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A Simulation Verification of the Asymptotic Refinement in the Bootstrap Cointegration Rank Test

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Abstract

The cointegration rank trace test using asymptotic critical value exhibits size distortion when the sample size is small. Bootstrap is an appealing method to get an actual test size closer to the nominal one. This short paper discusses the bootstrap cointegration test focusing on the asymptotic refinement. We expand the test statistic of a simplified model and an intensive Monte Carlo simulation is done to verify that the bootstrap test gives asymptotic refinement.

Keywords: Cointegration, Rank test, Monte Carlo simulation, Bootstrap

1. Introduction and Motivation

An important and widely used model in formulating the cointegrating long-run economic relations is the vector error correction model (VECM):

$$\Delta X_t = \alpha\beta' X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \Phi D_t + \varepsilon_t, \quad t = 1, ..., T, \tag{1}$$

where k is the number of lags in levels, ε_t are independent error terms, $E(\varepsilon_t) = 0$, $E(\varepsilon_t \varepsilon_t') = \Omega$ with Ω positive definite and D_t is the deterministic term. If: (a) the AR characteristic polynomial has $p-r$ roots equal to 1 and all other roots lie outside the unit circle and (b) α and β have full column rank r, then X_t satisfies the $I(1, r)$ condition, i.e. X_t is $I(1)$ with cointegration rank r. The cointegration relationship is β such that $\beta' X_t - E(\beta' X_t)$ is stationary. To determine the cointegration rank r, the most commonly used procedure is the likelihood ratio based trace test given by Johansen (1995). The test procedure is outlined as follows: first, do the regression of ΔX_t and X_{t-1} on $(\Delta X_{t-1}, ..., \Delta X_{t-k+1}, D_t)$ respectively and obtain the residuals R_{0t} and R_{1t};

second, construct the matrices $S_{ij} = T^{-1}\sum_{t=1}^{T} R_{it}R'_{jt}, i, j = 0, 1$; third, solve the eigenvalue problem

$$|\hat{\lambda}S_{11} - S_{10}S_{00}^{-1}S_{01}| = 0, \tag{2}$$

for the eigenvalues $\hat{\lambda}_1 > ... > \hat{\lambda}_p$; the test statistic to test $\mathcal{H}(r)$: rank $\leq r$ against $\mathcal{H}(p)$: rank $= p$ is

$$Q_{r,T} = -T\sum_{i=r+1}^{p}\log(1-\hat{\lambda}_i).$$

The distribution of $Q_{r,T}$ is unknown and the inference is based on the asymptotic distribution:

$$-2\log Q_{r,T} \xrightarrow{d} tr\big(Q(r,\infty)\big) \quad \text{as} \quad T \to \infty,$$

where

$$Q(r,\infty) = \int_0^1 W_{p-r}(u)dF_{p-r}(u)'\Big[\int_0^1 F_{p-r}(u)F_{p-r}(u)'du\Big]^{-1}\int_0^1 F_{p-r}(u)dW_{p-r}(u)',$$

with W_{p-r} a standard $p-r$ dimensional Wiener process, $F_{p-r} = W_{p-r}$ when there is no deterministic term; $F_{p-r} = (W'_{p-r}, 1)'$ when there is a restricted constant; $F_{p-r} = (W'_{p-r}, u|1)'$ when there is a restricted linear trend, for details see e.g. Johansen (1995). The distribution of $tr\big(Q(r,\infty)\big)$ is non standard and the critical values are tabulated by simulation. Usually a sequential test procedure is used to determine the cointegration rank. Starting with testing $\mathcal{H}(0)$ against $\mathcal{H}(p)$, if $\mathcal{H}(0)$ is accepted, the rank is determined as 0 (no cointegration). If $\mathcal{H}(0)$ is rejected, continue testing $\mathcal{H}(1)$ against $\mathcal{H}(p)$. If $\mathcal{H}(1)$ is accepted, the rank is determined as 1 (1 cointegration relationship) otherwise continue to test $\mathcal{H}(2)$ against $\mathcal{H}(p)$. The cointegration rank is determined as r until $\mathcal{H}(r)$ against $\mathcal{H}(p)$ is accepted. Once the rank is determined, $\hat{\beta}_{p\times r}$ is estimated as $(\beta_1, \beta_2, ..., \beta_r)$ with β_i being the eigenvectors corresponding to $\hat{\lambda}_i$ in the eigenvalue problem (2). The rest of the parameters α, Γ_i, and Φ can be estimated by standard OLS regression results.

