-t} \, dt \]
\[ = -\frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{\sin(u+x)}{u} \, du \]
\[ = -\frac{\cos(x)}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{\sin(u)}{u} \, du - \frac{\sin(x)}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{\cos(u)}{u} \, du \]
\[ = -\cos x. \]

By using that the Hilbert transform commutes with translation we get

\[ H(\cos x) = H(\sin(x + \pi/2)) = -\cos(x + \pi/2) = \sin x. \]
5.4. The Riesz inequality on $L^2$. Riesz’s inequality is that the Hilbert transform is a bounded linear operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for $1 < p < \infty$. The proof of this is hard and requires interpolation so we will prove it only in the special case where $p = 2$. The full proof can be read in [3].

**Theorem 5.5.** Let $f \in L^2(\mathbb{R})$. Then $\|Hf\|_2 = \|f\|_2$ and

$$\mathcal{F}Hf(x) = -i\text{sgn}(\omega)f.$$ 

**Proof.** Let $H_{\epsilon,\eta}$ be the double truncated Hilbert transform defined by $H_{\epsilon,\eta}f = f * k_{\epsilon,\eta}$, where

$$k_{\epsilon,\eta}(x) = \begin{cases} \frac{(\pi x)^{-1}}{\pi t} & \text{if } \epsilon \leq |x| \leq \eta, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $f \in L^2(\mathbb{R})$. Then $H_{\epsilon,\eta}f \in L^2(\mathbb{R})$ by Theorem 2.14 since $f \in L^2(\mathbb{R})$ and $k_{\epsilon,\eta} \in L^1(\mathbb{R})$.

If we take the Fourier transform of $H_{\epsilon,\eta}f$ and use Theorem 3.23 we get $\mathcal{F}H_{\epsilon,\eta}f = \mathcal{F}f\mathcal{F}k_{\epsilon,\eta}$, where $\mathcal{F}k_{\epsilon,\eta}$ can be calculated by

$$\mathcal{F}k_{\epsilon,\eta}(\omega) = \int_{\epsilon \leq |t| \leq \eta} e^{-i\omega t} \frac{dt}{\pi t}$$

We see that as $\epsilon \to 0$ and $\eta \to \infty$, $\mathcal{F}k_{\epsilon,\eta}(\omega) \to -i\text{sgn}(\omega)$ for every $\omega \in \mathbb{R}$. From that we can draw the conclusion that there exists a constant $C$ independent of $\epsilon$, $\eta$ and $\omega$ such that $|\mathcal{F}k_{\epsilon,\eta}(\omega)| \leq C$. By Remark 5.3 we have that $Hf = \lim_{\epsilon \to 0, \eta \to \infty} H_{\epsilon,\eta}f$, where the limits are to be taken in $L^2$. We can now show that

$$Hf = \lim_{\epsilon \to 0, \eta \to \infty} H_{\epsilon,\eta}f = \mathcal{F}^{-1}(-i\text{sgn}(\omega)\mathcal{F}f(\omega))$$

by the following observation:

$$\lim_{\epsilon \to 0, \eta \to \infty} \|H_{\epsilon,\eta}f - \mathcal{F}^{-1}(-i\text{sgn}(\omega)\mathcal{F}f(\omega))\|_2$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0, \eta \to \infty} \|\mathcal{F}k_{\epsilon,\eta}(\omega)\mathcal{F}f(\omega) - (-i\text{sgn}(\omega)\mathcal{F}f(\omega))\|_2$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0, \eta \to \infty} \|\mathcal{F}k_{\epsilon,\eta}(\omega) - (-i\text{sgn}(\omega))\mathcal{F}f(\omega)\|_2 = 0,$$

where we have used Plancherel’s theorem and the dominated convergence theorem. If we take the Fourier transform of both sides of equation (5.3) we get

$$\mathcal{F}Hf(x) = -i\text{sgn}(\omega)\mathcal{F}f(\omega).$$

By taking the norm on both sides we get

$$\|\mathcal{F}Hf\|_2 = \| -i\text{sgn}(\omega)\mathcal{F}f(\omega)\|_2 = \|\mathcal{F}f\|_2$$
and if we use Plancherel’s theorem we get
\[ \|Hf\|_2 = \|f\|_2. \]

