UPPSALA
UNIVERSITET

U.U.D.M. Project Report 2013:10

Classication of simple complex weight modules with finite-dimensional weight spaces over the Schrödinger algebra

Brendan Frisk Dubsky

Examensarbete i matematik, 30 hp
Handledare och examinator: Volodymyr Mazorchuk
Maj 2013

Department of Mathematics
Uppsala University

Classification of simple complex weight modules with finite-dimensional weight spaces over the Schrödinger algebra

Brendan Frisk Dubsky

Department of mathematics

Uppsala university

A thesis submitted for the degree of

Master of science

Advisor: Volodymyr Mazorchuk

Abstract

We complete the classification of the simple complex weight modules with finite-dimensional weight spaces over the (centrally extended) Schrödinger algebra in $(1+1)$-dimensional space-time, building upon the classification of those such modules which have a highest weight by Dobrev et al., the "twisting functors" of Mathieu, and a result of Wu and Zhu. In particular, any such module which is not of highest or lowest weight, nor an $\mathfrak{sl}_2$-module, is obtainable by "twisting" a highest weight module.

Contents

Chapter 1

Introduction

1.1 Background and goal

The Schrödinger Lie group is the group of symmetries of the free particle Schrödinger equation. The (centrally extended) Lie algebra of this group in the case of $(1+1)$-dimensional space-time we call the Schrödinger algebra[1], $\mathcal{S}$. The purpose of this thesis is to complete classification of simple complex weight modules with finite-dimensional weight spaces of $\mathcal{S}$, *i.e.* those simple complex $\mathcal{S}$-modules which are diagonalized by the Cartan subalgebra of $\mathcal{S}$[2] such that each common eigenspace of the Cartan generators is of finite dimension (cf. Wu and Zhu [2013]). We denote the category of such modules by $\mathcal{C}$.

Though not semisimple, $\mathcal{S}$ has an analogue of the standard triangular decomposition of semisimple Lie algebras, which is used by Dobrev et al. [1997] to classify those modules in $\mathcal{C}$ which have a highest or lowest weight. Our main result, Theorem 3.3.1, is the classification of the category, $\mathcal{N}$, of those modules in $\mathcal{C}$ which do not have any highest or lowest weight (and which do not reduce to the well studied case of $\mathfrak{sl}_2$-modules), thereby indeed completing the classification of $\mathcal{C}$. To the knowledge of the author, this case has not been treated previously.

[1]Due to this origin, the Schrödinger algebra is of interest in mathematical physics. Wu and Zhu [2013] mentions Barut and Raczka [1986], Ballesteros et al. [2000], Barut and Xu and Niederer [1972] as providing examples of applications.

[2]The Cartan subalgebra of $\mathcal{S}$ is generated by the elements h and z. We will see that z acts as a scalar on the modules in which we take interest, so that we may define weight spaces to be eigenspaces of h.

The proof of Theorem 3.3.1 heavily uses so-called "twisting functors", invented by Mathieu [2000] in order to classify the simple complex weight modules with finite-dimensional weight spaces over simple Lie algebras, as well as the above discussed classification by Dobrev et al. [1997]. In particular we show, using a result of Wu and Zhu [2013] concerning weight space dimensions of modules in $\mathcal{N}$, that any module in $\mathcal{N}$ can be obtained by "twisting" one of the modules subject to the classification of Dobrev et al. [1997].

1.2 Acknowledgements

My deepest thanks goes to my advisor Volodymyr Mazorchuk, for introducing me to this subject and providing me with the basic ideas behind the project, for swift help whenever I needed, and, last but not least, for his trust in me.

Chapter 2

Preparatory material and earlier results

In this chapter, basic definitions of the concepts treated are given, such as definitions of the Schrödinger Lie algebra and its weight modules. Some preparatory results are also given, including classifications of simple complex weight modules with finite-dimensional weight spaces over $\mathfrak{sl}(2)$, as well as those over the Schrödinger algebra which have neither heighest, nor lowest, weight.

Throughout this text, all vector spaces discussed will be assumed to be complex (so will the algebras and modules in particular). Also, any associative algebra will be identified with its underlying Lie algebra, *i.e.* the Lie algebra having the same vector space as the associative one, and Lie relations being the commutator ones of the associative algebra.

2.1 The Schrödinger algebra, its universal enveloping algebra and weight modules

Here we define the Schrödinger Lie algebra and its called weight modules, which are the objects of study in this text. The classification problem for these modules is also reduced to the one for such modules over the associative "universal enveloping algebra" of this Lie algebra.

Definition The (centrally extended) *Schrödinger algebra* for $(1+1)$-dimensional space-time, $\mathcal{S}$, is the Lie algebra with basis $\{f, q, h, z, p, e\}$ and relations

$$\begin{aligned} [h, e] &= 2e & [h, p] &= p \\ [e, q] &= p & [e, p] &= 0 \\ [h, f] &= -2f & [h, q] &= -q \\ [p, f] &= -q & [z, S] &= 0 \\ [e, f] &= h & [p, q] &= z \\ [f, q] &= 0. \end{aligned} \tag{2.1}$$

One may note that $\mathcal{S}$ contains one of the most studied Lie algebras as a subalgebra.

Definition The Lie algebra $\mathfrak{sl}_2$ is the subalgebra of $\mathcal{S}$ generated by f, h and e.

Definition Consider the free unital associative algebra, A, generated by f, q, h, z, p and e, and its ideal, I, generated by $[h, e] - 2e$, $[h, p] - p$, $[e, q] - p$, $[e, p]$, $[h, f] + 2f$, $[h, q] + q$, $[p, f] + q$, $[z, S]$, $[e, f] - h$, $[p, q] - z$ and $[f, q]$. The *universal enveloping algebra* of $\mathcal{S}$ is defined to be the quotient $U = A/I$. The same notation as for the elements of A will be used to denote their images under the quotient map.

It is clear from the definition that the generators of U satisfy the same Lie bracket relations as those of $\mathcal{S}$.

Definition The universal enveloping algebra, $U(\mathfrak{sl}_2)$, of $\mathfrak{sl}_2$ is the subalgebra of U generated by f, h and e.

A universal enveloping algebra satisfies a certain universal property, cf., e.g., [Jacobson, 1979, p. 151-156], which reduces the study of categories of Lie algebra modules to that of categories of modules over their universal enveloping algebra. In our case, we have the following lemma and proposition.

Lemma 2.1.1. *For s ranging over $\{f, q, h, z, p, e\}$ consider the Lie algebra morphism $\epsilon : \mathcal{S} \to U$ being the extension of $\mathcal{S} \ni s \mapsto \epsilon(s) = s \in U$. Then the following holds: Let $\phi : \mathcal{S} \to End(V)$ be any representation. Then there is a unique morphism of associative algebras $\overline{\phi} : U \to End(V)$ such that $\phi = \overline{\phi} \circ \epsilon$.*

Proposition 2.1.2. *With the notation of Lemma 2.1.1, the category of $\mathcal{S}$-representations and $\mathcal{S}$-representation morphisms is isomorphic to the category of U-representations and U-representation morphisms via the map which takes an object ϕ to $\overline{\phi}$, and is the identity on morphisms.*

Proof By Lemma 2.1.1, the map between objects, $\phi \mapsto \overline{\phi}$, is bijective, with inverse $\overline{\phi} \mapsto \overline{\phi} \circ \epsilon$. Also a linear map is a morphism in one of the categories if and only if it is so in the other, because the images of ϕ and $\overline{\phi}$ are generated by the same elements $\phi(s) = \overline{\phi}(s)$, and, for both representations, a linear map to another $\mathcal{S}$ and U module W is a morphism if and only if it preserves the actions of the $\phi(s) = \overline{\phi}(s)$.

The following is the *Poincaré-Birkhoff-Witt theorem* in the particular case of the Lie algebra $\mathcal{S}$. See [Jacobson, 1979, p. 156-160] for the general statement and its proof[1].

Proposition 2.1.3. *The set $\{q^i f^j p^k e^l h^m z^n\}_{i,j,k,l,m,n \in \mathbb{N}}$ is a basis for the vector space U.*

All modules studied in this text will be so-called weight modules.

Definition A module V over $\mathcal{S}$, $\mathfrak{sl}_2$, U or $U(\mathfrak{sl}_2)$ is called a *weight module* if it decomposes into a sum of *weight spaces*

$$V = \bigoplus_{\lambda \in \text{supp}(V)} V_\lambda \tag{2.2}$$

where $V_\lambda \neq 0$ and $hv = \lambda v$ for any $v \in V_\lambda$ and $\lambda \in \text{supp}(V)$. The elements λ of the *support of* V, $\text{supp}(V)$, are called the *weights* of V, and vectors in the spaces V_λ are called *weight vectors*.

The following easy proposition describes how the generators of U act on weight spaces. It will be used repeatedly and without further comment throughout the text.

[1] The proof given is for the case of a real Lie algebra, but works for the complex case as well.