The main drawback in rank testing when using the asymptotic critical values is the size distortion, i.e. the discrepancy between the empirical rejection probability and the nominal size is large and sometimes too large to make reliable inference. To overcome this drawback, the small sample correction by Johansen (2002) can be used. The bootstrap method, if used correctly, can also help improve the finite sample size properties of asymptotic tests, see, among others Swensen (2006) and Cavaliere et al. (2012). However, all previous work has been focusing on the consistency and no work has been done regarding the asymptotic refinement. The motivation of this paper is to make an attempt to study the asymptotic refinement of the bootstrap cointegration rank test. The rest of the paper is organized as follows: Section 2 introduces the bootstrap algorithm and the consistency of the bootstrap cointegration rank test. Section 3 discusses the asymptotic refinement and section 4 presents the result from an intensive Monte Carlo experiment and a conclusion ends the paper.

2. Bootstrap

When the exact distribution of a statistic is unknown or very difficult to derive, people usually turn to the asymptotic distribution, which can be considered as an approximation of the exact distribution. The bootstrap is another way to approximate the exact distribution. In a bootstrap framework, the sample is considered as the population and the functionals of the empirical distribution function are approximations to the counterparts of the functionals of the population distribution function. In bootstrap hypothesis testing, the distribution of the statistic under the null is approximated by resampling. The crucial point is that the bootstrap sample must mimic the data generating process (DGP) under the null. In the procedure of determining the cointegration rank, for each test $\mathcal{H}(r)$ against $\mathcal{H}(p)$, the bootstrap sample should satisfy the $I(1,r)$ condition regardless of the true cointegration rank r_0. Cavaliere et al. (2012) gives a valid bootstrap algorithm as follows:

(1) Estimate Model (1) under $\mathcal{H}(r)$ yielding the estimates $\hat{\beta}^{(r)}$, $\hat{\alpha}^{(r)}$, $\hat{\Gamma}_1^{(r)}$,...,$\hat{\Gamma}_{k-1}^{(r)}$ and $\hat{\Phi}^{(r)}$ together with the corresponding residuals, $\hat{\varepsilon}_{r,t}$;

(2) Check that the equation $|\hat{A}^{(r)}(z)| = 0$, with $\hat{A}^{(r)}(z) = (1-z)I_p - \hat{\alpha}^{(r)}\hat{\beta}^{(r)\prime}z - \sum_{i=1}^{k-1}\hat{\Gamma}_i^{(r)}(1-z)z^i$, has $p-r$ roots equal to 1 and all other roots lie outside the unit circle;

(3) Generate T bootstrap residuals $\varepsilon^*_{r,t}$ by i.i.d. sampling with replacement from the re-centered residuals $\{\hat{\varepsilon}_{r,t} - T^{-1}\sum_{i=1}^{T}\hat{\varepsilon}_{r,i}\}, t = 1, ..., T$. The recentering is to ensure that the mean of the bootstrap residuals is

zero and is only necessary when no constant term is included in the regression;

(4) Construct the bootstrap sample recursively by

$$\Delta X^*_{r,t} = \hat{\alpha}^{(r)}\hat{\beta}^{(r)\prime}X^*_{r,t-1} + \sum_{i=1}^{k-1}\hat{\Gamma}_i^{(r)}\Delta X_{t-i} + \hat{\Phi}^{(r)}D_t + \varepsilon^*_t;$$

for $t = 1, 2, \ldots, T$, starting from $X^*_{-k+1} = X_{-k+1}, \ldots, X^*_0 = X_0$.

(5) Using the bootstrap sample $\{X^*_{r,t}\}$, calculate the bootstrap analogue S^*_{ij}, $i, j = 0, 1$ and the corresponding eigenvalues $\hat{\lambda}^*_1 > ... > \hat{\lambda}^*_p$. Compute the bootstrap LR statistic $Q^*_{r,T} = -T\sum_{i=r+1}^{p}\log(1-\hat{\lambda}^*_i)$;

(6) Repeat (3)-(5) B times, obtain the empirical distribution of $Q^*_{r,T}$.

