G\textsubscript{a}-actions on Complex Affine Threefolds

Isac Hedén
This thesis consists of two papers and a summary. The papers both deal with affine algebraic complex varieties, and in particular such varieties in dimension three that have a non-trivial action of one of the one-dimensional algebraic groups $\mathbb{G}_a := (\mathbb{C}, +)$ and $\mathbb{G}_m := (\mathbb{C}^*, \cdot)$. The methods used involve: blowing up of subvarieties; viewing $\mathbb{G}_a$- and $\mathbb{G}_m$-actions on an affine variety $X$ as locally nilpotent derivations and $\mathbb{Z}$-gradings, respectively, on $\mathcal{O}(X)$; and, passing from a filtered algebra $A$ to its associated graded algebra $\text{gr}(A)$.

In Paper I, we study Russell’s hypersurface $X$, i.e. the affine variety in the affine space $\mathbb{A}^4$ given by the equation $x + x^2y + z^3 + t^2 = 0$. We prove by geometric means Makar-Limanov’s result which states that $X$ is not isomorphic to $\mathbb{A}^3$ – a result which was crucial to Koras-Russell’s proof of the linearization conjecture for $\mathbb{G}_m$-actions on $\mathbb{A}^3$. Our strategy consists in realizing $X$ as an open part of a blowup $M \rightarrow \mathbb{A}^3$, and showing that each $\mathbb{G}_a$-action on $X$ descends to $\mathbb{A}^3$. This follows from considerations of the graded algebra associated to the coordinate ring $\mathcal{O}(X)$ with respect to a certain filtration.

In Paper II, we study $\mathbb{G}_a$-threefolds $X$ which have as their algebraic quotient the affine plane $\mathbb{A}^2 = \text{Sp}(\mathbb{C}[x, y])$, and are a principal bundle above the punctured plane $\hat{\mathbb{A}}^2 := \mathbb{A}^2 \setminus \{0\}$. In other words, we study affine $\mathbb{G}_a$-varieties $\hat{P}$ that extend a principal bundle $P$ over $\hat{\mathbb{A}}^2$; i.e. they are $P$ together with an extra fiber over the origin in $\mathbb{A}^2$. First the trivial bundle is studied, and some examples of extensions are given (including smooth ones which are not isomorphic to $\mathbb{A}^2 \times \mathbb{A}$). The most basic among the non-trivial principal bundles over $\hat{\mathbb{A}}^2$ is $\text{SL}_2(\mathbb{C}) \rightarrow \mathbb{A}^2 \setminus \{0\}$, where $e_1$ denotes the first unit vector, and we show that any non-trivial bundle can be realized as a pullback of this bundle with respect to a morphism $\hat{\mathbb{A}}^2 \rightarrow \hat{\mathbb{A}}^2$. Therefore, attention is restricted to extensions of $\text{SL}_2(\mathbb{C})$: we find two families of such extensions via a study of the graded algebras associated with the coordinate rings $\mathcal{O}((\hat{P}) \leftrightarrow \mathcal{O}(P)$ with respect to a filtration which is defined in terms of the $\mathbb{G}_a$-actions on $P$ and $\hat{P}$.

Keywords: Complex affine varieties, algebraic group actions of $\mathbb{C}^+$ and $\mathbb{C}^*$, locally nilpotent derivations, graded algebras, principal bundles, Russell’s hypersurface, Makar-Limanov invariant.

Isac Hedén, Uppsala University, Department of Mathematics, Box 480, SE-751 06 Uppsala, Sweden.

©Isac Hedén 2013

ISSN 1401-2049
ISBN 978-91-506-2357-4
urn:nbn:se:uu:diva-203708 (http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-203708)
List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I  Hedén, I. (2011) Russell’s hypersurface from a geometric point of view

II Hedén, I. (2013) Extension of principal $\mathbb{G}_a$-bundles over $\mathbb{A}^2_*$
# Contents

1 Introduction ......................................................................................................................... 7  
1.1 Complex affine varieties .............................................................................................. 7  
1.2 Algebraic group actions ............................................................................................... 10  
1.3 Quotient morphisms .................................................................................................... 15  
1.4 Principal bundles ........................................................................................................ 19  

2 Summary of papers ............................................................................................................ 23  
2.1 Paper I .......................................................................................................................... 23  
2.2 Paper II ......................................................................................................................... 23  

3 Sammanfattning på svenska (Summary in Swedish) ...................................................... 25  

4 Acknowledgements .......................................................................................................... 28  

References .............................................................................................................................. 30
1. Introduction

This is a thesis in the field of complex affine algebraic geometry. The objects of study are affine varieties, i.e. subsets of affine \( n \)-space \( \mathbb{A}^n \) over the field \( \mathbb{C} \) of complex numbers that can be described as the zero set of a finite family of polynomials in \( n \) variables.

In these introductory notes we outline the basic theory of actions of the group \( \mathbb{G}_a \), the additive group of complex numbers, and the group \( \mathbb{G}_m \), the multiplicative group of nonzero complex numbers, on such affine varieties.

1.1 Complex affine varieties

We start with affine varieties: Such a variety is the set of common zeros

\[ X := V(\mathbb{C}^m; f_1, \ldots, f_r) := \{ z \in \mathbb{C}^m; f_1(z) = \ldots = f_r(z) = 0 \} \]

of finitely many polynomials \( f_1, \ldots, f_r \in \mathbb{C}[T_1, \ldots, T_m] \). Morphisms between affine varieties are restrictions

\[
\begin{array}{c}
X \\ \cap \\ \mathbb{C}^m \\
\phi \\ \cap \\ \mathbb{C}^n
\end{array} \rightarrow \begin{array}{c}
Y \\
\cap \\
\mathbb{C}^n
\end{array}
\]

of polynomial mappings \( \phi : \mathbb{C}^m \rightarrow \mathbb{C}^n \) with \( \phi(X) \subset Y \). In order to understand such affine varieties

\[ X \hookrightarrow \mathbb{C}^m \]

as objects in their own right, i.e. not necessarily together with a given embedding, one keeps in mind only the functions on \( X \) that are obtained as restrictions of polynomials on \( \mathbb{C}^m \): For an affine variety \( X \hookrightarrow \mathbb{C}^m \) its algebra of “regular functions” is

\[ \mathcal{O}(X) := \mathbb{C}[T_1, \ldots, T_m]|_X \cong \mathbb{C}[T_1, \ldots, T_m]/I(X). \]

Here

\[ I(X) := \{ g \in \mathbb{C}[T_1, \ldots, T_m]; g|_X = 0 \} \]

is the ideal of all polynomials vanishing on \( X \). From that point of view the condition for a map \( \phi : X \rightarrow Y \) to be a morphism is simply that the pullback of a regular function on \( Y \) gives a regular function on \( X \). Indeed even the
affine variety $X$ itself can be reconstructed from its ring of regular functions: The points $x \in X$ are in one-to-one correspondence with the maximal ideals of $O(X)$ via:

$$X \ni x \mapsto m_x := \{f \in O(X) ; f(x) = 0\}.$$ 

So instead of starting with $X \hookrightarrow \mathbb{C}^m$ one can simply take the ring $A = O(X)$ as point of departure and define the corresponding affine variety as its spectrum

$$\text{Sp}(A) := \{m ; A \supseteq m \text{ max. ideal}\}.$$ 

For this, one needs two conditions to be satisfied by $A$, namely that it is

1. a finitely generated $\mathbb{C}$-algebra, i.e. isomorphic to a quotient of a polynomial ring, and
2. reduced, i.e. does not contain nonzero nilpotents.

