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The Hales-Jewett theorem and its application to further generalisations of *m*, *n*, *k*-games

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Abstract

An overview is given of main results of Ramsey Theory and especially of the finite Hales-Jewett theorem, along with Saharon Shelah's proof, which implies and unites most of the content. With the aid of the Hales-Jewett theorem some ideas are discussed for applying the theory to refine the search for winning strategies in higher-dimensional generalisations of m, n, k-games. These games are in turn generalisations of the classic 3×3 board game known as TicTacToe.

1 Introduction to Ramsey Theory

1.1 Background

Frank Plumpton Ramsey only lived to be 27 years old, but his last name is now synonymous with a major area of combinatorics and graph theory. Still today (2013) this area presents many unsolved problems.

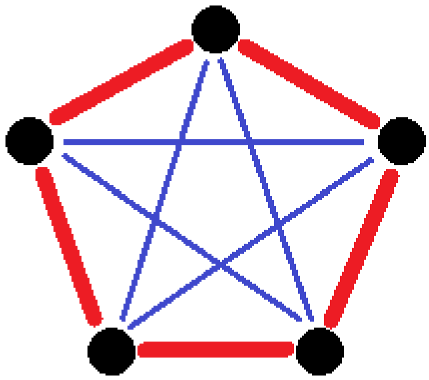
Ramsey Theory deals with finding order in chaos. Complete disorder is, in some sense, an utterly impossible feat. The classic example of Ramsey theory concerns a set of 6 people. Assume that the relation "one person A knows another B" is symmetric (if A knows B then B knows A) but not transitive. We can always, regardless of people, find a subset of three people where either everyone knows each other or no one knows another person (in the subset).

Proof. Name the people of the set A, B, C, D, E, F. Without loss of generality, A knows at least 3 people or does not know at least 3 people ($2 + 2 < 5$). Assume the first and that A knows B, C, D. If any pair (B,C), (C,D) or (B,D) know each other, we've found a set of three people who know each other. If this holds for no pair, then $\{B, C, D\}$ is a set of three people who do not know each other.

The case where A does not know 3 people is treated in the same way. $\square$

The least number of people required to guarantee having at least k people in the first set (knowing each other) or l people in the second set (not knowing each other) is called the

Ramsey number $R(k, l)$. A counterexample may be found with 5 people, so $R(3, 3) = 6$. The 'knowing relation' may be illustrated with graphs with an edge between each pair of n nodes (the *complete graph* K_n). The edges are coloured, for example, red (thick) = knowing and blue (thin) = not knowing. The following is a counterexample showing $R(3, 3) > 5$:



The Ramsey numbers may be extended from 2 classes to a finite set of n classes $k_1, ..., k_n$. We may then talk about the generalised Ramsey number $R(k_1, ..., k_n)$. It is the least positive integer N such that if the edges of the complete graph K_N are coloured with n colours, there is a colour i and a set of edges X_i such that the edges of X_i contain at least k_i nodes and every $x \in X_i$ has the colour i.[1] The following result about the Ramsey numbers is here provided without proof:

Theorem 1. *(Ramsey's theorem) For every choice of $n \in \mathbb{N}$ and $k_1, ..., k_n \in \mathbb{N}$, every generalised Ramsey Number $R(k_1, ..., k_n)$ exists (is finite).*

Some general theorems about the lower and upper bounds of the Ramsey numbers are known. But the only exactly computed 2-dimensional Ramsey numbers (as of 2013) are: $R(1, k) = 1$, $R(2, k) = k$, $R(3, 3) = 6$, $R(3, 4) = 9$, $R(3, 5) = 14$, $R(3, 6) = 18$, $R(3, 7) = 23$, $R(3, 8) = 28$, $R(3, 9) = 36$, $R(4, 4) = 18$, $R(4, 5) = 25$. A few other general numbers are known, for example $R(3, 3, 3) = 17$. [16]

The Ramsey numbers are computationally extremely complicated (practically impossible with today's computers) to calculate. Paul Erdös, a major contributor of Ramsey Theory, is attributed with the following (paraphrased) quote: *"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find R(5,5). We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the value of R(6,6), however, we would have no choice but to launch a preemptive attack."*[3]

Throughout this paper, the notation $[t]$ will be used to denote the set $\{1, ..., t\}$. Recall

[1] Another (equivalent) often used form of notation in the context of Ramsey theory is the *arrow notation*: $N \to (k_1, ...k_n)$.

that an arithmetic progression is a sequence of numbers $a, a+d, a+2d, ..., a+(t-1)d$ for $d \geq 1$. An "r-colouring" of a set A is a partition of A into r subsets, or equivalently a function $\chi : A \to [r]$.

B. L. van der Waerden published the following theorem regarding arithmetic progressions along with a proof in 1927:

Theorem 2. *(Van der Waerden's theorem) For any choice of integers r and t there exists a number $N = W(r,t)$ such that for any r-colouring of $[N]$ there will be at least one monochromatic arithmetic progression of length t in $[N]$.*

This theorem may also be generalized into an infinite version:

Theorem 3. *If $\mathbb{N}$ is partitioned into a finite set of classes, then at least one must contain arbitrarily long arithmetic progressions.*

These theorems will later on be given proofs. There are other theorems of this nature, for example:

Theorem 4 (Schur's theorem)**.** *If $\mathbb{N}$ is coloured with finitely many (r) colours, then there exist numbers x, y, z all in the same colour such that $x + y = z$. Stated equivalently, the set $\{x, y, x+y\}$ is a subset of some part of the partition of $\mathbb{N}$ induced by the given r-colouring.*

Proof. This proof is based on the existence of the Ramsey numbers, so we use Theorem 1. For r colours, let

$$n = R(\underbrace{3, ..., 3}_{r})$$

be the generalised Ramsey number with r number of 3:s. Assign, to each number $1, ..., n$, one of the r colours. Then for any a and b in the complete graph with n nodes numbered 1 to n, let the edge between them (ab) have the same colour as $|a-b|$. By definition of the Ramsey numbers, we can find a triple of nodes (i, j, k) where all the edges in between (ij, ik and jk) have the same colour. Without loss of generality, assume $i > j > k$. Then the numbers $x = |i-j| = i-j$, $y = |j-k| = j-k$ and $z = |i-k| = i-k$ will all have the same colour and satisfy the equation

$$x + y = i - j + j - k = i - k = z$$

This is true for the given value of n. Then, only assuming that r is finite, we must have that the proof certainly holds for a colouring of all of $\mathbb{N}$ since the corresponding Ramsey number always exists. □

Obviously we do not show how to find the numbers x, y or z, merely that they must exist. The same idea will return in the proof of the Hales-Jewett theorem described below.

The following theorem is of the same topic: disproving complete disorder. The entire field would perhaps better be called "Ramseyian theorems" were it not for A.W. Hales and R.I. Jewett. In 1963, they published their very general theorem:

Theorem 5. *Hales-Jewett (1963) For all values of $t, r \in \mathbb{N}$ there exists a number $HJ(r,t)$ such that, if $N \geq HJ(r,t)$ and points of $[t]^N$ is coloured with r colours, then $[t]^N$ contains at least one monochromatic line.*

The next chapter will focus on explaining and proving this theorem, from which many results of Ramsey theory follow as corollaries. We will see that Theorem 2 and Theorem 3 can be proven from this. Theorem 5 can, in a sense, be thought of as the uniting force behind a big part of the theory. It is more general than van der Waerden's theorem, which deals only with elements of $\mathbb{N}$.

