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Boundary estimates for non-negative solutions to non-linear parabolic equations

Kaj Nyström, Håkan Persson, & Olow Sande

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Abstract

This paper is mainly devoted to the boundary behavior of non-negative solutions to the equation

$$Hu = \partial_t u - \nabla \cdot A(x, t, \nabla u) = 0$$

in domains of the form $\Omega_T = \Omega \times (0, T)$ where $\Omega \subset \mathbb{R}^n$ is a bounded non-tangentially accessible (NTA) domain and $T > 0$. The assumptions we impose on A imply that H is a non-linear parabolic operator with linear growth. Our main results include a backward Harnack inequality, and the Hölder continuity up to the boundary of quotients of non-negative solutions vanishing on the lateral boundary. Furthermore, to each such solution one can associate a natural Riesz measure supported on the lateral boundary and one of our main result is a proof of the doubling property for this measure. Our results generalize, to the setting of non-linear equations with linear growth, previous results concerning the boundary behaviour, in Lipschitz cylinders and time-independent NTA-cylinders, established for non-negative solutions to equations of the type $\partial_t u - \nabla \cdot (A(x, t) \nabla u) = 0$, where A is a measurable, bounded and uniformly positive definite matrix-valued function. In the latter case the measure referred to above is essentially the caloric or parabolic measure associated to the operator and related to Green's function. At the end of the paper we also remark that our arguments are general enough to allow us to generalize parts of our results to general fully non-linear parabolic partial differential equations of second order.

2010 *Mathematics Subject Classification*: 35K59, 35K55

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1 Introduction

This paper is devoted to the boundary behaviour of non-negative solutions to certain non-linear parabolic equation in space-time cylinders $\Omega_T = \Omega \times (0, T)$, $T > 0$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, i.e., an open, connected and bounded set. Given p , $1 < p < \infty$, fixed, recall that general equations of p -parabolic type are equations of the form

$$Hu = \partial_t u - \nabla \cdot A(x, t, \nabla u) = 0 \tag{1.1}$$

where $A(x, t, \eta) = (A_1(x, t, \eta), \dots, A_n(x, t, \eta)) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is measurable, $A(x, t, \eta)$ is, for almost all fixed $(x, t) \in \mathbb{R}^n$, continuous in η_k , for every $k \in \{1, \dots, n\}$ and whenever $\eta \in \mathbb{R}^n$, and the following conditions are satisfied, whenever $(x, t, \eta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and for some β , $1 \leq \beta < \infty$:

$$\begin{aligned} (i) \quad & |A(x, t, \eta)| \leq \beta |\eta|^{p-1}, \\ (ii) \quad & (A(x, t, \eta) - A(x, t, \xi)) \cdot (\eta - \xi) \geq \beta^{-1}(|\eta| + |\xi|)^{p-2} |\eta - \xi|^2. \end{aligned} \tag{1.2}$$

In particular, the condition (ii) in (1.2) implies that

$$A(x, t, \eta) \cdot \eta \geq \beta^{-1} |\eta|^p \tag{1.3}$$

whenever x, t , and η are as above. In the special case when $p = 2$, $\nabla \cdot A(x, t, \eta) = \partial_{x_i}(a_{ij}(x, t)\eta_j)$ and the matrix $A(x, t) = \{a_{ij}(x, t)\}$ is real, symmetric, and such that $A(x, t, \eta)$ satisfies (1.2) (i) and (1.3), the problems studied in this paper has a long and rich history, see [G], [FGS], [FS], [FSY], [N], [SY]. On the contrary for $p \neq 2$, $1 < p < \infty$, very little is known concerning these problems and we refer the reader to [KMN] for an account of the current literature. In this paper we consider the case which is in between these two situations, i.e., in between the linear case and the non-linear case, as we consider general equations as in (1.1), assuming (1.2), but with the important extra assumption that $p = 2$. In particular, we consider non-linear parabolic equations with linear growth. In this special case we are able to give a complete account concerning boundary Harnack type inequalities for non-negative solutions vanishing on a portion of the lateral boundary of Ω_T , hence generalizing, in particular, [FGS], [FS], to the context non-linear parabolic equations with linear growth. Concerning Ω we simply assume that Ω is an NTA-domain in the sense of [JK]. Our results are new and we believe that this $p = 2$ case is the first situation to be considered beyond linear equation since already in this case several key estimates demand new proofs due to the non-linear character of the equation. The long-term goal is to understand the case p , $1 < p < \infty$, attempting to generalize, as far as possible, the elliptic estimates developed in [LLuN], [LN1], [LN3], [LN2], [LN4], to the parabolic setting.

1.1 Main results

We consider non-negative weak solutions to the equation in (1.1), in the case $p = 2$, in cylindrical domains $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $T > 0$, and we refer to the bulk of the paper for the definition of weak solutions. Concerning the domain Ω we will assume that Ω is a NTA-domain (non-tangentially accessible domain), with parameters M, r_0 , in the sense of Definition 2.10 below. We will often refer to Ω_T as a NTA-cylinder. Assuming that Ω is a NTA-domain one can prove, see Lemma 2.11 below, that all points on the parabolic boundary

$$\partial_p \Omega_T = S_T \cup (\overline{\Omega} \times \{0\}), \quad S_T = \partial\Omega \times [0, T],$$

of the cylinder Ω_T are regular for the Dirichlet problem for the operator H in (1.1). In particular, for any $f \in C(\partial_p \Omega_T)$, there exists a unique Perron-Wiener-Brelot-Bauer solution $u = u_f^{\Omega_T} \in C(\overline{\Omega_T})$ to the Dirichlet problem

$$Hu = 0 \text{ in } \Omega_T, \quad u = f \text{ on } \partial_p \Omega_T. \tag{1.4}$$

If Ω is a given NTA-domain in the sense that by Definition 2.10, with parameters M and r_0 , then there exists, for any $x_0 \in \partial\Omega$, $0 < r < r_0$, a non-tangential corkscrew point, i.e., a point $A_r(x_0) \in \Omega$, such that

$$M^{-1}r < d(x_0, A_r(x_0)) < r, \quad \text{and} \quad d(A_r(x_0), \partial\Omega) \geq M^{-1}r,$$

where $d(\cdot, \cdot)$ denotes the standard Euclidean distance. In the following we let $A_r(x_0, t_0) = (A_r(x_0), t_0)$ whenever $(x_0, t_0) \in S_T$ and $0 < r < r_0$. Given $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. For $(x, t) \in \mathbb{R}^{n+1}$ and $r > 0$ we let

$$C_r(x, t) = B(x, r) \times (t - r^2, t + r^2).$$

and

$$\Delta(x, t, r) = S_T \cap C_r(x, t). \tag{1.5}$$

We first establish the following theorem.

Theorem 1.1 *Let H be as in (1.1) and assume (1.2) with $p = 2$. Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. Let u be a non-negative solution of $Hu = 0$ in Ω_T vanishing continuously on S_T . Let $0 < \delta \ll \min\{r_0/2, \sqrt{T}\}$ be a fixed constant. Suppose $(x_0, t_0) \in S_T$ such that $\delta^2 < t_0 < T - \delta^2$ and assume that $r < \delta/2$. Then there exists $c = c(H, M, \text{diam}(\Omega), T, \delta)$, $1 \leq c < \infty$, such that*

$$u(x, t) \leq cu(A_r(x_0, t_0))$$

whenever $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$.

Let Ω, Ω_T, u be as in the statements of Theorem 1.1. Assume that u is continuous on the closure of Ω_T . Then $u(x, t) = 0$ whenever $(x, t) \in \partial\Omega \times [0, T]$. Extend u to $\mathbb{R}^n \times [0, T]$ by putting $u \equiv 0$ on $(\mathbb{R}^n \setminus \Omega) \times [0, T]$. Then there exists, see Lemma 2.7 below, a unique locally finite positive Borel measure μ on $\mathbb{R}^n \times [0, T]$, with support in $\partial\Omega \times [0, T]$, such that,

$$\begin{aligned} & - \int_{t'_1}^{t'_2} \int_{G'} A(x, t, \nabla u) \cdot \nabla \theta dx dt + \int_{t'_1}^{t'_2} \int_{G'} u \partial_t \theta dx dt \\ & - \int_{G'} u(x, t'_2) \theta(x, t'_2) dx + \int_{G'} u(x, t'_1) \theta(x, t'_1) dx = \iint \theta d\mu \end{aligned} \tag{1.6}$$

whenever $0 \leq t'_1 < t'_2 \leq T$, $\theta \in C_0^\infty(\mathbb{R}^n \times (0, T))$. We establish the following theorem concerning the doubling property of the measure μ .

Theorem 1.2 *Let H be as in (1.1) and assume (1.2) with $p = 2$. Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. Let u be a non-negative solution of $Hu = 0$ in Ω_T , assume that u is continuous on the closure of Ω_T and that u vanishes continuously on S_T . Let μ be the measure associated to u , with support in $\partial\Omega \times [0, T]$, as in (1.6). Let $0 < \delta \ll \min\{r_0/2, \sqrt{T}\}$ be a fixed constant. Suppose that $(x_0, t_0) \in S_T$ such that $\delta^2 < t_0 < T - \delta^2$ and assume that $r < \delta/2$. Then there exists $c = c(H, M, \text{diam}(\Omega), T, \delta)$, $1 \leq c < \infty$, such that*

$$\mu(\Delta(x_0, t_0, 2r)) \leq c\mu(\Delta(x_0, t_0, r)).$$

Due to the non-linear nature of the equation, to state our final result we have to impose more restrictive assumptions, more restrictive compared to the assumption that (1.2) holds with $p = 2$, on the operator defining the equations in (1.1). In the following we assume, in addition, that $A(x, t, \eta)$ is, for almost all $(x, t) \in \mathbb{R}^{n+1}$, continuously differentiable in η_k , for every $k \in \{1, \dots, n\}$ whenever $\eta \in \mathbb{R}^n \setminus \{0\}$, and that

$$(i) \quad \left| \frac{\partial A_i}{\partial \eta_j}(x, t, \eta) \right| \leq \beta, \quad (ii) \quad \sum_{i,j=1}^n \frac{\partial A_i}{\partial \eta_j}(x, t, \eta) \xi_i \xi_j \geq \beta^{-1} |\xi|^2, \\ (iii) \quad \lambda A(x, t, \xi) = A(x, t, \lambda \xi), \quad (1.7)$$

for some β , $1 \leq \beta < \infty$, whenever $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ and $\eta \in \mathbb{R}^n \setminus \{0\}$ and for all $\lambda \geq 0$. A straightforward calculation show that (1.7) (i) and (ii) imply (1.2) with $p = 2$. Furthermore, for technical reason we in addition assume that

$$(x, t) \rightarrow A(x, t, \eta) \text{ is Hölder continuous, with respect to } d_p, \text{ for each } \eta \in \mathbb{R}^n \setminus \{0\} \text{ fixed,} \quad (1.8)$$

and where $d_p((\cdot, \cdot), (\cdot, \cdot))$ denotes the parabolic distance between $(x, t), (y, s) \in \mathbb{R}^{n+1}$ introduced in (2.2) below. The assumption in (1.8) is only used qualitatively as discussed below. We prove the following theorems.

Theorem 1.3 *Let H be as in (1.1), assume (1.2) with $p = 2$, and assume (1.7). Assume (1.8). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. Let u, v be non-negative solutions of $Hu = 0$ in Ω_T vanishing continuously on S_T and assume in addition that $v \leq u$ in Ω_T . Let $0 < \delta \ll \min\{r_0/2, \sqrt{T}\}$ be a fixed constant. Suppose $(x_0, t_0) \in S_T$ such that $\delta^2 < t_0 < T - \delta^2$ and assume that $r < \delta/2$. Then there exists a $c = c(H, M, \text{diam}(\Omega), T, \delta)$, $1 \leq c < \infty$, such that*

$$c^{-1} \frac{u(A_r(x_0, t_0)) - v(A_r(x_0, t_0))}{v(A_r(x_0, t_0))} \leq \frac{u(x, t) - v(x, t)}{v(x, t)} \leq c \frac{u(A_r(x_0, t_0)) - v(A_r(x_0, t_0))}{v(A_r(x_0, t_0))},$$

whenever $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$.

Theorem 1.4 *Let H be as in (1.1), assume (1.2) with $p = 2$, and assume (1.7). Assume (1.8). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. Let u, v be non-negative solutions of $Hu = 0$ in Ω_T vanishing continuously on S_T . Let $0 < \delta \ll \min\{r_0/2, \sqrt{T}\}$ be a fixed constant. Suppose $(x_0, t_0) \in S_T$ such that $\delta^2 < t_0 < T - \delta^2$ and assume that $r < \delta/2$. Then u/v is Hölder continuous on the closure of $\Omega \times (\delta^2, T]$. Furthermore, there exist $c = c(H, M, r_0, \text{diam}(\Omega), T, \delta)$, $1 \leq c < \infty$, and $\alpha = \alpha(H, M, \text{diam}(\Omega), T, \delta)$, $\alpha \in (0, 1)$, such that*

$$\left| \frac{u(x, t)}{v(x, t)} - \frac{u(y, s)}{v(y, s)} \right| \leq c_1 \left(\frac{d_p((x, t), (y, s))}{r} \right)^\alpha \frac{u(A_r(x_0, t_0))}{v(A_r(x_0, t_0))}$$

whenever $(x, t), (y, s) \in \Omega_T \cap C_r(x_0, t_0)$.

Note that Theorem 1.1 – Theorem 1.4 have a global flavor in the sense that we assume that u and v are non-negative solutions of $Hu = 0$ in all of Ω_T vanishing continuously on the entire lateral boundary S_T . Naturally, local versions of these theorems can be also formulated. The following example shows that the class of operators considered in Theorem 1.3 and Theorem 1.4 is far from void.