Proposition 2.1.4. *Let V be a weight U-module with $\lambda \in \operatorname{supp}(V)$. Then the following hold*

$$\begin{aligned}
f &: V_\lambda \to V_{\lambda-2} \\
q &: V_\lambda \to V_{\lambda-1} \\
h &: V_\lambda \to V_\lambda \\
z &: V_\lambda \to V_\lambda \\
p &: V_\lambda \to V_{\lambda+1} \\
e &: V_\lambda \to V_{\lambda+2}.
\end{aligned} \tag{2.3}$$

Proof. Consider an arbitrary $v \in V_\lambda$. The desired result is an immediate consequence of the following calculations, which use the commutator relations of U (these are the same as the Lie bracket relations 2.1 of $\mathcal{S}$):

$$\begin{aligned}
&hfv = (fh - 2f)v = f(h-2)v = (\lambda - 2)fv \\
&qfv = (fq - f)v = f(h-1)v = (\lambda - 1)fv \\
&h^2v = \lambda hv \\
&hzv = zhv = \lambda zv \\
&hpv = (ph + p)v = p(h+1)v = (\lambda + 1)pv \\
&hev = (eh + 2e)v = e(h+2)v = (\lambda + 2)ev.
\end{aligned} \tag{2.4}$$

□

We may now define the category whose classification is our ultimate goal.

Definition We will denote by $\mathcal{C}$ the category with all simple weight U-modules with finite-dimensional weight spaces constituting the objects, and all non-zero U-module homomorphisms between these constituting the morphisms.

Because of Proposition 2.1.2, classifying all simple weight $\mathcal{S}$-modules with finite-dimensional weight spaces is equivalent to classifying $\mathcal{C}$. From this point on, all modules discussed will, unless otherwise stated, be assumed to be U-modules. In order to classify $\mathcal{C}$, we will use a decomposition which will depend on the following definition.

Definition Let V be a weight module and consider the preorder on $\text{supp}(V)$ defined by $\lambda_1 \leq \lambda_2$ if and only if $\text{Re}(\lambda_1) \leq \text{Re}(\lambda_2)$, with the ordering of $\mathbb{R}$ being the usual one. If there is a maximal element, λ, in $\text{supp}(V)$, then λ is called a *highest weight* of V, and any $v \in V_\lambda$ is called a *highest weight vector*. If V is generated by a highest weight vector, then it is called a *highest weight module*.

Lowest weights, lowest weight vectors and lowest weight modules are defined analogously.

On the modules in which we take interest, the following results give that z acts like a scalar. Whether this scalar is 0 or not will play a decisive role in the applicability of many results to come in the next chapter.

Proposition 2.1.5. *Let V be a U-module which is generated by a single vector, v, on which $z \in U$ acts like the scalar $c \in \mathbb{C}$. Then z acts like c on the entire V.*

Proof By assumption, an arbitrary element of V is of the form uv for some $u \in U$. Then we have $z(uv) = u(zv) = c(uv)$, since z commutes with the generators of U. □

Corollary 2.1.6. *Let $V \in \mathcal{C}$. Then z acts like a scalar on V.*

Proof. By Proposition 2.1.4, each weight space of V is invariant under the action of z. Each weight space is a finite-dimensional vector space over $\mathbb{C}$, and so z must have some eigenvector $v \neq 0$ in some weight space of V (so that z acts as a scalar on v). The module generated by v is a submodule of V, so by the simplicity of V, v in fact generates V. Hence Proposition 2.1.5 applies. □

The following (special case of a) famous lemma helps our understanding of the structure of $\mathcal{C}$.

Lemma 2.1.7. (Schur's lemma.) *Let $V, W \in \mathcal{C}$. Then either* $\text{Hom}_{\mathcal{C}}(V, W) \cong \mathbb{C}\backslash\{0\}$ *or* $\text{Hom}_{\mathcal{C}}(V, W) = \varnothing$.

Proof. Assume that $\text{Hom}_{\mathcal{C}}(V, W) \neq \varnothing$ and let $\phi \in \text{Hom}_{\mathcal{C}}(V, W)$ be arbitrary. Since ϕ is non-zero by definition of $\mathcal{C}$, we have that $\text{Im}(\phi) \subset W$ is a non-zero submodule, so since W is simple, ϕ must be surjective. Also $\ker(\phi) \subset V$ is a submodule, so since V is simple and ϕ non-zero, ϕ must be bijective, and therefore an isomorphism.

We then have an isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(V,V) &\to \mathrm{Hom}_{\mathcal{C}}(V,W) \\ \psi &\mapsto \phi \circ \psi, \end{aligned}$$

so we may without loss of generality assume that $\phi \in \mathrm{Hom}_{\mathcal{C}}(V,V)$. Let $\lambda \in \mathrm{supp}(V)$. Then $\phi_{|V_\lambda} : V_\lambda \to V_\lambda$ since ϕ is an isomorphism, and because $\dim(V_\lambda) < \infty$, the map $\phi_{|V_\lambda}$ has some non-zero eigenvalue, $\mu \in \mathbb{C}$. Let v be a corresponding eigenvector, so that $0 \neq v \in \ker(\phi - \mu \cdot \mathrm{Id})$. Assume, towards a contradiction, that $\phi - \mu \cdot \mathrm{Id}$ is not the zero map. Then $\phi - \mu \cdot \mathrm{Id}$ is a morphism in $\mathcal{C}$, so that it can not have non-zero kernel by the previous paragraph, a contradiction. Therefore we obtain that $\phi = \mu \cdot \mathrm{Id}$, and so $\mathrm{Hom}_{\mathcal{C}}(V,V) \subset \mathbb{C}\backslash\{0\} \cdot \mathrm{Id}$. The other inclusion is obvious, and the desired result follows. □

Definition By $\mathcal{C}_{\mathfrak{sl}_2}$ we denote the full subcategory of $\mathcal{C}$ with objects V fulfilling $pV = 0 = qV$.

By $\mathcal{H}$ we denote the full subcategory of $\mathcal{C}\backslash\mathcal{C}_{\mathfrak{sl}_2}$ with objects being highest, but not lowest, weight modules.

By $\mathcal{L}$ we denote the full subcategory of $\mathcal{C}\backslash\mathcal{C}_{\mathfrak{sl}_2}$ with objects being lowest, but not highest, weight modules.

By $\mathcal{B}$ we denote the full subcategory of $\mathcal{C}\backslash\mathcal{C}_{\mathfrak{sl}_2}$ with objects being both highest and lowest weight modules.

By $\mathcal{N}$ we denote the full subcategory of $\mathcal{C}\backslash\mathcal{C}_{\mathfrak{sl}_2}$ with objects being neither highest, nor lowest, weight modules.

Proposition 2.1.8. *We have the decomposition*

$$\mathcal{C} = \mathcal{C}_{\mathfrak{sl}_2} \sqcup \mathcal{H} \sqcup \mathcal{L} \sqcup \mathcal{B} \sqcup \mathcal{N}. \tag{2.5}$$

Proof. By Lemma 2.1.7, any morphism in $\mathcal{C}$ is an isomorphism, so that the decomposition follows immediately from the definitions. □

We end this section by giving the classification of $\mathcal{C}_{\mathfrak{sl}_2}$, which is an immediate consequence of the classical case of the corresponding $\mathfrak{sl}_2$-modules.

Proposition 2.1.9. *The category $\mathcal{C}_{\mathfrak{sl}_2}$ is classified by the set of modules of either of the forms:*

i $N_{\mathfrak{sl}_2}(n)$, *where* $n \in \mathbb{Z}_+$.

ii $M_{\mathfrak{sl}_2}(\lambda)$, *where* $\lambda \in \mathbb{C}\backslash\mathbb{N}$.

iii $M'_{\mathfrak{sl}_2}(\lambda)$, *where* $\lambda \in \mathbb{C}\backslash -\mathbb{N}$.

iv $L_{\mathfrak{sl}_2}(\lambda, \tau)$, *where* $\lambda \in \{a + bi | a, b \in \mathbb{R}$ *and* $a \in [0, 1)\}$, $\tau \in \mathbb{C}$ *and* $\tau \notin (\lambda + 1 + 2\mathbb{Z})^2$.

Here p, q *and* z *act like* 0 *on each module, and the modules are otherwise determined by the following:*

i $N_{\mathfrak{sl}_2}(n)$ *has basis* $\{v_i\}_{i\in\{0,\dots,n-1\}}$ *and actions:*

$$\begin{aligned} fv_i &= v_{i+1} \\ ev_i &= i(n-i)v_{i-1} \\ hv_i &= (n-1-2i)v_i, \end{aligned} \tag{2.6}$$

where we set $v_{-1} = 0$.

ii $M_{\mathfrak{sl}_2}(\lambda)$ *has basis* $\{v_i\}_{i\in\mathbb{N}}$ *and actions:*

$$\begin{aligned} fv_i &= v_{i+1} \\ ev_i &= i(\lambda - i + 1)v_{i-1} \\ hv_i &= (\lambda - 2i)v_i, \end{aligned} \tag{2.7}$$

where we set $v_{-1} = 0$.

iii $M'_{\mathfrak{sl}_2}(\lambda)$ *has basis* $\{v_i\}_{i\in\mathbb{N}}$ *and actions:*

$$\begin{aligned} fv_i &= v_{i+1} \\ ev_i &= -i(\lambda + i - 1)v_{i-1} \\ hv_i &= (\lambda + 2i)v_i, \end{aligned} \tag{2.8}$$

where we set $v_{-1} = 0$.

iv $L_{\mathfrak{sl}_2}(\lambda, \tau)$ *has basis* $\{v_i\}_{i \in 2\mathbb{Z}}$ *and actions:*

$$\begin{aligned} fv_i &= v_{i-2} \\ ev_i &= \frac{1}{4}(\tau - (\lambda - 2i + 1)^2)v_{i+2} \\ hv_i &= (\lambda - 2i)v_i. \end{aligned} \tag{2.9}$$

Proof. Let $\mathcal{D}$ be the category with the simple complex weight modules with finite-dimensional weight spaces over $\mathfrak{sl}_2$ as objects, and module isomorphisms between these as morphisms. Then the map $\Psi : \mathcal{C}_{\mathfrak{sl}_2} \to \mathcal{D}$ which restricts the actions on modules in $\mathcal{C}_{\mathfrak{sl}_2}$ to $U(\mathfrak{sl}_s)$ and leaves morphisms unchanged is an isomorphism of categories, with Ψ^{-1} extending the action of $U(\mathfrak{sl}_s)$ to an action of U by letting p, q and z act like 0 on every module. Indeed, the morphisms in both categories $\mathcal{C}_{\mathfrak{sl}_2}$ and $\mathcal{D}$ are the same, since every linear map preserves actions by 0. A classification of $\mathcal{D}$ can be found in Theorem 3.32 in [Mazorchuk, 2010, p. 72]. Applying Ψ^{-1} to this classification yields the classification in the statement of the proposition. □

2.2 Classification of simple highest and lowest weight U-modules with finite-dimensional weight spaces

In this section, we present the classification of $\mathcal{H}$, $\mathcal{L}$ and $\mathcal{B}$, in essence due to Dobrev et al. [1997]. The classification of $\mathcal{H}$ will also play the role of an important lemma for the classification of the remaining subcategory, *i.e.* that of $\mathcal{N}$.