2 The Hales-Jewett Theorem and Shelah's proof

2.1 The Hales-Jewett Theorem

Throughout this paper, t is assumed to be a positive integer and $[t]$ will be used to denote the set $\{1, ..., t\}$. That means $[t]^n = [t] \times ... \times [t]$ is the set of all n-tuples of $[t]$ or equivalently the n-dimensional cube (lattice) of side length t. The following terminology will also be common:

Definition 1. *A* **word** *is a finite sequence of symbols of some alphabet Σ, in this paper of $[t]$. We will also consider variable words, which are words over the extended alphabet $[t] \cup \{*\}$.*

Definition 2. *A* **point** *x in $[t]^n$ is an n-tuple $(x_1, ..., x_n)$ with $x_i \in [t]$ for all i. It may also be thought of as a word of length n: $x = x_1 x_2 ... x_n$.*

Definition 3. *A variable word will in this context be called a* **root** *- a point $x \in ([t] \cup \{*\})^n$ where $n \geq 1$ and at least one $x_i = *$.*

It is often useful to think of n-tuples as words of length n. Both versions may be used depending on the context.

Definition 4. *Two words a and b of length n_a and n_b, respectively, may be* **concatenated** *to form a word c of length $n_a + n_b$. The new word consists first of all the symbols of a followed by all the symbols of b. We write this as $c = ab$.*

Definition 5. *Let r be a positive integer. We may colour each point $x \in [t]^n$ with one of r colors, also called an* **r-colouring***. In other words, a colouring corresponds to a function $\chi : [t]^n \to [r]$. An r-colouring may be seen in a third way as a partition of $[t]^n$ into r equivalence classes.*

Definition 6. *Any root τ defines a* **combinatorial line** *$L_\tau = \{\tau(1), ..., \tau(t)\}$ where $\tau(i)$ is the word obtained from τ by substituting all occurences of $*$ with i. For example the root $\tau = 1 * 2$ defines the line $L_\tau = \{112, 122, 132\} \subset [3]^3$. A* **line** *will always be presumed to mean a combinatorial line, unless otherwise specified.*

This definition of a line is sufficient for the time being. It does not capture all possible geometric lines, for example not $\{131, 122, 113\} \subset [3]^3$, but this does not matter. Under this definition the set of lines is independent of the underlying set $[t]$ of symbols.

Definition 7. *A line L_τ is* **monochromatic** *(under the colouring χ) iff all points have the same colour, in other words iff $\chi(\tau(1)) = ... = \chi(\tau(t))$.*[2]

[2]Observe that for Ramsey's theorem of graphs, the colourings are instead of edges, not nodes.

Definition 8. *The* **cardinality** *of a set A is the number of elements in the set, and is denoted $|A|$. In particular, $|[t]| = |\{1, ..., t\}| = t$.*

Let us remind ourselves of the theorem at hand:

Theorem 5. *Hales-Jewett (1963): For all values of $t, r \in \mathbb{N}$ there exists a number $HJ(r,t)$ such that, if $N \geq HJ(r,t)$ and $[t]^N$ is coloured with r colours, then $[t]^N$ contains at least one monochromatic line.*

We call $HJ(r,t)$ the corresponding Hales-Jewett number. Most values are not known, but the proofs offer finite upper bounds to these values. Observe that although we allow $t = 0$ or $r = 0$, these are uninteresting cases. Either the cube consists of some cartesian product of empty sets, or we use a "0-colouring". The theorem still holds, but is only interesting for $t, r > 0$. The 0-colouring is treated as a 1-colouring (every point gets the same colour: none. It is equivalent to the definition $0! = 1$.

2.2 Shelah's proof

In the original paper by Hales and Jewett a proof was included that yielded extremely fast growing upper bounds for the value of $HJ(r,t)$. In this paper a completely different proof will be discussed, first given by Saharon Shelah. The upper bounds still grow very fast, but "only" primitively recursively[3] fast.

The following proof is made by induction. Hence we assume that $n := HJ(r, t-1)$ exists and want to show that $HJ(r,t)$ does. Define the sequence of numbers

$$N_1 := r^{t^n}$$

$$N_i := r^{t^{n+\sum_{j=1}^{i-1} N_j}}$$

for $i = 1, ..., n$. The sum of all N_i will be denoted N:

$$N := N_1 + ... + N_n = \sum_{i=1}^{n} r^{t^{n+\sum_{j=1}^{i-1} N_j}}$$

We will need the following definition and claim as provided in [9]:

Definition 9. *Two words a, b of length m are* **neighbours** *if they differ in exactly one coordinate, for example $a = a_1...a_{i-1}1a_{i+1}...a_m$, $b = a_1...a_{i-1}2a_{i+1}...a_m$.*

[3]A *primitive recursive* function can in computer science be described by only do-loops. It was believed that all recursive functions were primitive recursive, but this was disproven, for example, by the *Ackermann Function.*

Suppose $\tau_1, ..., \tau_m$ are words such that $\tau = \tau_1 ... \tau_m$ is a (concatenated) root. For any word $a = a_1 ... a_m$ we define $\tau(a) = \tau_1(a_1)...\tau_m(a_m)$, substituting all occurences of $*$ in τ_i by a_i.

Shelah first proved the following claim about neighbouring words:

Claim 1. *Given an r-coloring χ of $[t]^N$ there always exists some root $\tau = \tau_1 ... \tau_n$ such that:*

- *The length of τ_i equals $N_i = r^{t^{n+\sum_{j=1}^{i-1} N_j}}$ (hence τ has length N).*
- *The root τ has the property that $\chi(\tau(a)) = \chi(\tau(b))$ for any two neighbours $a, b \in [t]^N$.*

Proof. We will prove this by backward induction. Assume we have found the roots $\tau_{i+1} ... \tau_n$ and want to define the root τ_i. The induction base τ_n is trivially constructed from the choice of χ. Let

$$L_{i-1} = \sum_{j=1}^{i-1} N_j$$

be the length of the initial segment $\tau_1 ... \tau_{i-1}$. For all $0 \leq k \leq N_i$, let:

$$W_k = \underbrace{1...1}_{k}\underbrace{2...2}_{N_i - k}$$

Also, define an r-colouring χ_k of words in $[t]^{L_{i-1}+n-i}$ as:

$$\chi_k(x_1 ... x_{L_{i-1}} y_{i+1} ... y_n) = \chi(x_1 ... x_{L_{i-1}} W_k \tau_{i+1}(y_{i+1}) ... \tau_n(y_n))$$

