UPPSALA
UNIVERSITET

U.U.D.M. Project Report 2013:17

Morse homology of $\mathbb{RP}^n$

Sebastian Pöder

Examensarbete i matematik, 15 hp
Handledare och examinator: Ryszard Rubinsztein
Juni 2013

Department of Mathematics
Uppsala University

Morse homology of $\mathbb{RP}^n$

Sebastian Pöder

June 2013

Abstract

Given a real-valued function f on a manifold M, one can deduce topological information about M if f is a *Morse function*. One such procedure, following [5], is outlined here, leading to the so-called Morse homology of M. This is followed by an extensive example for $M = \mathbb{RP}^n$.

1 Introduction

Let M be a manifold and $f: M \to \mathbb{R}$ a function. In this paper we will describe a procedure that extracts the homology of M from the critical points of a Morse function f. This procedure was first used by Witten.

Relating the topology of M to the critical points of a function f on M is the subject of *Morse theory*, first studied by Morse, then by Bott, Smale, and others. Morse and Bott studied the energy functional on the loop space of a manifold, while Smale used these ideas in finite dimensions, culminating in the h-cobordism theorem.

In this paper we will first recall some relevant notions, then define the Morse complex over the integers $\mathbb{Z}$, which requires a study of orientations. After presenting an argument that indicate that this is a chain complex and has homology isomorphic to the singular homology of M, a detailed example for the real projective space is analyzed.

Acknowledgements

I would like to thank my advisor Ryszard Rubinsztein for his guidance, encouragement, and patience. I would also like to thank friends and family that have supported me, especially those of you who helped by giving comments on earlier drafts of this document. Thank you.

2 Preliminaries

Throughout this paper, manifolds will be smooth, Hausdorff, and second countable, and may sometimes have boundary. A function on a manifold will be smooth and real-valued unless otherwise specified. We begin by reminding of the central notions used below. Let M be a manifold of dimension n.

Non-degenerate critical points. Let $f\colon M \to \mathbb{R}$ be a function. A critical point a of f is called *non-degenerate* if the Hessian matrix

$$\left(\frac{\partial^2 f}{\partial u_i \partial u_j}(a)\right)_{ij}$$

is non-singular for some (and hence for any) coordinate system $\{u_i\}$. Such a critical point is assigned an *index*, denoted λ_a, equal to the maximum dimension of a subspace K_a^- of T_aM on which the Hessian is negative definite.

Definition. *The function f is a* Morse function *if each of its critical points is non-degenerate.*

Near a non-degenerate critical point a, there are (see e.g. [3, p. 6]) coordinates $\{u_i\}$ on M such that

$$f(u_1, \dots, u_n) = f(a) - u_1^2 - \cdots - u_{\lambda_a}^2 + u_{\lambda_a+1}^2 + \cdots + u_n^2.$$

Thus a is the only critical point in this chart. If M is compact and f is a Morse function, it follows that there are only finitely many critical points .

The Hessian can also be viewed as a bilinear operator on T_aM: for $v_a, w_a \in T_aM$, pick any extensions v, w and define

$$H_a(v_a, w_a) := v(w(f))(a) = (df(w))(v)(a).$$

Since a is a critical point, this is symmetric and independent of the extensions, see e.g. [3]. That a is a non-degenerate critical point of f can also be expressed by requiring that the map

$$H\colon T_aM \to T_a^*M, v \mapsto (w \mapsto H_a(v, w))$$

is an isomorphism. If M has a Riemannian metric we can furthermore identify $T_a^*M \cong T_aM$, so that H is a symmetric map on T_aM, and the subspace K_a^- above can now be chosen in a canonical way– the span of eigenvectors with negative eigenvalues.

Flows of vector fields. Let X be a (smooth) vector field on M. For $p \in M$ we can consider the differential equation

$$\frac{d}{dt}\gamma(t) = X(\gamma(t))$$
$$\gamma(0) = p.$$

If M is compact, or more generally X has compact support, then X generated a *flow* $\phi = \phi(p, t)$ such that $t \mapsto \phi_t(p) = \phi(p, t)$ is an integral curve for each $p \in M$, see [3].

If M has boundary, the flow might not be even locally defined. However if X always points inwards on ∂M, in addition to having compact support, the flow can be defined for $t \geq 0$.