Example 1.5 Let $M(x, t) = \{m_{ij}(x, t)\}$ be an $n \times n$ -matrix with bounded and measurable coefficients satisfying the ellipticity condition

$$\sum_{i,j=1}^n m_{ij} \xi_i \xi_j \geq \beta^{-1} |\xi|^2, \text{ for some } \beta, 1 \leq \beta < \infty,$$

for all $\xi \in \mathbb{R}^n$ and for all $(x, t) \in \mathbb{R}^{n+1}$. Assume also that $(x, t) \rightarrow m_{ij}(x, t)$ is Hölder continuous with respect to d_p . Let $F(\eta) = (F_1(\eta_1), \dots, F_n(\eta_n)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 mapping in a neighborhood of the unit sphere. Define, given $M(x, t) = \{m_{ij}(x, t)\}$ and F , the vector field $A(x, t, \cdot) = (A_1(x, t, \cdot), \dots, A_n(x, t, \cdot)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ through

$$A_i(x, t, \eta) := \begin{cases} \sum_{k=1}^n m_{ik}(x, t) \eta_k - \epsilon |\eta| F(x, t, \eta/|\eta|), & \eta \in \mathbb{R}^n \setminus \{0\}; \\ 0, & \eta = 0. \end{cases}$$

We claim that the vector field $A(x, t, \cdot)$ satisfies the conditions in (1.7) for $\epsilon > 0$ sufficiently small. Obviously $A(x, t, \cdot)$ satisfies (1.8). Indeed, with the dependence on x and t suppressed, we see, for $\eta \in \mathbb{R}^n \setminus \{0\}$, that

$$\frac{\partial A_i(\eta)}{\partial \eta_j} = m_{ij} + \epsilon \left(\sum_{k=1}^n \frac{\partial F_i}{\partial \eta_k} \left(\frac{\eta}{|\eta|} \right) \frac{\eta_j \eta_k}{|\eta|^2} - 2F_i(\eta/|\eta|) \frac{\eta_j}{|\eta|} - \frac{\partial F_i}{\partial \eta_j} \left(\frac{\eta}{|\eta|} \right) \right).$$

Since F_i is C^1 in a neighborhood of the unit sphere, $F_i(\eta/|\eta|)$ and $\frac{\partial F_i}{\partial \eta_j}(\eta/|\eta|)$ are all bounded, and hence assumption (i) of (1.7) is satisfied. By choosing $\epsilon > 0$ small enough we also see that assumption (ii) of (1.7) is satisfied. By construction $\lambda A(x, t, \xi) = A(x, t, \lambda \xi)$ and hence the vector field $A(x, t, \cdot)$ satisfies the conditions in (1.7), for $\epsilon > 0$ sufficiently small, for some $\beta = \beta(\epsilon)$, $1 \leq \beta < \infty$.

To put Theorem 1.1 - Theorem 1.4 in perspective we note, as briefly mentioned, that an important class of equations related to the equations studied in this paper are equations of the form

$$H u = \partial_t u - \nabla \cdot (A(x, t) \nabla u) = 0 \quad (1.9)$$

where the matrix-valued function $A = A(x, t) = \{a_{ij}(x, t)\}$ is assumed to be measurable, bounded, symmetric, and

$$\beta^{-1} |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \leq \beta |\xi|^2, \quad (1.10)$$

for some $\beta = \beta(\epsilon)$, $1 \leq \beta < \infty$, whenever $(x, t) \in \mathbb{R}^{n+1}$ and $\xi \in \mathbb{R}^n$. We here refer to these equations as linear symmetric uniformly parabolic equations in divergence form. In this setting one can conclude that there exists, for every $(x, t) \in \Omega_T$, a unique probability measure $\omega^{(x,t)} = \omega_H^{(x,t)}$ on $\partial_p \Omega_T$ such that the solution to the Dirichlet problem in (1.4) is given by

$$u(x, t) = \int_{\partial_p \Omega_T} f(y, s) d\omega^{(x,t)}(y, s). \quad (1.11)$$

In this case $\omega^{(x,t)}$ is referred to as the H -parabolic measure relative to (x, t) and Ω_T and in this special case Theorem 1.2 can be formulated for the H -parabolic measure. Note that for linear symmetric uniformly parabolic equations, in divergence as well as in non-divergence form, the study of the type of problems considered in this paper, and in particular Theorem 1.1 – Theorem 1.4, have a long and rich history which culminated with the celebrated papers of Fabes, Safonov and Yuan [FS], [FSY] and [SY]. In these works the authors proved versions of Theorem 1.1 – Theorem 1.4 for linear symmetric uniformly parabolic equations, both in divergence and non-divergence form. We remark that, while these authors work in Lipschitz cylinders, one can easily see that their proofs can be generalized to the setting of bounded NTA-cylinders. While the works in [FS], [FSY] and [SY] completed, for linear symmetric uniformly parabolic equations, the line of research considered in this paper, contributions by other researchers are contained in [FK], [FSt], [G], [KS], [FGS], [FS], [N]. For the corresponding elliptic theory we refer to [B], [C], [CFMS], [FGMS], [JK].

1.2 Brief discussion of proofs and the organization of the paper

Compared to linear symmetric uniformly parabolic equations, and in particular to linear symmetric uniformly parabolic equations in divergence form, the situation considered in this paper, (1.1) assuming in general (1.2) and $p = 2$, is considerably different as we consider non-linear equations, we make no a priori assumption on symmetry and in Theorem 1.1, Theorem 1.2, we make no assumption on the equation concerning homogeneity in the variable ∇u . In particular, in this case if u solves $Hu = 0$ and λ is a non-negative constant, then λu does in general not solve the same equation. However λu solves an equation having the same characteristics as the original equation. In the statements of Theorem 1.1 and Theorem 1.2 we only assume (1.2), and $p = 2$, and we hence develop proofs adapted to this genuinely non-linear context. In the case of Theorem 1.3 and Theorem 1.4 we assume more and in these cases the conditions in (1.7), and the qualitative condition in (1.8), enable us to make use of the theory for non-negative solutions to uniformly parabolic linear equations referred to above. To briefly elaborate a bit on this we first note that if u and v are as in the statement of Theorem 1.3 and Theorem 1.4, then by assuming (1.7) and (1.8) it follows, see [LSU], that ∇u and ∇v are locally Hölder continuous in Ω_T . In particular, $|\nabla u|$ and $|\nabla v|$ are well-defined and finite at almost all points in Ω_T . Using this, and to outline one general linearization approach, let $G \subset \mathbb{R}^n$ be a bounded domain, let $t_1 < t_2$ and $G_{t_1, t_2} := G \times \{t : t_1 < t < t_2\}$. Assume that $\hat{u}$ and $\hat{v}$ satisfy $H\hat{u} = 0 = H\hat{v}$ in G_{t_1, t_2} , with H as in (1.1), assume (1.7) and (1.8). Let $P := \{(x, t) \in G_{t_1, t_2} : |\nabla \hat{v}(x, t)| + |\nabla \hat{u}(x, t)| > 0\}$ and let $\hat{a}_{ij}(x, t) = \hat{a}_{ij}(x, t, \hat{u}(x, t), \hat{v}(x, t))$ be defined, for $i, j \in \{1, \dots, n\}$, as

$$\hat{a}_{ij}(x, t) := \begin{cases} \int_0^1 \frac{\partial A_i}{\partial \eta_j}(x, t, \tau \nabla \hat{u}(x, t) + (1 - \tau) \nabla \hat{v}(x, t)) d\tau, & \text{if } (x, t) \in P. \\ \delta_{ij}, & \text{if } (x, t) \in G_{t_1, t_2} \setminus \overline{P}, \end{cases} \quad (1.12)$$

where δ_{ij} denotes Kronecker's delta. Consider now the equation

$$\hat{H}u = \partial_t u - \nabla \cdot (\hat{A}(x, t) \nabla u) = 0 \quad (1.13)$$

where $\hat{A} = \hat{A}(x, t) = \{\hat{a}_{ij}(x, t)\}$ and $\hat{a}_{ij}(x, t)$ is defined as in (1.12). By the assumption in (1.8) the matrix $\hat{A}$ is well-defined and bounded almost everywhere in G_{t_1, t_2} . In particular, we see that

(1.13) is a linear uniformly parabolic equations in divergence form as discussed above, however the matrix $\hat{A} = \hat{A}(x, t) = \{\hat{a}_{ij}(x, t)\}$ is not necessarily symmetric. Suppose now that $(x, t) \in P$. Since $A_i(\cdot, \cdot, \eta)$ is C^1 in a neighborhood of each $\eta \in \mathbb{R}^n \setminus \{0\}$ we see that the function

$$s \mapsto \frac{\partial A_i}{\partial \eta_j}(x, t, s \nabla \hat{u}(x, t) + (1 - s) \nabla \hat{v}(x, t)),$$

has at most one singularity. Using the fundamental theorem of calculus and the continuity of A_i , we hence see that $\hat{A}(x, t)(\nabla \hat{u}(x, t) - \nabla \hat{v}(x, t)) = A(x, t, \nabla \hat{u}(x, t)) - A(x, t, \nabla \hat{v}(x, t))$. Letting $e := \hat{u} - \hat{v}$, we can use this observation, and the fact that $A(x, t, 0) = 0$, to conclude that

$$\begin{aligned} \iint_{G_{t_1, t_2}} \hat{A}(x, t) \nabla e \cdot \nabla \theta \, dx dt &= \iint_{G_{t_1, t_2} \setminus P} \hat{A}(x, t) (\nabla \hat{u} - \nabla \hat{v}) \cdot \nabla \theta \, dx dt \\ &= \iint_{G_{t_1, t_2} \setminus P} (A(x, t, \nabla \hat{u}) - A(x, t, \nabla \hat{v})) \cdot \nabla \theta \, dx dt \\ &= \iint_{G_{t_1, t_2}} (A(x, t, \nabla \hat{u}) - A(x, t, \nabla \hat{v})) \cdot \nabla \theta \, dx dt \\ &= \iint_{G_{t_1, t_2}} (\hat{u} - \hat{v}) \partial_t \theta \, dx dt = \iint_{G_{t_1, t_2}} e \partial_t \theta \, dx dt, \end{aligned}$$

whenever $\theta \in C_0^\infty(G_{t_1, t_2})$, $t_1 < t'_1 < t'_2 < t_2$ and $G' \Subset G$, and hence e is a weak solution in G_{t_1, t_2} to a linear non-symmetric uniformly parabolic equation $\hat{H}e = 0$ in divergence form as in (1.13). In particular, the equations in (1.13) falls within in the scope of the techniques in [FS], [FSY] and [SY] with the exception that we can not ensure that the matrix $\{\hat{a}_{ij}(x, t)\}$ is symmetric. However, reexamining the arguments in [FS], [FSY] and [SY] one realizes that the results we will use from these papers, and this is thoroughly discussed in the bulk of the paper, remain valid without the assumption on symmetry of $\{\hat{a}_{ij}(x, t)\}$. In particular, this implies that we have an effective toolbox at our disposal when studying solutions to (1.13) and in particular when studying the difference of solutions to our non-linear problem. This idea is used in the proof of Theorem 1.3. In general Theorem 1.3 is considered to be stronger than Theorem 1.4 and as such is expected to imply Theorem 1.4. This is also the case in our situation as we are able to deduce Theorem 1.4 using Theorem 1.3 and Theorem 1.1.

The rest of the paper is organized as follows. Section 2 is of preliminary nature and we here define weak solutions and consider solutions to general non-linear parabolic equations with linear growth, i.e., we consider (1.1) assuming only (1.2). Note that this general setting includes the equations considered in Theorem 1.1 – Theorem 1.4 as well as symmetric and non-symmetric linear uniformly parabolic equations. For these general equations we state and prove, or give reference to the proof of, fundamental principles as interior energy estimates, the Harnack inequality, interior Hölder continuity estimates, maximum principles, some of the Gaussian like decay estimates and existence of the Riesz measure referred to above. We here also discuss solvability of the Dirichlet problem and estimates using Harnack chains in the setting of NTA-cylinders. In Section 3 we prove a number of technical lemmas concerning the boundary behaviour of non-negative solutions to (1.1) assuming only (1.2). In Section 4 we prove Theorem 1.1 and Theorem 1.2 leaving the proof of Theorem 1.3 and Theorem 1.4 to Section 5. Since our proofs of Theorem 1.3 and Theorem 1.4 rely on corresponding results for associated linear

equations we in Section 5 initially consider more (refined) results for linear (non-symmetric) uniformly parabolic equations in divergence form following [FS], [FSY], [N]. In section 6 we discuss a generalization of Theorem 1.1 to general fully non-linear partial differential equations and this stresses that at least our proof of Theorem 1.1 is based on general principles. The discussion in Section 6 suggests a far reaching generalization of a recent and interesting result in [BG] where the stronger assumption that Ω should be $C^{1,1}$ -regular is imposed.