We thus have the colourings $\chi_0, ..., \chi_{N_i}$, a total of $N_i + 1$. But there are only at most N_i possible colourings of the words in $[t]^{L_{i-1}+n-i}$:

$$r^{t^{L_{i-1}+n-i}} \leq r^{t^{L_{i-1}+n}} = N_i$$

By the pigeonhole principle at least 2 such colourings must be identical. Without loss of generality, let them be $\chi_s = \chi_t$ (where $s < k$). Now we may define

$$\tau_i := \underbrace{1...1}_{s}\underbrace{*...*}_{k-s}\underbrace{2...2}_{N_i - k}$$

which satisfies the length condition (N_i) and, incidentally, all required conditions. To prove this, observe that:

$$\tau_i(1) = \underbrace{1...1}_{s}\underbrace{1...1}_{k-s}\underbrace{2...2}_{N_i - k} = \underbrace{1...1}_{k}\underbrace{2...2}_{N_i - k} = W_k$$

$$\tau_i(2) = \underbrace{1...1}_{s}\underbrace{2...2}_{k-s}\underbrace{2...2}_{N_i-k} = \underbrace{1...1}_{s}\underbrace{2...2}_{N_i-s} = W_s$$

Let a and b be two neighbours that differ (only) in the i:th coordinate:

$$a = a_1...a_{i-1}1a_{i+1}...a_n$$

$$b = a_1...a_{i-1}2a_{i+1}...a_n$$

Using these we have:

$$\tau(a) = \tau_1(a_1)...\tau_{i-1}(a_{i-1})\tau_i(1)\tau_{i+1}(a_{i+1})...\tau_n(a_n)$$
$$\tau(b) = \tau_1(a_1)...\tau_{i-1}(a_{i-1})\tau_i(2)\tau_{i+1}(a_{i+1})...\tau_n(a_n)$$

By definition of the colourings we get:

$$\begin{gathered}\chi(\tau(a)) \\ = \chi(\tau_1(a_1)...\tau_{i-1}(a_{i-1})W_k\tau_{i+1}(a_{i+1})...\tau_n(a_n)) \\ = \chi_k(\tau_1(a_1)...\tau_{i-1}(a_{i-1})a_{i+1}...a_n) \\ = \chi_s(\tau_1(a_1)...\tau_{i-1}(a_{i-1})a_{i+1}...a_n) \\ = \chi(\tau_1(a_1)...\tau_{i-1}(a_{i-1})W_s\tau_{i+1}(a_{i+1})...\tau_n(a_n)) \\ = \chi(\tau(b))\end{gathered}$$

which the claim asserts.

□

Using these tools we are now ready to tackle the full proof of the Hales-Jewett theorem.

Shelah's proof. The existence of $HJ(r,t)$ will be proven by induction. Let the number of colours r be fixed. The value $HJ(r,1)$=1 is trivial since $[1]^n$ contains just 1 point and whatever colour it has it constitutes a monochromatic line (of length 1). Hence $HJ(r,1) = 1$ can be used as the base case for the induction proof.

Assume the value $n = HJ(r,t-1)$ exists. We will prove that $HJ(r,t) \leq N = N_1 + ... + N_n$. In other words, we will prove that there is a χ-monochromatic line in $[t]^N$ for every r-colouring χ of $[t]^N$.

By Shelah's claim, there exists a root $\tau = \tau_1...\tau_n$ of length N such that $\chi(\tau(a)) = \chi(\tau(b))$ for any two neighbours $a, b \in [t]^n$. From the r-colouring χ define another r-colouring $\tilde{\chi}$ of $[t-1]^n$ by $\tilde{\chi}(a) := \chi(\tau(a))$.

Since $[t-1]$ has only $t-1$ symbols, under the induction hypothesis that $n = HJ(r, t-1)$ exists, there is a root $V = V_1...V_n \in ([t-1] \cup \{*\})^n$ where each V_i has length 1 and such that the line $L_V = (V(1), ...V(t-1))$ is monochromatic under $\tilde{\chi}$.

Consider now $\tau(V) = \tau_1(V_1)...\tau_n(V_n)$ which is of length N since each V_i is of length 1, meaning each $*$ in τ_i is replaced by a single symbol (V_i). Since V is a root, some V_i contains a $*$, hence some $\tau_i(V_i)$ contains a $*$, meaning that $\tau(V)$ is a root. Then the line

$$L_{\tau(V)} = (\tau(V(1)), ..., \tau(V(t))$$

is monochromatic under the arbitrary r-coloring χ. To see this, take any two adjacent points $\tau(V(i))$ and $\tau(V(i+1))$ from $L_{\tau(V)}$. If the root $\tau(V)$ contains only one $*$, then they differ in only one coordinate, hence these points are neighbours meaning $\chi(\tau(V(i))) = \chi(\tau(V(i+1)))$ by definition of τ.

Assume instead that $\tau(V)$ contains more than one $*$ so that $\tau(V(i))$ and $\tau(V(i+1))$ differ in more than one coordinate. Then, under χ and by definition of τ, they are still both the same colour. For example, if they differ in three coordinates, then we pass through a sequence of neighbours:

$$\begin{aligned}\chi(\tau(V(i))) &= \chi(...i...i...i...) = \chi(...(i+1)...i...i...) = \chi(...(i+1)...(i+1)...i...) \\ &= \chi(...(i+1)...(i+1)...(i+1)...) = \chi(\tau(V(i+1)))\end{aligned}$$

The same argument holds if $\tau(V(i))$ and $\tau(V(i+1))$ differ in 2, 4, or more coordinates.

With that we have shown that such a monochromatic line exists in $[t]^N$ (for any arbitrary r-colouring χ) and the proof is complete.

□

2.3 Known values

Like the Ramsey numbers, most Hales-Jewett numbers are practically impossible to find exact values of. The trivial case is with $r = 1$, then every combinatorial line is monochromatic and hence $HJ(1, k) = 1$. The only known non-trivial numbers are $HJ(2, 2) = 2$ and $HJ(2, 3) = 4$.

The upper bounds from Shelah's proof are incredibly large. There are, however, certain lower bounds related to using two colours: $HJ(2, 4) > 11$, $HJ(2, 5) > 59$, $HJ(2, 6) > 226$, $HJ(2, 7) > 617$, $HJ(2, 8) > 1069$, $HJ(2, 9) > 3389$.[2] The HJ-values of 2 colours will be interesting in chapter 3 discussing so called gTTT games.

2.4 Implications

With the Hales-Jewett theorem, we are ready to prove some other parts of Ramsey Theory. Let us recall, from the first chapter, what van der Waerden's theorem says:

Theorem 1. *(Van der Waerden's theorem) For any choice of integers r and t there exists a number $N = W(r,t)$ such that for any r-colouring of $[N]$ there will be at least one monochromatic arithmetic progression of length t.*

Proof. Take $M := nt + 1$ where $n = HJ(r,t)$. We will prove that this M satisfies the conditions of the theorem.

Define the map $f : [t]^n \to [M]$ by

$$x = (x_1, ..., x_n) \mapsto \sum_{i=1}^{n} x_i$$

Observe that the maximum value attained by $f(x)$ is $(\sum_{i=1}^{n} t) = tn < M$. Using r colours, define any r-colouring χ of the n-cube $[t]^n$. This induces a colouring $\tilde{\chi}$ of the set $[M]$ by letting $\chi(x) = \tilde{\chi}(f(x))$.