The stable/unstable manifold theorem. Consider a flow ϕ of a vector field X, near an isolated zero $a \in M$ of X. We can consider the stable and unstable sets of a,

$$S_a = \{x \in M | \lim_{t \to \infty} \phi_t(x) = a\}$$

and

$$U_a = \{x \in M | \lim_{t \to -\infty} \phi_t(x) = a\}.$$

Under certain conditions, for example that X is the gradient of a function f in some Riemannian metric, and a is a non-degenerate critical point of f, S_a and U_a are embedded manifolds near a. This is the content of the theorem on stable and unstable manifolds, see [2, Thm. 6.3.1]. In fact under the stated conditions S_a and U_a can in charts be realized as graphs of functions:

$$s \colon W \cap (K_a^-)^\perp \to K_a^-$$
$$u \colon W \cap K_a^- \to (K_a^-)^\perp$$

where the tangent space T_aM is identified with $\mathbb{R}^n$ so that $K_a^- \subset \mathbb{R}^n$, and W is a neighbourhood of $0 \in T_aM$. Furthermore, $u(a) = 0$, and the corresponding embedding $v \mapsto (v, u(v))$ is tangent to K_a^- at a, and similarly for s.

3 Morse homology

Let M be a compact, Riemannian manifold of dimension n, and let $f \colon M \to \mathbb{R}$ be a Morse function.

Denote by V the gradient vector field of f. We are interested in certain solutions of the downward gradient-flow equation:

$$\frac{d}{dt} x(t) = -V(x(t)). \tag{1}$$

The solutions of (1) give the flow $\phi \colon M \times \mathbb{R} \to M$, $\phi_t(x) = \phi(x, t)$, generated by $-V$. To each critical point a of f, we associate the *unstable* and *stable* manifolds of f at a:

$$U_a = \{x \in M | \lim_{t \to -\infty} \phi_t(x) = a\}$$

and

$$S_a = \{x \in M | \lim_{t \to \infty} \phi_t(x) = a\}.$$

The dimensions of U_a and S_a are equal to λ_a and $n - \lambda_a$, respectively. These are embedded open balls of the corresponding dimension; see sections 6.3 and 6.4 of [2].

For two critical points a and b, let $M(a,b)$ be the set of points in M which lie on trajectories of the flow that originate at a and tend to b, that is, $M(a,b) = U_a \cap S_b$. If the intersection $U_a \cap S_b$ is transversal, $M(a,b)$ is a manifold of dimension $\lambda_a - \lambda_b$ on which $\mathbb{R}$ acts freely via translation along trajectories. The quotient $M(a,b)/\mathbb{R}$ is the space of unparametrized trajectories from a to b and is denoted by $\hat{M}(a,b)$. If all such intersections are transversal, f is called *Morse-Smale* (with respect to the Riemannian metric).

3.1 Orientations

In this section we assume that f is Morse-Smale. For two critical points a and b with $\lambda_b = \lambda_a - 1$, we wish to count signed, unparametrized trajectories from a to b. To do this we need to orient $M(a,b)$ and $\hat{M}(a,b)$.

Choose an orientation of each of the subspaces $T_aU_a \subset T_aM$. Note that U_a and S_a are contractible. Moreover, $T_aU_a = N_aS_a$, where N_xS_a is the fiber at x of the normal bundle of $S_a \subset M$. Hence, the choices of orientations of T_aU_a yields an orientation of each U_a and a co-orientation of each S_a. Then for $p \in M(a,b)$ we can use the exact sequence

$$0 \to T_pM(a,b) \xrightarrow{i} T_pU_a \xrightarrow{\pi} N_pS_b \to 0 \tag{2}$$

to orient $M(a,b)$. Here i is the inclusion and π is the restriction of the orthogonal projection $\pi\colon T_pM \to N_pS_b$. This orients $M(a,b)$ as follows: choose vectors $v_1, \ldots, v_l \in T_pU_a$ such that $\pi(v_1), \ldots, \pi(v_l)$ is an oriented basis of N_pS_b. Then a basis $u_1, \ldots, u_k$ of $T_pM(a,b)$ is oriented whenever $i(u_1), \ldots, i(u_k), v_1, \ldots, v_l$ is an oriented basis of T_pU_a.

We can also orient $\hat{M}(a,b)$ in a similar manner, using the exact sequence

$$0 \to \mathbb{R} \xrightarrow{\iota} T_xM(a,b) \to T_{\hat{x}}\hat{M}(a,b) \to 0,$$

where $\iota(1) = -V(x)$. When $\lambda_b = \lambda_a - 1$, one shows (see [2, Lma 6.5.4]) that $\hat{M}(a,b)$ is a finite collection of signed points. Summing over these signs yields a number $n(a,b)$.