2 Preliminaries on non-linear parabolic equations with linear growth

Points in Euclidean $(n+1)$ -space $\mathbb{R}^{n+1}$ are denoted by $x = (x_1, \dots, x_n, t)$. Given a set $E \subset \mathbb{R}^n$, let $\overline{E}, \partial E$, $\text{diam } E$ be the closure, boundary, and diameter of E . Let $\cdot$ denote the standard inner product on $\mathbb{R}^n$, $|x| = (x \cdot x)^{1/2}$, the Euclidean norm of x , and let dx be Lebesgue n -measure on $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. Throughout the paper we will use the following notation:

$$\begin{aligned} C_r(x, t) &= B(x, r) \times (t - r^2, t + r^2), \quad C_r^+(x, t) = B(x, r) \times (t, t + r^2), \\ C_r^-(x, t) &= B(x, r) \times (t - r^2, t), \quad C_{r_1, r_2}(x, t) = B(x, r_1) \times (t - r_2^2, t + r_2^2), \\ C_{r_1, r_2}^+(x, t) &= B(x, r_1) \times (t, t + r_2^2), \quad C_{r_1, r_2}^-(x, t) = B(x, r_1) \times (t - r_2^2, t), \end{aligned} \quad (2.1)$$

for $(x, t) \in \mathbb{R}^{n+1}$ and $r, r_1, r_2 > 0$. Furthermore, we let

$$d_p((x, t), (y, s)) = d_p(x, t, y, s) = (|x - y|^2 + |t - s|)^{1/2} \quad (2.2)$$

denote the parabolic distance between $(x, t), (y, s) \in \mathbb{R}^{n+1}$. Given $E, F \subset \mathbb{R}^n$, let $d(E, F)$ be the Euclidean distance from E to F . In case $E = \{y\}$, we write $d(y, F)$. If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$, we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are q th power integrable on O . Let

$$\|f\|_{W^{1,q}(O)} = \|f\|_{L^q(O)} + \|\nabla f\|_{L^q(O)}$$

be the norm in $W^{1,q}(O)$ where $\|\cdot\|_{L^q(O)}$ denotes the usual Lebesgue q -norm in O . $C_0^\infty(O)$ is the set of infinitely differentiable functions with compact support in O and we let $W_0^{1,q}(O)$ denote the closure of $C_0^\infty(O)$ in the norm $\|\cdot\|_{W^{1,q}(O)}$. $W_{loc}^{1,q}(O)$ is defined in the standard way. By $\nabla \cdot$ we denote the divergence operator. Given $t_1 < t_2$ we denote by $L^q(t_1, t_2, W^{1,q}(O))$ the space of functions such that for almost every t , $t_1 \leq t \leq t_2$, the function $x \rightarrow u(x, t)$ belongs to $W^{1,q}(O)$ and

$$\|u\|_{L^q(t_1, t_2, W^{1,q}(O))} := \left(\int_{t_1}^{t_2} \int_O \left(|u(x, t)|^q + |\nabla u(x, t)|^q \right) dx dt \right)^{1/q} < \infty.$$

$L_{loc}^q(t_1, t_2, W^{1,q}(O))$, $L^q(t_1, t_2, W_0^{1,q}(O))$ and $L_{loc}^q(t_1, t_2, W_0^{1,q}(O))$ are defined analogously. We let $C(t_1, t_2, L^q(O))$ be the space of functions such that the function $t \rightarrow \|u(t, \cdot)\|_{L^q(O)}$ is continuous whenever $t_1 \leq t \leq t_2$. $C_{loc}(t_1, t_2, L^q(O))$ is defined analogously. Finally, if $\hat{O} \subset \mathbb{R}^{n+1}$ is open then we let $C_0^\infty(\hat{O})$ be the set of infinitely differentiable functions with compact support in $\hat{O}$.

Given a bounded domain $G \subset \mathbb{R}^n$ and $t_1 < t_2$ we let $G_{t_1, t_2} := G \times \{t : t_1 < t < t_2\}$. We say that u is a weak solution to (1.1) in G_{t_1, t_2} if, for all open sets $G' \subseteq G$ and $t_1 < t'_1 < t'_2 < t_2$, we have $u \in L^2(t'_1, t'_2, W^{1,2}(G'))$ and

$$\int_{t'_1}^{t'_2} \int_{G'} A(x, t, \nabla u) \cdot \nabla \theta dx dt - \int_{t'_1}^{t'_2} \int_{G'} u \partial_t \theta dx dt + \int_{G'} u(x, t'_2) \theta(x, t'_2) dx - \int_{G'} u(x, t'_1) \theta(x, t'_1) dx = 0 \quad (2.3)$$

whenever $\theta \in C_0^\infty(G_{t_1, t_2})$. Furthermore, we say that u is a *weak supersolution* to (1.1) if the left hand side of (2.3) is non-negative for all $\theta \in C_0^\infty(G_{t_1, t_2})$ with $\theta \geq 0$. If, instead the left hand side is non-positive then we say that u is a *weak subsolution*. Using (1.2) the existence of weak solutions follows from standard monotonicity arguments (see Lemma 2.11 below). Furthermore, by parabolic regularity theory (see Lemma 2.5 below), weak solutions are locally Hölder continuous after a redefinition on a set of measure zero. In particular, we will throughout the paper assume that any solution u is continuous in the interior of its domain of definition. Finally, when we in the following write that a constant c depends on the operator H , $c = c(H)$, we mean that c depends on the dimension n and the constant β in (1.2). In general all constants c will satisfy $1 \leq c < \infty$.

2.1 Fundamental principles

In this section we state a number of fundamental results for weak solutions to the equation in (1.1) assuming (1.2). We begin with a discussion concerning the structural invariance under certain transformations of the class of equations considered.

Suppose that u is a solution to (1.1), that $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an affine transformation and that λ is a positive real number. Due to the general character of the operator A , one can in general not expect $u \circ T$ or λu to remain solutions to (1.1). However, as most of our results do not depend on the specific operator A , but only on the constants in the structural conditions (1.2), it turns out that it often suffices that $u \circ T$ and λu are solutions to equations satisfying the same structural conditions. The following lemmas formalizes this and the proofs of the lemmas are omitted.

Lemma 2.1 *Let H be as in (1.1) and assume (1.2) with $p = 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$, and assume that u is a weak solution to (1.1) in Ω_T and $r > 0$. Then the function $\tilde{u}(x, t) := u(rx - x_0, r^2t - t_0)$, is a solution to the equation $\tilde{H}\tilde{u} = 0$ in the domain*

$$\tilde{\Omega}_T := \{r^{-1}(x_0 + x) : x \in \Omega\} \times (r^{-2}t_0, r^{-2}(T + t_0)),$$

where $\tilde{H}$ is given by

$$\tilde{H}u := u_t - \tilde{A}(x, t, \nabla u),$$

and where $\tilde{A}$ is an operator of the same type as A , satisfying the same structural conditions (1.2) with the same constant β .

Lemma 2.2 *Assumptions as in Lemma 2.1. Then the function $\tilde{u}(x, t) := \lambda u(x, t)$ is a solution to the equation $\tilde{H}\tilde{u} = 0$ where $\tilde{H}$ is given by*

$$\tilde{H}u := u_t - \tilde{A}(x, t, \nabla u),$$

and where $\tilde{A}$ is an operator of the same type as A , satisfying the same structural conditions (1.2) with the same constant β .

Lemma 2.3 (*Interior energy estimates for subsolutions*) Let H be as in (1.1) and assume (1.2) with $p = 2$. Suppose that u is a bounded non-negative weak subsolution to (1.1) in Ω_T . Let $(x_0, t_0) \in \mathbb{R}^{n+1}$, $r_1 > 0$, $r_2 > 0$. There exists a constant $\gamma = \gamma(H, n)$, $1 \leq \gamma < \infty$, such that the following holds. Assume that $C_{r_1, r_2}^-(x_0, t_0) \subset \Omega_T$, and that ϕ is a smooth function defined in $C_{r_1, r_2}^-(x_0, t_0)$ satisfying $\phi(x, t) = 0$ for x outside $B(x_0, r_1)$. Then

$$\begin{aligned} & \sup_{t_0 - r_2^2 < t < t_0} \int_{B(x_0, r_1)} u^2 \phi^2(x, t) dx + \gamma^{-1} \iint_{C_{r_1, r_2}^-(x_0, t_0)} |\nabla u|^2 \phi^2 dx d\tau \\ & \leq \int_{B(x_0, r_1)} u^2 \phi^2(x, t_0 - r_2^2) dx + \gamma \iint_{C_{r_1, r_2}^-(x_0, t_0)} u^2 |\nabla \phi|^2 dx d\tau + \gamma \iint_{C_{r_1, r_2}^-(x_0, t_0)} u^2 \phi \phi_t dx d\tau. \end{aligned}$$

Proof. Formally, to prove the lemma choose $\theta := u\phi^2$ as the test function in the weak formulation of subsolutions, apply the structure conditions (1.2) and the fundamental theorem of calculus to the weak formulation (2.3). For details, and the justification of these formal calculations, see for example Proposition II.3.1 in [DB2]. ■

Lemma 2.4 (*Harnack inequality*) Let H be as in (1.1) and assume (1.2) with $p = 2$. Let $(x_0, t_0) \in \mathbb{R}^{n+1}$, $r > 0$. Assume that u is a non-negative weak solution to (1.1) in $C_r^-(x_0, t_0)$. Then, given $0 < h_1 < h_2 < 1$, $0 < \gamma < 1$, there exists a constant $c = c(H, h_1, h_2, \gamma)$, $1 \leq c < \infty$, such that

$$u(x, t) \leq cu(x_0, t_0) \text{ whenever } (x, t) \in C_{\gamma r, h_2 r}^-(x_0, t_0) \setminus C_{\gamma r, h_1 r}^-(x_0, t_0).$$

Proof. Follows from repeated application of Theorem 3 in [AS] and Lemma 2.1. ■

Lemma 2.5 (*Interior Hölder continuity estimate*) Let H be as in (1.1) and assume (1.2) with $p = 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$, and assume that u is a weak solution to (1.1) in Ω_T . Then, after a redefinition on a set of measure zero, u is continuous. Furthermore, there exist constants $c = c(H)$, $1 \leq c < \infty$, $\alpha = \alpha(H)$, $0 < \alpha < 1$, such that if $(x_0, t_0) \in \Omega_T$, $r > 0$, $C_{2r}(x_0, t_0) \subset \Omega_T$, then

$$|u(x, t) - u(y, s)| \leq c \left(\frac{d_p(x, t, y, s)}{r} \right)^\alpha \sup_{C_{2r}(x_0, t_0)} |u|$$

whenever $(x, t), (y, s) \in C_{2r}(x_0, t_0)$.

Proof. See for example Theorem III.1.1 in [DB2]. ■

Lemma 2.6 (*Strong maximum principle*) Let H be as in (1.1) and assume (1.2) with $p = 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$, assume that u is a weak subsolution to (1.1) in Ω_T , bounded from above, and that v is a weak supersolution to (1.1) in Ω_T , bounded from below. Then, if $u \leq v$ on $\partial_p \Omega_T$, it follows that $u \leq v$ almost everywhere on Ω_T .

Proof. According to [Z], every lower (upper) bounded sub- (super-) solution to (1.1) has a upper (lower) semicontinuous representative. The lemma can now be proved along the same lines as Lemma 3.1 in [KL]. ■

2.2 Existence of Riesz measure for subsolutions

Lemma 2.7 *Let H be as in (1.1) and assume (1.2) with $p = 2$. Let u be a non-negative weak subsolution to (1.1) in $C_{r_1, r_2}(x_0, t_0)$. Then there exists a unique locally finite positive Borel measure μ on $C_{r_1, r_2}(x_0, t_0)$ such that*

$$-\iint_{t'_1 G'}^{t'_2} A(x, t, \nabla u) \cdot \nabla \theta \, dx dt + \iint_{t'_1 G'}^{t'_2} u \partial_t \theta \, dx dt - \int_{G'} u \theta \, dx \Big|_{t'_1}^{t'_2} = \iint \theta \, d\mu \quad (2.4)$$

whenever $t_0 - r_2^2 \leq t'_1 < t'_2 \leq t_0 + r_2^2$, $\theta \in C_0^\infty(C_{r_1, r_2}(x_0, t_0))$.

Proof. With u as in the statement of the theorem we let

$$\begin{aligned} T_u \theta &:= \theta \mapsto \iint u \theta_t \, dx dt - \iint A(x, t, \nabla u) \cdot \nabla \theta \, dx dt \\ &\quad - \int u(x, t'_2) \theta(x, t'_2) \, dx + \int u(x, t'_1) \theta(x, t'_1) \, dx \end{aligned}$$

whenever $\theta \in C_0^\infty(C_{r_1, r_2}(x_0, t_0))$. In particular, T_u is a linear functional on $C_0^\infty(C_{r_1, r_2}(x_0, t_0))$. Since every non-negative distribution can be represented by a Radon measure, (see Theorem 2.1.7 in [H]), we want to show that T_u is a non-negative distribution. However, since u is a subsolution we see that T_u is non-negative and it only remains to verify that T_u is a distribution. To prove that T_u is a distribution it suffices to prove that there exists, given a compact $K \subset C_{r_1, r_2}(x_0, t_0)$, a constant c_K such that

$$|T_u \theta| \leq c_K \left(\sum \sup_{C_{r_1, r_2}(x_0, t_0)} \theta_{x_i} + \sup_{C_{r_1, r_2}(x_0, t_0)} \theta_t \right),$$

whenever $\theta \in C_0^\infty(C_{r_1, r_2}(x_0, t_0))$ with support in K . However, this follows trivially from the definition of $T_u \theta$ and the structure conditions (1.2). ■

2.3 Gaussian type decay estimates

Lemma 2.8 *Let H be as in (1.1) and assume (1.2) with $p = 2$. Let $(x_0, t_0) \in \mathbb{R}^{n+1}$, $r > 0$. Then there exist constants $c = c(H)$, $1 \leq c < \infty$, and $\alpha = \alpha(H)$, $0 < \alpha < 1$, such that the following is true. Assume that u solves $Hu = 0$ in $C_{2r}^+(x_0, t_0)$, that u is continuous on the closure of $C_{2r}^+(x_0, t_0)$ and that $u = 0$ on $\partial_p C_{2r}^+(x_0, t_0) \cap \{(x, t) : t = t_0\}$. Then*

$$|u(x, t) - u(y, s)| \leq c \left(\frac{d_p(x, t, y, s)}{r} \right)^\alpha \sup_{C_{2r}^+(x_0, t_0)} |u|$$

whenever $(x, t), (y, s) \in C_r^+(x_0, t_0)$.