Consider any combinatorial line of length t:

$$L_\tau = \{\tau(1), ..., \tau(t)\}$$

For the sequence of points $\{\tau(1), ..., \tau(t)\}$, the difference $f(\tau(i+1)) - f(\tau(i))$ is precisely the number of $*$'s in the root τ. This number is constant. Then L_τ is mapped by f to an arithmetic progression of length t.

By the Hales-Jewett theorem, there exists a combinatorial line in $[t]^n$ that is monochromatic. The function f maps it to a monochromatic progression of length t and the proof is complete.

$\square$

Since upper bounds of $HJ(r,t)$ grow fast, so do those of the van der Waerden numbers.[4] Finding them is very difficult, just like with the Ramsey numbers. The infinite version then follows quite trivially since an r-colouring may be seen as a partitioning into r disjoint sets:

Theorem 2. *If $\mathbb{N}$ is partitioned into r disjoint sets $C_1, ..., C_r$ then at least one contains arbitrarily long arithmetic progressions.*

[4] This is because we (currently) define their upper bounds by upper bounds of Hales-Jewett numbers. With a stronger proof, this need not be true.

Proof. Theorem 1 says that we may find a set $C_j \subset [M]$ $(1 \leq j \leq r)$ containing an arithmetic progression of length t provided that M is large enough. We in fact proved that $M = nt + 1$ is large enough where $n = HJ(r, t)$.

For any t we may find M_t such that $C_j \cap [M_t]$ contains an arithmetic progression of length t. Letting $t = 1, 2, 3, ...,$ with r being finite, some C_j must satisfy this infinitely many times (by the infinite pigeon-hole principle).

Hence we can always be sure to find a set C_j with arbitrarily long arithmetic progressions when partitioning all of $\mathbb{N}$.

□

Definition 10. *For a vector space X over a field K, a set of vectors $W \subset X$ is a* **homothetic copy** *of $V \subset X$ if there is a vector $u \in X$ and a constant $\lambda \in K$ such that:*

$$W = w + \lambda V := \{w + \lambda v | v \in V\}$$

In other words, if there is a bijection between U and V that consists only of translation and dilation.

Theorem 6. *Colour each vector of $\mathbb{Z}^m$ with an r-colouring. Then every finite subset $V \subset \mathbb{Z}^m$ has a homothetic copy $W \subset \mathbb{Z}^m$ which is monochromatic.*

Proof. We are given a set of vectors $V = \{v_1, ..., v_t\} \subset \mathbb{Z}^m$. Let r denote the number of colours used and choose any r-colouring of $\mathbb{Z}^m$.

Set $n = HJ(r, t)$. Points $x \in V^n$ are of the form $(x_1, ..., x_n)$ where $x_i \in V$. Define the function $f : V^n \to \mathbb{Z}^m$ by $f(x) = x_1 + ... + x_n$ which induces an r-colouring of V^n by assigning $x = (x_1, ..., x_n)$ the same colour as $f(x) \in \mathbb{Z}^m$.

By Hales-Jewett there exists a root $\tau = \tau_1 ... \tau_n$ such that the line $L_\tau = \{\tau(v_1), ..., \tau(v_t)\} \subset V^n$ is monochromatic. Let $I := \{i | \tau_i = *\}$ and $\lambda := |I|$. Since τ is a root it is clear that $\lambda > 0$.

Define the following:

$$u = \sum_{i \in I} v_i.$$

By definition it follows that:

$$f(\tau(v_j)) = \underbrace{\sum_{i \in I} v_i}_{=u} + \underbrace{\sum_{i \notin I} v_j}_{=\lambda vj}.$$

Then we may conclude that

$$f(L_\tau) = \{f(\tau(v_1)), ..., f(\tau(v_t))\} = \{u + \lambda v_j | j = 1, ..., t\} \subset \mathbb{Z}^m$$

is a monochromatic, homothetic copy of $V = \{v_1, ..., v_t\}$. $\square$

This is one strength of the Hales-Jewett theorem: it not only applies to numbers. Here we used it for vectors in an arbitrary vector space $\mathbb{Z}^m$.

3 The generalized TicTacToe-game

3.1 Description

TicTacToe (or 3-in-a-row) is a traditional game by two players, each with their respective symbol (traditionally 'X' and 'O'), who take turns laying one symbol at a time on a 3x3 board. The first to achieve a complete line (vertical, horisontal or diagonal) with their symbol wins. The game has for a long time been generalised to k-in-a-row on an $m \times n$ board, known formally as m, n, k-games. The symbols may be seen as two distinct colours and playing the game is isomorphic to choosing a specific 2-colouring of the set of points $[m] \times [n]$. We may generalise this to the $m_1, ..., m_n, k$-game: Playing k-in-a-row on the n-dimensional board $[m_1] \times ... \times [m_n]$.

The generalised TicTacToe game[5] (from here on abbreviated gTTT) is this generalisation. However, for simplicity, we assume $m_1 = ... = m_n = t$. It consists of 2 players laying their own symbol, one per turn, on an n-dimensional lattice of side length t. In other words, the players take turns constructing a 2-colouring of the n-cube $[t]^n$. For example, regular TicTacToe is played on $[3]^2$. Although the game may be even further generalized to $r \geq 2$ players, this paper only discusses the 2-player case.

Once a symbol has been placed, it may not be moved or replaced. Any colouring of a point will remain the same during the rest of the game. We will call a combination of symbols on the board (a specific colouring of $[t]^n$) a *position*. A position that contains a monochromatic line is called a *winning position*, this means the game has been won. A position which is not a winning positions and no more moves can be played (all points on the n-cube have been coloured) is a *drawn position*, we say the game has been drawn. Not all positions are possible to reach through the rules, a position that is possible under the gTTT rules will be called a *legal position*. For legal positions, the number of colourings differs by at most 1 between any pair of players.

The gTTT game with two players is a game where all information about the current board position, previous moves and winning conditions is known. No information is hidden. In game theory this is known as a *2-person perfect information game*. The total number of moves (colouring of points) which are made during a game is t^n thus making the game *finite*.

Claim 2. *For any finite 2-person perfect information game with no draws, there exists a winning strategy for one player.*[6]

[5]Not to be confused with *Harary's generalized tic-tac-toe* which is a generalisation from only straight combinatorial lines to any *polyomino*, a figure connected by adjacent points (not diagonally). In this paper, we will consider gTTT to be the standard version and only include straight "in a row"-lines as winning combinations.[15]

[6]Also known as *Zermelo's theorem* in game theory, not to be confused with *the well-ordering theorem* of set theory equivalent to the axiom of choice.

Proof. Note that there cannot be winning strategies for both players, then these could be played against each other with the result that both players win, which is a contradiction.

The first player plays *not to lose*; that is, playing to make sure that player 2 does not have a winning strategy after each move. If player 1 cannot do this, then it means player 2 had a winning strategy from the beginning. On the other hand, if player 1 can play in this way, then he/she must win, because the game will be over after some finite number of moves and no draw is possible. □

When discussing a winning strategy we of course don't refer to the two players making completely random moves. A winning strategy guarantees reaching a won position (for the player following it) regardless of how the other player moves. Following such a winning strategy, or at least doing the optimal move in every situation, is known as *perfect play.* Hence player 2 may have perfect play without winning.