We are now ready to define the Morse complex. Let $C_* = C_*(M,f)$ be the abelian group freely generated by the set of critical points of f, graded by the indices of the critical points. The differential is defined by

$$\partial(a) = \sum_{b:\lambda_b = \lambda_a - 1} n(a,b)b.$$

It remains to check that $\partial^2 = 0$ and that the homology of this complex, the *Morse homology*, is isomorphic to the singular homology of M. An argument is given below under stronger assumtions on the relative positions of the stable and unstable manifolds of each critical point, the estimates (3) and (4) . This occupies the next section and is similar to [6, Sec. 2.5], building on the theory contained in [3] and [4]. The proof in the general case is harder, see [7].

3.2 Isomorphism in the special case

The proof in the special case compares the Morse complex with another complex which has known homology. Let $g\colon M \to \mathbb{R}$ be a function that satisfies

- g has the same critical points as f, each with the same index,
- $V(g) = dg(V) > 0$ except at the critical points,
- g equals f plus a constant in some neighbourhood of each critical point,
- if $a \in M$ is a critical point, then $g(a) = \lambda_a$.

That such a function exists follows from repeated application of the proof of [4, Thm. 4.1]. That proof goes through without change in our situation, since f decreases along flow lines and is assumed to be Morse-Smale.

Take a chart in a neighbourhood of each critical point a wherein f and g differ only by a constant. Furthermore take coordinates $x_1, \ldots, x_n$ in some smaller chart such that, if $\lambda_a = k$, then

$$g(x_1, \ldots, x_n) = k - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$

For brevity introduce the notation

$$\begin{aligned} x_- &= (x_1, \ldots, x_k, 0, \ldots, 0) \\ x_+ &= (0, \ldots, 0, x_{k+1}, \ldots, x_n) \\ x_-^2 &= x_1^2 + \cdots + x_k^2 \\ x_+^2 &= x_{k+1} + \cdots + x_n^2 \end{aligned}$$

depending on k, so that

$$g(x_-, x_+) = k - x_-^2 + x_+^2.$$

By shrinking, we can assume that this chart contains local stable and unstable manifolds; by shrinking further, we can assume that these are given as graphs of functions $s\colon \{0\} \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ and $u\colon \mathbb{R}^k \times \{0\} \to \mathbb{R}^{n-k}$, respectively.

Take $0 < \eta < 1/5$ such that each of the charts contain the corresponding coordinate ball

$$x_-^2 + x_+^2 \leq 2\eta.$$

Furthermore take $0 < \epsilon < \eta$ such that around each critical point a, the intersections $S_a \cap g^{-1}(k + \epsilon)$ and $U_a \cap g^{-1}(k - \epsilon)$ are contained in this ball.

Now *assume* that there exist $0 < \epsilon' < \frac{\epsilon}{9}$ such that if $x_+^2 < \epsilon'$, then

$$|s(x_+)|^2 < \frac{\epsilon}{9}, \tag{3}$$

and that if $x_-^2 < \frac{4}{9}\epsilon$, then

$$|u(x_-)|^2 < \frac{\epsilon'}{2}. \tag{4}$$

Put

$$M_k = g^{-1}((-\infty, k+1-\epsilon])$$

so that M_k contains the critical points of index less than or equal to k, and none of higher index. The topology of M_k is described in terms of M_{k-1} by theorem 3.2 of [3]: M_k is homotopy equivalent to M_{k-1} with one k-cell attached for each critical point of index k. A variant of this homotopy equivalence will be contructed below. These k-cells can be identified with the corresponding unstable discs: if a has index k, put

$$D(a) = U_a \cap g^{-1}([k-\epsilon, k+1/2]).$$

Note that $\partial D(a) = U_a \cap g^{-1}(k-\epsilon)$.

If $a_1, \dots, a_r$ are the critical points of index k, we have

$$\begin{aligned} H_s(M_k, M_{k-1}) &\cong H_s(M_k \cup D(a_1) \cup \dots \cup D(a_r), M_{k-1}) \\ &\cong H_s(D(a_1) \cup \dots \cup D(a_r), \partial D(a_1) \cup \dots \cup \partial D(a_r)) \\ &\cong H_s(D(a_1), \partial D(a_1)) \oplus \dots \oplus H_s(D(a_r), \partial D(a_r)). \end{aligned}$$

The first isomorphism is induced from the homotopy equivalence mentioned above, and the second is excision. Thus $H_s(M_k, M_{k-1})$ is either zero, if $s \neq k$, or free on the generators corresponding to the unstable discs of the critical points of index k, if $s = k$. Recall also that we oriented each unstable manifold, and the unstable discs inherit this orientation. These will form our preferred basis of $H_k(M_k, M_{k-1})$.