The crucial point in the bootstrap algorithm is that all parameters should be estimated under the null, i.e. imposing rank r in the underlying VECM model. In practice, this means that we need to estimate the parameters for each test $\mathcal{H}(r)$ against $\mathcal{H}(p)$. Cavaliere et al. (2012) shows that, asymptotically, $X^*_{r,t}$ satisfies the $I(1, r)$ condition if the parameters $\Gamma_i^{(r)}$ and $\Phi^{(r)}$ are estimated under $\mathcal{H}(r)$. However, if these parameters are estimated from the unrestricted model, i.e. under $\mathcal{H}(p)$, $X^*_{r,t}$ are not guaranteed to satisfy the $I(1, r)$ condition even asymptotically. Step (2) is necessary because the bootstrap algorithm only ensures that the bootstrap sample $X^*_{r,t}$ satisfies the $I(1, r)$ condition asymptotically and this could fail in finite samples.

The distribution of the bootstrap statistic is analyzed in the bootstrap probability space. To be specific, for each realization of $\{\hat{\varepsilon}_t\}$, it generates a bootstrap probability space $(\Omega^*, \mathcal{F}^*, P^*)$ conditional on the realization of $\{\hat{\varepsilon}_t\}$. The bootstrap residuals $\{\varepsilon^*_t\}$ can be regarded as random variables defined on this probability space and its distribution function is the empirical distribution function of $\{\hat{\varepsilon}_t\}$. Under this set up, we can discuss the expectation E^*, the variance Var^* and the convergence behavior $\xrightarrow{p^*}$, $\xrightarrow{d^*}$ for the (functionals of) bootstrap samples defined on $(\Omega^*, \mathcal{F}^*, P^*)$. Furthermore since $\{\hat{\varepsilon}_t\}$ is a realization of the process $\{\varepsilon_t\}$, if the convergence in $(\Omega^*, \mathcal{F}^*, P^*)$ occurs a.s. (almost surely) for all realizations of $\{\hat{\varepsilon}_t\}$, then we say, $\xrightarrow{p^*}$ a.s. and $\xrightarrow{d^*}$ a.s.. We can define $\xrightarrow{p^*}$ in probability and $\xrightarrow{d^*}$ in probability in the similar manner, see e.g. Park (2002).

Cavaliere et al. (2012) show the consistency of the bootstrap cointegration rank test, i.e. for any $r \leq r_0$, $Q^*_{r,T} \xrightarrow{d^*} tr\big(Q(r,\infty)\big)$ as $T \to \infty$ in probability.

3. Asymptotic refinement

3.1. Revisit the asymptotic refinement of the unit root test

We first review the asymptotic refinement of the unit root test which is highly related to our topic because each individual process in a cointegrated system is a unit root process. Park (2003) proved that the bootstrap gives asymptotic refinements in the Dickey-Fuller unit root test (both in the t-test and coefficient test) in the AR(p) model with p known:

$$Y_t = \rho Y_{t-1} + \sum_{i=1}^{p-1} \gamma_i \Delta Y_{t-i} + \varepsilon_t.$$

For a review of unit root t-test and unit root coefficient test, see Hamilton (1994). Take the coefficient test for example (the t-test follows in a completely same manner), the Dickey-Fuller coefficient test statistic G_T and the asymptotic distribution are

$$G_T = \frac{T(\hat{\rho}-1)}{1-\sum_{i=1}^{k-1}\hat{\gamma}_i} \xrightarrow{d} \frac{\frac{1}{2}[W^2(1)-1]}{\int_0^1 W^2(r)dr} = G,$$

where G represents the random variable with the asymptotic distribution. Under some technical moment conditions, the test statistic can be expanded as

$$G_T = G + T^{-\frac{1}{4}}G_1 + T^{-\frac{1}{2}}G_2 + o_p(T^{-\frac{1}{2}}). \tag{3}$$