Only if some kind of rule(s) allow the second player some advantage to compensate for not making the first move can there exist a winning second player strategy.[7] In gTTT games there is no compensatory advantage, which leads us to the following claim:

Claim 3. *In games where the set of moves always is symmetric (the same for all players), such as m, n, k-games and gTTT games in general, if a winning strategy exists, it belongs to the first player.*

Proof. Both players have the same available set of possible moves leading to the same results and a move is independent of time. If the second player has no winning strategy, by the previous claim, the first player must have one.

Assume that the second player has a winning strategy S. The first player will convert this into a winning strategy for him- or herself (known as *strategy stealing*) by making the first move at random (at position $x \in [t]^n$) and then adopt strategy S. Looking at the game as the second player and by assumption of S, this is a winning strategy for the first player. If S calls for a move where the first random move was played, player 1 may simply play at random again (at position $y \in [t]^n$). The game is now equivalent to player 1 starting by making a random move in y and then following the winning strategy S as before.

Player 1 must win with S, otherwise this is a contradiction to the original assumption that the second player has a winning strategy. Buf if player 1 does win, then player 2 has no winning strategy, which was the assumption. This logical inconsistency forces us to the conclusion that S does not exist. □

The goal of gTTT is to create a "monochromatic line" accross $[t]^n$ of length $t_0 \leq t$. If not otherwise specified, $t_0 = t$. By using the theorem of Hales-Jewett, we can make

[7]According to Hales-Jewett, there is a large enough dimensionality so that a winning strategy exists even for any number of $r \geq 2$ players. As stated before, this scenario will not be examined here.

predictions about the existence of a winning strategy (for the first player) by looking at when a draw is impossible.

If a strategy can be shown to exist, the game is at least *ultra-weak solved*. When the algorithm for winning is found and can be run in a reasonable amount of time, the game is called *weak* or *strong* solved, depending on how much information the algorithm provides.[8] Games that permit drawn positions even with perfect play are known as *futile games*. Standard TicTacToe, $[3]^2$, is a futile game where all optimal moves have been found[10]. Note that all versions of gTTT games can be "solved" by sheer brute force, creating game trees and testing each move. Because of the practical impossibility of this simple algorithm, owing to the huge demand for memory and computational speed, the game is not considered solved until a practical algorithm that can be run in a reasonable amount of time has been found.

3.2 Application of Hales-Jewett

Using this by now familiar theorem, we may conclude that for a given board size (side length) t, there exists $N = HJ(2,t)$ such that there must exist a monochromatic line of any length $t_0 \leq t$ in $[t]^N$. In other words, a draw is simply impossible for dimensionality N or higher, regardless of strategy. According to claim 3, there must exist a winning strategy for player 1.

The value $HJ(2,t)$ is however an upper bound for guaranteeing the existence of a winning strategy. There may exist winning strategies for player 1 even though a draw is possible with incorrect play. For example, extending $[3]^2$ to the three-dimensional game $[3]^3$ it has been proven that a winning strategy exists.[6] We shall call the minimum number of dimensions needed for a winning strategy to exist by $GHJ(t)$. In other words, $GHJ(3) = 3$.

Let us look at $HJ(2,3)$. Shelah's proof gives the upper bound $HJ(2,3) \leq 512 + 2^{3^{514}} \approx 5.24 \cdot 10^{244}$. It has however been shown[1] that the true value is $HJ(2,3) = 4$. In other words, playing three-in-a-row in four dimensions, a drawn position is impossible. Also worth noting is the extreme upper bound in comparison with the true value 4. Still, Shelah's estimate is much better than the values offered in the original paper by Hales and Jewett[4]. Even more interesting, there are no drawn *legal* positions already in three dimensions. We need some more notation: Let $HJ_{legal}(2,3) = 3$ mean what was just stated.[9]

Definition 11. *A* **layer** *in* $[t]^n$ *is a position in* $[t]^{n-1}$*. Any position in* $[t]^n$ *consists of*

[8] Weak solved: The algorithm shows how to play from start to a won position. Strong solved: The algorithm shows the optimal move for any given position, even positions that do not result from not playing optimal moves beforehand.

[9] Observe that a monochromatic line $\{131, 122, 113\}$ which may constitute a win is not a combinatorial line in the Hales-Jewett sense. Hence the discrepancy between $HJ_{legal}(2,t)$ and $HJ(2,t)$.

exactly t layers.

Claim 4. $HJ_{legal}(2,3) = 3$. *In other words, there are no legal drawn positions in the gTTT game of* $[3]^3$.

Proof. Let us denote the symbol (or colour) of player 1 'X', and the symbol of player 2 'O'. There are only two drawn positions, up to symmetries and inversion (switching X and O), in $[3]^2$:

$$(1)\begin{bmatrix} O & X & O \\ X & O & X \\ X & O & X \end{bmatrix} (2)\begin{bmatrix} X & O & X \\ O & X & X \\ O & X & O \end{bmatrix}$$

Step 1: Observe that $[3]^3$ contains three layers (1, 2, 3) of $[3]^2$. Each of these must be a draw, and they must all together still not permit a monochromatic line in $[3]^3$. Assume that layer 1 equals the inverted (1).

Step 2: The centre square of layer 2 must be either X or O. Assume it is O.

Then, to avoid a win by X, we are forced to select the following layer 3:

$$1\begin{bmatrix} X & O & X \\ O & X & O \\ O & X & O \end{bmatrix} 2\begin{bmatrix} & & \\ & O & \\ & & \end{bmatrix} 3\begin{bmatrix} X & O & X \\ X & O & X \\ O & X & O \end{bmatrix}$$

It now turns out to be impossible to place symbols in the middle layer without creating a won position. The same problem arises by assuming X in step 2 and/or starting with (2) in step 1. For a complete proof see [5].

□

We may generalise the above proof and make the following claim.

Claim 5. *Any legal draw in* $[t]^n$ *consists of t layers of legal draws of* $[t]^{n-1}$.

This may need clarification.

Definition 12. *Let $\sim$ be the equivalence relation of two board positions under rotation, reflection and inversion ($X \mapsto O, O \mapsto X$).*

Definition 13. *Let $D_{t,n}$ denote the set of all equivalence classes of legal drawn positions under $\sim$ in* $[t]^n$.

Definition 14. *For computational purposes let* $d_{t,n} = |D_{t,n}|$.

Then $d_{3,1} = d_{3,2} = 2$ and $d_{3,3} = 0$. Also, $d_{2,1} = 1$, $d_{2,2} = 0$ and $d_{1,1} = 0$. The Hales-Jewett theorem tells us that $d_{t,n}$ becomes 0 if n is large enough, regardless of t.

We can construct elements of $D_{t,n}$ just as we can construct positions in $[t]^n$, by combining layers. Pick t (not neccessarily distinct) elements from $D_{t,n-1}$ and arrange them in order 1 to t. This position may be tested to be a draw or not. If so, it is included in $D_{t,n}$, otherwise discarded.