In this way everything can be reduced to rings and ring homomorphisms: The category of affine varieties is anti-equivalent to the category of finitely generated reduced $\mathbb{C}$-algebras. In particular embeddings

$$X = \text{Sp}(A) \hookrightarrow \mathbb{C}^m$$

correspond to surjections

$$A \twoheadrightarrow \mathbb{C}[T_1, \ldots, T_m].$$

Indeed if $a \subset \mathbb{C}[T_1, \ldots, T_m]$ denotes the kernel of such an epimorphism, the affine variety $X \hookrightarrow \mathbb{C}^m$ is given as the zero set of $a$, i.e.

$$X = V(a) := \{x \in \mathbb{C}^m ; f(x) = 0 \ \forall f \in a\}.$$ 

Note that

$$V(a) = V(f_1, \ldots, f_r),$$

if $f_1, \ldots, f_r$ are generators of the ideal $a$.

**Example 1.1.1.**

1. Affine $m$-space

$$\mathbb{A}^m := \text{Sp}(\mathbb{C}[T_1, \ldots, T_m])$$

is finally nothing but $\mathbb{C}^m$, but we think of it as an affine variety rather than a vector space.

2. The Laurent algebra $A = \mathbb{C}[S, S^{-1}]$ admits the epimorphism

$$\mathbb{C}[T_1, T_2] \twoheadrightarrow A, T_1 \mapsto S, T_2 \mapsto S^{-1}$$

with kernel $a = (T_1 T_2 - 1) \hookrightarrow \mathbb{C}[T_1, T_2]$, the corresponding affine variety is the hyperbola $X = V(\mathbb{C}^2; T_1 T_2 - 1) \hookrightarrow \mathbb{A}^2$. 

8
3. For $X = \text{Sp}(A)$ and $Y = \text{Sp}(B)$ we have

$$X \times Y = \text{Sp}(A \otimes B)$$

with the tensor product taken over $\mathbb{C}$. In particular

$$X \times \mathbb{A}^1 = \text{Sp}(A[S]).$$

**Topologies:** Any complex affine variety carries two different topologies. Its *complex topology* is defined as the relative topology inherited from some ambient $\mathbb{A}^m$ – but finally is independent from the choice of the embedding $X \hookrightarrow \mathbb{A}^m$. A much coarser topology is the *Zariski topology*, whose closed subsets are of the form

$$V(X; g_1, \ldots, g_s) := \{x \in X; g_1(x) = \ldots = g_s(x) = 0\}$$

with regular functions $g_1, \ldots, g_s \in \mathcal{O}(X)$. When we speak about open and closed subsets of an affine variety, we always refer to the Zariski topology.

**Remark 1.1.2.** Any closed subspace $Y \hookrightarrow X$ of an affine variety $X$ is again an affine variety: Compose an embedding $X \hookrightarrow \mathbb{A}^m$ with the inclusion $Y \hookrightarrow X$.

In order to better understand the geometry of an affine variety $X$, we want to make any open subset an object in a broader category containing the affine varieties as a full subcategory. First of all, the notion of a regular function can be extended to open subsets: A “special open” subset is the complement

$$X_f := X \setminus V(f)$$

of the set

$$V(f) := \{x \in X; f(x) = 0\}$$

of zeros of a regular function $f \in \mathcal{O}(X)$. For such a set we take

$$\mathcal{O}(X_f) := \mathcal{O}(X)_f := \left\{ \frac{g}{f^m}; g \in \mathcal{O}(X), m \in \mathbb{N} \right\},$$

i.e. regular functions on $X_f$ are fractions, where the numerator is (the restriction of) a regular function on $X$ and the denominator a power of $f$. Since the special open subsets form a basis of the Zariski topology, we may then define for an arbitrary open subset $U \subset X$ its ring of regular functions as

$$\mathcal{O}(U) := \{h : U \longrightarrow \mathbb{C}, h|_{X_f} \in \mathcal{O}(X_f), \forall X_f \subset U\}.$$
Example 1.1.3.  
1. For an affine variety $X$ the quasi-affine variety $X_f$ is even affine: The map $X_f \rightarrow X \times \mathbb{A}^1; x \mapsto (x, f(x))$ induces an isomorphism 

$$X_f \cong V(X \times \mathbb{A}^1; 1 - f S) \hookrightarrow X \times \mathbb{A}^1 = \text{Sp}(A[S]).$$

2. In particular the quasi-affine variety $\mathbb{A}^1^* := \mathbb{A}^1 \setminus \{0\}$ is affine. It is isomorphic to the hyperbola in example 1.1.1.2.

3. On the other hand the quasi-affine variety $\mathbb{A}^2^* := \mathbb{A}^2 \setminus \{0\}$ is not affine: The inclusion $\mathbb{A}^2^* \rightarrow \mathbb{A}^2$ is obviously not an isomorphism, though the pullback homomorphism $\mathcal{O}(\mathbb{A}^2) \rightarrow \mathcal{O}(\mathbb{A}^2^*)$ – the restriction of regular functions – is.

1.2 Algebraic group actions

Definition 1.2.1. An affine algebraic group $G$ is a complex affine variety admitting a group law given by a morphism

$$G \times G \rightarrow G$$

of algebraic varieties, such that the inversion

$$G \rightarrow G, x \mapsto x^{-1},$$

is a morphism as well.

Example 1.2.2.  
1. The two most basic complex affine algebraic groups are the additive group

$$\mathbb{G}_a := (\mathbb{C}, +)$$

and the multiplicative group

$$\mathbb{G}_m := (\mathbb{C} \setminus \{0\}, \cdot)$$

of complex numbers. Though as varieties $\mathbb{G}_a \cong \mathbb{A}^1$ and $\mathbb{G}_m \cong \mathbb{A}^1^*$, a new notation is traditionally preferred whenever the group aspect is important.

2. The semidirect product

$$\mathbb{G}_a \rtimes \mathbb{G}_m,$$

where $\varphi : \mathbb{G}_m \rightarrow \text{Aut}(\mathbb{G}_a), \lambda \mapsto \varphi_\lambda$ with $\varphi_\lambda(t) = \lambda^{-n} t$, is an affine algebraic group.

3. The special linear group

$$\text{SL}_2(\mathbb{C}) := \{ A \in \mathbb{C}^{2,2} \cong \mathbb{A}^4; \det(A) - 1 = 0 \}. $$
Definition 1.2.3. An algebraic group action of an affine algebraic group $G$ on an affine variety $X$ is a morphism

$$\sigma : G \times X \longrightarrow X$$

of affine varieties such that $\sigma(e, x) = x$, and $\sigma(g_1, \sigma(g_2, x)) = \sigma(g_1g_2, x)$ for all $g_1, g_2 \in G$ and $x \in X$. We use the notation

$$gx := \sigma(g, x);$$

for $G = \mathbb{G}_a$ we write as well

$$t \ast x := \sigma(t, x).$$

Example 1.2.4. 1. Given a polynomial $f \in \mathbb{C}[x]$ we define an action

$$\mathbb{G}_a \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2, (t, (x, y)) \mapsto (x, y + tf(x)).$$

According to [32] any algebraic $\mathbb{G}_a$-action on $\mathbb{A}^2$ is conjugate to such an action.

2. Given integers $p, q \in \mathbb{Z}$ we define an action

$$\mathbb{G}_m \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2, (\lambda, (x, y)) \mapsto (\lambda^p x, \lambda^q y).$$

According to [12] any algebraic $\mathbb{G}_m$-action on $\mathbb{A}^2$ is conjugate to such an action.