The first order term G follows the asymptotic distribution as expected. Let G_{TT} denote the second order terms $T^{-\frac{1}{4}}G_1 + T^{-\frac{1}{2}}G_2$, G_{TT} involves various functionals of Brownian motions and various model parameters: $G_{TT} = f\big(\theta, (W, W^1, W^2, S)\big)$, where $\theta = (v, \sigma^2, \mu^3, \kappa^4, \tau^4)$, and W^1 and W^2 are standard Brownian motions independent of W, S is a normal random vector, v represents the starting value, $\sigma^2 = E\varepsilon_t^2$, $\mu^3 = E\varepsilon_t^3$, $\kappa^4 = E\varepsilon_t^4$ and τ^4 can be regarded as the non-normality parameter. $\tau = 0$ if and only if ε_t is normal. The distributional effect of G_{TT} is of order $O(T^{-\frac{1}{2}})$ because the characteristic function of G_1 can be expanded in powers of $T^{-\frac{1}{2}}$, for more details, see Park

(2003). We have

$$P(G_T \leq x) = P(G \leq x) + O(T^{-\frac{1}{2}}),$$
$$= P(G + G_{TT} \leq x) + o(T^{-\frac{1}{2}}).$$

From the above result it can be seen that if the asymptotic critical value c_α is used in the test, then $P(G_T \leq c_\alpha) = \alpha + O(T^{-\frac{1}{2}})$. The discrepancy between the rejection probability and the nominal size α is of order $O(T^{-\frac{1}{2}})$.

For the bootstrap statistic G_T^*, the expansion is analogous to the expansion G_T in (3),

$$G_T^* = G + G_{TT}(\theta_T) + o_p^*(T^{-\frac{1}{2}}).$$

where $G_{TT}(\theta_T) = f\big(\theta_T, (W, W^1, W^2, S)\big)$, $\theta_T = (v, \sigma_T^2, \mu_T^3, \kappa_T^4, \tau_T^4)$ are the sample analogue estimators of $\theta = (v, \sigma^2, \mu^3, \kappa^4, \tau^4)$. For example $\sigma_T^2 = \hat{\sigma}^2 = E^* \varepsilon_t^{*2}$. Due to the law of iterative logarithm for i.i.d. sequences, we have for any $\epsilon > 0$,

$$\theta_T = \theta + o_p^*(T^{-\frac{1}{2}+\epsilon}),$$

and therefore,

$$G_T^* = G + G_{TT}(\theta) + o_p^*(T^{-\frac{1}{2}}). \tag{4}$$

Define c_α^* as the bootstrap critical value for testing G_T with nominal rejection probability α, i.e.

$$P^*(G_T^* \leq c_\alpha^*) = \alpha,$$

then we have

$$P(G_T \leq c_\alpha^*) = \alpha + o(T^{-\frac{1}{2}}),$$

i.e. the discrepancy between the rejection probability of the bootstrap test and the nominal size is of order $o(T^{-\frac{1}{2}})$.

To sum up, the result of bootstrap unit root test, under the condition that $E(\varepsilon_t) = 0$ and $E|\varepsilon_t^r| < \infty$, for some $r > 12$, the critical values obtained by the bootstrap resampling are correct up to the second-order terms, and the errors in rejection probabilities are of order $o(T^{-\frac{1}{2}})$ if the tests are based upon the bootstrap critical values.

3.2. Asymptotic refinement of the cointegration rank test

For the trace statistic of the cointegration rank test, consider the VECM model:

$$\Delta X_t = \Pi X_{t-1} + \varepsilon_t, \quad t = 1, ..., T \tag{5}$$

with $X_0 = 0$ and ε_t are iid $N(0,\Omega)$. Model (5) is quite restrictive, as we exclude the short run dynamics and deterministic terms which are important in real life macro-econometric analysis. However, the focus of this paper is to verify the asymptotic refinement of cointegration rank test. Therefore, the simplest framework is addressed at this first stage.

Under this framework, the residuals R_{0t} and R_{1t} are reduced to ΔX_t and X_{t-1}, the S_{ij} matrices in (2) are simplified as $S_{00} = \frac{1}{T}\sum_{t=1}^{T}\Delta X_t \Delta X_t'$, $S_{10} = \frac{1}{T}\sum_{t=1}^{T} X_{t-1}\varepsilon_t'$, $S_{11} = \frac{1}{T}\sum_{t=1}^{T} X_{t-1}X_{t-1}'$. The expansions of these matrices are given by the following proposition,