Proof. Using Lemma 2.1, Lemma 2.2, and making the transformation

$$\tilde{u}(x, t) := \frac{u(rx - x_0, r^2 t - t_0)}{\|u\|_\infty},$$

we see that it is enough to consider the case when $\tilde{u}$ solves $\tilde{H}\tilde{u} = 0$ in $C_2^+(0, 0)$, $\tilde{u}$ is continuous on the closure of $C_2^+(0, 0)$, $\tilde{u} = 0$ on $\partial_p C_2^+(0, 0) \cap \{(x, t) : t = 0\}$, and

$$\sup_{C_2^+(0,0)} |\tilde{u}| = 1. \quad (2.5)$$

However, now applying Theorem 2 in [DB] we see that there exist constants $c = c(H)$, $1 \leq c < \infty$, and $\alpha = \alpha(H)$, $0 < \alpha < 1$, such that

$$|\tilde{u}(x, t) - \tilde{u}(y, s)| \leq c(d_p(x, t, y, s))^\alpha \text{ whenever } (x, t), (y, s) \in C_1^+(0, 0).$$

Scaling back we get the conclusion stated in the lemma. ■

Lemma 2.9 (*Gaussian decay estimates*) *Let H be as in (1.1) and assume (1.2) with $p = 2$. There exists a $\hat{K} \gg 1$, $\hat{K} = \hat{K}(H)$ such that the following is true whenever $(x_0, t_0) \in \mathbb{R}^{n+1}$, $r > 0$, $K \geq \hat{K}$. Let u be a non-negative solution to $Hu = 0$ in $C_{Kr, 2r}^-(x_0, t_0)$ vanishing continuously on $\partial_p C_{Kr, 2r}^-(x_0, t_0) \cap \left(B(x_0, (K-2)r) \times \{t_0 - 4r^2\}\right)$. Then there exists a constant $c = c(H)$, $1 \leq c < \infty$, such that*

$$\sup_{C_r^-(x_0, t_0)} u \leq c \exp(-K/c) \sup_{C_{Kr, 2r}^-(x_0, t_0)} u.$$

Proof. In the following we let $\hat{K} \gg 1$ be a constant to be chosen, we consider $K \geq \hat{K}$, and we note that we can without loss of generality assume that $(x_0, t_0) = (0, 0)$ and $r = 1$. Let $\tilde{x}_0 \in \mathbb{R}^n$ and $R > 2$ be such that $B(\tilde{x}_0, 2R) \subset B(0, 2K)$. Furthermore, assume that v solves $Hv = 0$ in $C_{2R}^+(\tilde{x}_0, -4)$, that v is continuous on the closure of $C_{2R}^+(\tilde{x}_0, -4)$ and that $v = 0$ on $\partial_p C_{2R}^+(\tilde{x}_0, -4) \cap \{(x, t) : t = -4\}$. Then, using Lemma 2.8, we have that

$$|v(x, t) - v(y, s)| \leq c \left(\frac{d_p(x, t, y, s)}{R} \right)^\alpha \sup_{C_{2R}^+(\tilde{x}_0, -4)} |v| \quad (2.6)$$

whenever $(x, t), (y, s) \in C_R^+(\tilde{x}_0, -4)$. In particular, using (2.6) with $(y, s) \in B(\tilde{x}_0, 2) \times \{t = -4\}$, we have that

$$|v(x, t)| \leq cR^{-\alpha} \sup_{C_{2R}^+(\tilde{x}_0, -4)} |v| \quad (2.7)$$

whenever $(x, t) \in C_2^+(\tilde{x}_0, -4)$. Based on (2.7) we in the following we let R be such that $cR^{-\alpha} = 1/2$. Using this R we let $\hat{K} = 2(R+1) + R$. Hence, $\hat{K} = \hat{K}(H)$. Let now $x_1 \in \partial B(0, K - 2(R+1))$ and using the solution u we introduce an auxiliary function v as follows. We let v be a solution to $Hu = 0$ in $C_{2R}^+(x_1, -4)$, continuous on the closure of $C_{2R}^+(x_1, -4)$, $v = u$ on $\partial_p C_{2R, 2}^+(x_1, -4)$ and $v(x, t) = u(x, 0)$ whenever $(x, t) \in \partial_p (C_{2R}^+(x_1, -4)) \setminus \partial_p C_{2R, 2}^+(x_1, -4)$. Then, by the maximum principle (see Lemma 2.6), we have that

$$\sup_{C_{2R}^+(x_1, -4)} |v| \leq \sup_{C_{2R, 2}^+(x_1, -4)} u \leq \sup_{C_{K, 2}^-(0, 0)} u. \quad (2.8)$$

Furthermore, using (2.7), (2.8), and the definition of θ , we see that

$$|v(x, t)| \leq \theta \sup_{C_{K,2}^-(0,0)} u \quad (2.9)$$

whenever $(x, t) \in C_2^+(x_1, -4)$. Note that $v \equiv u$ on $C_{2R,2}^+(x_1, -4)$ and hence

$$|u(x, t)| \leq \theta \sup_{C_{K,2}^-(0,0)} u \quad (2.10)$$

whenever $(x, t) \in C_2^+(x_1, -4)$. Since x_1 is arbitrary in this argument, it then follows from the maximum principle that (2.9) holds whenever $(x, t) \in C_{K-2(R+1),2}^-(0,0)$. Repeating this argument we can conclude, by induction, that

$$|u(x, t)| \leq \theta^j \sup_{C_{K,2}^-(0,0)} u \text{ whenever } (x, t) \in C_{K-2j(R+1),2}^-(0,0), \quad (2.11)$$

as long as $K - 2j(R + 1) > 0$. Let j_0 be the largest j such that $K - 2j(R + 1) \geq R$ and note that by construction of K we know that $j_0 \geq 1$. Then $j_0 \approx ((K - R)/(R + 1))$ and we have, in particular, that

$$|u(x, t)| \leq ce^{-K/c} \sup_{C_{K,2}^-(0,0)} u \quad (2.12)$$

whenever $(x, t) \in C_1^-(0,0)$. Hence the proof of the lemma is complete. ■

2.4 Further results in NTA-cylinders

Given a rectifiable curve $\gamma \subset \mathbb{R}^n$ we let $H^1(\gamma)$ denote the length, 1-dimensional Hausdorff measure, of γ . Throughout the paper Ω will be a NTA-domain (non-tangentially accessible domain), with parameters M, r_0 , in the following sense, see [JK] and [BL].

Definition 2.10 *A bounded domain Ω is called non-tangentially accessible (NTA) if there exist $M \geq 2$ and $r_0 > 0$ such that the following are fulfilled:*

1. *corkscrew condition: for any $w \in \partial\Omega, 0 < r < r_0$, there exists $A_r(w) \in \Omega$ satisfying*

$$M^{-1}r < |A_r(w) - w| < r, d(A_r(w), \partial\Omega) > M^{-1}r,$$

2. *$\mathbb{R}^n \setminus \Omega$ satisfies the corkscrew condition,*

3. *uniform condition: if $w_1, w_2 \in \Omega$, then there exists a rectifiable curve $\gamma : [0, 1] \rightarrow \Omega$ with $\gamma(0) = w_1, \gamma(1) = w_2$, and such that*

$$(a) \quad H^1(\gamma) \leq M |w_1 - w_2|,$$

$$(b) \quad \min\{H^1(\gamma([0, t])), H^1(\gamma([t, 1]))\} \leq M d(\gamma(t), \partial\Omega).$$

Recall from the introduction that we have introduced $A_r(x_0, t_0) = (A_r(x_0), t_0)$ whenever $(x_0, t_0) \in S_T$ and $0 < r < r_0$. In the following we will also use the notation

$$A_r^+(x_0, t_0) := (A_r(x_0), t_0 + 2r^2), \quad A_r^-(x_0, t_0) := (A_r(x_0), t_0 - 2r^2), \quad (2.13)$$

whenever $(x_0, t_0) \in S_T$ and $0 < r < r_0$, where r_0 is the constant in Definition 2.10.

Lemma 2.11 (*Solvability of the classical Dirichlet problem in NTA-cylinders*) *Let H be as in (1.1) and assume (1.2). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. Let $f \in C(\partial_p \Omega_T)$. Then there exists a unique solution $u \in C(\bar{\Omega}_T)$ to (1.1) such that $u = f$ on $\partial_p \Omega_T$.*

Proof. The existence and uniqueness of weak solutions follows from standard arguments based on the monotonicity and boundedness of A , see (1.2), see for example Proposition III.4.1 in [Sh]. The continuity of solutions up to the boundary is proved, for example, in Theorem III.1.2 in [DB2], assuming that $\mathbb{R}^n \setminus \Omega$ satisfies a uniform positive density condition. In the case of NTA-domains, the latter condition is a consequence of condition (ii) in Definition 2.10. ■

Lemma 2.12 *Let H be as in (1.1) and assume (1.2). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M , and $T > 0$. Let u be a non-negative solution to the equation $Hu = 0$ in Ω_T . Furthermore, let $(y_1, s_1), (y_2, s_2) \in \Omega_T$, suppose that $s_2 > s_1$, $(s_2 - s_1)^{1/2} \geq \eta^{-1}d(y_1, y_2)$ for some $\eta > 1$, that $d(y_1, \partial\Omega) > \epsilon$, $d(y_2, \partial\Omega) > \epsilon$, $T > s_2$, $s_1 > \epsilon^2$ and that $d_p(y_1, s_1, y_2, s_2) < c\epsilon$ for some $\epsilon > 0$. Then there exists a constant $\hat{c} = \hat{c}(H, \eta, c)$, $1 \leq \hat{c} < \infty$, such that*

$$u(y_1, s_1) \leq \hat{c}u(y_2, s_2).$$

Proof. For a complete proof, in an analogous context, see for instance [FGGMN]. ■

3 More technical lemmas

The purpose of this section is to establish a number of basic technical estimates that will be used in the proof of Theorems 1.1-1.2. Several of these estimates can be proved, using the notion of NTA-domain and Lemma 2.12, along the lines of the corresponding proofs previously established in the literature in the context linear uniformly parabolic equation. As a consequence, wherever appropriate, we will either omit details or be quite brief. In general, $c(a_1, \dots, a_m)$ denotes a positive constant ≥ 1 , which may depend only on H, M and $a_1, \dots, a_m$, and which is not necessarily the same at each occurrence. When we write $A \approx B$ we mean that A/B is bounded from above and below by constants which, unless otherwise stated, only depend on H, M . Throughout the section we assume that

$$H \text{ is as in (1.1) and (1.2) holds with } p = 2. \quad (3.1)$$

The following lemmas are typical applications of the NTA geometry and the Harnack chain lemma (Lemma 2.12). For proofs of Lemma 3.1 and Lemma 3.2 we refer to [N].

Lemma 3.1 Assume (3.1). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. Let $(x_0, t_0) \in S_T$ and

$$r < \min\{r_0/2, \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\}.$$

Let u be a non-negative solution to $Hu = 0$ in $\Omega_T \cap C_{2r}(x_0, t_0)$. Then there exist $c = c(H, M)$, $1 \leq c < \infty$, and $\gamma = \gamma(H, M) > 0$, such that

$$u(x, t) \leq c \left(\frac{r}{d_p(x, t, S_T)} \right)^\gamma u(A_{r/c}^+(x_0, t_0)), \quad (3.2)$$

whenever $(x, t) \in \Omega_T \cap C_{r/c}(x_0, t_0)$.

Lemma 3.2 Assume (3.1). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. Let $(x_0, t_0) \in S_T$ and

$$r < \min\{r_0/2, \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\}.$$

Let u be a non-negative solution to $Hu = 0$ in $\Omega_T \cap C_{2r}(x_0, t_0)$. Then there exist $c = c(H, M)$, $1 \leq c < \infty$, and $\gamma = \gamma(H, M) > 0$, such that

$$u(A_{r/c}^-(x_0, t_0)) \leq c \left(\frac{r}{d_p(x, t, \partial_p \Omega_T)} \right)^\gamma u(x, t),$$

whenever $(x, t) \in \Omega_T \cap C_{r/c}(x_0, t_0)$.

Lemma 3.3 Assume (3.1). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. There exists a $\hat{K} \gg 1$, $\hat{K} = \hat{K}(H, M)$, such that the following is true whenever $(x_0, t_0) \in S_T$ and

$$r < \min\{r_0/(2\hat{K}), \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\}.$$

Let u be a non-negative solution to $Hu = 0$ in $\Omega_T \cap C_{\hat{K}r, 2r}^-(x_0, t_0)$ vanishing continuously on $S_T \cap C_{\hat{K}r, 2r}^-(x_0, t_0)$. Then there exists a constant $\theta = \theta(H, M)$, $0 < \theta < 1$, such that

$$\sup_{\Omega_T \cap C_r^-(x_0, t_0)} u \leq \theta \sup_{\Omega_T \cap C_{\hat{K}r, 2r}^-(x_0, t_0)} u.$$

Proof. This is an obvious consequence of the NTA character of Ω and the results in, for example, chapter III in [DB]. We omit further details. ■

Lemma 3.4 Assume (3.1). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 and $T > 0$. Let $(x_0, t_0) \in S_T$ and let $r < \min\{r_0/2, \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\}$. Let u be a non-negative solution to $Hu = 0$ in $\Omega_T \cap C_{2r}(x_0, t_0)$ which vanishes continuously on $\Delta(x_0, t_0, 2r)$. Then there exist constants $c = c(H, M)$, $1 \leq c < \infty$, and $\alpha = \alpha(H, M) \in (0, 1)$, such that

$$u(x, t) \leq c \left(\frac{d_p(x, t, x_0, t_0)}{r} \right)^\alpha \sup_{\Omega_T \cap C_{2r}(x_0, t_0)} u \quad (3.3)$$

whenever $(x, t) \in \Omega_T \cap C_{r/c}(x_0, t_0)$.

Proof. This follows from repeated use of Lemma 3.3. We omit the details. ■

Lemma 3.5 Assume (3.1). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. Let $(x_0, t_0) \in S_T$ and let $r < \min\{r_0/2, \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\}$. Let u be a non-negative solution to $Hu = 0$ in $\Omega_T \cap C_{2r}(x_0, t_0)$ vanishing continuously on $\Delta(x_0, t_0, 2r)$. Then there exists a constant $c = c(H, M)$, $1 \leq c < \infty$, such that

$$u(x, t) \leq cu(A_r^+(x_0, t_0))$$

whenever $(x, t) \in \Omega_T \cap C_{r/c}(x_0, t_0)$.