Proposition 2.2.1. *The category* $\mathcal{H}$ *is classified by*

$$\{M(\lambda, c)\}_{\lambda \in \mathbb{C} \setminus (-\frac{1}{2} + \mathbb{N}), c \in \mathbb{C} \setminus \{0\}} \cup \{N(\lambda, c)\}_{\lambda \in -\frac{1}{2} + \mathbb{N}, c \in \mathbb{C} \setminus \{0\}} \tag{2.10}$$

where $M(\lambda, c)$ *(called a* Verma module*) has basis* $\{v_{i,j}\}_{i,j \in \mathbb{N}}$ *on which the action* *of U is given by*

$$qv_{i,j} = v_{i+1,j} \tag{2.11}$$

$$fv_{i,j} = v_{i,j+1} \tag{2.12}$$

$$zv_{i,j} = cv_{i,j} \tag{2.13}$$

$$hv_{i,j} = (\lambda - i - 2j)v_{i,j} \tag{2.14}$$

$$pv_{i,j} = -jv_{i+1,j-1} + civ_{i-1,j} \tag{2.15}$$

$$ev_{i,j} = j(\lambda + 1 - i - j)v_{i,j-1} + \frac{1}{2}ci(i-1)v_{i-2,j}, \tag{2.16}$$

and $N(\lambda, c)$ *has basis* $\{v_{i,j}\}_{i,j\in\mathbb{N}, j\leq \lambda+\frac{1}{2}}$ *on which the action of U is given by*

$$qv_{i,j} = v_{i+1,j} \tag{2.17}$$

$$fv_{i,j} = \begin{cases} v_{i,j+1} & \text{if } j < \lambda + \frac{1}{2} \\ -\sum_{s=0}^{\lambda+\frac{1}{2}} \frac{1}{(2c)^{\lambda+\frac{3}{2}-s}} \binom{\lambda+\frac{3}{2}}{s} v_{i+2\lambda+3-2s,s} & \text{if } j = \lambda + \frac{1}{2} \end{cases} \tag{2.18}$$

$$zv_{i,j} = cv_{i,j} \tag{2.19}$$

$$hv_{i,j} = (\lambda - i - 2j)v_{i,j} \tag{2.20}$$

$$pv_{i,j} = -jv_{i+1,j-1} + civ_{i-1,j} \tag{2.21}$$

$$ev_{i,j} = j(\lambda + 1 - i - j)v_{i,j-1} + \frac{1}{2}ci(i-1)v_{i-2,j}. \tag{2.22}$$

Also, the category $\mathcal{B}$ *is empty.*

Proof These results are easy to derive from Theorem 1 in Dobrev et al. [1997]: Consider the Lie algebra $\mathcal{S}(1)$ defined in Dobrev et al. [1997]. It is readily checked that

$$\begin{aligned} \psi : \mathcal{S}(1) &\to \mathcal{S} \\ D &\mapsto -h \\ P_t &\mapsto -e \\ P_x &\mapsto p \\ K &\mapsto f \\ G &\mapsto -q \\ m &\mapsto -z \end{aligned} \tag{2.23}$$

extends to a Lie algebra isomorphism[1]. Let $\mathcal{H}_\mathcal{S}$ and $\mathcal{B}_\mathcal{S}$ be the categories of $\mathcal{S}$-modules isomorphic to $\mathcal{H}$ and $\mathcal{B}$ respectively obtained via Proposition 2.1.2, and $\mathcal{H}_{\mathcal{S}(1)}$ and $\mathcal{B}_{\mathcal{S}(1)}$ the categories with objects being simple weight $\mathcal{S}(1)$-modules with lowest, but not heighest, weight (as defined in Dobrev et al. [1997]) and simple weight $\mathcal{S}(1)$-modules with both lowest and highest weight respectively, for $m \neq 0$, and morphisms being module isomorphisms in both cases. Then $\Psi_1 : \mathcal{H}_\mathcal{S} \to \mathcal{H}_{\mathcal{S}(1)}$ and $\Psi_2 : \mathcal{B}_\mathcal{S} \to \mathcal{B}_{\mathcal{S}(1)}$ acting on objects by pullback along ψ and on morphisms like the identity are invertible functors (with inverse on objects obtained via pullback along ψ^{-1}). Bijectivity on morphisms follows from the actions of the elements of $\mathcal{S}(1)$ being the same as the corresponding ones in $\psi(\mathcal{S})$. The desired result is then obtained upon applying Ψ_1 and Ψ_2 to the classifications of $\mathcal{H}_{\mathcal{S}(1)}$ and $\mathcal{B}_{\mathcal{S}(1)}$ respectively, and setting $\lambda = d$. □

The classification of $\mathcal{H}$ now gives the classification of $\mathcal{L}$ as an easy corollary.

Corollary 2.2.2. *The category $\mathcal{L}$ is classified by*

$$\{M'(\lambda, c)\}_{\lambda \in \mathbb{C} \setminus (-\frac{1}{2}+\mathbb{N}), c \in \mathbb{C}\setminus\{0\}} \cup \{N'(\lambda, c)\}_{\lambda \in -\frac{1}{2}+\mathbb{N}, c \in \mathbb{C}\setminus\{0\}} \tag{2.24}$$

where $M'(\lambda, c)$ has basis $\{v_{i,j}\}_{i,j \in \mathbb{N}}$ on which the action of U is given by

$$pv_{i,j} = v_{i+1,j} \tag{2.25}$$

$$ev_{i,j} = v_{i,j+1} \tag{2.26}$$

$$zv_{i,j} = -cv_{i,j} \tag{2.27}$$

$$hv_{i,j} = -(\lambda - i - 2j)v_{i,j} \tag{2.28}$$

$$qv_{i,j} = -jv_{i+1,j-1} + civ_{i-1,j} \tag{2.29}$$

$$fv_{i,j} = j(\lambda + 1 - i - j)v_{i,j-1} + \frac{1}{2}ci(i-1)v_{i-2,j}, \tag{2.30}$$

[1] Here the element m is identified with the scalar with which it acts on the modules in question (as follows from Proposition 2.1.6)

and $N'(\lambda, c)$ has basis $\{v_{i,j}\}_{i,j\in\mathbb{N}, j\leq\lambda+\frac{1}{2}}$ on which the action of U is given by

$$pv_{i,j} = v_{i+1,j} \tag{2.31}$$

$$ev_{i,j} = \begin{cases} v_{i,j+1} & \text{if } j < \lambda + \frac{1}{2} \\ -\sum_{s=0}^{\lambda+\frac{1}{2}} \frac{1}{(2c)^{\lambda+\frac{3}{2}-s}} \binom{\lambda+\frac{3}{2}}{s} v_{i+2\lambda+3-2s,s} & \text{if } j = \lambda + \frac{1}{2} \end{cases} \tag{2.32}$$

$$zv_{i,j} = -cv_{i,j} \tag{2.33}$$

$$hv_{i,j} = -(\lambda - i - 2j)v_{i,j} \tag{2.34}$$

$$qv_{i,j} = -jv_{i+1,j-1} + civ_{i-1,j} \tag{2.35}$$

$$ev_{i,j} = j(\lambda + 1 - i - j)v_{i,j-1} + \frac{1}{2}ci(i-1)v_{i-2,j}. \tag{2.36}$$

Proof It is readily checked that

$$\begin{aligned} U &\to U \\ -h &\mapsto h \\ f &\mapsto e \\ q &\mapsto p \\ e &\mapsto f \\ p &\mapsto q \\ -z &\mapsto z \end{aligned} \tag{2.37}$$

extends to an algebra automorphism. In the same way as in the proof of Proposition 2.2.1, the pullback of this automorphism gives rise to a category isomorphism from $\mathcal{H}$ to $\mathcal{L}$. The desired result is obtained upon application of this functor to the classification of $\mathcal{H}$ found in Proposition 2.2.1. □

Chapter 3

Classification of the simple weight U-modules with finite-dimensional weight spaces and neither highest nor lowest weight

In this chapter, the classification of $\mathcal{C}$ will be completed using the classification of $\mathcal{H}$ presented in the previous chapter. What remains is to classify $\mathcal{N}$. The strategy is as follows: We will consider the localization of U with respect to f, denoted $U^{(f)}$, and then introduce a one-parameter family of certain $U^{(f)}$-automorphisms, each member of which will then induce a "twisting functor" on the category of U-modules. It will be shown that each module in $\mathcal{N}$ can be "twisted" into a U-module which is generated by some member of $\mathcal{H}$. Therefore, we will see, every module in $\mathcal{N}$ may be obtained by twisting back a module in $\mathcal{H}$. At this point, we will still have some redundancy though, which prevents us from completing the classification. However, our work will in particular have given us the knowledge that z can not act like 0 on modules in $\mathcal{N}$, and using this, we may repeat the entire procedure, only now we localize with respect to q instead. This time the redundancy will be easily dealt with.