3. The standard action of $\text{SL}_2(\mathbb{C})$ on $\mathbb{A}^2$.

4. We can think of $\mathbb{G}_a$ as a closed subgroup of $\text{SL}_2(\mathbb{C})$ via the embedding

$$\mathbb{G}_a \longrightarrow \text{SL}_2(\mathbb{C}), t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

and get a $\mathbb{G}_a$-action from the right:

$$\begin{pmatrix} x & u \\ y & v \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} x & u + tx \\ y & v + ty \end{pmatrix} \right).$$

Orbits and stabilizers, fixed points:

1. For any point $x \in X$ we have the morphism $\sigma_x : G \longrightarrow X, g \mapsto \sigma(g, x)$. The fibre of this map at $x$ is denoted by $G_x$, and is called the stabilizer (or isotropy subgroup) of $x$ with respect to the action $\sigma : G \times X \longrightarrow X$. It is a closed subgroup of $G$, being the inverse image of the diagonal of the morphism $G \times X \longrightarrow X \times X, (g, x) \mapsto (gx, x)$, hence an affine algebraic group.
When \( \mathbb{G}_a \) acts on an affine variety, there can be no nontrivial stabilizers since \( \mathbb{G}_a \) has no finite nontrivial subgroups. For \( \mathbb{G}_m \), the possible stabilizers \( \neq \mathbb{G}_m \) are groups of unity:

\[
C_n := \{ \eta \in \mathbb{G}_m; \eta^n = 1 \}.
\]

2. Orbits are locally closed, i.e. open in their closure, so in particular they are algebraic varieties, and the orbit map \( G \longrightarrow Gx := \sigma(G,x) \) is a morphism. But they are not necessarily affine; for instance when \( \text{SL}_2(\mathbb{C}) \) acts on \( \mathbb{A}^2 \) in the standard way, there are only two orbits: \( \{(0,0)\} \) and \( \mathbb{A}^2_\mathbb{C} \).

A \( \mathbb{G}_a \)-orbit in an affine variety \( X \) is closed: Since \( \mathbb{G}_a \) has no finite nontrivial subgroups, it is either a point or 

\[
\mathbb{G}_a \cdot x \cong \mathbb{A}^1,
\]

a maximal irreducible affine variety. For an affine \( \mathbb{G}_m \)-variety \( X \ni x \) we have \( \mathbb{G}_m \cdot x \cong \mathbb{A}^1 \) (though the orbit map need not be an isomorphism, but rather is an \( n \)-sheeted covering, if the stabilizer of a point on the orbit is \( C_n \)) or \( \mathbb{G}_m \cdot x = \{ x \} \). In particular, a \( \mathbb{G}_m \)-orbit is closed or its closure consists of the orbit and one additional point, a fixed point.

**Remark 1.2.5.** For one-dimensional groups there is also an easy description of an action on the level of regular functions:

1. An action

\[
\mathbb{G}_a \times X \longrightarrow X, (t,x) \mapsto t \cdot x
\]

of the additive group \( \mathbb{G}_a \) can be thought of as the flow of a vector field, whose action on the algebra \( A \) of regular functions on \( X \) provides a derivation

\[
D : A \longrightarrow A,
\]

i.e. a linear map satisfying the Leibniz rule

\[
D(fg) = D(f)g + fD(g),
\]

the differentiation with respect to the acting parameter

\[
Df(x) = \frac{df(t \cdot x)}{dt} \bigg|_{t=0}.
\]

Since \( f(t \cdot x) \) is a polynomial in \( t \), it follows that \( D \) is “locally nilpotent”, i.e. for all \( f \in A \) there is a natural number \( n = n(f) \), such that \( D^n f = 0 \). On the other hand, given a locally nilpotent derivation \( D : A \longrightarrow A \), a group action

\[
\mathbb{G}_a \times X \longrightarrow X
\]
can be constructed from the algebra homomorphism

\[ A \longrightarrow A[T], f \mapsto \sum_{n \geq 0} \frac{D^n(f)}{n!} T^n, \]

a finite sum, \( D \) being locally nilpotent.

2. An action

\[ \mathbb{G}_m \times X \longrightarrow X, (\lambda, x) \mapsto \lambda x, \]

of the multiplicative group \( \mathbb{G}_m \) induces a \( \mathbb{G}_a \)-action by applying the exponential homomorphism \( \exp: \mathbb{G}_a \longrightarrow \mathbb{G}_m \), leading to the derivation

\[ Df(x) = \left. \frac{df(e^t x)}{dt} \right|_{t=0}. \]

Though doing that we leave the category of affine varieties, the derivation \( D: A \longrightarrow A \) is well defined nevertheless. But it is no longer locally nilpotent: For example for \( \mathbb{G}_m = \text{Sp}(\mathbb{C}[\Lambda, \Lambda^{-1}]) \) acting on itself, we find

\[ D(\Lambda^n) = n\Lambda^n. \]

Indeed the linear map \( D: A \longrightarrow A \) turns out to be semisimple with integer eigenvalues, giving rise to a direct sum decomposition

\[ A = \bigoplus_{n=-\infty}^{\infty} A_n \]

with eigenspaces \( A_n \) satisfying

\[ A_n = \{ f \in A; f(\lambda x) = \lambda^n f(x), \forall \lambda \in \mathbb{G}_m \}. \]

The components \( f_n \in A_n \) of a given function \( f \in A \) then can be obtained as Fourier coefficients \( f_n(x) \) of the function \( \partial E \longrightarrow \mathbb{C}, z \mapsto f(zx) \) (where \( E \subset \mathbb{C} \) denotes the unit disk), i.e.

\[ f_n(x) = \frac{1}{2\pi i} \int_{\partial E} f(\lambda x) \lambda^{-(n+1)} d\lambda. \]

So, the knowledge of \( D \) corresponds to a \( \mathbb{Z} \)-decomposition of \( A \). And on the other hand, given a \( \mathbb{Z} \)-decomposition, we can produce an action

\[ \mathbb{G}_m \times X \longrightarrow X; \]

it is obtained from the algebra homomorphism

\[ A \longrightarrow A[\Lambda, \Lambda^{-1}], f \mapsto \sum_n f_n \Lambda^n. \]

On affine \( n \)-space \( \mathbb{A}^n \) the most basic actions are linear actions:
Definition 1.2.6.  1. An algebraic action $G \times \mathbb{A}^n \rightarrow \mathbb{A}^n$ is called linear if the corresponding group homomorphism $G \rightarrow \text{Aut}(\mathbb{A}^n)$ factors through $\text{GL}_n(\mathbb{C})$:

$$
\begin{array}{ccc}
G & \rightarrow & \text{Aut}(\mathbb{A}^n) \\
\downarrow & & \downarrow \\
\text{GL}_n(\mathbb{C}) & \rightarrow & \\
\end{array}
$$

and

2. it is called linearizable if it becomes linear after a polynomial change of coordinates.

Remark 1.2.7. Obviously an action can only be linearizable if its fixed point set is connected, and therefore it is easy to present $\mathbb{G}_a$-actions on $\mathbb{A}^n$ which are not linearizable – take for instance the action on $\mathbb{A}^2$ which is defined by the locally nilpotent derivation $D(x) = 0$ and $D(y) = x^2 - x$; its fixed point set is given by $V(x) \cup V(x - 1)$.

Example 1.2.8.  1. Any linear action $\mathbb{G}_m \times \mathbb{A}^n \rightarrow \mathbb{A}^n$ can be diagonalized, i.e. there is a basis $u_1, \ldots, u_n$ of the vector space $\mathbb{A}^n$ consisting of simultaneous eigenvectors for the maps $x \mapsto \lambda \cdot x$, i.e. there are integers $k_1, \ldots, k_n$, such that

$$
\lambda \cdot u_i = \lambda^{k_i} u_i
$$

holds for all $\lambda \in \mathbb{G}_m$.