Proof. Using Lemma 2.12, Lemma 3.1, Lemma 3.2 and Lemma 3.4 one can prove Lemma 3.5 by standard arguments along the lines of [Sa]. ■

Remark 3.5 Note that if u is a non-negative solution to $Hu = 0$ in all of Ω_T then Lemma 3.5 can be improved in the following way. Let $(x_0, t_0) \in S_T$ and r be as in the statement of Lemma 3.5. Let u be a non-negative solution to $Hu = 0$ in Ω_T vanishing continuously on $\Delta(x_0, t_0, 2r)$. Then there exists a constant $c = c(H, M)$, $1 \leq c < \infty$, such that

$$u(x, t) \leq cu(A_r^+(x_0, t_0))$$

whenever $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$. In fact, the restriction $(x, t) \in \Omega_T \cap C_{r/c}(x_0, t_0)$ in Lemma 3.5 is simply a result of the fact that we in Lemma 3.5 are only assuming that u is a non-negative solution in $\Omega_T \cap C_{2r}(x_0, t_0)$.

Lemma 3.6 Assume (3.1). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 and $T > 0$. Let u be a non-negative solution to $Hu = 0$ in Ω_T which vanishes continuously on S_T . Let $0 < \delta \ll \min\{\sqrt{T}, r_0\}$ be given. Then there exists a constant $c = c(H, M, \text{diam}(\Omega), T, \delta)$, $1 \leq c < \infty$, such that

$$\sup_{(x,t) \in \Omega^\delta \times (\delta^2, T)} u(x, t) \leq c \inf_{(x,t) \in \Omega^\delta \times (\delta^2, T)} u(x, t),$$

where $\Omega^\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$.

Proof. This follows from Lemma 3.5, Lemma 2.12 and Lemma 2.6. See Theorem 1.3 in [FGS] or Lemma 2.7 in [N] for details. ■

Lemma 3.7 Assume (3.1). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 and $T > 0$. There is a $\hat{K} = \hat{K}(H, M) \gg 1$ such that the following is true. Let $K > \hat{K}$ be given and consider $(x_0, t_0) \in S_T$. Furthermore, assume that $r < \min\{r_0/(2K), \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\}$. Let $\gamma = \gamma(H, M) \in (0, 1)$ be as in Lemma 3.1 and Lemma 3.2. Assume that u is a non-negative solution to $Hu = 0$ in $\Omega_T \cap C_{Kr, 2r}^-(x_0, t_0)$ vanishing continuously on $S_T \cap C_{Kr, 2r}^-(x_0, t_0)$ and that

$$\sup_{\Omega_T \cap C_r^-(x_0, t_0)} u \geq (2K)^{-\gamma} \sup_{\Omega_T \cap C_{Kr, 2r}^-(x_0, t_0)} u.$$

Then

$$\sup_{\Omega_T \cap (B(x_0, Kr) \times \{t_0 - 4r^2\})} u \geq \frac{1}{2} \sup_{\Omega_T \cap C_r^-(x_0, t_0)} u. \quad (3.4)$$

Proof. We may without loss of generality assume that $r = 1$. Let K be a degree of freedom to be fixed at a later point of the proof. Obviously (3.4) is true if

$$\sup_{\Omega_T \cap C_{K,2}^-(x_0, t_0)} u = \sup_{\Omega_T \cap (B(x_0, K) \times \{t_0 - 4\})} u. \quad (3.5)$$

Hence we in the following assume that (3.5) does not hold. Based on this assumption we see, by continuity and the maximum principle, that the supremum over $\Omega_T \cap C_{K,2}^-(x_0, t_0)$ is attained at some point on the lateral boundary of $\Omega_T \cap C_{K,2}^-(x_0, t_0)$. Let $\phi \in C_0^\infty(B(x_0, K))$ be a function such that $0 \leq \phi \leq 1$ and $\phi = 1$ on $B(x_0, K - 2)$. Let h be the solution to the Dirichlet problem in $C_{K,2}^-(x_0, t_0)$ with the boundary data

$$h(x, t) = \begin{cases} 0, & (x, t) \in \partial_p C_{K,2}^-(x_0, t_0) \setminus \Omega_T; \\ (1 - \phi(x))u(x, t), & (x, t) \in B(x_0, K) \times \{t = t_0 - 4\} \cap \Omega_T; \\ u(x, t), & (x, t) \in \partial B(x_0, K) \times \{t : t_0 - 4 < t < t_0\} \cap \Omega_T. \end{cases}$$

We now assume, in order to reach a contradiction, that

$$\sup_{\Omega_T \cap C_K^-(x_0, t_0) \cap \{(x, t) : t = t_0 - 4\}} u \leq \frac{1}{2} \sup_{C_1^-(x_0, t_0)} u. \quad (3.6)$$

Let $\tilde{u}$ be defined by

$$\tilde{u}(x, t) = \begin{cases} u(x, t), & \text{if } (x, t) \in C_{K,2}^-(x_0, t_0) \cap \Omega_T, \\ 0, & \text{if } (x, t) \in C_{K,2}^-(x_0, t_0) \setminus \Omega_T. \end{cases}$$

Then $\tilde{u}$ is easily seen to be subparabolic in the sense of [KKP]. Furthermore, since $\tilde{u}$ is continuous, it also follows from Theorem 5.8 in [KKP] that $\tilde{u}$ is a subsolution to (1.1). Hence, using the maximum principle, it follows that

$$\tilde{u}(x, t) - \frac{1}{2} \sup_{C_1^-(x_0, t_0)} u \leq h(x, t), \text{ whenever } (x, t) \in C_{K,2}^-(x_0, t_0).$$

Now using the Gaussian decay estimate of Lemma 2.9 and the assumptions from the statement of the lemma, we deduce that

$$\begin{aligned} \frac{1}{2} \sup_{C_1^-(x_0, t_0) \cap \Omega_T} u &\leq \sup_{C_1^-(x_0, t_0)} h \leq ce^{-cK} \sup_{C_{K,2}^-(x_0, t_0)} h \\ &= ce^{-cK} \sup_{C_{K,2}^-(x_0, t_0) \cap \Omega_T} u \leq ce^{-cK} (2K)^\gamma \sup_{C_1^-(x_0, t_0) \cap \Omega_T} u. \end{aligned} \quad (3.7)$$

We now choose $K = K(H, M)$ large enough to ensure that

$$ce^{-cK} (2K)^\gamma < \frac{1}{2}. \quad (3.8)$$

The estimates of equations (3.7) and (3.8) lead to a contradiction and thus the lemma is proved. $\blacksquare$

4 Proof of Theorem 1.1 and Theorem 1.2

The purpose of this section is to prove Theorem 1.1 and Theorem 1.2.

4.1 Proof of Theorem 1.1

Let ρ be the largest number $r \leq \rho \leq \delta/2$ satisfying the inequality

$$\sup_{\Omega_T \cap C_r^-(x_0, t_0)} u(x, t) \leq (r/\rho)^\gamma \sup_{\Omega_T \cap C_\rho^-(x_0, t_0)} u(x, t), \quad (4.1)$$

where γ is the constant appearing in Lemma 3.1. Since $r < \delta/2$, it follows that

$$r < \min\{r_0/2, \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\},$$

and hence we can use Lemma 3.2 and the definition of the point $A_r^-(x_0, t_0)$ to see that

$$u(A_\rho^-(x_0, t_0)) \leq c(\rho/r)^\gamma u(A_r^-(x_0, t_0)). \quad (4.2)$$

In the following we prove that

$$\sup_{\Omega_T \cap C_\rho^-(x_0, t_0)} u(x, t) \leq cu(A_\rho^-(x_0, t_0)) \quad (4.3)$$

for this particular choice of ρ . Combining (4.1), (4.2) and (4.3) we see that

$$\sup_{\Omega_T \cap C_r^-(x_0, t_0)} u(x, t) \leq cu(A_r^-(x_0, t_0)). \quad (4.4)$$

Since the same argument holds with t_0 replaced by $t_0 + r^2$, Theorem 1.1 follows from (4.3) and Lemma 2.12. To prove (4.3), we let $K > \hat{K} \gg 1$ where $\hat{K}$ is given as in Lemma 3.7. We then divide the proof into two cases. First, we assume that $\delta/(2K) < \rho$. In this case ρ is large and combining Lemma 3.5 and Lemma 3.6 with $\delta/(2K)$ playing the part of δ in the latter lemma, we see that

$$\sup_{\Omega_T \cap C_\rho^-(x_0, t_0)} u(x, t) \leq cu(A_\rho^+(x_0, t_0)) \leq cu(A_\rho^-(x_0, t_0)), \quad (4.5)$$

for some $c = c(H, M, \text{diam}(\Omega), T, \delta)$, $1 \leq c < \infty$. Hence, the proof is complete in this case. Next, we assume that $r \leq \rho \leq \delta/(2K)$ and we then first note, by the definition of ρ , that

$$\sup_{\Omega_T \cap C_\rho^-(x_0, t_0)} u \geq (K)^{-\gamma} \sup_{\Omega_T \cap C_{K\rho}^-(x_0, t_0)} u. \quad (4.6)$$

Obviously (4.6) implies

$$\sup_{\Omega_T \cap C_\rho^-(x_0, t_0)} u \geq (2K)^{-\gamma} \sup_{\Omega_T \cap C_{K\rho, 2\rho}^-(x_0, t_0)} u,$$

and hence we can use Lemma 3.7 to conclude that

$$\sup_{\Omega_T \cap C_{K\rho}^-(x_0, t_0) \cap \{(x, t) : t = t_0 - 4\rho^2\}} u \geq \frac{1}{2} \sup_{\Omega_T \cap C_{K\rho, \rho}^-(x_0, t_0)} u. \quad (4.7)$$

Using Lemma 3.5 and the Harnack inequality, we can now use (4.7) to conclude (4.3). This completes the proof of the theorem. $\square$

4.2 Proof of Theorem 1.2

We here prove the following lemma.

Lemma 4.1 *Let H be as in (1.1) and assume (1.2). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. Let u be a non-negative solution to $H u = 0$ in Ω_T , vanishing continuously on S_T . Then there exists a unique locally finite positive Borel measure μ on $\mathbb{R}^n \times [0, T]$, with support in S_T , such that*

$$\begin{aligned} & - \int_{t'_1}^{t'_2} \int_{G'} A(x, t, \nabla u) \cdot \nabla \theta dx dt + \int_{t'_1}^{t'_2} \int_{G'} u \partial_t \theta dx dt \\ & - \int_{G'} u(x, t'_2) \theta(x, t'_2) dx + \int_{G'} u(x, t'_1) \theta(x, t'_1) dx = \iint \theta d\mu \end{aligned} \quad (4.8)$$

whenever $0 \leq t'_1 < t'_2 \leq T$ and $\theta \in C_0^\infty(\mathbb{R}^n \times (0, T))$. Furthermore, let $0 < \delta \ll \min\{r_0/2, \sqrt{T}\}$ be a fixed constant. Suppose $(x_0, t_0) \in S_T$ such that $\delta^2 < t_0 < T - \delta^2$ and assume that $r < \delta/2$. Then there exists a $c = c(H, M, \text{diam}(\Omega), T, \delta)$, $1 \leq c < \infty$, such that

$$c^{-1} u(A_r(x_0, t_0)) \leq r^{-n} \mu(\Delta(x_0, t_0, 2r)) \leq c u(A_r(x_0, t_0)). \quad (4.9)$$

Proof. Extend u to $\mathbb{R}^n \times [0, T]$ by putting $u \equiv 0$ on $(\mathbb{R}^n \times [0, T]) \setminus \Omega_T$. Arguing as in the proof of Lemma 3.7, one sees that this function is indeed a subsolution. The existence of the measure μ in (4.8) then follows from Lemma 2.7. Hence it only remains to prove (4.9).

To establish the right hand side inequality in (4.9), let $\psi \in C_0^\infty(B(x_0, 2r))$ be a standard cut-off function such that $0 \leq \psi \leq 1$, ψ is identically 1 on $B(x_0, r)$, and $|\nabla \psi| \leq c/r$. For brevity of notation, we let $t_0^+ := t_0 + 4r^2$, and $t_0^- := t_0 - 4r^2$. Now using that

$$\mu(\Delta(x_0, t_0, r)) \leq \iint_{C_{2r}(x_0, t_0)} \psi(x) d\mu,$$

and the definition of μ , we see that

$$\begin{aligned} \mu(\Delta(x_0, t_0, r)) & \leq \int_{B(x_0, 2r)} \left(u(x, t_0^-) \psi(x, t_0^-) - u(x, t_0^+) \phi(x, t_0^+) \right) dx - \iint_{C_{2r}(x_0, t_0)} A(x, t, \nabla u) \cdot \nabla \psi dx dt \\ & \leq cr^n \sup_{C_{2r}(x_0, t_0) \cap \Omega_T} u + \left| \iint_{C_{2r}(x_0, t_0)} A(x, t, \nabla u) \cdot \nabla \psi dx dt \right|. \end{aligned}$$

Next, using the definition of ψ , the structure conditions on A , the Hölder inequality and the energy estimates (Lemma 2.3), we can conclude that

$$\begin{aligned}
\left| \iint_{C_{2r}(x_0, t_0)} A(x, t, \nabla u) \cdot \nabla \psi \, dx dt \right| &\leq c \iint_{C_{2r}(x_0, t_0)} |\nabla u| |\nabla \psi| \, dx dt \\
&\leq c \left(\iint_{C_{2r}(x_0, t_0)} |\nabla u|^2 \, dx dt \right)^{1/2} \left(\iint_{C_{2r}(x_0, t_0)} |\nabla \psi|^2 \, dx dt \right)^{1/2} \\
&\leq cr^n \sup_{C_{2r}(x_0, t_0) \cap \Omega_T} u.
\end{aligned}$$

Here $c = c(n, \beta)$, where β is the constant appearing in (1.2). Putting these estimates together and applying Theorem 1.1 and Lemma 2.12, we can conclude that

$$\begin{aligned}
\mu(\Delta(x_0, t_0, r)) &\leq cr^n \sup_{C_{2r}(x_0, t_0) \cap \Omega_T} u \\
&\leq cu(A_{8r}^-(x_0, t_0)) \leq cu(A_r(x_0, t_0)).
\end{aligned} \tag{4.10}$$

Hence the proof of the right hand side inequality in (4.9) is complete.