Proposition 1. *If $\varepsilon_t \sim N(0,\Omega)$, X_t is generated by Model (5) with $X_0 = 0$, then*

$$\frac{1}{T}S_{11} = \frac{1}{T}\sum_{t=1}^{T} X_{t-1}X_{t-1}' = \Omega^{\frac{1}{2}}\int_0^1 WW'dr\Omega^{\frac{1}{2}} + O_p(T^{-1}),$$

$$S_{10} + S_{01} = \frac{1}{T}(\sum_{t=1}^{T} X_{t-1}\varepsilon_t' + \sum_{t=1}^{T}\varepsilon_t X_{t-1}') = \Omega^{\frac{1}{2}}[W(1)W'(1) - \frac{1}{\sqrt{T}}U - I]\Omega^{\frac{1}{2}} + o_p(T^{-\frac{1}{2}}),$$

$$S_{00} = \frac{1}{T}\sum_{t=1}^{T}\Delta X_t \Delta X_t' = \Omega^{\frac{1}{2}}(I + \frac{1}{\sqrt{T}}U)\Omega^{\frac{1}{2}} + o_p(T^{-\frac{1}{2}}),$$

where $W(1)$ is a standard p-dimensional vector random variable $N_p(0,I)$, U is a normally distributed random matrix with mean zero and $Var[vec(U)] = I_{p\times p} + K$, where K is the commutation matrix. The expansions of S_{01} and S_{10} have the forms

$$S_{01} = \Omega^{\frac{1}{2}}\big(S_1 - \frac{1}{\sqrt{T}}S_2\big)\Omega^{\frac{1}{2}} + o_p(T^{-\frac{1}{2}})$$
$$S_{10} = \Omega^{\frac{1}{2}}\big(S_1' - \frac{1}{\sqrt{T}}S_2'\big)\Omega^{\frac{1}{2}} + o_p(T^{-\frac{1}{2}})$$

where S_1 and S_2 are random matrices of stochastic order $O_p(1)$, $S_1 + S_1' = W(1)W'(1) - \Omega$ and $S_2 + S_2' = U$.

Corollary 1.

$$TS_{11}^{-1} = \Omega^{-\frac{1}{2}}\Big(\int_0^1 WW'dr\Big)^{-1}\Omega^{-\frac{1}{2}} + O_p(T^{-1}),$$
$$S_{00}^{-1} = \Omega^{-\frac{1}{2}}(I - \frac{1}{\sqrt{T}}U)\Omega^{-\frac{1}{2}} + o_p(T^{-\frac{1}{2}}).$$

With these results, the trace statistic of the rank zero test $Q_{0,T}$ can be expanded as:

Theorem 1.

$$Q_{0,T} = -T\sum_{t=1}^{p}\log(1-\hat{\lambda}_i) = R - T^{-\frac{1}{2}}R_1 + o_p(T^{-\frac{1}{2}}),$$

where

$$\begin{aligned} R =& tr\Big\{\Big(\int_0^1 WW'dr\Big)^{-1}\int_0^1 W(dW)'\int_0^1 W'(dW)\Big\},\\ R_1 =& tr\Big\{\Big(\int_0^1 WW'dr\Big)^{-1}S_1\sqrt{2}US_1'\\ &+\Big(\int_0^1 WW'dr\Big)^{-1}S_1S_2'\\ &+\Big(\int_0^1 WW'dr\Big)^{-1}S_2S_1'\Big\}. \end{aligned}$$

The asymptotic expansion has the leading term R representing its asymptotic distribution. The second order term R_1, which involves a standard p-dimensional vector Brownian motions W and a normal random matrix U, corresponds to G_1 and G_2 in the expansion (3). The remainder term in the expansion is of stochastic order $o_p(T^{-\frac{1}{2}})$. As a result of Theorem 1, when using the asymptotic critical value in testing $\mathcal{H}(0)$ against $\mathcal{H}(p)$, the error in the rejection probability is of order $O(T^{-\frac{1}{2}})$, i.e. $P(Q_{0,T} \leq x) = P(R \leq x) + O(T^{-\frac{1}{2}})$.