2. For $n \leq 3$ any algebraic action $\mathbb{G}_m \times \mathbb{A}^n \rightarrow \mathbb{A}^n$ is linearizable. This is obvious for $n = 1$, has been proved for $n = 2$ by Gutwirth in 1962, see [12], and for $n = 3$ by M. Koras and P. Russell in 1999, see [21]. Russell’s hypersurface $X \hookrightarrow \mathbb{A}^4$, which is the matter of interest in our first paper, has played an important role in their argument.

3. Any linear action $\mathbb{G}_a \times \mathbb{A}^n \rightarrow \mathbb{A}^n$, $(t, x) \mapsto t \cdot x$, is of the form

$$(t, x) \mapsto t \cdot x := e^{tN} x$$

with a nilpotent matrix $N \in \mathbb{C}^{n \times n}$. The locally nilpotent derivation

$$
D : \mathbb{C}[x_1, \ldots, x_n] \rightarrow \mathbb{C}[x_1, \ldots, x_n], f \mapsto Df,
$$

looks as follows

$$
Df(x) = \nabla f(x) \cdot Nx.
$$

where $u \cdot v = \sum_{i=1}^n u_i v_i$ for $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n)$. The fixed point set is the linear subspace $\ker(N) \hookrightarrow \mathbb{A}^n$.

4. The action

$$
\mathbb{G}_a \times \mathbb{A}^2 \rightarrow \mathbb{A}^2, (t, (x, y)) \mapsto (x, y + tf(x))
$$

is linear if and only if $f(x) = ax, a \in \mathbb{C}$, and linearizable if and only if $\deg f = 1$. 

14
1.3 Quotient morphisms

In order to understand the orbit structure of a group action

\[ G \times X \longrightarrow X, \]

on a “geometric object” \( X \), one tries to make the set \( X/G \) of all orbits a geometric object itself. Given a group action on an affine variety \( X = \text{Sp}(A) \) as above, we obtain a topological space together with

\[ \mathcal{O}(X/G) := \{ f : X/G \longrightarrow \mathbb{C}; f \circ \pi \in \mathcal{O}(X) \} \]

as algebra of its regular functions. Here \( \pi : X \longrightarrow X/G \) denotes the map \( x \mapsto Gx \). But regular functions then do not separate points in \( X/G \), since the algebra

\[ A^G := \{ f \in A; f(gx) = f(x) \ \forall g \in G, x \in X \} \cong \mathcal{O}(X/G) \]

of \( G \)-invariant regular functions on \( X \) does not separate orbits: As continuous functions, they are not only constant on orbits, but also on orbit closures, and non-closed orbits may occur. But there is a solution to our problem: The algebraic quotient \( X//G \) is defined as

\[ X//G := \text{Sp}(A^G), \]

if one knows that \( A^G \) is a finitely generated \( \mathbb{C} \)-algebra. Then the quotient morphism

\[ X \longrightarrow X//G \]

is constant on orbit closures. Indeed, we call any morphism \( X \longrightarrow Z \) a quotient morphism, if the pullback of functions is injective and \( \mathcal{O}(Z) \cong A^G \) via pullback (and hence \( Z \cong X//G \)).

Remark 1.3.1. The algebra of invariant functions can be determined as follows:

1. For a multiplicative action \( \mathbb{G}_m \times X \longrightarrow X \) on \( X := \text{Sp}(A) \) we have

\[ A^{\mathbb{G}_m} = A_0, \]

if \( A = \bigoplus_{n \in \mathbb{Z}} A_n \) is the corresponding decomposition of \( A \).

2. For an additive action \( \mathbb{G}_a \times X \longrightarrow X \) on \( X := \text{Sp}(A) \) we have

\[ A^{\mathbb{G}_a} = \ker D \]

with the corresponding locally nilpotent derivation \( D : A \longrightarrow A \).

Example 1.3.2. We describe the quotient morphism \( X \longrightarrow X//G \) for some examples where it is easy to compute:
1. The action
\[ \mathbb{G}_a \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2, (t, (x, y)) \mapsto (x, y + tf(x)), \]
has quotient morphism
\[ \mathbb{A}^2 \longrightarrow \mathbb{A}^1, (x, y) \mapsto x. \]

2. For the action
\[ \mathbb{G}_m \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2, (\lambda, (x, y)) \mapsto (\lambda^p x, \lambda^q y) \]
we have to distinguish several cases:
   a) If \( pq > 0 \), we have \( A_0 = \mathbb{C} \), hence \( \mathbb{A}^2 // \mathbb{G}_m \) has only one point.
      The geometric reason for this is that the origin lies in the closure of every orbit, and the quotient morphism is constant on orbit closures.
   b) If \( p = 0, q \neq 0 \), we obtain again
      \[ \mathbb{A}^2 \longrightarrow \mathbb{A}^1, (x, y) \mapsto x. \]
      Every fiber consists of one nontrivial orbit and a fixed point.
   c) If \( p > 0 > q \) and \( \gcd(p, q) = 1 \), we find
      \[ A_0 = \mathbb{C}[x^{-q}y^p], \]
      so the quotient morphism is
      \[ \mathbb{A}^2 \longrightarrow \mathbb{A}^1, (x, y) \mapsto x^{-q}y^p. \]
      The zero fiber consists of a fixed point, the origin, and two nontrivial orbits, the punctured coordinate axes, while the remaining fibers are orbits.

3. Consider the \( \mathbb{G}_a \)-action
\[ \mathbb{G}_a \times \mathbb{A}^3 \longrightarrow \mathbb{A}^3, (t, (x, y, z)) \mapsto e^{tN} \]
with the nilpotent matrix
\[ N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \]
The algebraic quotient morphism is given by
\[ \pi : \mathbb{A}^3 \longrightarrow \mathbb{A}^2, \]
\[ (x, y, z) \mapsto (z, xz - \frac{y^2}{2}). \]
Now
\[ \pi^{-1}(0, 0) = \mathbb{A}^1 \times 0 \times 0 \] (fixed points),
\[ \pi^{-1}(0, b) = \mathbb{A}^1 \times \{ \pm \sqrt{2b} \} \times 0 \] (two orbits), \( \text{if } b \neq 0 \),
\[ \pi^{-1}(a, b) \cong \mathbb{A}^1, \] (one orbit, a parabola), \( \text{if } b \neq 0 \neq a \).
Thus over the origin there is a line of fixed points, over other points $(0,b)$, $b \neq 0$, there are two orbits, while over $V := \mathbb{A}^1 \times \mathbb{A}^1$ fibers are orbits. Even better, the isomorphism of the third line depends nicely on the base point $(a,b) \in V$, so that we get a “trivialization” over $V$, i.e. an equivariant isomorphism

$$\pi^{-1}(V) \cong V \times \mathbb{G}_a$$

with $\mathbb{G}_a$ acting by translation on the second factor. There are (up to isomorphy) two further linear $\mathbb{G}_a$-actions on $\mathbb{A}^3$: the trivial one, and that one coming from a nonzero $3 \times 3$ nilpotent matrix with $N^2 = 0$. The algebraic quotient morphism of the latter is in standard coordinates just the projection $\mathbb{A}^3 \rightarrow \mathbb{A}^2$, $(x,y,z) \mapsto (y,z)$.

4. The $\mathbb{G}_a$-action on $X = \text{SL}_2(\mathbb{C})$ by right multiplication with $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{G}_a$, has as quotient morphism the orbit map

$$\text{SL}_2(\mathbb{C}) \rightarrow \mathbb{A}^2$$

with the first unit vector $e_1 \in \mathbb{A}^2$. Its image is the punctured plane $\mathbb{A}^2_* \subset \mathbb{A}^2$; it is, in particular, not surjective; but fortunately its non-empty fibers are exactly the $\mathbb{G}_a$-orbits, since $\begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix}$ is the stabilizer of $e_1$ with respect to the natural action of $\text{SL}_2(\mathbb{C})$ on $\mathbb{A}^2$. Thus an invariant function is the pullback of a function on $\mathbb{A}^2_*$, while on the other hand $\mathcal{O}(\mathbb{A}^2_*) = \mathcal{O}(\mathbb{A}^2)$.