To establish the left hand side inequality in (4.9), let h solve the continuous Dirichlet problem on $C_{2r}(x_0, t_0)$ with the boundary values of u . Then it follows from the Harnack inequality and the maximum principle that there exists $\tilde{c} = \tilde{c}(H, c) = \tilde{c}(H, M, r_0)$ such that

$$h(x, t) \geq \tilde{c}^{-1} h(A_r^-(x_0, t_0)) \geq \tilde{c}^{-1} u(A_r^-(x_0, t_0)),$$

whenever $(x, t) \in C_r(x_0, t_0)$. Next, given r and $0 < \theta < 1$ we see, using Lemma 3.4, that there exists $0 < \rho < 1$, $\rho = \rho(H, \theta, r_0, M)$, such that

$$\sup_{C_{\rho r}(x_0, t_0)} u \leq \theta \sup_{C_r(x_0, t_0)} u.$$

Hence, given $(x, t) \in C_{\rho r}(x_0, t_0)$, we have that

$$\begin{aligned}
h(x, t) - u(x, t) &\geq \tilde{c}^{-1} u(A_r^-(x_0, t_0)) - \sup_{C_{\rho r}(x_0, t_0)} u \\
&\geq \tilde{c}^{-1} u(A_r^-(x_0, t_0)) - \theta \sup_{C_r(x_0, t_0)} u.
\end{aligned}$$

However, using Theorem 1.1 we can conclude that

$$u(A_r^-(x_0, t_0)) \geq c^{-1} u(A_r(x_0, t_0)), \tag{4.11}$$

and using the same theorem together with Lemma 2.12, we conclude that

$$\sup_{C_r(x_0, t_0)} u \leq cu(A_{2r}^-(x_0, t_0)) \leq cu(A_r(x_0, t_0)). \tag{4.12}$$

Hence, after choosing θ small enough (and thus indirectly choosing ρ small), we can conclude that

$$h(x, t) - u(x, t) \geq \hat{c}^{-1}u(A_r(x_0, t_0)) \quad \text{whenever } (x, t) \in C_{\rho r}(x_0, t_0).$$

Let $\phi := h - u$ on $C_{2r}(x_0, t_0)$ and note that ϕ is non-negative on $C_{2r}(x_0, t_0)$ and vanishes on the parabolic boundary of $C_{2r}(x_0, t_0)$. Furthermore, ϕ is continuous and satisfies $\phi \geq \hat{c}^{-1}u(A_r(x_0, t_0))$ on $C_{\rho r}(x_0, t_0)$. Let $\kappa = n/(n-2)$ for $n > 2$ and let $\kappa = 2$ if $n = 2$. Using the Hölder inequality and the Sobolov inequality it then follows that

$$\begin{aligned} \hat{c}^{-2}u(A_r(x_0, t_0))^2(\rho r)^{n+2} &\leq \int_{t_0^-}^{t_0^+} \left(\int_{B_{2r}(x_0, t_0)} \phi(x, t)^2 dx \right) dt \\ &\leq \int_{t_0^-}^{t_0^+} \left(r^n \left(r^{-n} \int_{B_{2r}(x_0, t_0)} \phi(x, t)^{2\kappa} dx \right)^{1/\kappa} \right) dt \\ &\leq cr^2 \iint_{C_{2r}(x_0, t_0)} |\nabla \phi|^2 dx dt, \end{aligned} \tag{4.13}$$

where the constant c only depends on the dimension n . Furthermore, using (1.2) we see that

$$\iint_{C_{2r}(x_0, t_0)} |\nabla \phi|^2 dx dt \leq \beta \int_{C_{2r}(x_0, t_0)} (A(x, t, \nabla h) - A(x, t, \nabla u)) \cdot \nabla \phi dx dt. \tag{4.14}$$

In the following we let, given a function $f = f(x, t)$,

$$f_\epsilon(x, t) = \int_{\mathbb{R}} f(x, t - s) \psi_\epsilon(s) ds, \tag{4.15}$$

where ψ_ϵ is a standard mollifier having support contained in $(-\epsilon, \epsilon)$. Then

$$\begin{aligned} \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{C_{2r}(x_0, t_0)} (A(x, t, \nabla h) - A(x, t, \nabla u)) \cdot \nabla (\phi_\epsilon * \psi_\epsilon) dx dt \\ = \int_{C_{2r}(x_0, t_0)} (A(x, t, \nabla h) - A(x, t, \nabla u)) \cdot \nabla \phi dx dt. \end{aligned}$$

We also see that

$$\begin{aligned} \iint_{C_{2r}(x_0, t_0)} A(x, t, \nabla h) \cdot \nabla (\phi_\epsilon * \psi_\epsilon) dx dt &= \iint_{C_{2r}(x_0, t_0)} h \partial_t ((\phi_\epsilon * \psi_\epsilon)) dx dt \\ &\quad - \int_{B_{2r}(x_0, t_0)} h(x, t_0 + 4r^2) (\phi_\epsilon * \psi_\epsilon)(x, t_0 + 4r^2) dx \\ &\quad + \int_{B_{2r}(x_0, t_0)} h(x, t_0 - 4r^2) (\phi_\epsilon * \psi_\epsilon)(x, t_0 - 4r^2) dx, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
\iint_{C_{2r}(x_0, t_0)} A(x, t, \nabla u) \cdot \nabla(\phi_\epsilon * \psi_\epsilon) dx dt &= \iint_{C_{2r}(x_0, t_0)} u \partial_t(\phi_\epsilon * \psi_\epsilon) dx dt \\
&- \int_{B_{2r}(x_0, t_0)} u(x, t_0 + 4r^2)(\phi_\epsilon * \psi_\epsilon)(x, t_0 + 4r^2) dx \\
&+ \int_{B_{2r}(x_0, t_0)} u(x, t_0 - 4r^2)(\phi_\epsilon * \psi_\epsilon)(x, t_0 - 4r^2) dx \\
&- \iint_{C_{2r}(x_0, t_0)} (\phi_\epsilon * \psi_\epsilon) d\mu.
\end{aligned} \tag{4.17}$$

Hence,

$$\int_{C_{2r}(x_0, t_0)} (A(x, t, \nabla h) - A(x, t, \nabla u)) \cdot \nabla(\phi_\epsilon * \psi_\epsilon) dx dt = I_1^\epsilon + I_2^\epsilon + I_3^\epsilon + I_4^\epsilon, \tag{4.18}$$

where

$$\begin{aligned}
I_1^\epsilon &= \iint_{C_{2r}(x_0, t_0)} (h - u) \partial_t(\phi_\epsilon * \psi_\epsilon) dx dt \\
I_2^\epsilon &= - \int_{B_{2r}(x_0, t_0)} (h - u)(x, t_0 + 4r^2)(\phi_\epsilon * \psi_\epsilon)(x, t_0 + 4r^2) dx \\
I_3^\epsilon &= \int_{B_{2r}(x_0, t_0)} (h - u)(x, t_0 - 4r^2)(\phi_\epsilon * \psi_\epsilon)(x, t_0 - 4r^2) dx \\
I_4^\epsilon &= \iint_{C_{2r}(x_0, t_0)} (\phi_\epsilon * \psi_\epsilon) d\mu.
\end{aligned} \tag{4.19}$$

By construction

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} I_2^\epsilon &= - \int_{B_{2r}(x_0, t_0)} (h - u)(x, t_0 + 4r^2)(h - u)(x, t_0 + 4r^2) dx, \\
\lim_{\epsilon \rightarrow 0} I_3^\epsilon &= 0, \\
\lim_{\epsilon \rightarrow 0} I_4^\epsilon &= \iint_{C_{2r}(x_0, t_0)} (h - u) d\mu.
\end{aligned} \tag{4.20}$$

Furthermore, using a change of variables and the Fubini theorem we have that

$$\begin{aligned}
I_1^\epsilon &= \frac{1}{2} \iint_{C_{2r}(x_0, t_0)} \partial_t (\phi_\epsilon)^2 dx dt \\
&= \frac{1}{2} \int_{B_{2r}(x_0, t_0)} \phi_\epsilon(x, t_0 + 4r^2) \phi_\epsilon(x, t_0 + 4r^2) dx \\
&\quad - \frac{1}{2} \int_{B_{2r}(x_0, t_0)} \phi_\epsilon(x, t_0 - 4r^2) \phi_\epsilon(x, t_0 - 4r^2) dx,
\end{aligned} \tag{4.21}$$

and hence, again by construction,

$$\lim_{\epsilon \rightarrow 0} I_1^\epsilon = \frac{1}{2} \int_{B_{2r}(x_0, t_0)} \phi(x, t_0 + 4r^2) \phi(x, t_0 + 4r^2) dx. \tag{4.22}$$

In particular, putting these deductions together we see that

$$\begin{aligned}
&\int_{C_{2r}(x_0, t_0)} (A(x, t, \nabla h) - A(x, t, \nabla u)) \cdot \nabla \phi dx dt \\
&= \lim_{\epsilon \rightarrow 0} \int_{C_{2r}(x_0, t_0)} (A(x, t, \nabla h) - A(x, t, \nabla u)) \cdot \nabla (\phi_\epsilon * \psi_\epsilon) dx dt \\
&= -\frac{1}{2} \int_{B_{2r}(x_0, t_0)} \phi(x, t_0 + 4r^2) \phi(x, t_0 + 4r^2) dx \\
&\quad + \iint_{C_{2r}(x_0, t_0)} (h - u) d\mu.
\end{aligned} \tag{4.23}$$

Starting from the trivial estimate

$$h - u \leq \sup h - \inf u = \sup h,$$

we can now use Lemma 3.5, Theorem 1.1, and the Harnack inequality, (4.14) and (4.23), to deduce that

$$\begin{aligned}
\iint_{C_{2r}(x_0, t_0)} |\nabla \phi|^2 dx dt &\leq c \iint_{C_{2r}(x_0, t_0)} (h - u) d\mu \\
&\leq cu(A_r(x_0, t_0)) \mu(C_{2r}(x_0, t_0)).
\end{aligned} \tag{4.24}$$

Furthermore, using this estimate and (4.13), we conclude that

$$\begin{aligned}
\hat{c}^{-2} \left(u(A_r(x_0, t_0)) \right)^2 \rho^{n+2} r^{n+2} &\leq cr^2 \iint_{C_{2\bar{r}}(x_0, t_0)} |\nabla \phi|^2 dx dt \\
&\leq cr^2 u(A_r(x_0, t_0)) \mu(C_{2r}(x_0, t_0)).
\end{aligned} \tag{4.25}$$

Hence,

$$c^{-1}u(A_r(x_0, t_0))r^n \leq \mu(C_{2r}(x_0, t_0)), \quad (4.26)$$

and the proof of the lemma is complete. ■

Proof of Theorem 1.2. Using Theorem 1.1 and the Harnack inequality, we see that Theorem 1.2 is an immediate consequence of Lemma 4.1. ■

5 Proof of Theorem 1.3 and Theorem 1.4

The purpose of this section is to prove Theorem 1.3 and Theorem 1.4. Let H be as in (1.1), assume (1.2) with $p = 2$, and assume (1.7). Assume (1.8). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$, and let u, v , be non-negative solutions of $Hu = 0 = Hv$ in Ω_T vanishing continuously on S_T . Then (1.8) implies, see [LSU], that

$$\nabla u \text{ and } \nabla v \text{ are locally Hölder continuous in } \Omega_T. \quad (5.1)$$

In particular, $|\nabla u|$ and $|\nabla v|$ are well-defined and finite at almost all points in Ω_T .

5.1 Refined estimates for uniformly parabolic equations

In this section we consider linear (non-symmetric) uniformly parabolic equations in divergence form. In particular, we consider equations $\hat{H}u = 0$ as defined in (1.13) and we assume that

$$\hat{A} = \hat{A}(x, t) = \{\hat{a}_{ij}(x, t)\} \text{ is measurable, bounded and that (1.10) holds.} \quad (5.2)$$

We emphasize that in (5.2) there is no assumption on symmetry of the matrix $\{\hat{a}_{ij}(x, t)\}$. We will need the following results.