The asymptotic expansion of the bootstrap analogue is very difficult to derive. This is because the bootstrap residuals $\{\varepsilon_t^*\}$ are sampled from the estimated residuals $\{\hat{\varepsilon}_t\}$ and hence are not normal even if the error term ε_t is from $N(0,\Omega)$. In the univariate unit root test, the non-normality is treated by a non-normality parameter τ. The non-normality parameter τ appears in $G_{TT}(\theta)$ in (4) and can not be generalized to vector form directly. This is of interest to the author for further study. However, it can be expected that each component in the bootstrap asymptotic expansion will converge to the counterpart in the non-bootstrap version, i.e. the non-normality parameter in the bootstrap expansion will converge to zero. Hence the bootstrap will provide asymptotic refinement such that the discrepancy between the nominal size and the bootstrap test rejection probability is of order $o(T^{-\frac{1}{2}})$. In the next section, we present an intensive Monte Carlo experiment to verify this property.

4. A Monte Carlo Simulation

In our simulation we consider a 2-dimensional VECM model. The DGP is an 2-dimensional random walk with no cointegration relation, i.e. $\alpha\beta'$ is a zero matrix. The data is generated from

$$\Delta X_t = \varepsilon_t, \tag{6}$$

where $\varepsilon_t \sim N(0, I)$, and $X_0 = 0$. The sample size T is from $(25, 50, ..., 400)$ and the experiment was run with 1,000,000 replications. The asymptotic critical value is the 0.05 quantile of the 100,000 evaluated statistics simulated from model (6) with sample size $T = 1000$. A Warp-Speed bootstrap[1] is used to reduce the heavy computational cost. To be specific, for each replication, only one bootstrap statistic is calculated and the bootstrap distribution is taken as the empirical distribution of the whole 1,000,000 bootstrap statistics. For each replication and each sample size T, we test $\mathcal{H}(0)$ against $\mathcal{H}(2)$ at significance level $\alpha = 0.05$ using both the asymptotic critical value and the bootstrap critical value. After 1,000,000 replications, we get $\hat{\alpha}_{A,T}$ and $\hat{\alpha}_{B,T}$, which are the rejection probability of the asymptotic test and the bootstrap test respectively when the sample size is T. The 95% pointwise confidence interval is constructed as $\hat{\alpha} \pm 1.96\sqrt{0.05(1-0.05)/1000000}$.

Table 1 reports $\hat{\alpha}_{A,T}$ and $\hat{\alpha}_{B,T}$ and it is clear that the bootstrap test has smaller error in rejection probability than the test using asymptotic critical value. To investigate the property of asymptotic refinement, we calculate and plot $\sqrt{T}(\hat{\alpha}_{A,T} - 0.05)$ and $\sqrt{T}(\hat{\alpha}_{B,T} - 0.05)$ together with their pointwise confidence intervals as shown in Figure 1, panel(a). For the test using asymptotic critical value, $\sqrt{T}(\hat{\alpha}_{A,T} - 0.05)$ tends to a nonzero constant while $\sqrt{T}(\hat{\alpha}_{B,T} - 0.05)$ tends to zero. This verifies that the bootstrap gives asymptotic refinements, i.e. $(\hat{\alpha}_{A,T} - \alpha)$ is of order $O(T^{-\frac{1}{2}})$ and $(\hat{\alpha}_{B,T} - \alpha)$ is of

[1]The Warp-Speed bootstrap only draws one bootstrap replication for each simulation and hence largely reduces the computational cost. Giacomini et al. (2013) show that the Warp-Speed bootstrap is valid when calibrating the finite sample coverage of bootstrap confidence intervals if the bootstrap is asymptotically valid. Because of the close relation between confidence intervals and hypothesis testing the method should be valid in our simulation.