3.1 Properties of modules in $\mathcal{N}$

The next few general results on the modules in $\mathcal{N}$ will be of crucial importance in the next section and, ultimately, in Theorem 3.3.1, where $\mathcal{N}$ is classsified.

Lemma 3.1.1. *1. Let V be a U-module, $v \in V$, $u, s \in U$ and $n, m \in \mathbb{N}$. Assume that $\operatorname{ad}_s^n u = 0$. If $s^m v = 0$ holds, then also we have that $s^{nm} uv = 0$. In particular, the action of ad_s on U being locally nilpotent implies that local nilpotency of the action of s on vectors of V is preserved by the action of U.*

2. Additionally, for $s \in \{p, q, e, f\}$, the action of ad_s on U is indeed locally nilpotent, so that the previous part may be applied.

Proof 1. Assume that $s^m v = 0$. For some $r_n \in \operatorname{span}(\{s^i u s^j\}_{i,j\in\mathbb{N}})s$, we have $\operatorname{ad}_s^n u = s^n u + r_n$, as follows from induction with base case $\operatorname{ad}_s u = su - us$ and, assuming for $k \in \mathbb{N}$ that $\operatorname{ad}_s^k u = s^k u + r_k$ where $r_k \in \operatorname{span}(\{s^i u s^j\}_{i,j\in\mathbb{N}})s$, induction step $\operatorname{ad}_s^{k+1} u = s^{k+1} u + s r_k - s^k u s - r_k s$, so we may take $r_{k+1} = s r_k - s^k u s - r_k s \in \operatorname{span}(\{s^i u s^j\}_{i,j\in\mathbb{N}})s$. Also $\operatorname{ad}_s^n u = 0$ by assumption, so $s^n u = -r_n$.

Using this repeatedly, and finally the nilpotency of s on v, we obtain

$$\begin{aligned} s^{nm} uv = s^{n(m-1)}(-r_n v) &\in s^{n(m-1)} \operatorname{span}(\{s^i u s^j\}_{i,j\in\mathbb{N}}) sv \\ &\subset s^{n(m-2)} \operatorname{span}(\{s^i u s^j\}_{i,j\in\mathbb{N}}) s^2 v \\ &\vdots \\ &\subset \operatorname{span}(\{s^i u s^j\}_{i,j\in\mathbb{N}}) s^m v = \{0\}. \end{aligned} \tag{3.1}$$

Thus s acts nilpotently on uv as well.

2. Assume that for some $u \in U$ and $k \in \mathbb{N}$, $\operatorname{ad}_s^k(u) = 0$ and consider an arbitrary $b_1 \in \{f, q, h, z, p, e\}$. If we can show that $\operatorname{ad}_s^{k+2}(b_1 u) = 0$, then by induction (with trivial base case $u = 1$) and the fact that U is spanned by products of elements in $\{1, f, q, h, z, p, e\}$, we get that ad_s acts locally nilpotently on U.

By use of formulae 2.1 we see that in case $s = e$, there is a sequence $b_2 \in \operatorname{span}(e, p, h)$, $b_3 \in \operatorname{span}(e)$, $\{b_i\}_{3<i\in\mathbb{N}} = \{0\}$, in case $s = f$ a sequence $b_2 \in \operatorname{span}(f, q, h)$, $b_3 \in \operatorname{span}(f)$, $\{b_i\}_{3<i\in\mathbb{N}} = \{0\}$, and in the case $s \in \{p, q\}$ a sequence $b_2 \in \operatorname{span}(p, q, z)$, $b_3 \in \operatorname{span}(z)$, $\{b_i\}_{3<i\in\mathbb{N}} = \{0\}$, such that for all $i, j \in \mathbb{N}$

$$\operatorname{ad}_s(b_i \operatorname{ad}_s^j(u)) = sb_i \operatorname{ad}_s^j(u) - b_i \operatorname{ad}_s^j(u)s = b_{i+1} \operatorname{ad}_s^j(u) + b_i \operatorname{ad}_s^{j+1}(u) \tag{3.2}$$

holds, and, by using the above formula repeatedly, that we get

$$\begin{aligned}\operatorname{ad}_s^{k+2}(b_1 u) &\in \operatorname{ad}_s^{k+1} \operatorname{span}(b_2 u, b_1 \operatorname{ad}_s(u)) \\ &\subset \operatorname{ad}_s^{k} \operatorname{span}(b_3 u, b_2 \operatorname{ad}_s(u), b_1 \operatorname{ad}_s^2(u)) \\ &\subset \operatorname{ad}_s^{k-1} \operatorname{span}(b_3 \operatorname{ad}_s(u), b_2 \operatorname{ad}_s^2(u), b_1 \operatorname{ad}_s^3(u)) \\ &\vdots \\ &\subset \operatorname{span}(b_3 \operatorname{ad}_s^k u, b_2 \operatorname{ad}_s^{k+1}(u), b_1 \operatorname{ad}_s^{k+2}(u)) \\ &= \{0\}\end{aligned} \tag{3.3}$$

and the desired result follows.

□

Lemma 3.1.1 can be used, together with other results, to prove the following theorem due to Wu and Zhu [2013].

Theorem 3.1.2. *For any $L \in \mathcal{N}$ with $\lambda \in \operatorname{supp}(L)$, we have $\operatorname{supp}(L) = \lambda + \mathbb{Z}$ and $\dim(L_{\lambda+i}) = \dim(L_{\lambda+j})$ for all $i, j \in \mathbb{Z}$.*

The next lemma and the proposition to follow will, via the twisting functors of the next section, to great gain allow us to relate the members of $\mathcal{N}$ to those of $\mathcal{H}$.

Lemma 3.1.3. *Let V be a weight U-module with finite-dimensional weight spaces.*

1. *If e acts locally nilpotently on V, then V is a highest weight module. Similarly, if f acts locally nilpotently on V, then V is a lowest weight module.*

2. *Assume in addition that z acts like some non-zero $c \in \mathbb{C}$ on V. Then if p acts locally nilpotently on V, then V is a highest weight module, and that if q acts locally nilpotently on V, then V is a lowest weight module.*

Proof. Let $s_{1,1} = q = t_{2,1}$, $s_{2,1} = p = t_{1,1}$, $s_{1,2} = f = t_{2,2}$, $s_{2,2} = e = t_{1,2}$, and $i, j \in \{1, 2\}$. Assume, towards a contradiction, that $s_{i,j}$ acts locally nilpotently on V, but that V is not a lowest weight module in case $i = 1$ and not a highest weight module in case $i = 2$.

By assumption, there exists a $\lambda' \in \mathbb{C}$ with $\mathrm{Re}(\lambda') > 0$ such that $\lambda := (-1)^i\lambda' \in \mathrm{supp}(V)$. From the same assumption, together with local nilpotency of $s_{i,j}$ on V (also using that V has finite-dimensional weight spaces), follows that there is some $m \in \mathbb{N}$ such that $s_{i,j}^m V_\lambda = 0$, $\lambda + j \cdot (-1)^i m \in \mathrm{supp}(V)$ and some $0 \neq v \in V_{\lambda + j\cdot(-1)^i m}$ fulfilling $s_{i,j}v = 0$ exists. Using formulae 2.1, we may argue by induction as follows:

1. Assume that for some $k \in \mathbb{N}$ we have that $s_{i,2}t_{i,2}^{k+1}v = \sum_{l=0}^{k} (\lambda' + 2(m - l))t_{i,2}^k v$. Then we have

$$
\begin{aligned}
s_{i,2}t_{i,2}^{k+2}v &= (t_{i,2}s_{i,2} + (-1)^i h)t_{i,2}^{k+1}v \\
&= (t_{i,2}s_{i,2} + (-1)^i((-1)^i\lambda' + 2(-1)^i(m - k - 1)))t_{i,2}^{k+1}v \\
&= (t_{i,2}s_{i,2} + \lambda' + 2(m - k - 1))t_{i,2}^{k+1}v \\
\text{(by induction assumption)} &= \sum_{l=0}^{k} (\lambda' + 2(m - l))t_{i,2}^k v + (\lambda' + 2(m - k - 1))t_{i,2}^{k+1}v \\
&= \sum_{l=0}^{k+1} (\lambda' + 2(m - l))t_{i,2}^{k+1}v.
\end{aligned}
\tag{3.4}
$$

By induction, with base case given by $s_{i,2}t_{i,2}v = (t_{i,2}s_{i,2} + (-1)^i h)v = (\lambda' + 2m)v$, we obtain that $s_{i,2}t_{i,2}^{n+1}v = \sum_{l=0}^{n} (\lambda' + 2(m - l))t_{i,2}^n v$ for any $n \in \mathbb{N}$.