Some direct application of the last theorem include the following:

**Theorem 5.6.** Let \( f \in L^2(\mathbb{R}) \) be a differentiable function with \( f' \in L^2(\mathbb{R}) \) and let \( Hf \) be differentiable with \( (Hf)' \in L^2(\mathbb{R}) \) then \( H[f'] = (Hf)'. \)

**Proof.** The technique of this proof is to compare the Fourier transform of \( H[f'] \) and of \( (Hf)' \), if they are equal we can take the inverse Fourier transform to get \( H[f'] = (Hf)' \).

\[ \hat{Hf}'(\omega) = -i \text{sgn}(\omega) \hat{f}'(\omega) = -i \text{sgn}(\omega)(i\omega)\hat{f}(\omega) = \text{sgn}(\omega)\hat{f}(\omega) \]
and
\[ (Hf)'(\omega) = i\omega \hat{Hf} = i\omega(-i \text{sgn}(\omega))\hat{f}(\omega) = \text{sgn}(\omega)\hat{f}(\omega). \]

\[ \square \]

**Theorem 5.7.** If \( f \in L^2(\mathbb{R}) \), then \( HHf = -f \).

**Proof.** Let \( T \) be the operator multiplying function with \(-i \text{sgn}(x)\), \( Tf(x) = -i \text{sgn}(x)f(x) \). Then
\[ HHf = \mathcal{F}^{-1}TF\mathcal{F}^{-1}TF f = \mathcal{F}^{-1}TTF f = -f. \]

One reason why the Hilbert transform is so important in mathematics is because it is almost the only translation and dilation invariant bounded linear operator from \( L^p(\mathbb{R}) \) to \( L^p(\mathbb{R}) \). Let us state it as a theorem without proof. The proof can be found in [3].

**Theorem 5.8.** Let \( T : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \) be a translation and dilation invariant bounded linear operator, then
\[ T = aH + bI, \]
where \( I \) is the identity operator.

## 6. Normconvergence of Fourier series

In this section we will see that the Hilbert transform can be used to study the convergence of Fourier series. We will see that the Fourier series of \( f \) converges to \( f \) in \( L^p \)-norm. In formulas that can be written as
\[ \lim_{n \to \infty} \left( \int_0^{2\pi} |f(t) - S_n f(t)|^p dt \right)^{1/p} = 0. \]

One of the key ideas is that if can show that the operators \( S_n \) are uniformly bounded in \( L^p(\mathbb{T}) \), then \( \lim_{n \to \infty} S_n f \) exists and \( \lim_{n \to \infty} S_n f = f \).

**Theorem 6.1.** Let \( 1 \leq p < \infty \) and let \( f \in L^p(\mathbb{T}) \) and assume that the partial summation operators \( S_n \) are uniformly bounded. Then \( \lim_{n \to \infty} S_n f \) exists and \( \lim_{n \to \infty} S_n f = f \).
Proof. Assume \(\|S_n\|_{op} \leq C\) for some \(0 < C < \infty\). Let \(\epsilon > 0\) and choose a trigonometric polynomial \(g\) such that \(\|f - g\|_p < \epsilon/(2 + 2C)\) (this is possible by Theorem 3.2), and let \(n\) be greater then the degree of \(g\). Then

\[
\|f - S_nf\|_p \leq \|f - g\|_p + \|g - S_ng\|_p + \|S_ng - S_nf\|_p
\]

\[
\leq \frac{\epsilon}{2 + 2C} + 0 + C \frac{\epsilon}{2 + 2C}
\]

\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Since this statement holds for arbitrary \(\epsilon > 0\) the statement follows. \(\square\)

6.1. The Hilbert transform as a multiplier. We can get one variant of the Fourier series if we in front of every coefficient in the Fourier series add a weight from the sequence \(\Lambda = \{\lambda_k\}_{-\infty}^{\infty}\). The linear multiplier operator \(M_\Lambda\) is formally defined in the following way.