Remark 5.0 *Theorem 1.1 applies to solutions to $\hat{H}u = 0$. In particular, Theorem 1.1 remains valid, as stated, with the operator H replaced by $\hat{H}$.*

Lemma 5.1 *Let $\hat{H}$ be as in (1.13) and assume (5.2). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. Then there exists $K = K(\hat{H}, M)$, $K \gg 1$, and $c = c(\hat{H}, M, r_0) \geq 2$, such that the following is true. Suppose that $(x_0, t_0) \in S_T$ and that*

$$r < \min\{r_0/(2cK), \sqrt{(T - t_0)/(4c^2K^2)}, \sqrt{t_0/(4c^2K^2)}\}.$$

Suppose that u and v are two non-negative solutions to $\hat{H}u = 0$ in $\Omega_T \cap C_{cKr}(x_0, t_0)$, and assume that $u = 0$ continuously on $\Delta(x_0, t_0, 2Kr)$. Then there exists a constant $\hat{c} = \hat{c}(\hat{H}, M, r_0)$ such that

$$\sup_{\Omega_T \cap C_r(x_0, t_0)} \frac{u}{v} \leq \hat{c} \frac{u(A_{Kr}^+(x_0, t_0))}{v(A_{Kr}^-(x_0, t_0))}.$$

Proof. In the following we let $\Omega_T^r = \{x \in \Omega : d(x, \partial\Omega) > r\} \times (0, T)$. We first note that if we choose K large enough then, since $(\Omega_T \setminus \Omega_T^r) \cap C_{Kr}(x_0, t_0) \subset \Omega_T \cap C_{Kr}(x_0, t_0)$, we can use Remark 3.5 to conclude that

$$u(x, t) \leq c_1 u(A_{Kr}^+(x_0, t_0)) \text{ whenever } (x, t) \in (\Omega_T \setminus \Omega_T^r) \cap C_{Kr}(x_0, t_0). \quad (5.3)$$

Furthermore, by the Harnack inequality we have that

$$v(x, t) \geq c_2^{-1} v(A_{Kr}^-(x_0, t_0)),$$

for every $(x, t) \in \partial_p((\Omega_T \setminus \Omega_T^r) \cap C_{Kr}(x_0, t_0)) \cap \partial_p \Omega_T^r$. Let

$$\begin{aligned} \hat{u}(x, t) &= c_1^{-1} \frac{u(x, t)}{u(A_{Kr}^+(x_0, t_0))} - c_2 \frac{v(x, t)}{v(A_{Kr}^-(x_0, t_0))}, \\ \hat{v}(x, t) &= c_2 \frac{v(x, t)}{v(A_{Kr}^-(x_0, t_0))}, \end{aligned}$$

whenever $(x, t) \in (\Omega_T \setminus \Omega_T^r) \cap C_{Kr}(x_0, t_0)$. We are now essentially in the setting of Lemma 3.12 in [FGGMN]. Note however that we have only, compared to [FGGMN], shifted the roles of u and v , here $\hat{u}$ and $\hat{v}$ take on the roles of u and v , and that we work with $C_r(x_0, t_0)$, $C_{Kr}(x_0, t_0)$, instead of $C_r^-(x_0, t_0)$, $C_{Kr}^-(x_0, t_0)$. Furthermore, the only important difference compared to [FGGMN] is that in [FGGMN] the authors assume symmetry of the coefficient matrix. We emphasize though that Lemma 3.12 in [FGGMN], which is modeled on the corresponding lemmas in [FSY] and [SY], is proved using quite weak principles. In particular, the proof boils down to the use of the linearity of the operator, the solvability of the Dirichlet problem, the Harnack inequality and Hölder decay of solutions, and since all this is true in the setting of Lemma 5.1, (see Lemma 2.11, Lemma 2.12, Lemma 3.2 and Remark 5.0), we can reuse the proof of Lemma 3.12 in [FGGMN] to conclude that the appropriate version of this lemma also holds in our situation. Hence we can conclude that $\hat{u}(x, t) \leq \hat{v}(x, t)$, for $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$, and therefore that

$$\frac{u(x, t)}{v(x, t)} = \frac{u(A_{Kr}^+(x_0, t_0))}{v(A_{Kr}^-(x_0, t_0))} \frac{c_2}{c_1} \left(\frac{\hat{u}(x, t)}{\hat{v}(x, t)} + 1 \right) \leq c_3 \frac{u(A_{Kr}^+(x_0, t_0))}{v(A_{Kr}^-(x_0, t_0))}$$

whenever $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$. This completes the proof of Lemma 5.1. ■

5.2 Proof of Theorem 1.3

We first prove the left-hand inequality of Theorem 1.3. We do this by proving the existence of Λ , $1 \leq \Lambda < \infty$, and $\hat{c} \geq 1$, such that if $r' = r/\hat{c}$ and if

$$e(x, t) = \Lambda \left(\frac{u(x, t) - v(x, t)}{u(A_r(x_0, t_0)) - v(A_r(x_0, t_0))} \right) - \frac{v(x, t)}{v(A_r(x_0, t_0))} \quad (5.4)$$

for $(x, t) \in \Omega_T \cap C_{4r}(x_0, t_0)$, then

$$e(x, t) \geq 0 \text{ whenever } (x, t) \in \Omega_T \cap C_{2r'}(x_0, t_0). \quad (5.5)$$

To do this, we initially allow $\Lambda, \hat{c} \geq 1$ to vary in (5.4). The constants Λ and $\hat{c}$ are then fixed near the end of the argument. Put

$$\begin{aligned}\hat{u}(x, t) &= \frac{\Lambda u(x, t)}{u(A_r(x_0, t_0)) - v(A_r(x_0, t_0))}, \\ \hat{v}(x, t) &= \frac{\Lambda v(x, t)}{u(A_r(x_0, t_0)) - v(A_r(x_0, t_0))} + \frac{v(x, t)}{v(A_r(x_0, t_0))}.\end{aligned}$$

Observe from (5.4) that $e = \hat{u} - \hat{v}$. Let $\hat{H}$ be defined as in (1.13) using $\hat{u}, \hat{v}$, and let e_1 and e_2 be the unique weak solutions to $\hat{H}e_i = 0, i = 1, 2$, in $\Omega_T \cap C_{4r}(x_0, t_0)$, with continuous boundary values

$$\begin{aligned}e_1(x, t) &= \frac{u(x, t) - v(x, t)}{u(A_r(x_0, t_0)) - v(A_r(x_0, t_0))}, \\ e_2(x, t) &= \frac{v(x, t)}{v(A_r(x_0, t_0))},\end{aligned}\tag{5.6}$$

whenever $(x, t) \in \partial_p(\Omega_T \cap C_{4r}(x_0, t_0))$. Now, using Lemma 5.1, we can conclude, for some $\hat{c} \geq 1$ and $r' = r/\hat{c}$, that

$$\hat{c}^{-1} \frac{e_1(A_r^-(x_0, t_0))}{e_2(A_r^+(x_0, t_0))} \leq \frac{e_1(x, t)}{e_2(x, t)} \leq \hat{c} \frac{e_1(A_r^+(x_0, t_0))}{e_2(A_r^-(x_0, t_0))}\tag{5.7}$$

whenever $(x, t) \in \Omega_T \cap C_{2r'}(x_0, t_0)$. We now put

$$\Lambda = \hat{c} \frac{e_2(A_r^+(x_0, t_0))}{e_1(A_r^-(x_0, t_0))},$$

and we observe from (5.7) that

$$\Lambda e_1(x, t) - e_2(x, t) \geq 0 \text{ whenever } (x, t) \in \Omega_T \cap C_{2r'}(x_0, t_0).\tag{5.8}$$

Let $\hat{e} = \Lambda e_1 - e_2$ and note from linearity of $\hat{H}$ that $\hat{e}$ and e both satisfy the same linear uniformly parabolic equation in $\Omega_T \cap C_{4r}(x_0, t_0)$ and also that these functions have the same continuous boundary values on $\partial_p(\Omega_T \cap C_{4r}(x_0, t_0))$. Hence, using the (weak) maximum principle for the operator $\hat{H}$ it follows that $e = \hat{e}$ and then by (5.8) that $e(x, t) \geq 0$ in $\Omega_T \cap C_{2r'}(x_0, t_0)$. To complete the proof of the left-hand inequality in Theorem 1.3 we prove that

$$\Lambda \leq c(H, M).\tag{5.9}$$

However, using the maximum principle on e_2 and v , and Theorem 1.1 and Lemma 2.12 applied to v , we see that there is a constant c such that

$$\begin{aligned}e_2(A_r^+(x_0, t_0)) &\leq \sup_{\partial_p(\Omega_T \cap C_{4r}(x_0, t_0))} e_2(x, t) \\ &\leq c \sup_{\Omega_T \cap C_{4r}(x_0, t_0)} \frac{v(x, t)}{v(A_r(x_0, t_0))} \leq \frac{v(A_{4r}(x_0, t_0))}{v(A_r(x_0, t_0))} \leq c,\end{aligned}$$

and hence we only have to prove a bound from below for $e_1(A_r^-(x_0, t_0))$. To proceed we let $\tilde{H}$ be defined as in (1.13) using u, v , instead of $\hat{u}, \hat{v}$. We note that $\tilde{H}(u - v) = 0$ in Ω_T , that $u - v$ is non-negative, and that $u - v$ vanishes continuously on S_T . Hence, applying Lemma 3.2 and Theorem 1.1, now in the context of $\tilde{H}$ and $u - v$, we can conclude that

$$e_1(x, t) = \frac{u(x, t) - v(x, t)}{u(A_r(x_0, t_0)) - v(A_r(x_0, t_0))} \geq \frac{\rho^\gamma}{c} \frac{u(A_r^-(x_0, t_0)) - v(A_r^-(x_0, t_0))}{u(A_r(x_0, t_0)) - v(A_r(x_0, t_0))} \geq \frac{1}{c}, \quad (5.10)$$

whenever $(x, t) \in \partial_p(\Omega_T \cap C_{4r}(x_0, t_0)) \cap \{(x, t) \in \Omega_T : d(x, \partial\Omega) \geq \rho r\}$, and $\rho \in (0, 1)$. Let

$$(\zeta, \tau) \in \partial_p(\Omega_T \cap C_{4r}(x_0, t_0)) \cap \{(x, t) \in \Omega_T : d(x, \partial\Omega) \geq \rho r\}$$

with $\tau = t_0 - (3r/2)^2$. Now from (5.10) and the Hölder continuity of e_1 we can conclude that

$$e_1(x, t) \geq \tilde{c}^{-1}, \quad (5.11)$$

whenever $(x, t) \in (\Omega_T \cap C_{4r}(x_0, t_0)) \cap C_{r/\tilde{c}}(\zeta, \tau)$. Using (5.11) and the Harnack inequality of Lemma 2.12 it then follows that $e_1(A_r^-(x_0, t_0)) \geq c^{-1}$ and hence the proof of (5.9) is complete. The proof of the right-hand inequality in Theorem 1.3 is similar and hence the proof of Theorem 1.3 is complete. $\square$

We here also note the following local and weaker version of Theorem 1.3, whose proof is implicit in the proof of Theorem 1.3.

Lemma 5.2 *Let H be as in (1.1), assume (1.2) with $p = 2$, and assume (1.7). Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 , and $T > 0$. Then there exists $K = K(H, M)$, $K \gg 1$, and $c = c(H, M, r_0) \geq 2$, such that the following is true. Let $(x_0, t_0) \in S_T$ and assume that*

$$r < \min\{r_0/(2cK), \sqrt{(T - t_0)/(4c^2K^2)}, \sqrt{t_0/(4c^2K^2)}\}.$$

Let u and v be two non-negative solutions to $Hu = 0$ in $\Omega_T \cap C_{cKr}(x_0, t_0)$, $v \leq u$ in $\Omega_T \cap C_{cKr}(x_0, t_0)$, and assume that $u = 0 = v$ continuously on $\Delta(x_0, t_0, 2Kr)$. Then there exists a constant $\hat{c} = \hat{c}(H, M, r_0)$ such that

$$\hat{c}^{-1} \frac{u(A_{Kr}^-(x_0, t_0)) - v(A_{Kr}^-(x_0, t_0))}{v(A_{Kr}^+(x_0, t_0))} \leq \frac{u(x, t) - v(x, t)}{v(x, t)} \leq \hat{c} \frac{u(A_{Kr}^+(x_0, t_0)) - v(A_{Kr}^+(x_0, t_0))}{v(A_{Kr}^-(x_0, t_0))},$$

whenever $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$.

5.3 Proof of Theorem 1.4

Let u, v be non-negative solutions of $Hu = 0$ in Ω_T vanishing continuously on S_T . Let $0 < \delta \ll \min\{r_0/2, \sqrt{T}\}$ be a fixed constant. Suppose $(x_0, t_0) \in S_T$ such that $\delta^2 < t_0 < T - \delta^2$ and assume that $r < \delta/2$. We then first note that Theorem 1.3 implies that there exists $c = c(H, M, r_0, \text{diam}(\Omega), T, \delta)$, $1 \leq c < \infty$, such that

$$c^{-1} \frac{u(A_r(x_0, t_0))}{v(A_r(x_0, t_0))} \leq \frac{u(x, t)}{v(x, t)} \leq c \frac{u(A_r(x_0, t_0))}{v(A_r(x_0, t_0))}, \quad (5.12)$$

whenever $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$ and $v \leq u$ on Ω_T . To remove the condition that $v \leq u$, we let $\hat{u}, \hat{v}$ be solutions of $Hu = 0$ in Ω_T with boundary data $\max\{u, v\}$ and $\min\{u, v\}$, respectively, on $\partial_p \Omega_T$. Then $\hat{v} \leq v$, $u \leq \hat{u}$ and $\hat{v} \leq \hat{u}$ on Ω_T and Theorem 1.3 applies to all these pairs. Writing

$$\frac{u(x, t)}{v(x, t)} = \frac{u(x, t)}{\hat{u}(x, t)} \frac{\hat{u}(x, t)}{\hat{v}(x, t)} \frac{\hat{v}(x, t)}{v(x, t)} \quad (5.13)$$

we see that (5.12) is a consequence of Theorem 1.3. Next, note that (5.12) implies that

$$\frac{u(x_1, t_1)}{v(x_1, t_1)} \leq c \frac{u(x_2, t_2)}{v(x_2, t_2)} \quad (5.14)$$

whenever $(x_1, t_1), (x_2, t_2) \in \Omega_T \cap C_r(x_0, t_0)$. We also note, simply using the Harnack inequality, the maximum principle, (5.14) and Lemma 3.6, that there exist constants $\Lambda_i = \Lambda_i(H, M, r_0, \text{diam}(\Omega), T, \delta)$, $0 \leq \Lambda_i < \infty$, $i \in \{1, 2\}$, such that

$$\Lambda_1 \leq \frac{u(x, t)}{v(x, t)} \leq \Lambda_2 \text{ whenever } (x, t) \in \Omega \times (\delta^2, T]. \quad (5.15)$$

To avoid trivialities in the following argument we from now on assume that u and v are defined in $\Omega \times (0, \infty)$. This is easily achieved by continuing u and v beyond $t = T$ in the natural way.

Given (x, t) in the closure of $\Omega \times (\delta^2, T]$ and $\rho > 0$ we define

$$\omega^{u,v}(x, t, \rho) = \sup_{\Omega \times (\delta^2, T] \cap C_\rho(x, t)} \frac{u}{v} - \inf_{\Omega \times (\delta^2, T] \cap C_\rho(x, t)} \frac{u}{v}. \quad (5.16)$$

Then, to start with, we note that (5.15) implies that

$$\omega^{u,v}(x_0, t_0, 2r) \leq \Lambda_2 - \Lambda_1 < \infty \text{ whenever } (x_0, t_0) \in \partial\Omega \times (\delta^2, T] \text{ and } r \text{ as above.} \quad (5.17)$$

In the following we let (x, t) be an arbitrary point in $(\Omega \times (\delta^2, T]) \cap C_r(x_0, t_0)$ and we let $d := d(x, \partial\Omega) = d_p(x, t, S_T)$. Given $0 < \rho \leq r$ we consider two cases: $\rho \leq d$ (interior case) and $\rho > d$ (boundary case).