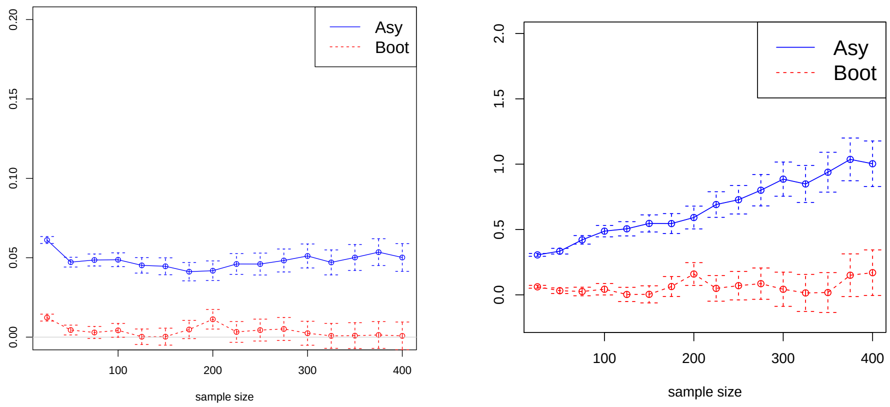
order $o(T^{-\frac{1}{2}})$. We also plot $T(\hat{\alpha}_{A,T} - 0.05)$ and $T(\hat{\alpha}_{B,T} - 0.05)$ in panel(b), if we multiply the discrepancy by T, $T(\hat{\alpha}_{A,T} - 0.05)$ diverges as expected while $T(\hat{\alpha}_{B,T} - 0.05)$ does not. However, it is difficult to see whether $T(\hat{\alpha}_{B,T} - 0.05)$ converges to zero or a nonzero constant.

Table 1: Empirical rejection probability in percent.

T	50	100	150	200	250	300	350	400
$\hat{\alpha}_{A,T}$	5.6679	5.4874	5.3643	5.2958	5.2911	5.2951	5.2681	5.2508
$\hat{\alpha}_{B,T}$	4.9367	5.0428	5.0025	4.9204	4.9719	5.0142	5.0050	4.9576

5. Conclusion

In this paper, we study the asymptotic refinement of the bootstrap cointegration rank trace test. An asymptotic expansion is derived for the test statistic of testing $\mathcal{H}(0)$ against $\mathcal{H}(p)$ in the simplest model framework with normal errors. However, the bootstrap analogue of the expansion is very difficult to derive, mainly because normality of the bootstrap residuals is violated. We verify that the bootstrap cointegration rank test will give asymptotic refinement by an intensive Monte Carlo experiment and the results exhibit the evidence that the error in bootstrap rejection probability is of smaller order than the asymptotic rejection probability. Due to the extremely heavy computational cost we did not do the simulation for the model with lag terms and deterministic terms. We conjecture that when it comes to more complicated models, the bootstrap will probably also give asymptotic refinement because the test statistic is still pivotal. In further studies it is still very interesting to find a complete theoretical result of the asymptotic refinement.



(a) $\sqrt{T}(|\hat{\alpha} - 0.05|)$ (b) $T(|\hat{\alpha} - 0.05|)$

Figure 1: Asymptotic properties of $\hat{\alpha}_A$ and $\hat{\alpha}_B$.

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6. Appendix

Proof of Proposition 1

First, if we define $X_T(u) = \frac{1}{\sqrt{T}}\sum_{t=1}^{[Tu]}\varepsilon_t$, and let $W(u)$, $0 \leq u \leq 1$, denote the p-dimension standard Brownian motion. Then given T and r, $X_T(u) \sim \Omega^{\frac{1}{2}}N_p(0, \frac{[Tu]}{T}I_p)$. Since for given u, $W(u) \sim N_p(0, uI_p)$, we also have

$$\begin{aligned} X_T(u) &= \Omega^{\frac{1}{2}}W(u)(\frac{[Tu]}{Tu})^{\frac{1}{2}} \\ &= \Omega^{\frac{1}{2}}W(u)(1 - \frac{Tu - [Tu]}{Tu})^{\frac{1}{2}} \\ &= \Omega^{\frac{1}{2}}W(u)(1 - \frac{1}{2}\frac{Tu - [Tu]}{Tu}) + o_p(T^{-1}). \end{aligned}$$

It follows that $X_T(u) = \Omega^{\frac{1}{2}}W(u) + O_p(T^{-1})$ for large T. It is then straightforward to show that

$$\frac{1}{T}S_{11} = \frac{1}{T^2}\sum_{t=1}^{T} X_{t-1}X'_{t-1} = \int_0^1 X_T(r)X'_T(r)dr = \Omega^{\frac{1}{2}}\int_0^1 WW'dr\Omega^{\frac{1}{2}} + O_p(T^{-1}).$$