Definition 6.2. Let \(\Lambda = \{\lambda_k\}_{-\infty}^{\infty}\). Then the multiplier operator \(M_\Lambda\) is defined by

\[
M_\Lambda f(x) = \sum_{k=-\infty}^{\infty} \lambda_k \hat{f}(k)e_k,
\]

where \(e_k(x) = e^{ikx}\). Observe one also has \(\overline{M_\Lambda f(k)} = \lambda_k \hat{f}(k)\). This follows from the uniqueness of the Fourier expansion.

Of course there is no guarantee that all multiplier operators will converge for all functions, so we have to be a little careful which multiplier operator we work with. One quick observation is that the operators \(S_n\) for the partial sums is a very easy multiplier operator coming from the sequence \(\Lambda = \{\lambda_k\}\), where \(\lambda_k = 1\) if \(|k| \leq n\) and \(\lambda_k = 0\) if \(|k| > n\).

Let us define the Hilbert transform as a multiplier operator.

Definition 6.3. Let \(\Lambda = \{\lambda_k\}\), where \(\lambda_k = -\text{sgn}(k)\) and \(\text{sgn}(k)\) is defined by

\[
\text{sgn}(k) = \begin{cases} 
1 & \text{if } k < 0, \\
0 & \text{if } k = 0, \\
-1 & \text{if } k > 0.
\end{cases}
\]

Then we define the Hilbert transform \(H\) by \(Hf = M_\Lambda f\).

Remark 6.4. By an argument similar to the one in the proof of Theorem 5.5, it is possible to show that this definition of the Hilbert transform coincides with Definition 5.2.

The Hilbert transform is closely related to partial summation in the following way.

Theorem 6.5. Let \(f \in L^p(\mathbb{T})\) and let \(e_n(x) = e^{inx}\) for \(n \in \mathbb{Z}\). Then

\[
S_nf = \frac{1}{2} \left( ie_{-n}H[e_nf] - ie_nH[e_{-n}f] + \hat{f}(-n)e_{-n} + \hat{f}(n)e_n \right).
\]

Proof. First note that \(\overline{e_Rf(k + R)} = \hat{f}(k)\) since

\[
\overline{e_Rf(k + R)} = \int_0^{2\pi} e^{iRt} f(t)e^{-i(k+R)t} dt = \int_0^{2\pi} f(t)e^{-ikt} dt = \hat{f}(k).
\]
Now we get
\[
\frac{1}{2} \left( ie^{-n}H[e_nf] - ie^{-n}H[e_{-n}f] + \hat{f}(-n)e_{-n} + \hat{f}(n)e_n \right)
\]
\[
= \frac{1}{2} \left( ie^{-n} \sum_{k=-\infty}^{\infty} -\text{sgn}(k)e_{-n}f(k)e_k - ie^{-n} \sum_{k=-\infty}^{\infty} -\text{sgn}(k)e_{-n}f(k)e_k + \hat{f}(-n)e_{-n} + \hat{f}(n)e_n \right)
\]
\[
= \frac{1}{2} \left( \sum_{k=-\infty}^{\infty} \text{sgn}(k + n)e_{-n}f(k + n)e_k - \sum_{k=-\infty}^{\infty} \text{sgn}(k - n)e_{-n}f(k - n)e_k + \hat{f}(-n)e_{-n} + \hat{f}(n)e_n \right)
\]
\[
= \frac{1}{2} \left( \sum_{k=-\infty}^{\infty} \text{sgn}(k + n)\hat{f}(k)e_k - \sum_{k=-\infty}^{\infty} \text{sgn}(k - n)\hat{f}(k)e_k + \frac{1}{2}(\hat{f}(-n)e_{-n} + \hat{f}(n)e_n) \right)
\]
\[
= \sum_{k=-\infty}^{\infty} \frac{1}{2} \left( (\text{sgn}(k + n) - \text{sgn}(k - n) + \chi_{-n}(k) + \chi_n(k)) \hat{f}(k)e_k \right)
\]
\[
= \sum_{k=-\infty}^{\infty} \lambda_k \hat{f}(k)e_k,
\]
where \(\chi_R(k)\) is 0 if \(k \neq R\) and 1 if \(k = R\). This is the same multiplier operator as the one for partial summation since we get from an easy calculation that for \(k < -n\) or \(k > n\) it yields \(\lambda_k = 0\) and for \(-n < k < n\) we have that \(\lambda_k = 1\) and for \(k = -n\) or \(k = n\) we find that \(\lambda_k = 1\). \(\square \)