Let $C'_k = H_k(M_k, M_{k-1})$ form the groups of a new complex C'_* with the differential given by the composition

$$\partial' \colon H_k(M_k, M_{k-1}) \to H_{k-1}(M_{k-1}) \to H_{k-1}(M_{k-1}, M_{k-2}),$$

where the first map is the boundary map from the long exact sequence of (M_k, M_{k-1}), and the second is induced by inclusion.

This complex C'_* is closely related to a cellular complex and has homology isomorphic to the singular homology of M. The proof is the same as the proof for the cellular homology, see [1, Thm. 2.35].

There is an isomorphism $C_k \to C'_k$ sending a critical point to the class of its unstable disc. It remains to check that this map is a chain map, and hence an isomorphism of chain complexes.

Let a be a critical point of index $k+1$ and let $b_1, \dots, b_l$ be the critical points of index k. We need to check the coefficient of $D(b_i)$ when representing $\partial D(a) \in H_k(M_k, M_{k-1}) = C'_k$ in the preferred basis $\{D(b_j)\}_{j=1}^{l}$.

We can modify V into a vector field $\tilde{V}$ on M_k by putting $\tilde{V} = \frac{-1}{V(g)} V$. However near the critical points we need to introduce a cut-off function. Do this in the coordinate balls

$$x_-^2 + x_+^2 < \delta < \epsilon' \tag{5}$$

in each of the chosen charts around the critical points b_j and those of lower index. $\delta > 0$ will be defined by a condition given below. Note that $\tilde{V}$ points inwards along the boundary of M_k, so the flow $\tilde{\phi}_\tau \colon M_k \to M_k$ along $\tilde{V}$ is well defined for positive τ. Moreover, since

$$\frac{d}{d\tau}\Big|_{\tau=0} g(\tilde{\phi}_\tau(p)) = \langle \tilde{V}, \mathrm{grad}(g)\rangle_p = \left\langle \frac{-1}{V(g)} V, \mathrm{grad}(g)\right\rangle_p = -1$$

as long as $p \in g^{-1}(k+\epsilon', k+1-\epsilon)$, we have that

$$g(\tilde{\phi}_\tau(p)) = g(p) - \tau$$

as long as both p and $\tilde{\phi}_\tau(p)$ lie in $g^{-1}(k+\epsilon', k+1-\epsilon)$ or more generally, as long as the trajectory of p from p to $\tilde{\phi}_\tau(p)$ avoids the neighbourhoods $x_-^2 + x_+^2 < \delta$ of each critical point.

A function $\varphi \colon \partial D(a) \to \tilde{D}(b_i)/\partial\tilde{D}(b_i)$, where $\tilde{D}(b_i)$ is a smaller disc inside $D(b_i)$, will be defined in four steps, as follows. The first three of these will define a homotopy with a parameter t that runs from zero to one in each step.

First, apply the flow $\tilde{\phi}_{t\tau}$ to M_k, where

$$\tau(x) = \begin{cases} (g(x) - (k+\epsilon)) & \text{if } x \in g^{-1}([k+\epsilon, k+1-\epsilon]) \\ 0 & \text{if } x \in g^{-1}((-1, k+\epsilon)) \end{cases}$$

so that at $t = 1$, $\tilde{\phi}_\tau$ is a retraction of M_k onto $g^{-1}((-1, k+\epsilon))$. In particular the image $\tilde{\phi}_\tau(\partial D(a))$ of $\partial D(a)$ lies in the level surface $g^{-1}(k+\epsilon)$ when $t = 1$.

Second, define a function $\tau' \colon g^{-1}(k+\epsilon) \to [\epsilon, 2\epsilon]$, satisfying: outside some neighbourhood A of the intersections

$$\bigcup_j S_{b_j} \cap g^{-1}(k+\epsilon),$$

τ' is constantly equal to 2ϵ. On some smaller neighbourhood $A' \subset A$ of $\cup_j S_{b_j} \cap g^{-1}(k+\epsilon)$, τ' is equal to ϵ. Given A, we can choose δ from (5) such that $\tilde{\phi}_{t\tau}(g^{-1}(k+\epsilon) \setminus A)$ never intersects the sets (5), for $0 \le t \le 1$. Hence

$$g\big(\tilde{\phi}_\tau(g^{-1}(k+\epsilon) \setminus A)\big) = k - \epsilon.$$

Consider the region R traced out by $\tilde{\phi}_{t\tau'}(g^{-1}(k+\epsilon))$ as t runs from zero to one. We can extend τ' to R in such a way that $\tilde{\phi}_{\tau'}$ maps the region onto $\tilde{\phi}_{\tau'}(g^{-1}(k+\epsilon))$, and each trajectory in R is mapped to a single point. Then necessarily $\tau' = 0$ on $\tilde{\phi}_{\tau'}(g^{-1}(k+\epsilon))$, so τ' extends to a continuous function on $g^{-1}((-1, k+\epsilon))$ by setting it equal to zero on $g^{-1}((-1, k+\epsilon)) \setminus R$. A deformation retraction

$$g^{-1}((-1, k+\epsilon)) \to g^{-1}((-1, k+\epsilon))$$

onto the set

$$\tilde{\phi}_{\tau'}(g^{-1}(-1, k+\epsilon))$$

can now be defined by $(x,t) \mapsto \tilde{\phi}_{t\tau'}(x)$.