The multiplicative group $\mathbb{G}_m$ is the complexified unit circle, and the unit circle is a compact real Lie group. Affine algebraic groups, which are the complexification of a compact real Lie group, are called “reductive”. For such groups the following classical result holds:

**Theorem 1.3.3.** Let $G$ be a reductive algebraic group acting on $X = \text{Sp}(A)$. Then we have:

1. The fixed algebra $A^G$ is a finitely generated $\mathbb{C}$-algebra.
2. The quotient morphism $\pi : X \rightarrow X//G$ maps $G$-invariant closed sets onto closed sets, so it is in particular onto.
3. In any $\pi$-fiber there is exactly one closed $G$-orbit, and thus,
4. if all orbits are closed (e.g. if $G$ is a finite group), then $X//G$ is the orbit space of the action.

**Remark 1.3.4.** Since for dimension reasons any orbit closure contains a closed orbit, it follows that for a reductive group $G$ and $x,y \in X$ we have

$$\pi(x) = \pi(y) \iff \overline{Gx} \cap \overline{Gy} \neq \emptyset.$$
For general affine algebraic groups, in particular for $\mathbb{G}_a$, there are examples of actions where the fixed algebra is not finitely generated. But in low dimensions there is no problem:

**Theorem 1.3.5.** Let the affine algebraic group $G$ act on the normal affine variety $X = \text{Sp}(A)$. Then the fixed algebra $A^G$ is finitely generated, whenever $\dim X \leq 3$.

**Proof.** According to Zariski, [38], if $A$ is a normal, complex, affine algebra and $\mathbb{C} \subset F \subset \mathbb{Q}(A)$ is a subfield of its field of fractions of transcendence degree at most two, then $A \cap F$ is an affine algebra. We may assume that there is an orbit of positive dimension, and then by [24] $\text{trdeg}_{\mathbb{C}}(\mathbb{Q}(A^G)) \leq 2$. Hence, by Zariski’s result, with $F := \mathbb{Q}(A^G)$ we see that $\text{Sp}(A^G)$ is finitely generated. \hfill $\square$

For dimension $> 3$ the following is known:

**Remark 1.3.6.**

1. By a result of Maurer and Weitzenböck the quotient $\mathbb{A}^n//\mathbb{G}_a$ exists for all linear $\mathbb{G}_a$-actions on $\mathbb{A}^n$.

2. There are examples of affine algebraic group actions where $A^G$ is not finitely generated; the first example was constructed by Nagata in 1958 (c.f. [4, pp. 52-61]).

3. His famous example has then been modified and simplified considerably: In 1994 Annette A’Campo-Neuen was able to construct the first example of a linear action (the additive group $(\mathbb{G}_a)^{12}$ acts on affine 19-space $\mathbb{A}^{19}$) whose ring of invariant functions is not finitely generated (c.f. [1]).

4. In 1999, G. Freudenberg and D. Daigle showed in [3] that there are $\mathbb{G}_a$-actions on $\mathbb{A}^5$, where the algebraic quotient does not exist.

In Ex.1.3.2.3 we have seen an example where above some dense open subset the quotient morphism looks like a product. That is true in general:

**Theorem 1.3.7.**

1. Any affine $\mathbb{G}_a$-variety $X = \text{Sp}(A)$ admits a dominant $\mathbb{G}_a$-invariant morphism $\pi : X \longrightarrow Y$ together with a dense open subset $V \subset Y$, such that

$$\pi^{-1}(V) \cong V \times \mathbb{G}_a.$$ 

If it exists, we may take $Y = X//\mathbb{G}_a$.

2. If the algebraic quotient $X//\mathbb{G}_a$ exists, we have

$$\dim X//\mathbb{G}_a = \dim X - 1.$$
Proof. Choose $f \in A$ with $g = D(f) \neq 0, D^2 f = 0$. Then $U := X_g = \text{Sp}(A_g)$ is an invariant affine open subset, and we take $Z = V(f) \cap U$. Then

$$Z \times G_a \longrightarrow U, (z, t) \mapsto t \ast z$$

is an isomorphism. We have $(A^G_a)_g \cong O(Z)$ and may take $Y = \text{Sp}(B)$, where $B \subset A^G_a$ is a finitely generated subalgebra with $g \in B$ and $B_g \cong (A^G_a)_g$. Finally let $V = Y_g$. □

So it remains to study the quotient morphism near points in $X // G \setminus V$. For $X // G_a \cong \mathbb{A}^2$ and $V = \mathbb{A}^2_*$ this is done in the third section of our second paper.

1.4 Principal bundles

Let us finally discuss some interesting geometry related to affine $G_a$-surfaces. We return to the quotient morphism

$$\pi: \mathbb{A}^3 \longrightarrow \mathbb{A}^2, w = (x, y, z) \mapsto (u, v) = (y^2 - 2xz, z)$$

of Ex.1.3.2.3 (here composed by the automorphism $\mathbb{A}^2 \longrightarrow \mathbb{A}^2, (a, b) \mapsto (-2b, a)$ for convenience). For a horizontal line $\mathbb{A}^1 \times b \hookrightarrow \mathbb{A}^2$ with $b \in \mathbb{C}^*$, the inverse image

$$Y_b := \pi^{-1}(\mathbb{A}^1 \times b) \hookrightarrow \mathbb{A}^3$$

is an affine $G_a$-surface with

$$\pi|_{Y_b}: Y_b \longrightarrow \mathbb{A}^1 \times b \cong \mathbb{A}^1$$

as quotient morphism. It is a product above $\mathbb{A}^1$, while the zero fiber is the disjoint union of two affine lines. In order to understand that phenomenon better, we shall introduce the notion of a $G_a$-principal bundle. But before we can do so, we have to admit more general (pre)varieties:

**Definition 1.4.1.** A $\mathbb{C}$-ringed space is a topological space, with a $\mathbb{C}$-algebra $\mathcal{O}(U)$ of continuous “regular functions” for every open subset, such that

1. $\mathcal{O}(U)|_{V} \subset \mathcal{O}(V)$ for $V \subset U$, and
2. regularity is a local notion.

Morphisms $\varphi: X \longrightarrow Y$ are continuous maps, such that the pullback of regular functions is regular again, i.e.

$$f \in \mathcal{O}(V) \implies f \circ \varphi \in \mathcal{O}(\varphi^{-1}(V))$$

should hold for all open sets $V \subset Y$.

**Remark 1.4.2.** 1. The affine and quasi-affine varieties each form a full subcategory of the category of $\mathbb{C}$-ringed spaces.
2. An open subset of a $\mathbb{C}$-ringed space is itself a $\mathbb{C}$-ringed space.

**Definition 1.4.3.** An algebraic prevariety is a $\mathbb{C}$-ringed space admitting an open cover consisting of open subspaces isomorphic to affine varieties.

**Definition 1.4.4.** A $G$-principal bundle is a $G$-prevariety $P$ (with a right action) together with an invariant morphism

$$\pi : P \longrightarrow B$$

onto a prevariety $B$, such that there is an open cover $B = \bigcup_{i=1}^{r} U_i$ and equivariant isomorphisms (also called “trivializations”)

$$\pi^{-1}(U_i) \cong U_i \times G$$

for all $i = 1, \ldots, r$. Here $G$ acts by right translation on the second factor of the right hand side.