2. Assume that for some $k \in \mathbb{N}$ we have that $s_{i,1}t_{i,1}^{k+1}v = (-1)^i(k + 1)ct_{i,1}^k v$.

Under the assumption that z acts like c, we then have

$$\begin{aligned} s_{i,1}t_{i,1}^{k+2}v &= (t_{i,1}s_{i,1} + (-1)^i z)t_{i,1}^{k+1}v \\ &= (t_{i,1}s_{i,1} + (-1)^i c)t_{i,1}^{k+1}v \\ \text{(by induction assumption)} &= (-1)^i(k+1)ct_{i,2}^{k+1}v + (-1)^i ct_{i,1}^{k+1}v = (-1)^i(k+2)ct_{i,1}^{k+1}v. \end{aligned} \tag{3.5}$$

By induction, with base case given by $s_{i,1}t_{i,1}v = (t_{i,1}s_{i,1} + (-1)^i c)v = (-1)^i cv$, we obtain that $s_{i,1}t_{i,1}^{n+1}v = (-1)^i(n+1)ct_{i,1}^n v$ for any $n \in \mathbb{N}$.

From these calculations we see that

$$s_{i,j} : \text{span}(t_{i,j}^{n+1}v) \to \text{span}(t_{i,j}^n v), \tag{3.6}$$

and furthermore this map is injective (and therefore bijective) for $n \leq m$ in case $j = 2$, and for z acting like c in case $j = 1$. In these cases, we in particular see that $V_\lambda \ni t_{i,j}^m v \neq 0$ (otherwise we would get $v = 0$), as well as $s_{i,j}^m t_{i,j}^m v \neq 0$. This, however, contradicts $s_{i,j}^m V_\lambda = 0$. □

Proposition 3.1.4. *Let V be a weight U-module with finite-dimensional weight spaces. Assume that there is some $0 \neq v \in V$ such that $ev = 0$ or $pv = 0$. Then V has a simple highest weight submodule.*

Proof. By assumption,

$$W_e := \{v \in V | e^n v = 0 \text{ for some } n \in \mathbb{N}\} \neq 0 \tag{3.7}$$

or

$$W_p := \{v \in V | p^n v = 0 \text{ for some } n \in \mathbb{N}\} \neq 0. \tag{3.8}$$

By Lemma 3.1.1, W_e and W_p are submodules, and by Lemma 3.1.3, one of them is a highest weight module. Since V has finite-dimensional weight spaces, it is of finite length, so there is a simple highest weight submodule of V. □

Corollary 3.1.5. *Let $L \in \mathcal{N}$. Then both e and f act bijectively on L.*

If in addition z does not act like 0 on L, then also both p and q act bijectively on L.

Proof The actions of e, f, p and q decompose into their restrictions to the weight spaces of L, and by Theorem 3.1.2, these restrictions are linear maps between finite-dimensional vector spaces of equal dimension. Therefore it suffices to show that e, f act injectively on L, and that in case z does not act like 0, p and q do as well.

Let $s \in \{e, f\}$. Assume, towards a contradiction, that there is some $v \neq 0$ in L such that $sv = 0$. Since L is simple , $Uv = L$, so by Lemma 3.1.1, s acts locally nilpotently on L. This, however, contradicts Lemma 3.1.3. If $s \in \{p, q\}$ and z does not act like 0 on L, then the same argument gives the second part of the statement.

3.2 The twisting functor

The definition of the twisting functors hinges on the localizations of U with respect to f and q respectively. While such localizations, as can be defined in a very general sense by a certain universal property (cf. [Lam, 1999, p. 289]), are guaranteed to exist for any ring, we will require additional properties, similar to those which hold in the case of a domain, of our localizations.

Definition In this text, *the localization,* $U^{(u)}$*, of* U *with respect to* $u \in U$ is defined to be the unique algebra such that $U \subset U^{(u)}$ is a ring extension in which u is invertible, and every element can be written on the form $u^{-n}s$, for some $s \in U$ and $n \in \mathbb{N}$, if such an algebra exists.

The next proposition shows that these requirements prove not to constitute any problems.

Proposition 3.2.1. *Let* $u \in \{q, f\}$*. Then the localization* $U^{(u)}$ *exists.*

Proof By Lemma 3.1.1, ad_u is locally nilpotent on U. Then by Lemma 4.2 in Mathieu [2000], q and f satisfy Ore's localizability conditions, so that $U^{(u)}$ exists, as proved in §10A in Lam [1999]. □

Thanks to the extra requirements, we have the following analogue to the Poincaré-Birkhoff-Witt theorem.

Proposition 3.2.2. *The set* $\{q^{i_1}f^{i_2}p^{i_3}e^{i_4}h^{i_5}z^{i_6}\}_{i_1\in\mathbb{Z} \text{ and } i_2\dots,i_6\in\mathbb{N}}$ *is a basis for* $U^{(q)}$*, and* $\{q^{i_1}f^{i_2}p^{i_3}e^{i_4}h^{i_5}z^{i_6}\}_{i_2\in\mathbb{Z} \text{ and } i_1,i_3,\dots,i_6\in\mathbb{N}}$ *is a basis for* $U^{(f)}$.

Proof. We will use that, by Proposition 2.1.3, $\{q^i f^j p^k e^l h^m z^n\}_{i,j,k,l,m,n\in\mathbb{N}}$ is a basis for U. By definition of the localization, $\{q^{i_1}f^{i_2}p^{i_3}e^{i_4}h^{i_5}z^{i_6}\}_{i_1\in\mathbb{Z} \text{ and } i_2\dots,i_6\in\mathbb{N}}$ spans $U^{(q)}$. Next, if there were linearly dependent elements

$$u_1,\dots,u_n \in \{q^{i_1}f^{i_2}p^{i_3}e^{i_4}h^{i_5}z^{i_6}\}_{i_1\in\mathbb{Z} \text{ and } i_2\dots,i_6\in\mathbb{N}}$$

such that $a_1u_1+\cdots+a_nu_n=0$, for some $0\neq a_1,\dots,a_n\in\mathbb{C}$, then for some $m\in\mathbb{N}$, we would have that $q^m u_1,\dots,q^m u_n$ were basis elements of U, but also that $a_1q^mu_1+\cdots+a_nq^mu_n=0$, a contradiction.

Because q and f commute, it is easily seen that $\{q,q^{-1},f,f^{-1}\}$ has pairwise commuting elements, and so an analogous argument works for the case of $U^{(f)}$. □

Next we construct and give basic properties of the automorphisms of $U^{(q)}$ and $U^{(f)}$ which will be used to define the twisting functors.

Proposition 3.2.3. *There are families of algebra automorphisms* $\{\Theta_x^{(q)}\}_{x\in\mathbb{C}}$ *and* $\{\Theta_x^{(f)}\}_{x\in\mathbb{C}}$*, with* $\Theta_x^{(q)}: U^{(q)}\to U^{(q)}$ *being the unique extension of*

$$\begin{aligned}
\Theta_x^{(q)}(q^{\pm1}) &= q^{\pm1}\\
\Theta_x^{(q)}(f) &= f\\
\Theta_x^{(q)}(z) &= z\\
\Theta_x^{(q)}(h) &= h-x\\
\Theta_x^{(q)}(p) &= p+xq^{-1}z\\
\Theta_x^{(q)}(e) &= e+xq^{-1}p+\frac{1}{2}x(x-1)q^{-2}z.
\end{aligned}\tag{3.9}$$

and $\Theta_x^{(f)} : U^{(f)} \to U^{(f)}$ *being the unique extension of*

$$\begin{aligned}
\Theta_x^{(f)}(q) &= q \\
\Theta_x^{(f)}(f^{\pm 1}) &= f^{\pm 1} \\
\Theta_x^{(f)}(z) &= z \\
\Theta_x^{(f)}(h) &= h - 2x \\
\Theta_x^{(f)}(p) &= p - xqf^{-1} \\
\Theta_x^{(f)}(e) &= e + x(h - 1 - x)f^{-1}.
\end{aligned} \tag{3.10}$$

Proof If there is an algebra morphism fulfilling the formulae 3.9 or 3.10, then it is uniquely determined, since $\{q, q^{-1}, f, z, h, p, e\}$ is a generating set for $U^{(q)}$ and $\{q, f, f^{-1}, z, h, p, e\}$ is a generating set for $U^{(f)}$. Let $u \in \{q, f\}$.