Denote $N = \tilde{\phi}_{\tau'}(g^{-1}(-1, k+\epsilon))$, so that $\partial N = \tilde{\phi}_{\tau'}(g^{-1}(k+\epsilon))$. By taking A, and hence δ, sufficiently small we can ensure that the intersection of $\partial N \setminus g^{-1}(k-\epsilon)$ with our chosen chart around each b_j is contained in the coordinate ball

$$x_-^2 + x_+^2 \leq 2\eta$$

Moreover, because of the estimate (4), we can by shrinking A further ensure that in each chart the intersection $N \cap \{x_-^2 < \frac{4}{9}\epsilon\}$ is contained in the strip $x_+^2 < \epsilon'$.

Third, define a map r_1 from N onto a set to be defined, and a homotopy r_t from the identity to r_1. Outside each of the chosen charts around b_j, let r_t be the identity. Inside these charts, r_t is defined by three different expressions below.

Case 1. In the region $x_-^2 \leq \epsilon$, let r_t be

$$\begin{aligned} r_t(x_-, x_+) = {} & (x_- - tB(|x_-|)s(x_+), 0, \dots, 0) + \\ & (0, \dots, 0, (1-t)x_+ + tB\left(|x_- - B(|x_-|)s(x_+)|\right) u\left(x_- - B(|x_-|)s(x_+)\right)) \end{aligned}$$

where s and u are the functions defining the local stable and unstable manifolds for b_j in this chart. Here B is a smooth, nonincreasing function of one variable with $B(z) = 1$ if $z \leq \frac{\sqrt{\epsilon}}{3}$ and $B(z) = 0$ if $z \geq \frac{2\sqrt{\epsilon}}{3}$. This is the reason for the requirement (3), since on $N \cap \{|x_-| < \frac{2}{3}\sqrt{\epsilon}\}$ we have $x_+^2 \leq \epsilon'$. Hence this region is mapped to itself, at least when $t = 1$, and the points mapped to zero is exactly the stable manifold. At $t = 1$, this region is mapped into the graph

$$\{\big(x_-, B(|x_-|)u(x_-)\big) : x_-^2 \leq \epsilon\}.$$

Case 2. In the region

$$\epsilon \leq x_-^2 \leq x_+^2 + \epsilon$$

define r_t as

$$r_t(x_-, x_+) = (x_-, c_t x_+)$$

where c_t is the expression

$$c_t = (1-t) + t\sqrt{\frac{x_-^2 - \epsilon}{x_+^2}}.$$

Case 3. In the region M_{k-1}, given in coordinates by the inequality

$$-x_-^2 + x_+^2 \leq -\epsilon,$$

let r_t be the identity.

Taken together, in this chart r_1 is a continuous map from N onto the union M'_{k-1} of M_{k-1} and the graph of $B(|x_-|)u(x_-)$, which is equal to the unstable disc of b_j close to 0. Furthermore, the preimage of 0 under r_1 is exactly the set $S_{b_j} \cap N$, and r_1 is smooth in a neighbourhood of this set. However r_1

may not be the identity on M'_{k-1}. To correct this we can apply the homotopy $H_t\colon M'_{k-1} \to M'_{k-1}$ defined in each region $x_-^2 \leq \epsilon$ by

$$\begin{aligned} H_t(x_-, x_+) = &(x_- - tB(|x_-|)s(x_+), 0, \dots, 0) + \\ &(0, \dots, 0, B\left(|x_- - tB(|x_-|)s(x_+)|\right) u\left(x_- - tB(|x_-|)s(x_+)\right)) \end{aligned}$$

end extended to M'_{k-1} by the identity.

The composition of these first three steps defines a homotopy from the identity on M_k to a retraction onto M'_{k-1}.

Fourth, to define the map φ, compose the retraction with the quotient map that collapses

$$M'_{k-1} \setminus C \subset M'_{k-1}$$

to a point. Here C is the set $x_-^2 \leq \frac{\epsilon}{9}$ in the chart around b_i, which is a neighbourhood of b_i in $D(b_i)$.