For a gTTT game, there are a total of 2^{t^n} positions. However, most of them are not possible under the standard rules of gTTT. Only

$$\binom{t^n}{\lfloor (t^n)/2 \rfloor}$$

positions fall into the category of legal positions. This is easily derived from the more general formula of a legal positions with r players (colours) by colouring a legal amount ($\lfloor (t^n)/r \rfloor$) for $r-1$ players. The remaining points are coloured for the last (r:th) player.

$$\binom{t^n}{\lfloor (t^n)/r \rfloor}\binom{t^n - \lfloor (t^n)/r \rfloor}{\lfloor (t^n)/r \rfloor} \cdot \ldots \cdot \binom{t^n - (r-2)\lfloor (t^n)/r \rfloor}{\lfloor (t^n)/r \rfloor} = \prod_{i=0}^{r-2} \binom{t^n - i \cdot \lfloor (t^n)/r \rfloor}{\lfloor (t^n)/r \rfloor}$$

If n is the first integer for which no legal positions end in a draw, then $n = HJ(2,t)$ and, as stated, $GHJ(t) \leq n$. Even this number, however, is too fast growing to provide estimates for $GHJ(t)$ upper bounds. Already $t = 4, n = 4$ gives approximately $5.77 \cdot 10^{75}$ legal positions.

Hales and Jewett, in their original paper[4], proved that gTTT played on $[t]^n$ has no winning strategy for either player if:

For odd t:

$$t \geq 3^n - 1 \Leftrightarrow n \leq \frac{ln(t+1)}{ln(3)}$$

For even t:

$$t \geq 2^{n+1} - 2 \Leftrightarrow n \leq \frac{ln(t+2)}{ln(2)} - 1$$

Claim 6. *For all $t > 2$ we have:*

$$\frac{ln(t+1)}{ln(3)} < \frac{ln(t+2)}{ln(2)} - 1$$

Proof. For $t = 2$ we see that:

$$\underbrace{\frac{ln(2+1)}{ln(3)}}_{=1} = \underbrace{\frac{ln(2+2)}{ln(2)} - 1}_{=1}$$

Since $ln(2) < 1$ but $ln(3) > 1$ we observe that $\frac{ln(t+1)}{ln(3)}$ grows slower than $\frac{ln(t+2)}{ln(2)}$ and the inequality is verified. □

This claim means that $d_{t,n} > 0$ whenever $n \leq \frac{ln(t+2)}{ln(2)} - 1$. Limiting the claim to $t > 2$ is unimportant since $D_{2,n} = \emptyset$ for all $n > 1$. $[2]^2$ has only two different positions under $\sim$, both won:

$$1 \begin{bmatrix} X & X \\ O & O \end{bmatrix} 2 \begin{bmatrix} X & O \\ O & X \end{bmatrix}$$

As stated earlier, a winning strategy exists when a draw is impossible. It is however conceivable that a winning strategy exists even though a drawn position is possible with incorrect play. In other words, $GHJ(t) \leq HJ(2,t)$. Very little is known about the true value of $GHJ(t)$ and the inequality need not be sharp.

We may summarise the information provided in the following theorem:

Theorem 7. *For a gTTT game of $[t]^n$, there exists a winning strategy whenever $n = GHJ(t)$, where:*

$$\frac{\ln(t+1)}{\ln(3)} < GHJ(t) \leq HJ_{legal}(2,t) \leq HJ(2,t) \leq N < \infty$$

Proof. Follows by definition of the three functions, Shelah's proof and the calculations in this chapter. □

We may even conjecture that $GHJ(t) = HJ_{legal}(2,t) = HJ(2,t)$ for all t except $t = 3$ but we simply do not know.

It is trivial to show both that $HJ(2,2) = 2$ and $GHJ(2) = 2$, as well as $HJ(2,1) = 1$ and $GHJ(1) = 1$. A simple 7-step winning strategy for $[3]^3$ has been found.[6]

3.3 Algorithms

When, by sheer brute force, we analyse positions of gTTT in order to determine drawn positions, we are not interested in all positions of $[t]^n$. We may in fact limit ourselves to drawn legal positions of $[t]^n$. Furthermore, according to claim 5, we need only look at the different n-combinations of elements in $D_{t,n-1}$. The most interesting cases are when $d_{t,n} = 0$ meaning that no draws exist and hence a winning strategy does exist.

An example of finding draws in $[3]^3$ is a good start. We know that $d_{3,2} = 2$. There are 8 possible ways of rotating and reflecting points on a square, and if we include inversion there are at most $8 \cdot 2 = 16$ different positions given from each unique position. Combinations do not yield more variations than the ones considered since the set of all rotations, reflections and inversions is a finite algebraic group. Thus, with the three layers of $[3]^3$ we have at most

$$(2 \cdot 16)^3 = 32768$$

legal drawn positions possible in $[3]^3$. It is easy for a computer to verify that none is in fact drawn and we can conclude that a draw is impossible here. It is certainly easier to consider only these positions instead of the total of

$$\binom{3^3}{\lfloor \frac{3^3}{2} \rfloor} = 20\,058\,300$$

legal positions. It is much better than the total of 2^{27} (legal and non-legal) positions in $[3]^3$.

Now let us generalise by borrowing the concept of the *hyperoctahedral group*[8] from abstract algebra. The hyperoctahedral group C_n has order $(2^n)(n!)$ and consists of the various rotations and reflections possible of the n-dimensional cube. Counting inversions, there is a maximum of $(2^{n+1})(n!)$ ways of using the elements of $D_{t,n-1}$ to form a layer in $[t]^n$, which in total has t layers.

We are now in a position to make the following assertion about the size of $D_{t,n}$:

$$0 \leq d_{t,n} \leq (d_{t,n-1} \cdot 2^{n+1} \cdot n!)^t$$

The new set $D_{t,n}$ is then further reduced by choosing only one representative from each equivalence class under $\sim$ (up to rotation, reflection and inversion), in order to simplify calculation of possible draws in $n+1$ dimensions.[10] Of course since $d_{3,3} = 0$ we know that $d_{3,4} = 0$, but this method works for other values of t as well. At the same time, this helps us calculate lower bounds of $HJ(2,t)$ by claim 7.

We can formalise a simple brute force algorithm for attempting to find Hales-Jewett numbers.

1. Input board size t and colours r.

[10] In a mathematical sense, only the number of equivalence classes is interesting. When actually computing $d_{t,n}$ it may be more efficient to store all representatives of $D_{t,n-1}$, assuming it can be done efficiently. This is because some combinations of rotations, reflections and inversion may be identical.

2. Start at $N = 2$.
   (a) Recursively loop through each colouring of $[t]^N$.
   (b) For each colouring, check for monochromatic line.
   (c) If none, break and go to (2) but with $N = N + 1$.
   (d) If every possible colouring has a monochromatic line: stop and print N.

This algorithm, as simple as it appears, is computationally difficult even for small values of t and r. It requires at least $(r-1)^{t^N}$ steps to generate the board. In the context of gTTT games it requires t^N steps.