The key point in showing the norm convergence is to show that \(S_n\) is uniformly bounded, since if \(S_n\) is uniformly bounded we can then use Theorem 6.1 to get convergence. Now when we have showed that \(S_n\) can be written in terms of the Hilbert transform we wish to estimate \(\|S_n\|_{op}\) in terms of the \(\|H\|_{op}\). This type of estimate is possible by the following theorem.

**Theorem 6.6.** It yields that \(\|S_n\|_{op} \leq \|H\|_{op} + 1\), with the operator norm taken in \(L^p\). In particular \(S_n\) is uniformly bounded if \(H\) is bounded.

**Proof.** First note that the operator \(Lf = e_nf\) is an isometry since
\[
\|Lf\|_p = \left( \int_T |e_nf|^p d\lambda \right)^{1/p} = \left( \int_T |f|^p d\lambda \right)^{1/p} = \|f\|_p,
\]
which specifically means that \(\|L\|_{op} = 1\), also note that for the projection operators \(Lf = \hat{f}(n)e_n\) we have \(\|L\|_{op} = 1\) since
\[
\|\hat{f}(n)e_n\|_p = \frac{1}{2\pi} \left| \int_T fe_{-n} d\lambda \right| \|e_n\|_p
\]
\[
\leq \frac{1}{2\pi} \|f\|_p \|e_{-n}\|_q \left( \int_T |e_n|^p d\lambda \right)^{1/p}
\]
\[
= \frac{1}{2\pi} \|f\|_p (2\pi)^{1/q} (2\pi)^{1/p}
\]
\[
= (2\pi)^{1/q+1/p-1} \|f\|_p
\]
\[
= \|f\|_p
\]
by Hölder’s inequality where \(1/p + 1/q = 1\).
Now we can estimate $\|S_n\|$ by
\[
\|S_n f\|_p = \frac{1}{2}\|ie^{-n}H[e_n f] - ie_n H[e_{-n} f] + \hat{f}(-n)e_{-n} + \hat{f}(n)e_n\|_p
\leq \frac{1}{2}(\|Hf\|_p + \|Hf\|_p + 1 + 1) = \|Hf\|_p + 1.
\]

Thus if we could show that the Hilbert transform is bounded on $L^p(\mathbb{T})$, then we will get that the partial sums of the Fourier series are uniformly bounded and therefore we can use Theorem 6.1 to show that the Fourier series of $f$ will converge to $f$ for all functions $f$ in $L^p(\mathbb{T})$. We formulate this as a theorem.

**Theorem 6.7.** (Marcel Riesz) If the Hilbert transform is a bounded operator from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$, then norm convergence is valid in $L^p(\mathbb{T})$.

6.2. **The Hilbert transform is bounded on $L^2$.** In this section we will show that the Hilbert transform is bounded on $L^2(\mathbb{T})$. However, in [3] there is a proof that for $1 < p < \infty$, the Hilbert transform is bounded on $L^p(\mathbb{T})$.