**Example 1.4.5.**

1. If $P \cong B \times G$ with an equivariant isomorphism, we say that $P$ is the trivial bundle.
2. We state without proof: If $B$ is affine, any $\mathbb{G}_a$-principal bundle $P \longrightarrow B$ is trivial.
3. The morphism

$$\text{SL}_2(\mathbb{C}) \longrightarrow \mathbb{A}^2_{\mathbb{C}}, C \mapsto Ce_1,$$

is a non-trivial $\mathbb{G}_a$-principal bundle. Namely $\text{SL}_2(\mathbb{C}) \not\cong \mathbb{A}^2_{\mathbb{C}} \times \mathbb{G}_a$, since the left hand side is affine, but the right hand side is not.
4. Let us return to the morphisms

$$Y_b \xrightarrow{\pi} \mathbb{A}^1, b \neq 0.$$

Doubling the origin in the target yields the line $\mathbb{A}^1$ with two origins, a (non-separated) prevariety, and we obtain a bundle projection

$$P := Y_b \longrightarrow B := \mathbb{A}^1$$

by associating to the two lines in the zero fiber different values in $\mathbb{A}^1$.
5. If $X$ is an affine $\mathbb{G}_m$-variety, the orbits with trivial stabilizer form an open subset and the total space $P$ of some $\mathbb{G}_m$-principal bundle $P \longrightarrow B$ with a not necessarily separated base $B$. If $P = X$, then we have $B = X//\mathbb{G}_m = X/\mathbb{G}_m$.
6. Even if a $\mathbb{G}_a$-action on an affine variety $X$ is free and the quotient map $\pi : X \longrightarrow X//G$ has connected fibres, it need not be a $\mathbb{G}_a$-principal bundle over an open subset $V \subset X//\mathbb{G}_a$. There are free affine $\mathbb{G}_a$-surfaces, where $\pi$ has multiple fibres.
It is easy to see that the varieties $Y_b, b \in C^*$, are isomorphic to one another: There is a commutative diagram
\[
\begin{array}{ccc}
Y_b & \xrightarrow{\cong} & \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta \\
\downarrow & & \downarrow \\
\mathbb{A}^1 & \overset{}{=} & \mathbb{A}^1 \\
\end{array}
\]
where $\mathbb{P}^1$ denotes the projective line and $\Delta \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ the diagonal. The left vertical arrow is the quotient morphism, while the right one is in affine coordinates on $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ the inverse of the difference of the coordinates:
\[
(x, y) \mapsto \frac{1}{x - y},
\]
and the $\mathbb{G}_a$-action on $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$ is
\[
t \ast (x, y) = (x + bt, y + bt).
\]
But there is also a sequence $(S_n)_{n > 0}$ of $\mathbb{G}_a$-bundles over the line $\mathbb{A}^1$ with double origin of particular interest: The affine varieties $S_n$ are pairwise non-isomorphic, even non-homeomorphic, but nevertheless we find
\[
S_n \times \mathbb{A}^1 \cong S_m \times \mathbb{A}^1
\]
for all $n, m \in \mathbb{N}_{>0}$. In other words: For $n \neq m$ we have
\[
\mathcal{O}(S_m) \not\cong \mathcal{O}(S_n),
\]
while
\[
\mathcal{O}(S_m)[T] \cong \mathcal{O}(S_n)[T].
\]

**Danielewski surfaces**: For integers $n \geq 1$ we define the $n$:th Danielewski surface by
\[
S_n := V(\mathbb{A}^3; x^n z - y^2 + 1).
\]
These surfaces have a $\mathbb{G}_a$-action defined by the locally nilpotent derivation which is given by $D(x) = 0, D(y) = x^n$, and $D(z) = 2y$; and the algebraic quotient morphism is given by
\[
\pi_n : S_n \longrightarrow \mathbb{A}^1, (x, y, z) \mapsto x.
\]
For the fibres we have
\[
\pi_n^{-1}(x) \cong \begin{cases} 
\mathbb{G}_a & \text{if } x \neq 0 \\
\mathbb{G}_a \times \{\pm 1\} & \text{if } x = 0 
\end{cases}
\]
and $S_n$ is a $\mathbb{G}_a$-principal bundle over $\mathbb{A}^1$. The Danielewski surfaces provide counterexamples to the following cancellation conjecture for $n = 1$: if $X$ and $Y$ are affine varieties such that $X \times \mathbb{A}^n \cong Y \times \mathbb{A}^n$, then $X \cong Y$. 

21
Proof. Consider the following cartesian diagram

\[
\begin{array}{ccc}
S_n \times \mathbb{G}_a & \xleftarrow{\cong} & S_n \times \Delta^1 S_m \xrightarrow{\cong} S_m \times \mathbb{G}_a \\
\downarrow \pi_n & & \downarrow \pi_m \\
S_n & \xleftarrow{\cong} & S_m
\end{array}
\]

where the skew arrows are $\mathbb{G}_a$-principal bundles. Since the upper ones have an affine base, they are trivial, and we get the isomorphisms on the top row. On the other hand one shows $S_n \not\cong S_m$ for $n \neq m$ by computing their first homology groups at infinity (the inverse limit of the first homology groups $H_1(S_n \setminus K)$ of complements $S_n \setminus K$ of compact subsets $K \subset S_n$):

\[H_1^\infty(S_n) \cong \mathbb{Z}_{2n}.\]
2. Summary of papers

2.1 Paper I
The hypersurface
\[ X := \{(x, y, z, t) \in \mathbb{A}^4; x + x^2y + z^3 + t^2 = 0\}, \]
known as “Russell’s hypersurface”, has played an important role in the proof of the fact that any action
\[ \mathbb{G}_m \times \mathbb{A}^3 \longrightarrow \mathbb{A}^3 \]
on affine three space is linearizable, i.e. there is a polynomial change of coordinates in \( \mathbb{A}^3 \), such that the conjugate action is of the form
\[ \mathbb{G}_m \times \mathbb{A}^3 \longrightarrow \mathbb{A}^3, (\lambda, (u, v, w)) \mapsto (\lambda^p u, \lambda^q v, \lambda^r w). \]
Russell’s hypersurface looks very much like affine three space, and if it were actually isomorphic to \( \mathbb{A}^3 \), the natural action
\[ \mathbb{G}_m \times X \longrightarrow X, (\lambda, (x, y, z, t)) \mapsto (\lambda^6 x, \lambda^{-6} y, \lambda^2 z, \lambda^3 t) \]
would provide via conjugation a non-linearizable action on \( \mathbb{A}^3 \). The first to prove that this is not the case was Makar-Limanov; then there was a more general theorem by S. Kaliman about morphisms \( \mathbb{A}^3 \longrightarrow \mathbb{A}^1 \), which implies \( X \not\sim \mathbb{A}^3 \). Here we investigate the geometric background of the original proof, which was rather based on algebraic arguments. The main point is to observe that \( X \) does not admit as many \( \mathbb{G}_a \)-action as \( \mathbb{A}^3 \) does: The possible \( \mathbb{G}_a \)-orbits are always contained in the fibers of the morphism \( X \longrightarrow \mathbb{A}^1, (x, y, z, t) \mapsto x \).

2.2 Paper II
Here we study affine \( \mathbb{G}_a \)-varieties \( X \) with algebraic quotient
\[ X / / \mathbb{G}_a \cong \mathbb{A}^2, \]
such that the quotient map \( \pi : X \longrightarrow \mathbb{A}^2 \) induces a \( \mathbb{G}_a \)-principal bundle
\[ \pi|_{X_v} : X_v := \pi^{-1}(\mathbb{A}^2_v) \longrightarrow \mathbb{A}^2_v. \]
We show the following:
1. If the bundle $X_s \to \mathbb{A}^2_s$ is trivial, there is a natural morphism $\mathbb{A}^2 \times \mathbb{G}_a \to X$: it is either an open embedding (but not necessarily an isomorphism) or contracts $0 \times \mathbb{G}_a$ to a singular point of $X$. The zero fiber is the union of one orbit and a purely two dimensional subvariety in the first case and purely two dimensional in the second case.