The sought coefficient is equal to the degree of φ. Note that b_i is a regular value for φ, and each $a' \in \varphi^{-1}(b_i)$ lies on a trajectory for the original flow ϕ from a to b_i.

Let $v_1, \dots, v_k$ be a positive basis of $T_{a'}\partial D(a)$, so that $-V(a'), v_1, \dots, v_k$ is a positive basis of $T_{a'}D(a) = T_{a'}U_a$. Then $d_{a'}\varphi$ maps $v_1, \dots, v_k$ to a basis $w_1, \dots, w_k$ of $T_{b_i}D(b_i)$. This last basis is positive exactly when (2) orients $M(a, b_i)$ in the same way as $-V$ does.

This shows that a' contributes the same sign to the degree of φ as it does to $n(a, b)$, which implies that the Morse complex is isomorphic to the cellular complex.

4 Example: $\mathbb{RP}^n$

In this section we study the Morse homology of $\mathbb{RP}^n$ arising from Morse functions of a particular kind. This is inspired by [5] where similar functions on $\mathbb{C}P^n$ are studied. Here $\mathbb{RP}^n$ is regarded as $\mathbb{R}^{n+1} \setminus \{0\}$ modulo multiplication by nonzero constants. The metric comes from the unit sphere $S^n \subset \mathbb{R}^{n+1}$, which in turn has a metric induced from $\mathbb{R}^{n+1}$.

Let A be a symmetric $(n+1) \times (n+1)$-matrix with real entries, and let $\Lambda\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ be the function defined by

$$\Lambda(x) = \frac{\langle x, Ax \rangle}{\|x\|^2}.$$

Such a function is invariant under the scalar multiplication by nonzero constants, and so descends to a function on $\mathbb{RP}^n$. This function will be denoted Λ^*.

The function

$$f(x) = \frac{1}{2}\langle x, Ax \rangle,$$

defined on $\mathbb{R}^{n+1}$ will be useful to help us understand the flow lines of Λ^*. To begin, choose an ON-basis $w_0, \dots, w_n$ of $\mathbb{R}^{n+1}$ that diagonalizes A, with $x_0, \dots, x_n$

the corresponding coordinates and $\lambda_0, \dots, \lambda_n$ the corresponding eigenvalues. Let $x = x(t)$ be a solution of the downward gradient flow equation,

$$\frac{dx}{dt} = -\operatorname{grad}(f)(x), \tag{6}$$

of the function f in $\mathbb{R}^{n+1}$. We calculate

$$\frac{\partial f}{\partial x_i}(x) = \frac{1}{2}(\langle w_i, Ax\rangle + \langle x, Aw_i\rangle) = \langle x, \lambda_i w_i\rangle = x_i\lambda_i$$

so the equation (6) becomes

$$\frac{dx}{dt} = -\operatorname{grad}(f)(x) = -\sum_{i=0}^{n} \frac{\partial f}{\partial x_i} w_i = -\sum_{i=0}^{n} x_i\lambda_i w_i = -Ax. \tag{7}$$

Lemma 4.1. *The trajectories of the negative gradient flow of $\frac{1}{2}\Lambda^*$ on $\mathbb{RP}^n$ are the images under the quotient map*

$$\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n, p \mapsto [p]$$

of nonzero solutions $x\colon \mathbb{R} \to \mathbb{R}^{n+1}$ of the linear equation (7).

Proof. At a point y along the trajectory $x(t)$ in $\mathbb{R}^{n+1} \setminus \{0\}$, we can decompose $-\operatorname{grad}(f)(y) = -Ay$ into parts parallel and orthogonal to y. The parallel part is $\langle -Ay, \frac{y}{\|y\|}\rangle \frac{y}{\|y\|} = -\Lambda(y)y$, so the orthogonal part is $-Ay + \Lambda(y)y$. When applying the radial projection $\tilde{\pi}\colon \mathbb{R}^{n+1} \setminus \{0\} \to S^n$, $d\tilde{\pi}$ acts on the orthogonal part as multiplication by $\frac{1}{\|y\|}$ while the parallel part lies in its kernel. Hence the image $\tilde{x}(t)$ in S^n solves the equation

$$\frac{d\tilde{x}}{dt} = -A\tilde{x} + \Lambda(\tilde{x})\tilde{x} = -\operatorname{grad}(f|_{S^n})(\tilde{x}),$$

which is the downward gradient-flow equation for f restricted to the sphere. The second equality holds since the metric on S^n is induced from $\mathbb{R}^{n+1}$: when $y \in S^n$ and $v \in T_yS^n$,

$$\begin{aligned}\langle \operatorname{grad}(f|_{S^n}), v\rangle_{S^n} &= d(f|_{S^n})(v) = df(v) = \langle \operatorname{grad} f, v\rangle_{\mathbb{R}^{n+1}} \\ &= \langle \operatorname{proj}_{T_yS^n}(\operatorname{grad} f), v\rangle_{\mathbb{R}^{n+1}} = \langle Ay - \Lambda(y)y, v\rangle_{\mathbb{R}^{n+1}}.\end{aligned}$$