Note that the second equality sign holds exactly. Now we consider S_{10},

$$\begin{aligned}
S_{10} + S_{01} &= \frac{1}{T}\sum_{t=1}^{T} X_{t-1}\varepsilon_t' + \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t X_{t-1}' \\
&= \frac{1}{T}\sum_{t=1}^{T}(\sum_{i=1}^{t-1}\varepsilon_i\varepsilon_t' + \varepsilon_t\sum_{i=1}^{t-1}\varepsilon_i') \\
&= \frac{1}{T}\sum_{i\neq j}\varepsilon_i\varepsilon_j' \\
&= \frac{1}{T}\sum_{t=1}^{T}(X_tX_t' - X_{t-1}X_{t-1}' - \varepsilon_t\varepsilon_t') \\
&= \frac{1}{T}(X_TX_T' - \sum_{t=1}^{T}\varepsilon_t\varepsilon_t') \\
&= \frac{1}{T}\big(X_TX_T' - \sum_{t=1}^{T}(\varepsilon_t\varepsilon_t' - \Omega) - T\Omega\big).
\end{aligned}$$

Let $Z_t = \Omega^{-\frac{1}{2}}X_t$ and $\eta_t = \Omega^{-\frac{1}{2}}\varepsilon_t$, then

$$S_{10} + S_{01} = \frac{1}{T}\Omega^{\frac{1}{2}}\Big[Z_tZ_t' - \sum_{t=1}^{T}(\eta_t\eta_t' - I) - TI\Big]\Omega^{\frac{1}{2}}.$$

Note that $\eta_t \sim N_p(0, I)$ and $\frac{1}{\sqrt{T}}Z_t = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\eta_t \sim N_p(0, I) = W(1)$. By the central limit theorem

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\eta_t\eta_t' - I) \xrightarrow{d} N(0, \Sigma),$$

where $\Sigma = I_{p\times p} + K_p$ (K_p is the commutation matrix),

$$S_{10} + S_{10} = \Omega^{\frac{1}{2}}[W(1)W'(1) - \frac{1}{\sqrt{T}}U - I]\Omega^{\frac{1}{2}} + o_p(T^{-\frac{1}{2}}),$$

where U is a random symmetric $p \times p$ matrix distributed as $N(0, \Sigma)$. The expansion of S_{00} is straightforward,

$$\begin{aligned} S_{00} &= \frac{1}{T}\sum_{t=1}^{T} \Delta X_t \Delta X_t' \\ &= \frac{1}{T}\sum_{t=1}^{T} \varepsilon_t \varepsilon_t' \\ &= \frac{1}{T}\sum_{t=1}^{T} (\varepsilon_t \varepsilon_t' - \Omega) + \Omega \\ &= \Omega^{\frac{1}{2}}(I + \frac{1}{\sqrt{T}}U)\Omega^{\frac{1}{2}} + o_p(T^{-\frac{1}{2}}), \end{aligned}$$

where $U \sim N(0, \Sigma)$.

Proof of Corollary 1

If a random variable A_T is of order $A_T = A + O_p(T^{-1})$, then $A_T^{-1} = A^{-1} + O_p(T^{-1})$, so

$$TS_{11}^{-1} = \Omega^{-\frac{1}{2}}\Big(\int_0^1 WW'dr\Big)^{-1}\Omega^{-\frac{1}{2}} + O_p(T^{-1}).$$

$$S_{00}^{-1} = \Omega^{-\frac{1}{2}}(1 - \frac{1}{\sqrt{T}}U)\Omega^{-\frac{1}{2}} + o_p(T^{-\frac{1}{2}}).$$

Proof of Theorem 1

We note that S_{11} is $O_p(T)$ whereas S_{10} and S_{00} are $O_p(1)$, so the roots $\hat{\lambda}_i$ of equation (2) is $O_p(T^{-1})$. This implies that

$$\begin{aligned} -T\sum_{i=1}^{p} \log(1 - \hat{\lambda}_i) &= T\sum_{i=1}^{p} \hat{\lambda}_i - T\sum_{i=1}^{p} \hat{\lambda}_i^2 + o_p(T^{-1}) \\ &= T\sum_{i=1}^{p} \hat{\lambda}_i + O_p(T^{-1}). \end{aligned}$$

Note that $T\sum_{i=1}^{p} \hat{\lambda}_i = Ttr\{S_{11}^{-1}S_{10}S_{00}^{-1}S_{01}\}$ and therefore Theorem 1 is proved.