A combinatorial line is defined by a root $\tau = (x_1, ..., x_N)$ with $x_i \in ([t] \cup \{*\})$ where at least one $x_i = *$, meaning there are $N \cdot (t+1)^{N-1}$ combinatorial lines of length t. At worst they must all be checked out for monochromaticity.

The previous arguments have provided ideas on a more efficient algorithm when dealing with gTTT games. Let us formalise it.

1. Start with a set $D_{t,n-1} = \{X_1, ..., X_m\}$ of draws in $[t]^{n-1}$.
2. Assume a function (modify(X_i,j)) exists that takes a board position X_i and where $0 \leq j \leq (2^{n+1})(n!)$ is a possible rotation/reflection/inversion or combination of the three. The function returns a new board which is the result of position X_i and modification j.
3. Create a $[t]^n$ board by selecting t layers. For each i and j and each layer k:
   - Select element X_i from $D_{t,n-1}$.
   - Place (modify(X_i, j)) as layer k.
   - For each colouring of the board, check for monochromatic line. If none, add this to a new list Y.
4. Remove elements from Y until exactly one representative of each equivalence class under $\sim$ remains.
5. $D_{t,n}$ has been created as the quotient group $(Y/\sim)$.

This algorithm requires $(d_{t,n-1} \cdot 2^{n+1} \cdot n!)^t$ steps to find Y which is the most costly computation. Again, there are $N \cdot (t+1)^{N-1}$ combinatorial lines. Removing identical elements of Y under $\sim$ is a much simpler task.

The newer algorithm is certainly simpler, (complexity of algorithm 1) $\geq$ (complexity of algorithm 2), since

$$2^{t^n} \geq (d_{t,n-1} \cdot 2^{n+1} \cdot n!)^t$$

is equivalent to

$$t^n \log(2) \geq t \log(d_{t,n-1} \cdot 2^{n+1} \cdot n!) = t[\log(d_{t,n-1}) + (n+1)\log(2) + \log(n!)]$$

which simplifies to:

$$t^{n-1} \geq \frac{1}{\log(2)}[\log(d_{t,n-1}) + (n+1)\log(2) + \log(n!)]$$

This inequality holds for large enough t and n.

3.4 Using the algorithm

Let us see what these algorithms may tell us about the game $[4]^3$, sometimes known as *Qubic*. We want to examine $D_{4,n}$ and specifically $D_{4,3}$. It is easily verified that the cardinality of $D_{4,1}$ is 5, the complete list is as follows:

- $XXXO$
- $XXOX$
- $XXOO$
- $XOXO$
- $XOOX$

To create $D_{4,2}$ we will require $(5 \cdot 2^2 \cdot 2!)^4 \approx 2\,560\,000$ calculations.[11] This number is certainly within the realm of today's computers. The problem arises later as there are 865 distinct drawn positions of $[4]^2$ under $\sim$. Due to the large amount, these are not listed here.

We may try to proceed to find $D_{4,3}$ with the same method. Observe that with $d_{4,2} = 865$ the total number of calculations needed is:

$$(865 \cdot 2^3 \cdot 3!)^4 = 41\,520^4 = 2\,971\,867\,067\,228\,160\,000 \approx 2.97 \cdot 10^{18}$$

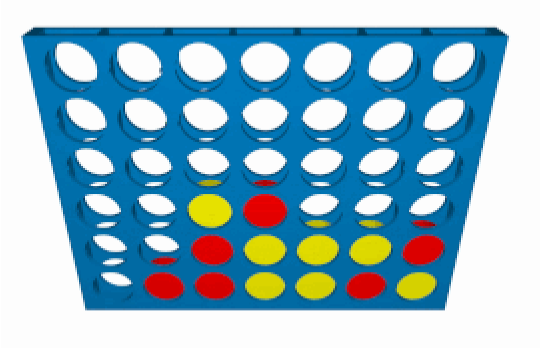
Still less than the original algorithm that requires $2^{4^3} = 2^{64} \approx 1.8 \cdot 10^{19}$ steps, it is still too large to be practically calculable.

It turns out that $[4]^3$ is a win for player 1. It was strongly solved in 1994.[14]

[11] Observe that in 1 dimension, rotation and reflection are the exact same operations.

3.5 gTTT games with gravity

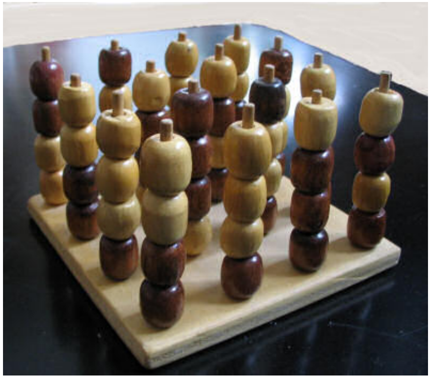
Another adaptation of $[4]^2$ is known as *Connect 4* (other popular names include *Captain's Mistress*, *Four Up*, *Plot Four*, *Find Four*, *Fourplay*, *Four in a Row* and *Four in a Line*), usually played on a 6×7 board, making it a variant of the m, n, k-game: a 6,7,4-game. The board is traditionally not flat on a surface but placed upright and the symbols placed obey physical gravity. A symbol can only be placed at a certain place if there is a symbol directly below it.



We may formalise this as follows: A colouring of (x, y) may only be determined if $y = 1$ *or* a colouring of $(x, y-1)$ has been determined. We will denote this by saying that the dimension y *has gravity* or generally that Connect 4 is the gTTT game of $[6] \times [7]$ with gravity.

The game was solved in 1988 by Victor Allis[12]. If the first player starts in the middle column, a win can be forced by perfect play. However, starting in the four outer-most columns, the second player can actually force a win! This does not mean that the game is more balanced than the equivalent gTTT game without gravity, since the first player playing in the outer edges obviously does not constitute perfect play. There is still only a winning strategy for the second player if the first player makes a bad first move. Several drawn positions also exist. But the concept of strategy stealing is not as obvious with gravity, as will be discussed in the following paragraphs.

Consider the game $[4]^3$ where one dimension has gravity (known as *Score four*), a three-dimensional version of Connect 4. In other words, a colouring of (x, y, z) may only be determined if $z = 1$ or if $(x, y, z-1)$ has been coloured. Saying specifically that dimension z has gravity is irrelevant, we could without loss of generality let the x or y dimension have gravity, the resulting game would be isomorphic to the first.



Score Four has not been solved, but research is progressing in this area, for example a self-playing reinforcement based learning AI.[11]

Observe that the rule of gravity, regardless of how many dimensions it applies to (we need not stop at one) does not change the set of possible positions. $[3]^3$ still has no draws, even if all three dimensions have gravity. The type of game (a 2-person finite perfect information game) remains the same, meaning that claim 2 still holds and a winning strategy must exist.