As for the preservation of the algebra relations, we will to begin with show that $\Theta_x^{(u)}$ is an automorphism when $x \in \mathbb{N}$. This will be done by showing that for $s \in \{q, f, z, h, p, e, u^{-1}\}$, we have $\Theta_x^{(u)}(s) = u^{-x} s u^x$. The extension of this conjugation to $U^{(u)}$ is clearly again conjugation by u^x, and conjugation with an invertible element gives rise to an algebra automorphism. For $s \in \{q, f, z, u^{-1}\}$ the result is obvious, since then u commutes with s. For other s we proceed by induction, where the base case, when $x = 0$, is immediate. In case $u = q$, assume that the formulae 3.9 hold true for $x = k \in \mathbb{N}$. The induction step in the case $s = h$ is given by

$$\begin{aligned}
\Theta_{k+1}^{(q)}(h) &= q^{-1}\Theta_k(h)q \\
&= q^{-1}(h - k)q \\
&= q^{-1}(hq - kq) \\
&= q^{-1}(qh - q - kq) \\
&= h - (k + 1),
\end{aligned} \tag{3.11}$$

in the case $s = p$ by

$$\begin{aligned}\Theta_{k+1}^{(q)}(p) &= q^{-1}\Theta_k(p)q \\ &= q^{-1}(p + kq^{-1}z)q \\ &= q^{-1}pq + kq^{-1}z \\ &= q^{-1}(qp + z) + kq^{-1}z \\ &= p + q^{-1}z + kq^{-1}z \\ &= p + (k+1)q^{-1}z\end{aligned} \tag{3.12}$$

and in the case $s = e$ by

$$\begin{aligned}\Theta_{k+1}^{(q)}(e) &= q^{-1}\Theta_k(e)q \\ &= q^{-1}(e + kq^{-1}p + \frac{1}{2}k(k-1)q^{-1}z)q \\ &= q^{-1}(eq + kq^{-1}pq + \frac{1}{2}k(k-1)q^{-1}z \\ &= q^{-1}(qe + p + kq^{-1}(qp + z) + \frac{1}{2}k(k-1)q^{-1}z) \\ &= e + q^{-1}p + kq^{-1}p + kq^{-1}z + \frac{1}{2}k(k-1)q^{-2}z \\ &= e + (k+1)q^{-1}p + \frac{1}{2}(k+1)(k+1-1)q^{-2}z.\end{aligned} \tag{3.13}$$

In case $u = f$, assume that the formulae 3.9 hold true for $x = k \in \mathbb{N}$. The induction step in the case $s = h$ is given by

$$\begin{aligned}\Theta_{k+1}^{(f)}(h) &= f^{-1}\Theta_k(h)f \\ &= f^{-1}(h - 2k)f \\ &= f^{-1}(hf - 2kf) \\ &= f^{-1}(fh - 2f - 2kf) \\ &= h - 2(k+1),\end{aligned} \tag{3.14}$$

in the case $s = p$ by

$$\begin{aligned}
\Theta_{k+1}^{(f)}(p) &= f^{-1}\Theta_k(p)f \\
&= f^{-1}(p - kqf^{-1})f \\
&= f^{-1}pf - kf^{-1}qf^{-1}f \\
&= f^{-1}(fp - q) - kf^{-1}q \\
&= p - (k+1)qf^{-1}
\end{aligned} \tag{3.15}$$

and in the case $s = e$ by

$$\begin{aligned}
\Theta_{k+1}^{(f)}(e) &= f^{-1}\Theta_k(e)f \\
&= f^{-1}(e + k(h-1-k)f^{-1})f \\
&= f^{-1}(ef + k(h-1-k)) \\
&= f^{-1}(fe + h + k(h-1-k)) \\
&= e + f^{-1}(k+1)(h-k) \\
&= e + f^{-1}(k+1)(h-k)ff^{-1} \\
&= e + f^{-1}f(k+1)(h-2-k)f^{-1} \\
&= e + (k+1)(h-1-(k+1))f^{-1}.
\end{aligned} \tag{3.16}$$

To show that $\Theta_x^{(u)}$ preserves the relations of $U^{(u)}$ for general $x \in \mathbb{C}$, it suffices to show that each $\Theta_x^{(u)}$ preserves commutators of the generators of $U^{(u)}$, *i.e.* that $\Theta_x^{(u)}(s_1s_2 - s_2s_1) = \Theta_x^{(u)}(s_1)\Theta_x^{(u)}(s_2) - \Theta_x^{(u)}(s_2)\Theta_x^{(u)}(s_1)$ for all $s_1, s_2 \in \{q, f, z, h, p, e, u^{-1}\}$. It is clear from the defining formulae 3.9 and 3.10 respectively that for abitrary $s_1, s_2 \in \{q, f, z, h, p, e, u^{-1}\}$

$$\Theta_x^{(u)}(s_1s_2 - s_2s_1) - (\Theta_x^{(u)}(s_1)\Theta_x^{(u)}(s_2) - \Theta_x^{(u)}(s_1)\Theta_x^{(u)}(s_2)) \tag{3.17}$$

is a polynomial in x, with each coefficient belonging to $U^{(u)}$ and therefore being a linear combination of basis elements of $U^{(u)}$, so that for some (finite) indexing set F and subset $\{b_i\}_{i\in F}$ of a basis of $U^{(u)}$ we have

$$\Theta_x^{(u)}(s_1s_2 - s_2s_1) - (\Theta_x^{(u)}(s_1)\Theta_x^{(u)}(s_2) - \Theta_x^{(u)}(s_1)\Theta_x^{(u)}(s_2)) = \sum_{i\in F} g_i(x)b_i \tag{3.18}$$

where the $g_i(x)$ are polynomials in x. But we have seen that $\Theta_x^{(u)}$ is an automorphism, so that $\sum_{i \in F} g_i(x) b_i = 0$, for $x \in \mathbb{N}$, so that $g_i(x) = 0$ for all $i \in F$ and $x \in \mathbb{N}$. From the fact that a nonzero polynomial in one variable can have only finitely many zeros now follows that $g_i \equiv 0$ for all $i \in F$. This in turn implies that $\Theta_x^{(u)}(s_1 s_2 - s_2 s_1) = \Theta_x^{(u)}(s_1)\Theta_x^{(u)}(s_2) - \Theta_x^{(u)}(s_1)\Theta_x^{(u)}(s_2)$ for all $x \in \mathbb{C}$, as desired.

Finally, it is an immediate consequence of the next proposition that $\Theta_x^{(u)} \circ \Theta_{-x}^{(u)} = \Theta_0^{(u)} = \mathrm{Id}$, so that $\Theta_x^{(u)}$ is invertible and thus an automorphism. □

Proposition 3.2.4. *For all $x, y \in \mathbb{C}$ and $u \in \{q, f\}$, we have $\Theta_x^{(u)} \circ \Theta_y^{(u)} = \Theta_{x+y}^{(u)}$.*

Proof This is a consequence of the defining formulae 3.9 and 3.10 applied to the generators of $U^{(u)}$. The only non-immediate case is $\Theta_x^{(u)} \circ \Theta_y^{(u)}(e) = \Theta_{x+y}^{(u)}(e)$, but this is given readily by direct calculation, and is left to the reader. □

For any $x \in \mathbb{C}$ and $u \in \{q, f\}$, we may view $U^{(u)}$ as a U-bimodule, denoted $U_x^{(u)}$, where $s \in U$ acts on $U^{(u)}$ from the left by multiplication with $\Theta_x^{(u)}(s)$, and from the right by multiplication with s.

Definition Denote by U-Mod the category of all U-modules and their morphisms. For $u \in \{q, f\}$, the functors

$$\begin{aligned} B_x^{(u)} : U\text{-Mod} &\to U\text{-Mod} \\ V &\mapsto U_x^{(u)} \otimes V, \end{aligned} \tag{3.19}$$

where $x \in \mathbb{C}$, are called *Mathieu's twisting functors* (with respect to U and u), and were first used by Mathieu [2000].

The next two results will in a straightforward way let us compose and invert the twisting functors when applied to $\mathcal{N}$.

Lemma 3.2.5. *For $x, y \in \mathbb{C}$ and $u \in \{q, f\}$, we have $B_x^{(u)} \circ B_y^{(u)} \simeq B_{x+y}^{(u)}$.*

Proof Let $V \in U$-Mod, $v \in V$, $s \in U$, $s_1 \in U_x^{(u)}$ and $s_2 \in U_y^{(u)}$. Then s acting on $s_1 \otimes (s_2 \otimes v) = 1 \otimes (\Theta_y^{(u)}(s_1)s_2 \otimes v) \in B_x^{(u)} \circ B_y^{(u)}(V)$ gives

$$\begin{aligned} s(1 \otimes \Theta_y^{(u)}(s_1)s_2 \otimes v) &= \Theta_x^{(u)}(s) \otimes (\Theta_y^{(u)}(s_1)s_2 \otimes v) \\ (\text{by Proposition 3.2.4}) &= 1 \otimes \Theta_{x+y}^{(u)}(s)\Theta_y^{(u)}(s_1)s_2 \otimes v \end{aligned} \tag{3.20}$$

so the map

$$\begin{aligned} B_x^{(u)} \circ B_y^{(u)}(V) &\to B_{x+y}^{(u)}(V) \\ s_1 \otimes (s_2 \otimes v) &\mapsto \Theta_y^{(u)}(s_1) s_2 \otimes v \end{aligned} \tag{3.21}$$

is clearly an isomorphism of U-modules, from which follows that $B_x^{(u)} \circ B_y^{(u)}$ and $B_{x+y}^{(u)}$ are isomorphic functors. $\square$

Lemma 3.2.6. *Let $u \in \{q, f\}$ and let V be a U-module on which u acts bijectively. Then $B_0^{(u)}(V) \simeq V$.*

Proof Since u acts bijectively on V, every element of $B_0^{(u)}(V)$ may be written uniquely on the form $1 \otimes v$, where $v \in V$. Then

$$\begin{aligned} V &\to B_0^{(u)}(V) \\ v &\mapsto 1 \otimes v \end{aligned} \tag{3.22}$$

defines an isomorphism. $\square$

Applying the twisting functors to various modules of the form $N(\lambda, c)$ will in the next few results turn out to exhaust $\mathcal{N}$. The modules $B_x^{(q)}(N(\lambda, c))$ are described explicitly in the following proposition.