2. Otherwise, $X_s$ is affine and the zero fiber purely two dimensional. We obtain an embedding $X_s \hookrightarrow \mathbb{A}^4$ as hypersurface in affine four space and see that it can be obtained as a pullback of the standard bundle $SL_2(\mathbb{C}) \to \mathbb{A}^2, C \mapsto C e_1$, with respect to a morphism $\mathbb{A}^2 \to \mathbb{A}^2$. Hence we may concentrate on that bundle and construct two interesting series of affine $\mathbb{G}_a$-varieties $X$ over the plane with $X_s \cong SL_2(\mathbb{C})$. They have one member in common, the only smooth example, and the zero fiber coincides with the fixed point set. But that need not always be the case: by applying equivariant affine modifications to their common member, we can even construct $\mathbb{G}_a$-varieties $X$ with one dimensional or even empty fixed point sets.
3. Sammanfattning på svenska (Summary in Swedish)

Denna avhandling behandlar frågor inom den delen av algebraisk geometri som brukar benämnas komplex affin algebraisk geometri. De objekt som man huvudsakligen intresserar sig för där är affina varieteter, det vill säga sådana delvarieteter i något affint rum \(\mathbb{A}^n\) över kroppen \(\mathbb{C}\) av alla komplexa tal, som kan beskrivas med hjälp av polynomiella ekvationer. Betraktad som mängd är \(\mathbb{A}^n\) ingenting annat än \(\mathbb{C}^n\), men \(\mathbb{A}^n\) utrustas även med en topologi, Zariski-topologin, och en kärve \(\mathcal{O}_{\mathbb{A}^n}\) av reguljära funktioner. De affina varieteterna bildar en kategori där morfismerna från ett objekt \(X \subset \mathbb{A}^n\) till ett objekt \(Y \subset \mathbb{A}^m\) ges av polynomiella avbildningar \(\varphi : \mathbb{A}^n \rightarrow \mathbb{A}^m\) sådana att \(\varphi(X) \subset Y\). Kategorin av affina varieteter är antiekvivalent med kategorin av reducerade affina algebror, det vill säga ändligt genererade och reducerade algebror över \(\mathbb{C}\), via funktorn \(\mathcal{O}\) som till varje affin varietet \(X\) associerar dess algebra av reguljära funktioner \(\mathcal{O}_X(X)\). Med denna dualitet kan geometriska observationer leda till resultat som kan formuleras i termer av kommutativ algebra och vice versa.

Vi är särskilt intresserade av affina \(\mathbb{G}_a\)- respektive \(\mathbb{G}_m\)-varieteter, dvs. sådana varieteter som har en (icke-trivial) gruppverkan av \(\mathbb{G}_a\) respektive \(\mathbb{G}_m\). Den första av dessa två är den additiva gruppen av komplexa tal, och den andra är den multiplikativa gruppen av nollskilda komplexa tal. I själva verket är dessa två de enda endimensionella sammanhängande affina algebraiska grupperna som finns. Verkningar av \(\mathbb{G}_a\) respektive \(\mathbb{G}_m\) på en affin varietet \(X\) motsvaras på den algebraiska sidan av lokalt nilpotenta deriveringar, respektive \(\mathbb{Z}\)-graderingar på \(\mathcal{O}_X(X)\). Dessa två korresponderar är grundläggande, och gör att vi fritt kan växla synsätt, och ibland betrakta en \(\mathbb{G}_a\)-verkan som en lokalt nilpotent derivering, och en \(\mathbb{G}_m\)-verkan som en \(\mathbb{Z}\)-gradering på \(\mathcal{O}_X(X)\), och vice versa.

I artikel I behandlas Russell’s hyperyta, som är den hyperyta i \(\mathbb{A}^4\) som ges av ekvationen

\[x + x^2y + z^3 + t^2 = 0,\]

och målet är att skilja den från \(\mathbb{A}^3\) i kategorin av affina varieteter i en "geometrisk anda": Makar-Limanovs ursprungliga bevis bygger istället på olika slags graderingar. Svårigheten med att hitta en invariant som skiljer dem åt ligger i att Russell’s hyperyta i så många avseenden är lik \(\mathbb{A}^3\) – den är glatt och till och med difféomorf med \(\mathbb{A}^3\), så alla de vanliga topologiska invarianterna gör bet på uppgiften. Inte heller går det att skilja dem åt med de klassiska algebraiska invarianterna, bland annat eftersom Russell’s hyperyta
har en ring av reguljära funktioner som är faktoriell, och den har en domi-
nant avbildning \( \mathbb{A}^3 \rightarrow X \). Som kontrast nämner vi att en glatt affin faktoriell
yta \( Y \) med en dominant avbildning \( \mathbb{A}^2 \rightarrow Y \) nödvändigtvis är isomor med
\( \mathbb{A}^2 \). Istället uppfanns av Makar-Limanov en ny invariant, \( ML(X) \) som kan
ordnas till varje affin varietet \( X \). Det är den delring av \( \mathcal{O}_X(X) \) som består
av alla funktioner som ligger i kärnan till samtliga lokalt nilpotenta deriva-
tioner på \( \mathcal{O}_X(X) \). En ekvivalent formulering är att den består av funktioner
som är invarianta med avseende på varje algebraisk \( G_a \)-verkan. Vi visar att
\( ML(X) \cong \mathbb{C}[x] \not\cong \mathbb{C} \cong ML(\mathbb{A}^3) \) med hjälp av rent geometriska argument, där
\( X \) betecknar Russell’s hyperyta.

I artikel II behandlas frågan om affina utvidgningar av \( G_a \)-principalknippen
över det punkerade planet \( \mathbb{A}^2 \). Givet ett \( G_a \)-principalknippe \( P \rightarrow \mathbb{A}^2 \), söker
vi alltså efter affina \( G_a \)-varieteter \( \hat{P} \) tillsammans med kommutativa diagram
\[
\begin{array}{c}
P \\
\downarrow \\
\mathbb{A}^2
\end{array} \quad \begin{array}{c}
\hat{P} \\
\downarrow \\
\hat{\mathbb{A}^2}
\end{array}
\]
där den övre horisontella avbildningen är en dominant \( G_a \)-ekvivariant öppen
inbäddning som identifierar \( P \) med den delen av \( \hat{P} \) som ligger ovanför \( \hat{\mathbb{A}^2} \).

Förutom det triviala knippet \( \mathbb{A}^2 \times G_a \), är \( SL_2(\mathbb{C}) \) i någon mening det mest
grundläggande exemplet på \( G_a \)-principalknippen över \( \mathbb{A}^2 \). Standardverkan av
\( G_a \) på \( SL_2(\mathbb{C}) \) definieras genom
\[
SL_2(\mathbb{C}) \times G_a \rightarrow SL_2(\mathbb{C})
\]
\[
\left( \begin{pmatrix} x & u \\ y & v \end{pmatrix}, t \right) \mapsto \begin{pmatrix} x & u \\ y & v \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},
\]
och projektionen på \( \mathbb{A}^2 \) ges av \( SL_2(\mathbb{C}) \rightarrow \mathbb{A}^2, A \mapsto Ae_1 \), där \( e_1 \) betecknar
den första enhetsvektorn. Det triviala knippet har uppenbarligen inte ett affint
totalrum, men det visar sig att samtliga icke-triviala knippen över \( \mathbb{A}^2 \) har
totalrum som är affina. Som bekant beskrivs ju till exempel \( SL_2(\mathbb{C}) \subset \mathbb{A}^4 \) som
nollställemängden till polynomet \( xv - yu - 1 \in \mathbb{C}[x,y,u,v] \). Utvidgningar till
det triviala knippet och icke-triviala knippen studeras var för sig.