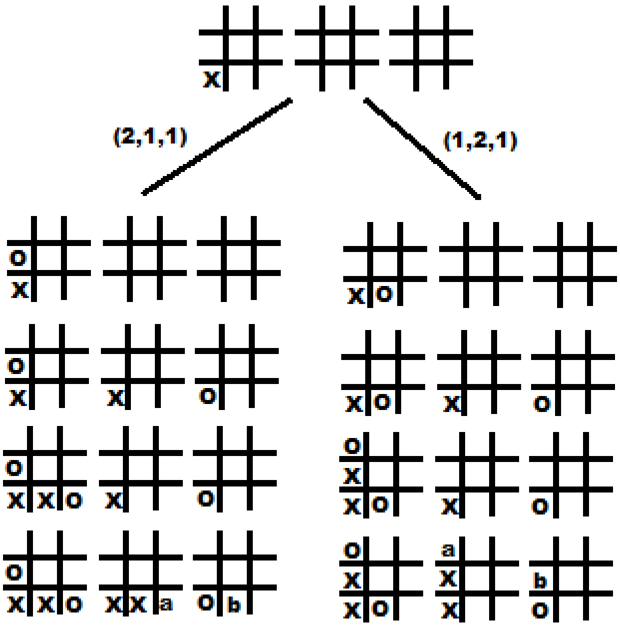
However, the game is no longer symmetric since the set of current possible moves depends on the ones previously played. Hence the concept of strategy stealing no longer works, claim 3 does not apply to gTTT games with gravity.

One winning strategy is included as an example:

Theorem 8. *There exists a winning strategy for $gTTT$ of $[3]^3$ where all dimensions have gravity.*

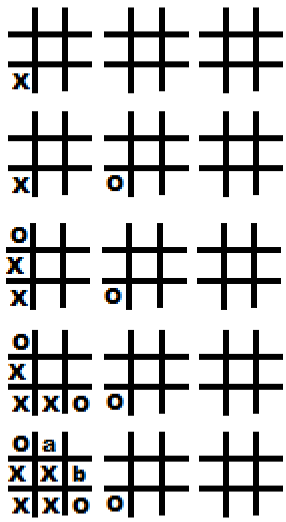
Proof. Let us denote the first player X and the second player O. The first move must be placed in (1,1,1). Player O has three options, to play (2,1,1), (1,2,1) or (1,1,2). Let us examine each one and notice that (2,1,1) and (1,2,1) lead to isomorphic games (reflected across the diagonal).

Two moves, one for each player, are displayed at once if the second move is forced, i.e. must be played to keep the other player from directly winning. Read the game tree from the top down.



Player X now has winning moves both at a and b. If O blocks a then X plays b and wins, or vice versa.

Playing (1,1,2) is no better:



Again X has two winning moves a and b. Wherever O chooses to block, X may simply play the other and win.

$\square$

The computational gain of the rule of gravity is that the number of possible positions gets severly limited. In the just mentioned game where all dimensions have gravity, there is only one possible starting move $(1, 1, 1)$ and only three possible second moves $(2, 1, 1)$, $(1, 2, 1)$ and $(1, 1, 2)$.

Another game idea is to play $[t]^n$ one layer at a time. In other words, $(x_1, ..., x_{n-1}, 2)$ may only be coloured if there is a colouring at all $(x_1, ..., x_{n-1}, 1)$ for any $x_i \in [t]$. The idea may be generalised in the obvious way. Existence of a winning strategy here is not equivalent with existence in the similar gTTT game with one dimension with gravity, and neither is it equivalent with the standard gTTT game.

3.6 Avoiding the strategy stealing argument

There are other ways than gravity for avoiding the fact that the second player cannot win. *Renju* is a variant of 5-in-a-row on $[15]^2$ which includes a special opening rule that compensates for the disadvantage of not making the first move. An example of such a special rule, called *RIF*, works as follows. The colours of the two players are traditionally represented by white and black stones:

1. The first player places 2 black stones and 1 white stone on the board.
2. The second player chooses whether to play black or white.
3. White places one more stone on the board.
4. Black places 2 stones on the board.
5. White removes 1 of the 2 new black stones.
6. White places a white stone.
7. The game proceeds as normal, one stone at a time.

Another way of using recolouring is found in a game known as *Pentago*. It is 5-in-a-row played on $[6]^2$. The board is split into four quadrants. After each move one quadrant must be rotated either clockwise or anticlockwise, allowing up to $4 \cdot 2 = 8$ different recolourings after each move.

Looking at statistics of more complex games which allow re-colouring (like Chess and Go) the first player wins more often than the second. It is conjectured that a winning strategy may exist for player 1, but neither game has been solved.

At any rate, some form of recolouring, either in the beginning as a special opening move rule or during the game as rotating quadrants, is neccessary in order to avoid strategy stealing in games where a move may be done in any empty location.

Using the previously discussed notion of ”gravity” things change. Suddenly situations may arise where a player is forced to make a move even though it aids the other player. This is known as *Zugzwang* (German: *Forced move*). Without gravity, a move may be placed anywhere and thus always in the optimal position. With gravity, this can no longer be guaranteed to be possible. The set of moves is no longer symmetric, previous moves influence possible future moves.

Chess, where pieces are moved in certain patterns, also has the concept of Zugzwang. Go, where a player may pass (choose to make no move), does not.

3.7 Other Ramsey related games

The gTTT game may be transformed into the following very different looking but actually isomorphic game (for $[3]^2$): Two players take turns saying the numbers 1-9. Each number may be said only once. The first player to have said three numbers which sum up to 15 wins.

This game is isomorphic to gTTT by observing that it corresponds to the arithmetic magic square

$$\begin{bmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{bmatrix}$$

with the sum 15: any column, row or diagonal add up to 15.

Since normal magic squares may be generalised into n dimensions, we may extend this game for $[t]^n$ by having the players say the numbers $1, ..., t^n$. The winner is the first player able to reach $M_n(t)$, the sum of the magic n-square of size t.[7]

$$M_n = \frac{t(t^n + 1)}{2}$$

The Ramsey numbers mentioned in the beginning may also be used for a certain type of game construction, known as *Sim*. The board then consists of the complete graph with 6 nodes (and $(6-1)! = 120$ edges). With two players and their respective colour (traditionally red and blue) they colour one edge per turn. The first to achieve a monochromatic triangle in their colour wins.

Since $R(3,3) = 6$ one triangle of either colour must be drawn eventually and so the game has no drawn positions. Like gTTT, this is a 2-player perfect information game with no draws. The set of moves is symmetric, hence the first player has a winning strategy (according to claim 3). The game may well be extended to creating quadrilaterals with $R(4,4) = 18$ nodes or pentagons with $R(5,5)$ nodes. Since the exact value of $R(5,5)$ is unknown, this game has an additional research-oriented benefit. It is known that $43 \leq R(5,5) \leq 49$ [13] so playing on 43 nodes, it might be possible to draw and in that case a counterexample has been found, proving that $R(5,5) \geq 44$ and mathematical research has advanced.[12] Either way the game must be somehow modified to be made playable in practice since the board contains no less than $(43-1)! \approx 1.4 \cdot 10^{51}$ edges and reaching a drawn position will require as many moves.

Possibly, with extended work, Ramsey theory could indeed be adapted to describe more complex pattern-generating games that are finite but allow for recolouring, such as Go. *Harary's generalised TicTacToe* is concerned with more than just straight lines and has some results in this area of research.[15]

[12] Of course, it may very well be that $R(5,5) = 43$. In that case there is no breakthrough to be made in this manner.

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