Proposition 3.2.7. *For $\lambda \in -\frac{1}{2} + \mathbb{N}$, $x \in \mathbb{C}$ and $0 \neq c \in \mathbb{C}$, the U-module $B_x^{(q)}(N(\lambda, c))$ has basis $\{v_{i,j}\}_{i,j \in \mathbb{Z}, 0 \leq j \leq \lambda + \frac{1}{2}}$, where $v_{i,j} = q^i f^j \otimes v_{0,0}$, and $v_{0,0} = v$ is a highest weight vector of $N(\lambda, c)$. The action is given by the following formulae:*

$$qv_{i,j} = v_{i+1,j} \tag{3.23}$$

$$fv_{i,j} = \begin{cases} v_{i,j+1} & \text{if } j < \lambda + \frac{1}{2} \\ -\sum_{s=0}^{\lambda+\frac{1}{2}} \frac{1}{(2c)^{\lambda+\frac{3}{2}-s}} \binom{\lambda+\frac{3}{2}}{s} v_{i+2\lambda+3-2s,s} & \text{if } j = \lambda + \frac{1}{2} \end{cases} \tag{3.24}$$

$$zv_{i,j} = cv_{i,j} \tag{3.25}$$

$$hv_{i,j} = (\lambda - (i + x) - 2j)v_{i,j} \tag{3.26}$$

$$pv_{i,j} = -jv_{i+1,j-1} + c(i + x)v_{i-1,j} \tag{3.27}$$

$$ev_{i,j} = j(\lambda + 1 - (i + x) - j)v_{i,j-1} + \frac{1}{2}c(i + x)((i + x) - 1)v_{i-2,j} \tag{3.28}$$

Proof The vector space $B_x^{(q)}(N(\lambda,c))$ is, by definition, Proposition 3.2.2 and the fact that z,h,p and e all act like scalars on v, spanned by $\{q^i f^j \otimes v\}_{i\in\mathbb{Z},j\in\mathbb{N}}$. For $j>\lambda+\frac{1}{2}$, however, $q^i f^j \otimes v = q^i f^{j-\lambda-\frac{3}{2}}(f v_{0,\lambda+\frac{1}{2}})$, which is a linear combination of elements of the form $q^{i'} f^{j'} \otimes v$, where $0\le j'<j$, so by induction, $B_x^{(q)}(N(\lambda,c))$ is spanned by $\{q^i f^j \otimes v\}_{i,j\in\mathbb{Z},0\le j\le\lambda+\frac{1}{2}}$ as well.

The elements of this set are also linearly independent, since if for some $m\in\mathbb{N}$ and $c_{-m},\ldots,c_m\in\mathbb{C}$ we would have $\sum_{|i|<m,0\le j\le\lambda+\frac{1}{2}} c_i(q^i f^j\otimes v)=0$, then it would follow that

$$\begin{aligned}0 &= \sum_{|i|<m,0\le j\le\lambda+\frac{1}{2}} c_i(q^{i+m} f^j \otimes v)\\ &= \sum_{|i|<m,0\le j\le\lambda+\frac{1}{2}} c_i(1\otimes q^{i+m} f^j v)\\ &= 1\otimes \sum_{|i|<m,0\le j\le\lambda+\frac{1}{2}} c_i q^{i+m} f^j v,\end{aligned}\tag{3.29}$$

which is impossible since the members of $\{q^{i+m} f^j v\}_{|i|<m,0\le j\le\lambda+\frac{1}{2}}$ are linearly independent.

Let us to begin with determine the action of U on the submodule of $B_0^{(q)}(N(\lambda,c))$ generated by $\{v_{i,j}\}_{i,j\in\mathbb{N}}$. This module is clearly isomorphic to $N(\lambda,c)$, though, and the action is given by formulae 2.17 through 2.22, which is the same as formulae 3.23 through 3.28 when $x=0$.

On the subset $\{v_{i,j}\}_{i,j\in\mathbb{N}}$ of $B_x^{(q)}(N(\lambda,c))$, $u\in U$ acts, by definition, as $\Theta_x^{(q)}(u)$ acts on $\{v_{i,j}\}_{i,j\in\mathbb{N}}\subset B_0(N(\lambda,c))$. This latter action is from Proposition 3.9 again seen to be given by the formulae 3.23 through 3.28, the nontrivial cases $u\in\{h,p,e\}$ of which via the following direct calculations[1]:

$$\begin{aligned}\Theta_x^{(q)}(h)v_{i,j} &= (h-x)v_{i,j}\\ &= (\lambda-i-2j-x)v_{i,j}\\ &= (\lambda-(i+x)-2j)v_{i,j},\end{aligned}$$

[1]One could also note that for $x\in\mathbb{Z}$, where $\Theta_x^{(q)}$ is an inner automorphism of $U^{(q)}$, $\Theta_x^{(q)}(u)$ acts on $v_{i,j}$ like u acts on $v_{i-x,j}$ and then argue along the lines of the proof of formulae 3.9 to obtain the same result.

$$\begin{aligned}\Theta_x^{(q)}(p)v_{i,j} &= (p + xq^{-1}z)v_{i,j}\\ &= -jv_{i+1,j-1} + (ci + xc)v_{i-1,j}\\ &= -jv_{i+1,j-1} + c(i + x)v_{i-1,j},\end{aligned}$$

and

$$\begin{aligned}\Theta_x^{(q)}(e)v_{i,j} &= (e + xq^{-1}p + \frac{1}{2}x(x-1)q^{-2})v_{i,j}\\ &= \frac{1}{2}ci(i-1)v_{i-2,j} + j(\lambda + 1 - j - i)v_{i,j-1} + x(-jv_{i,j-1} + civ_{i-2,j}) + \frac{1}{2}x(x-1)cv_{i-2,j}\\ &= \frac{1}{2}c(i(i-1) + x(x-1) + 2ix)v_{i-2,j} + j(\lambda + 1 - j - i - x)v_{i,j-1}\\ &= \frac{1}{2}c(i+x)((i+x)-1)v_{i-2,j} + j(\lambda + 1 - j - (i+x))v_{i,j-1}.\end{aligned}$$

Now, note that

$$uv_{i,j} = q^i\Theta_i^{(q)}(u)v_{0,j}$$

from which the action of U on $v_{i,j} \in B_0^{(q)}(N(\lambda, c))$ when $i < 0$ is immediately obtained. Finally, the action of U on $v_{i,j} \in B_x^{(q)}(N(\lambda, c))$ for general $x \in \mathbb{C}$ and $i \in \mathbb{Z}$ can now be calculated verbatim in the same way as for $i \geq 0$. □

Lemma 3.2.8. *Let $L \in \mathcal{N}$.*

1. *There is an $x \in \mathbb{C}$ and $0 \neq v \in B_x^{(f)}(L)$ such that $ev = 0$.*

2. *Assume that z does not act like 0 on L. Then similarly there is an $x \in \mathbb{C}$ and $0 \neq v \in B_x^{(q)}(L)$ such that $pv = 0$.*

Proof 1. It suffices to show that e does not for all $x \in \mathbb{C}$ act injectively on $B_x^{(f)}(L)$, which by the definition of $B_x^{(f)}$ and formulae 3.10 is equivalent to showing that for some $x \in \mathbb{C}$, $e + x(h - 1 - x)f^{-1}$ does not act injectively on L. Let $\lambda \in \operatorname{supp}(L)$, and pick bases in L_λ and $L_{\lambda+2}$. By Theorem 3.1.2, these spaces have the same (finite) dimension, say n, so $e_{|L_\lambda}$ and $f^{-1}_{|L_\lambda}$ are given by $n \times n$-matrices, E and F^{-1}, while $h_{|L_\lambda} = \lambda$. It now suffices to show that $\det(E + x(\lambda - 1 - x)F^{-1})$ is a non-constant polynomial in x, so that it, by the fundamental theorem of algebra, has some zero. By Leibniz' formula,

$$
\begin{aligned}
\det(E + x(\lambda - 1 - x)F^{-1}) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i^n (E + x(\lambda - 1 - x)F^{-1})_{i,\sigma_i} \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i^n (E_{i,\sigma_i} + x(\lambda - 1 - x)(F^{-1})_{i,\sigma_i}) \\
&= (x(\lambda - 1 - x))^n \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i^n (F^{-1})_{i,\sigma_i} + g(x) \\
&= \det(F^{-1}) x^n (\lambda - 1 - x)^n + g(x),
\end{aligned}
$$

where $g(x)$ is some polynomial fulfilling $\deg(g(x)) < 2n$. The coefficient of the highest-degree term, $(-1)^n \det(F^{-1})$, is non-zero since F^{-1} is invertible, so the polynomial is not constant.

2. Analogously to the first part, the second part is proved if we can, under the assumption that z acts like $c \neq 0$, show that $\Theta_x^{(q)}(p) = p - xzq^{-1}$ does not for all $x \in \mathbb{C}$ act injectively on L. Let the actions $p_{|L_\lambda}$ and $q^{-1}_{|L_\lambda}$ be given by the $n \times n$-matrices, P and Q^{-1} respectively. By Leibniz' formula again,

$$
\begin{aligned}
\det(P - xcQ^{-1}) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i^n (P - cxQ^{-1}_{i,\sigma_i}) \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i^n (P_{i,\sigma_i} - cx(Q^{-1})_{i,\sigma_i}) \\
&= x^n \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i^n (-c(Q^{-1})_{i,\sigma_i}) + g(x) \\
&= (-c)^n \det(Q^{-1}) x^n + g(x),
\end{aligned}
$$

where $g(x)$ this time is some polynomial fulfilling $\deg(g(x)) < n$. The coefficient of the highest-degree term, $(-c)^n \det(Q^{-1})$, is non-zero since $c \neq 0$ and Q^{-1} is invertible, so the polynomial is not constant. The desired result follows.