In $L^2(\mathbb{T})$ we get as a consequence of Parseval’s identity that we have an explicit formula for the operator norm in $L^2(\mathbb{T})$ of any multiplier operator.

**Theorem 6.8.** Let $\Lambda = \{\lambda_k\}$ be a sequence and assume that the corresponding multiplier operator $M_{\Lambda}: L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is well defined. Then $\|M_{\Lambda}\| = \sup |\lambda_k|$. In particular, $M_{\Lambda}$ is bounded if and only if $\Lambda$ is bounded.

**Proof.** It yields that
\[
\|M_{\Lambda}f\|_2^2 = 2\pi \sum_{k=-\infty}^{\infty} |\hat{M_{\Lambda}f}(k)|^2
= 2\pi \sum_{k=-\infty}^{\infty} |\lambda_k \hat{f}(k)|^2
\leq 2\pi \sup |\lambda_k|^2 \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2
= \sup |\lambda_k|^2 \|f\|_2^2.
\]

Here we have used the Parseval’s identity 3.7 at the first and last equality. On the other hand, let $\{n_k\}$ be a sequence of integers such that $\lim_{k \to \infty} |\lambda_{n_k}| = \sup |\lambda_n|$. Then $\|M_{\Lambda}e_{n_k}\| = \|\lambda_{n_k} e_{n_k}\| = |\lambda_{n_k}| \|e_{n_k}\|$ so $\|M_{\Lambda}e_{n_k}\|/\|e_{n_k}\| \to \sup |\lambda_n|$ and that shows equality.

Using Theorem 6.8, we note that the Hilbert transform is a multiplier operator with $\sup |\lambda_n| = 1$ and is therefore bounded on $L^2(\mathbb{T})$. This means by Theorem 6.7 that norm convergence is valid in $L^2(\mathbb{T})$.

As we mentioned earlier, it can be shown that the Hilbert transform is bounded on $L^p(\mathbb{T})$ for $1 < p < \infty$. Taking this fact for granted, Theorem 6.7 yields that norm convergence of the Fourier series is valid in $L^p(\mathbb{T})$. We formulate this as a theorem.
Theorem 6.9. Let \( f \in L^p(\mathbb{T}) \), where \( 1 < p < \infty \). Then
\[
\lim_{n \to \infty} \left( \int_0^{2\pi} |f(t) - S_n f(t)|^p dt \right)^{1/p} = 0.
\]

6.3. A counterexample in \( L^\infty(\mathbb{T}) \). The endpoints \( p = 1 \) and \( p = \infty \) are excluded in the previous theorem. The reason is that the result of the theorem turns out to be false in these two cases. We provide a counterexample in the case of \( L^\infty(\mathbb{T}) \).

Let \( f \) be defined by
\[
f(x) = \begin{cases} 
1 & \text{if } 0 < x < \pi, \\
0 & \text{if } \pi < x < 2\pi.
\end{cases}
\]

Then
\[
\hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt = \frac{1}{2}
\]
and
\[
\hat{f}(n) + \hat{f}(-n) = \frac{1}{2\pi} \int_0^{2\pi} f(t)(e^{-int} + e^{int}) dt
\]
\[
= \frac{1}{\pi} \int_0^{\pi} f(t) \cos(nt) dt
\]
\[
= \frac{1}{\pi} \int_0^{\pi} \cos(nt) dt = 0.
\]

This means that
\[
S_n f(0) = \sum_{k=-n}^{n} \hat{f}(k) = 1/2.
\]

Since the trigonometric polynomials are continuous, there is for every \( \epsilon > 0 \) an interval \([0, \delta]\) of nonzero measure where \( |f(x) - S_n f(x)| \geq 1/2 - \epsilon \), and hence \( \|f - S_n f\|_\infty \geq 1/2 \) so \( S_n f \not\to f \) in \( L^\infty(\mathbb{T}) \).

References