On S^n we also find $f = \frac{1}{2}\Lambda$, so when we pass to the quotient $\mathbb{RP}^n$ the image $[x(t)]$ of the trajectory $x(t)$ is a trajectory of the negative gradient flow of $\frac{1}{2}\Lambda^*$. □

Hence, if $x = x(t)$ is a solution of equation (7), then $[x(2t)]$ is a trajectory of Λ^*. We also find that if $[p] \in \mathbb{RP}^n$ is a critical point of Λ^*, then the solutions of (7) with initial condition $x(0) = p$ must stay in $\mathbb{R}p$. So we have

Lemma 4.2. *The critical points of Λ^* on $\mathbb{RP}^n$ are images of eigenvectors in $\mathbb{R}^{n+1}$ of the operator A.*

We want Λ^* to be a Morse function, so it must have isolated critical points. This happens only if each eigenspace of A is one-dimensional: from now on, let us assume this. Write the eigenvalues of A as

$$\lambda_0 < \lambda_1 < \ldots < \lambda_n$$

and reorder our ON-basis accordingly.

Lemma 4.3. *Each critical point $[w_i]$ of Λ^* is non-degenerate, with* $\operatorname{index}(w_i) = i$*. The unstable and stable manifolds of $[w_i]$ are*

$$U_i = \{[x_0{:}\ldots{:}x_i{:}0{:}\ldots{:}0] | x_i \neq 0\}$$

and

$$S_i = \{[0{:}\ldots{:}0{:}x_i{:}\ldots{:}x_n] | x_i \neq 0\}.$$

Proof. To investigate the critical point $[w_i]$, use the parametrization given by

$$\phi_i : B^n \to V_i \subset \mathbb{RP}^n, (y_1, \ldots, y_n) \mapsto [y_1, \ldots, y_i, \sqrt{1 - \sum_{j=1}^{n} y_j^2}, y_{i+1}, \ldots, y_n]$$

where B^n is the open unit ball in $\mathbb{R}^n$ and $V_i = \{[x_0{:}\ldots{:}x_n] \in \mathbb{RP}^n | x_i \neq 0\}$. Then

$$\Lambda^* \circ \phi_i(y_1, \ldots, y_n) = \sum_{j=1}^{i} y_j^2(\lambda_{j-1} - \lambda_i) + \lambda_i + \sum_{j=i+1}^{n} y_j^2(\lambda_j - \lambda_i).$$

We find that $\frac{\partial^2}{\partial y_k \partial y_l}\Lambda^* \circ \phi$ is zero unless $k = l$, in which case it is negative for $k \leq i$ and positive for $k > i$. Hence $[w_i]$ is non-degenerate of index i.

As for the description of the stable and unstable manifolds, take a point $x = x_k w_k + \cdots + x_l w_l \in \mathbb{R}^{n+1}$ with x_k and x_l nonzero. This point's trajectory in $\mathbb{R}^{n+1}$ under $-\operatorname{grad} f$ is

$$x(t) = x_k e^{-\lambda_k t} w_k + \cdots + x_l e^{-\lambda_l t} w_l.$$

As $t \to +\infty$ or $t \to -\infty$, the dominating terms are $x_k e^{-\lambda_k t}$ or $x_l e^{-\lambda_l t}$, so the image of this trajectory in $\mathbb{RP}^n$ tends to $[w_k]$ or $[w_l]$, respectively. □

Corollary 4.4. *The Morse complex $C_*(\mathbb{RP}^n, \Lambda^*)$ has*

$$C_k(\mathbb{RP}^n, \Lambda^*) = \begin{cases} \mathbb{Z}[w_k] & 0 \leq k \leq n \\ 0 & \textit{otherwise}, \end{cases}$$

and $M(i, i-1) := M([w_i], [w_{i-1}]) = U_i \cap S_{i-1}$ is a copy of $\mathbb{R}P^1$ with two points removed.