□

Proposition 3.2.9. *Let $L \in \mathcal{N}$. There are $x \in \mathbb{C}$, $c \in \mathbb{C}\backslash\{0\}$ and $\lambda \in -\frac{1}{2} + \mathbb{N}$ such that $L \simeq B_{-x}^{(f)}(N)$.*

Proof For any $x \in \mathbb{C}$, one sees from Proposition 3.2.3 that $B_x^{(f)}(L)$ is a weight module, the weight spaces of which have the same, finite, dimension as those of L (all weight spaces of L have the same dimension by Theorem 3.1.2). By Lemma 3.2.8, there is an $x \in \mathbb{C}$ and $0 \neq v \in B_x^{(f)}(L)$ such that $ev = 0$. Then by Proposition 3.1.4, there is a simple highest weight submodule, N, of $B_x^{(f)}(L)$. Then the module $B_{-x}^{(f)}(N)$ is a submodule of $B_{-x}^{(f)}(B_x^{(f)}(L))$, which by Lemmas 3.2.5 and 3.2.6 is isomorphic to L. But L simple then implies that $L \simeq B_{-x}^{(f)}(N)$.

Also we know from Proposition 2.2.1 that one of the following cases holds:

i $pN = 0 = qN$.

ii $N \simeq M(\lambda, c)$ for some $\lambda \in \mathbb{C}\backslash(-\frac{1}{2} + \mathbb{N})$ and $c \in \mathbb{C}\backslash\{0\}$.

iii $N \simeq N(\lambda, c)$ for some $\lambda \in -\frac{1}{2} + \mathbb{N}$ and $c \in \mathbb{C}\backslash\{0\}$.

Now, in the first case, p and q would act on $L \simeq B_{-x}^{(f)}(N)$ like $\Theta_{-x}^{(f)}(p)$ and $\Theta_{-x}^{(f)}(q)$ respectively act on N. From Proposition 3.2.3 we see that these elements would act like zero, so we would have that $pL = 0 = qL$, but this contradicts $L \in \mathcal{N}$. In the second case, the dimensions of the weight spaces would be unbounded, so the same would be true for $B_{-x}^{(f)}(N) \simeq L$, again contradicting $L \in \mathcal{N}$. Therefore the third alternative must hold, so that $L \simeq B_{-x}^{(f)}(N(\lambda, c))$. □

Proposition 3.2.10. *On an arbitrary $L \in \mathcal{N}$, z does not act like 0.*

Proof. By Proposition 3.2.9, we have $L \simeq B_{-x}^{(f)}(N(\lambda, c))$ for certain $x, c, \lambda \in \mathbb{C}$ where $c \neq 0$ (so that z does not act like 0 on $N(\lambda, c)$). The action of z is not altered by $B_{-x}^{(f)}$, however, so the claim follows. □

Proposition 3.2.10 finally lets us apply to modules in $\mathcal{N}$ several of our previous results in full force, without having to worry about the possible special case of z acting like 0.

3.3 Classification of $\mathcal{N}$

We can now classify $\mathcal{N}$, and thereby complete the classification of $\mathcal{C}$.

Theorem 3.3.1. *The category $\mathcal{N}$, as defined immediately before Proposition 2.1.8, is classified by*

$$\{B_x^{(q)}(N(\lambda,c))\}_{\lambda\in-\frac{1}{2}+\mathbb{N},c\in\mathbb{C}\backslash\{0\},x\in\{a+bi|a,b\in\mathbb{R} \text{ and } a\in(0,1)\}}, \tag{3.30}$$

where the modules $B_x^{(q)}(N(\lambda,c))$ are as defined in Proposition 3.2.7.

Proof Let $L \in \mathcal{N}$. By Corollary 3.2.10, we know that z does not act like 0 on L. Therefore the very same argument used in the proof of Proposition 3.2.9, but with f exchanged for q and e for p, gives that $L \simeq B_{-x}^{(q)}(N(\lambda,c))$, for some $x \in \mathbb{C}$, $c \in \mathbb{C}\backslash\{0\}$ and $\lambda \in -\frac{1}{2}+\mathbb{N}$.

Conversely, let $\lambda \in -\frac{1}{2}+\mathbb{N}$, $c \in \mathbb{C}\backslash\{0\}$ and $x \in \mathbb{C}$ be arbitrary. Let also $S \subset B_x^{(q)}(N(\lambda,c))$ be a simple (not necessarily proper) submodule. Then by Lemma 3.2.6, $B_{-x}^{(q)}(S) \subset B_0^{(q)}(N(\lambda,c))$ is a submodule as well. In addition, $B_0^{(q)}(N(\lambda,c))$ clearly has a submodule which we may identify with $N(\lambda,c)$. It is from this easily seen that $B_{-x}^{(q)}(S) \cap N(\lambda,c) \subset N(\lambda,c)$ is yet another submodule. This submodule is non-zero, because the injective action of q is unchanged by the twisting functors, so that $B_{-x}^{(q)}(S)$ has vectors of arbitrarily low weight, some of which must then lie in $N(\lambda,c)$ as well. From simplicity of $N(\lambda,c)$ follows that $B_{-x}^{(q)}(S) \cap N(\lambda,c) = N(\lambda,c)$, and therefore $B_0^{(q)}(S) = B_x^{(q)}(N(\lambda,c))$, so that either $S = B_x^{(q)}(N(\lambda,c))$, in which case $B_x^{(q)}(N(\lambda,c))$ is simple, or S is a highest weight module with maximal weight space dimension equal to that of $N(\lambda,c)$. From Proposition 3.2.7 we in this latter case get that in fact $S \simeq N(\lambda,c)$, so that $B_0^{(q)}(S) \simeq B_0^{(q)}(N(\lambda,c))$. All in all, we see that either $B_x^{(q)}(N(\lambda,c))$ is simple, or $B_0^{(q)}(N(\lambda,c)) \simeq B_x^{(q)}(N(\lambda,c))$. The latter case is investigated together with possible redundancy as follows.

Assume that $B_{x_1}^{(q)}(N(\lambda,c)) \simeq B_{x_2}^{(q)}(N(\lambda',c'))$. That $c'=c$ is obvious. The isomorphy also implies equality of weight space dimensions, so it follows from Proposition 3.2.7 that $\lambda'=\lambda$, and we furthermore get, by Lemma 3.2.5, that

$B_x^{(q)}(N(\lambda, c)) \simeq B_0^{(q)}(N(\lambda, c))$, where $x = x_1 - x_2$. Then

$$\mathbb{Z} + \lambda - x = \operatorname{supp}(B_x^{(q)}(N(\lambda, c))) = \operatorname{supp}(B_0^{(q)}(N(\lambda, c))) = \mathbb{Z} + \lambda \tag{3.31}$$

so that $x \in \mathbb{Z}$. Whenever $x \in \mathbb{Z}$, one conversely easily sees from formulae 3.23-3.28 that

$$\Phi : B_0^{(q)}(N(\lambda, c)) \to B_x^{(q)}(N(\lambda, c))$$
$$v_{i+x,j} \mapsto v_{i,j}$$

defines an isomorphism. Thus $B_0^{(q)}(N(\lambda, c))$ and $B_x^{(q)}(N(\lambda, c))$ are isomorphic if and only if $x \in \mathbb{Z}$, so that in total (via another application of Lemma 3.2.5) we have that $B_{x_1}^{(q)}(N(\lambda, c))$ is isomorphic to $B_{x_2}^{(q)}(N(\lambda', c'))$ if and only if $\lambda = \lambda'$, $c' = c$ and $x_1 - x_2 \in \mathbb{Z}$.

Therefore we see that the proposed classifying set in the statement of the theorem indeed is a maximal set of pairwise non-isomorphic members of $\mathcal{N}$. $\square$

References

Angel Ballesteros, Francisco J Herranz, and Preeti Parashar. (1+1) schrödinger lie bialgebras and their poisson-lie groups. *Journal of Physics A: Mathematical and General*, 33(17):3445, 2000. URL http://stacks.iop.org/0305-4470/33/i=17/a=304. 1

A.A.O. Barut and R. Raczka. *Theory of group representations and applications*. World Scientific Publishing Company Incorporated, 1986. ISBN 9789971502171. URL http://books.google.se/books?id=Mi1fy70II8cC. 1

A.A.O. Barut and B.W Xu. Conformal covariance and the probability interpretation of wave equations. *Phys. Lett.*, 82:218–220. 1

V.K. Dobrev, H.-D. Doebner, and Ch. Mrugalla. Lowest weight representations of the schrödinger algebra and generalized heat/schrödinger equations. *Reports on Mathematical Physics*, 39:201–218, 1997. 1, 2, 10, 11, 12

N. Jacobson. *Lie algebras*. Dover Books on Mathematics Series. Dover Publications, Incorporated, 1979. ISBN 9780486638324. URL http://books.google.se/books?id=hPE1Mmm7SFMC. 4, 5

T.Y. Lam. *Lectures on modules and rings*. Graduate texts in mathematics. Springer Verlag, 1999. ISBN 9780387984285. 19

O. Mathieu. Classification of irreducible weight modules. *Annales de l'institut Fourier*, 50:537–592, 2000. 2, 19, 24

V. Mazorchuk. *Lectures on $\mathfrak{sl}_2$-modules*. Imperial College, 2010. ISBN 9781848165175. 10

U. Niederer. The maximal kinematical invariance group of the free Schrodinger equation. *Helv.Phys.Acta*, 45:802–810, 1972. 1

Y. Wu and L. Zhu. Simple weight modules for schrödinger algebra. *Linear Algebra and its Applications*, 438:559–563, 2013. 1, 2, 16