När det gäller det triviala knippet \( \mathbb{A}^2 \times G_a \), har vi naturligtvis den ”up-
penbara” utvidgningen \( \mathbb{A}^2 \times G_a \), men det finns fler. Vi visar att om \( \hat{P} \) är en
utvidgning av det triviala knipet, så finns det alltid en \( G_a \)-ekvivariant mor-
fism \( \mathbb{A}^2 \times G_a \rightarrow \hat{P} \), och för den har vi två möjligheter: antingen är den en
öppen inbäddning eller så kollapsar den nollfibern \( \{0\} \times G_a \) till en singuljär
punkt. Det första fallet är nog det som leder till mest förvånande exempel:
det finns glatta utvidgningar av \( \mathbb{A}^2 \times G_a \) som inte är isomorfa med \( \mathbb{A}^2 \times G_a \).
Dessa konstrueras som \( G_a \)-principalknippen över en icke-separerad tvådimen-
sionell bas som i sin tur har en naturlig morfism till den affina linjen med två
origon. I det andra fallet ges en fullständig klassificering av de möjliga affina
utvidgningarna för vilka den ”vertikala” \( G_m \)-verkan på \( \hat{P} \) kan utvidgas till hela
\( \hat{P} \).
Vad beträffar icke-triviala $\mathbb{G}_a$-principalknippen, visar det sig att givet ett godtyckligt icke-trivialt knippe $P \rightarrow \mathbb{A}^2$, kan man alltid hitta en morfism $\varphi: \mathbb{A}^2 \rightarrow \mathbb{A}^2$ med $\varphi(\mathbb{A}^*_2) \subset \mathbb{A}^*_2$ som har egenskapen att $P$ är tillbakadragningen av $\text{SL}_2(\mathbb{C})$ med avseende på den morfismen: $P = \varphi^*(\text{SL}_2(\mathbb{C}))$. Konstruktionen av utvidgningar till $\text{SL}_2(\mathbb{C})$ leder alltså automatiskt till utvidgningar för varje icke-trivialt $\mathbb{G}_a$-principalknipp över $\mathbb{A}^*_2$. De först nämnda klassificeras i sin tur av delalgebror $B \subset A := \mathcal{O}(\text{SL}_2(\mathbb{C}))$ med särskilda algebraiska egenskaper som motsvarar den geometriska situationen.

För att förenkla studiet av sådana delalgebror, går vi över till graderade algebror, med avseende på en filtrering som definieras i termer av den lokalt nilpotenta derivering $D$ till $\text{SL}_2(\mathbb{C})$’s standard-$\mathbb{G}_a$-verkan:

$$\text{gr}_D(B) \subset \text{gr}_D(A) \cong \bigoplus_{\nu \geq 0} (x, y)^\nu S^\nu,$$

och ser att en utvidgning $P \hookrightarrow \hat{P}$ ger upphov till en graderad algebra på formen

$$\text{gr}_D(B) = \bigoplus_{\nu \geq 0} m_\nu S^\nu$$

för en följd av ideal $m_\nu \subset (x, y)^\nu \subset \mathbb{C}[x, y]$ som måste uppfylla några särskilda villkor, så som till exempel att den måste vara avtagande.

Någon fullständig klassificering av utvidgningar av $\text{SL}_2(\mathbb{C})$ når vi inte fram till och kan väl inte förväntas, utan vi presenterar två intressanta familjer: $\hat{P}_n$, där $n \in \mathbb{N}$ och $\hat{P}(p, q)$, där $(p, q)$ är ett par av relativt prima naturliga tal $p, q \in \mathbb{N}$.

Den första familjen realiserar, för givet $n \in \mathbb{N}$, idealföljden $m_\nu = (x, y)^{(n+2)\nu}$, och det visas att dessa utvidgningar bestäms entydigt av sina idealföljder på den graderade nivån. Anledningen är att de graderade algebrorna är Reesalgebror, dvs. algebror som genereras i grad 1.

Den andra familjen av utvidgningar realiserar idealföljder som inte alltid ger upphov till Reesalgebror, och i synnerhet inte alltid entydigt bestämmer någon utvidgning av $\text{SL}_2(\mathbb{C})$.

För samtliga utvidgningar av icke-triviala buntar gäller det i alla fall att nollfibern $\pi^{-1}(0) \hookrightarrow \hat{P}$ är rent tvådimensionell, och i exemplen ovan karakteriseras den som fixpunktsmängden till den utvidgade verkningen av $\mathbb{G}_a$. Så behöver det dock inte alltid vara: itererade uppbäskningar av $\hat{P}_0 \cong \hat{P}(1, 1)$ ger upphov till utvidgningar där den utvidgade $\mathbb{G}_a$-verkan är fri respektive har liten fixpunktsmängd.
First of all, I would like to thank my supervisor, Karl-Heinz Fieseler, for sharing his amazingly deep and broad knowledge of mathematics with me, and for helping me ask good questions. I am very grateful for his positive attitude towards the work, for him always being available to consult, and for his engagement and warmth. His input over the past years has been crucial for the completion of this work. Thank you!

Adrien Dubouloz and the BirPol-work group have invited me to many inspirational workshops and courses, and provided a great ambiance for working and discussing mathematics with really skilled mathematicians.

Of the many teachers that I’ve been privileged to meet, I would like to mention a few that have become role models for me due to their contagious enthusiasm for their subject, and for their efforts to prepare and lecture so well: Ernst Dieterich, Karl-Heinz Fieseler, Dennis Hejhal, Wolodymyr Mazorchuk and Ryszard Rubinsztain. Thank you!

Gunnar Berg, Magnus Jacobsson and Inger Sigstam have all been constructive and helpful discussion-partners when it comes to teaching, which is also a considerable part of life as a PhD-student. Thank you for sharing your experience with me and for being such fantastic colleagues!

I am also grateful to a large number of PhD-friends. I would especially like to mention a few of you: Johan for many helpful mathematical discussions in the early stages of this work (Bon courage pour tes deux ans à Paris!), Anders (cross country skiing with you is really fun), Tomas (biking and running with you is very... special), Fredrik (you’re the best ceiling-painter I know).

The love, appreciation, support and encouragement I get from my family (including extended family and in-laws) is amazing. I am so thankful for you all!

Others that have contributed to the great time during my PhD-studies are my church- and Credo-friends. Some that I would like to thank in particular are Berit, Daniel, David, Johan, Lauri, Likoko, Markus, Mattias and Patrik. You are all in different ways great sources of inspiration for me, and being involved in Credo with you has been rewarding and joyful.

I wish to thank the Graduate School in Mathematics and Computing (FMB) for supporting this work, and the mathematics department for their generosity shown in many ways.

Finally, the turn comes to Su-lin who at the same time is my favourite person in the whole world, my very best friend and my wife. That such a combination exists is almost too good to be true. But it is true, and for everything about her I am so thankful.
Mathematics is great fun, but it is from spending time and sharing ideas and thoughts with people that the joy comes – each contribution acknowledged above (and the many not mentioned here) is for me an expression of God’s goodness, and for this I am a truly grateful.
References


[27] L. Makar-Limanov, *On the hypersurface x + x²y + z² + t³ = 0 in C⁴ or a C³-like threefold which is not C³*, Israel J. Math. 96 (1996), no. part B, 419–429. MR 1433698 (98a:14052)


[34] ______, *Basic algebraic geometry. 2*, second ed., Springer-Verlag, Berlin, 1994, Schemes and complex manifolds, Translated from the 1988 Russian edition by Miles Reid. MR 1328834 (95m:14002)