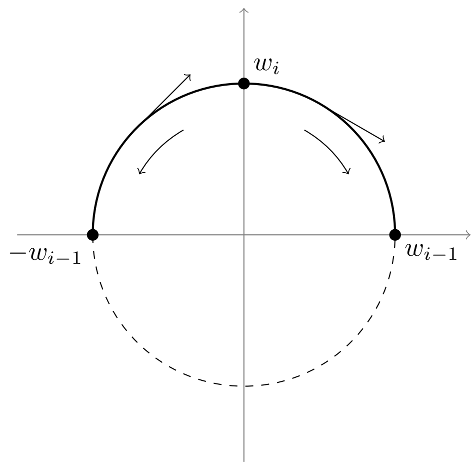

Figure 1: The part of S^n corresponding to $M(i, i-1)$. The curved arrows indicate the direction of the flow, and the drawn tangent vectors are the vectors denoted by v in the text in the two cases.

The next problem is that of orienting $M(i, i-1)$. We need to choose an oriented basis of each K_j^-, and from the description of U_i given in lemma 4.3 it follows that we can choose the push-forward of the oriented basis $w_0, \dots, w_{j-1} \in T_{w_j} S^n$ under the quotient map. Note that the corresponding basis at $-w_j \in S^n$ is $-w_0, \dots, -w_{j-1}$. For $x \in M(i, i-1)$, choose a representation $x = [0{:} \dots {:}x_{i-1}{:}x_i{:} \dots {:}0]$ such that $x_{i-1}^2 + x_i^2 = 1$ and $x_{i-1}, x_i > 0$ or $x_{i-1} < 0, x_i > 0$. The oriented tangent space $T_x M(i, i-1)$ has as an oriented basis the push-forward of $v = x_i w_{i-1} - x_{i-1} w_i$ under the projection $S^n \to \mathbb{RP}^n$.

If $x_{i-1}, x_i > 0$, v points in the direction of the flow on $M(i, i-1)$. The third map of (2) maps $w_0, \dots, w_{i-2} \in T_x U_i$ to the oriented basis $w_0, \dots, w_{i-2}$ of $N_x S_{i-1}$. Hence v is a positively oriented basis of $T_x M(i, i-1)$ if and only if $v, w_0, \dots, w_{i-2}$ is a positively oriented basis of $T_x U_i$: this happens exactly when i is odd. Hence, the sign corresponding to this trajectory is $(-1)^{i-1}$.

If $x_{i-1} < 0$ and $x_i > 0$, $-v$ is a basis of $T_x M(i, i-1)$ with the same orientation as the flow. The third map in (2) now maps $-w_0, \dots, -w_{i-2} \in T_x M(i, i-1)$ to the oriented basis $-w_0, \dots, -w_{i-2} \in N_x S_{i-1}$. Our conclusion is that $-v$ is a positive orientation of $T_x M(i, i-1)$ only if $-v, -w_0, \dots, -w_{i-2}$ is a positive basis of $T_x U_i$, but this is never true. The sign corresponding to this trajectory is always -1.

We summarize

Theorem 4.5. *The differential in $C_*(\mathbb{RP}^n, \Lambda^*)$ acts as*

$$\partial_i([w_i]) = (-1)^{i-1}[w_{i-1}] + (-1)[w_{i-1}] = \begin{cases} -2[w_{i-1}] & \text{if } i \text{ is even and } 2 \le i \le n \\ 0 & \text{otherwise,} \end{cases}$$

and so the Morse homology of the pair $(\mathbb{RP}^n, \Lambda^*)$ *is isomorphic to*

$$H_i(C_*(\mathbb{RP}^n, \Lambda^*)) \cong \begin{cases} \mathbb{Z}_2 & \text{if } i \text{ is odd and } 0 < i < n \\ \mathbb{Z} & \text{if } i = n \text{ and } n \text{ is odd, or if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Corollary 4.4 implies that $\hat{M}(i, i-1)$ has only two points. The corresponding signs are calculated above. Together with the description of the chain complex in corollary 4.4, this yields the stated homology. □

References

[1] A. Hatcher, *Algebraic Topology.* Cambridge, Cambridge University Press (2002).

[2] J. Jost, *Riemannian geometry and geometric analysis.* 5. ed. Berlin, Springer (2008).

[3] J. Milnor, *Morse Theory.* Princeton, Princeton University Press (1963).

[4] J. Milnor, *Lectures on the h-cobordism theorem.* Princeton, Princeton University Press (1965).

[5] P.B. Kronheimer, T. Mrowka, *Monopoles and three-manifolds.* Cambridge, Cambridge University Press (2007).

[6] L. Nicolaescu, *An invitation to Morse theory.* New York, Springer (2007).

[7] M. Schwarz, *Morse homology.* Basel, Birkhäuser (1993).