We first consider the case $\rho \leq d$. We first assume, in addition, that $\rho \leq d/2$ (strictly interior case). Let

$$\begin{aligned} \hat{u}(y, s) &:= (\omega^{u,v}(x, t, \rho))^{-1} \left(u(y, s) - \left(\inf_{(\Omega \times (\delta^2, T]) \cap C_\rho(x, t)} u/v \right) v(y, s) \right), \\ \hat{v}(y, s) &:= v(y, s), \end{aligned}$$

and note that we then have

$$\begin{aligned} (i) \quad & 0 \leq \frac{\hat{u}(y, s)}{\hat{v}(y, s)} \leq 1, \text{ whenever } (y, s) \in (\Omega \times (\delta^2, T]) \cap C_\rho(x, t), \\ (ii) \quad & \omega^{\hat{u}, \hat{v}}(x, t, \rho) = 1. \end{aligned} \quad (5.18)$$

In addition, assume first that

$$\frac{\hat{u}(x, t - \rho^2/2)}{\hat{v}(x, t - \rho^2/2)} \geq \frac{1}{2}. \quad (5.19)$$

Now, note that $\hat{u} = e$ in $C_\rho(x, t)$ where $e = \tilde{u} - \tilde{v}$ is realized as the difference between the solutions

$$\tilde{u} = (\omega^{u,v}(x, t, \rho))^{-1}u,$$

and

$$\tilde{v} = (\omega^{u,v}(x, t, \rho))^{-1} \left(\inf_{(\Omega \times (\delta^2, T]) \cap C_\rho(x, t)} u/v \right) v.$$

Let $\hat{H}$ be defined as in (1.13) using $\tilde{u}$ and $\tilde{v}$. Then, using the Harnack inequality for the operator $\hat{H}$, we note that

$$\hat{u}(x, t - \rho^2/2) \leq c\hat{u}(y, s) \text{ whenever } (y, s) \in C_{\rho/2}(x, t). \quad (5.20)$$

Similarly, using the Harnack inequality for the operator H , we note that

$$\hat{v}(y, s) \leq c\hat{v}(x, t + \rho^2/2) \text{ whenever } (y, s) \in C_{\rho/2}(x, t). \quad (5.21)$$

Moreover, using Theorem 1.1, we see that

$$\hat{v}(x, t + \rho^2/2) \leq c\hat{v}(x, t - \rho^2/2). \quad (5.22)$$

Thus, combining (5.19)–(5.22) we can conclude that

$$\frac{1}{2} \leq \frac{\hat{u}(x, t - \rho^2/2)}{\hat{v}(x, t - \rho^2/2)} \leq c \frac{\hat{u}(y, s)}{\hat{v}(y, s)} \leq c, \quad (5.23)$$

whenever $(y, s) \in C_{\rho/2}(x, t)$, and hence

$$\omega^{\hat{u}, \hat{v}}(x, t, \rho/2) \leq \theta \quad (5.24)$$

where $\theta = 1 - 1/(2c) \in (0, 1)$. Rewriting (5.24) we see that

$$\omega^{u,v}(x, t, \rho/2) \leq \theta \omega^{u,v}(x, t, \rho). \quad (5.25)$$

Assume now, on the contrary, that (5.19) does not hold and instead that

$$\frac{\hat{u}(x, t - \rho^2/2)}{\hat{v}(x, t - \rho^2/2)} < \frac{1}{2}. \quad (5.26)$$

In this case we let $\bar{u} = \hat{v} - \hat{u}$, $\bar{v} = \hat{v}$. Then (5.18) and (5.19) holds with $\hat{u}, \hat{v}$ replaced by $\bar{u}, \bar{v}$. We can then first conclude $\omega^{\bar{u}, \bar{v}}(x, t, \rho/2) \leq \theta$ and subsequently again that (5.25) holds. Next, iterating the estimate in (5.25) we deduce that

$$\omega^{u,v}(x, t, \rho) \leq \left(\frac{2\rho}{d} \right)^{\sigma_1} \omega(x, t, d) \quad (5.27)$$

whenever $\rho \leq d/2$ and for $\sigma_1 = -\log_2 \theta$. This estimate also holds whenever $d/2 < \rho \leq d$, in which case the estimate is trivial.

We now consider the case $\rho > d$. Let $\tilde{x}_0 \in \partial\Omega$ be such that $d = d(x, \tilde{x}_0)$. It then holds that $C_\rho(x, t) \subset C_{2\rho}(\tilde{x}_0, t)$, and hence that $\omega^{u,v}(x, t, \rho) \leq \omega^{u,v}(\tilde{x}_0, t, 2\rho)$. Let c and K be as in the statement of Lemma 5.2. We first assume in addition, that $cK\rho < r/2$. In this case we put

$$\begin{aligned}\hat{u}(y, s) &:= (\omega^{u,v}(\tilde{x}_0, t, 2cK\rho))^{-1} \left(u(y, s) - \left(\inf_{(\Omega \times (\delta^2, T]) \cap C_{2cK\rho}(\tilde{x}_0, t)} u/v \right) v(y, s) \right), \\ \hat{v}(y, s) &:= v(y, s),\end{aligned}$$

and note that we then have

$$\begin{aligned}(i) \quad & 0 \leq \frac{\hat{u}(y, s)}{\hat{v}(y, s)} \leq 1, \text{ whenever } (y, s) \in (\Omega \times (\delta^2, T]) \cap C_{2cK\rho}(\tilde{x}_0, t), \\ (ii) \quad & \omega^{\hat{u}, \hat{v}}(\tilde{x}_0, t, 2cK\rho) = 1.\end{aligned}\tag{5.28}$$

We now first assume that

$$\frac{\hat{u}(A_{2K\rho}^-(\tilde{x}_0, t))}{\hat{v}(A_{2K\rho}^-(\tilde{x}_0, t))} \geq \frac{1}{2}.\tag{5.29}$$

Let

$$\begin{aligned}\tilde{u}(y, s) &:= (\omega^{u,v}(\tilde{x}_0, t, 2cK\rho))^{-1} u(y, s), \\ \tilde{v}(y, s) &:= (\omega^{u,v}(\tilde{x}_0, t, 2cK\rho))^{-1} \left(\inf_{(\Omega \times (\delta^2, T]) \cap C_{cK\rho}(\tilde{x}_0, t)} u/v \right) v(y, s).\end{aligned}$$

Then $\tilde{u}$ and $\tilde{v}$ are two non-negative solutions to $Hu = 0$ in $\Omega_T \cap C_{2cK\rho}(\tilde{x}_0, t)$, $\tilde{v} \leq \tilde{u}$ in $\Omega_T \cap C_{2cK\rho}(\tilde{x}_0, t)$, and $\tilde{u} = \tilde{v} = 0$ continuously on $\Delta(\tilde{x}_0, t, 2Kr)$. We can therefore apply Lemma 5.2 to $\tilde{u}$ and $\tilde{v}$ to deduce that

$$\frac{\hat{u}(A_{2K\rho}^-(\tilde{x}_0, t))}{\hat{v}(A_{2K\rho}^+(\tilde{x}_0, t))} \leq c \frac{\hat{u}(y, s)}{\hat{v}(y, s)} \leq c,\tag{5.30}$$

whenever $(y, s) \in (\Omega \times (\delta^2, T]) \cap C_{2\rho}(\tilde{x}_0, t)$. Applying Theorem 1.1 to $\hat{v}$, we also see that

$$\frac{\hat{u}(A_{2K\rho}^-(\tilde{x}_0, t))}{\hat{v}(A_{2K\rho}^-(\tilde{x}_0, t))} \leq c \frac{\hat{u}(A_{2K\rho}^-(\tilde{x}_0, t))}{\hat{v}(A_{2K\rho}^+(\tilde{x}_0, t))},\tag{5.31}$$

whenever $(y, s) \in (\Omega \times (\delta^2, T]) \cap C_{2\rho}(\tilde{x}_0, t)$. We can now deduce from (5.30), (5.31) and (5.29) that

$$\frac{1}{2} \leq \frac{\hat{u}(y, s)}{\hat{v}(y, s)} \leq c^2,$$

and therefore

$$\omega^{\hat{u}, \hat{v}}(\tilde{x}_0, t, 2\rho) \leq \theta,\tag{5.32}$$

where $\theta = 1 - 1/(2c^2) \in (0, 1)$. Rewriting this expression we see that

$$\omega^{u,v}(\tilde{x}_0, t, 2\rho) \leq \theta \omega^{u,v}(\tilde{x}_0, t, 2cK\rho).\tag{5.33}$$

Assume now, on the contrary, that (5.29) does not hold and instead that

$$\frac{\hat{u}(A_{2K\rho}^-(\tilde{x}_0, t))}{\hat{v}(A_{2K\rho}^-(\tilde{x}_0, t))} < \frac{1}{2}. \quad (5.34)$$

In this case we let $\bar{u} = \hat{v} - \hat{u}$, $\bar{v} = \hat{v}$. Then (5.28) and (5.29) holds with $\hat{u}, \hat{v}$ replaced by $\bar{u}, \bar{v}$. We can then first conclude $\omega^{\bar{u}, \bar{v}}(\tilde{x}_0, t, 2\rho) \leq \theta$ and subsequently again that (5.33) holds. Iterating (5.33) we see that

$$\begin{aligned} \omega^{u,v}(x, t, \rho) &\leq \omega^{u,v}(\tilde{x}_0, t, 2\rho) \leq \theta \omega^{u,v}(\tilde{x}_0, t, 2cK\rho) \\ &\leq \left(\frac{2cK\rho}{r}\right)^{\sigma_2} \omega^{u,v}(\tilde{x}_0, t, r) \leq \left(\frac{2cK\rho}{r}\right)^{\sigma_2} \omega^{u,v}(x_0, t_0, 4r), \end{aligned} \quad (5.35)$$

where $\sigma_2 = -\ln(\theta)/\ln(cK)$. Obviously this also holds if $cK\rho \geq r/2$. Letting $\alpha := \min\{\sigma_1, \sigma_2\}$ and combining (5.27) and (5.35) we also see that if $\rho \leq d < r$, then

$$\omega^{u,v}(x, t, \rho) \leq \left(\frac{2\rho}{d}\right)^\alpha \omega^{u,v}(x, t, d) \leq \left(\frac{2\rho}{d}\right)^\alpha \left(\frac{2cKd}{r}\right)^\alpha \omega^{u,v}(x_0, t_0, 4r). \quad (5.36)$$

We now see that (5.35) and (5.36) imply that

$$\omega^{u,v}(x, t, \rho) \leq c \left(\frac{\rho}{r}\right)^\alpha \omega^{u,v}(x_0, t_0, 4r) \quad (5.37)$$

for all $\rho \leq r$. Given $(y, s) \in \Omega_T \cap C_r(x_0, t_0)$ we now let $\hat{\rho} := d_p(x, t, y, s)$. It then follows from (5.37) and (5.12) that

$$\left| \frac{u(x, t)}{v(x, t)} - \frac{u(y, s)}{v(y, s)} \right| \leq \omega^{u,v}(x, t, \hat{\rho}) \leq c \left(\frac{\hat{\rho}}{r}\right)^\alpha \omega^{u,v}(x_0, t_0, 4r) \leq c \left(\frac{\hat{\rho}}{r}\right)^\alpha \frac{u(A_{4r}(x_0, t_0))}{v(A_{4r}(x_0, t_0))}.$$

The theorem now follows from an straightforward application of Lemma 2.12 and Theorem 1.1. $\square$

6 A remark on fully non-linear parabolic equations

The purpose of this remark, and we refer to [N1] for the complete proof of Theorem 6.2 stated below, is to indicate that Theorem 1.1 actually also remains valid for general uniformly parabolic fully non-linear parabolic equations in time-independent cylinders of the form $\Omega_T = \Omega \times (0, T)$ where Ω is a Lipschitz domain. So far, see [BG], this result has only been establish in cylinders $\Omega_T = \Omega \times (0, T)$ where Ω is a $C^{1,1}$ -regular domain. In particular, we consider fully non-linear parabolic equations of the form

$$Hu = F(D^2u(x, t), Du(x, t), x, t) - \partial_t u = 0 \quad (6.1)$$

in $\mathbb{R}^{n+1}$ assuming that F satisfies $F(0, 0, x, t) = 0$. Let $\mathcal{M}_n$ denote the set of all real $n \times n$ -matrices and given $M \in \mathcal{M}_n$ we let $\|M\| = \sup_{|x|=1} |Mx|$. We say that the equation in (6.1) is uniformly parabolic in $\mathbb{R}^{n+1}$ if there exist $\lambda, \Lambda, 0 < \lambda \leq \Lambda < \infty$ and $\eta > 0$, such that

$$n\lambda\|N\| - \eta|p - q| \leq F(M + N, p, x, t) - F(M, q, x, t) \leq \Lambda\|N\| + \eta|p - q| \quad (6.2)$$

whenever $M, N \in \mathcal{M}_n$, $p, q \in \mathbb{R}^n$, $(x, t) \in \mathbb{R}^{n+1}$. The fundamental theory for fully non-linear uniformly parabolic equations was developed in the papers of Lihe Wang, see [W1], [W2], but we also refer to [CKS]. Let $D \subset \mathbb{R}^{n+1}$ be a bounded open set. We let $C(D)$ and $C^2(D)$ denote the set of functions defined in D which are continuous and have continuous partial derivatives up to order two, with respect to x and t , respectively in D . By a parabolic neighborhood of a point $(x_0, t_0) \in \mathbb{R}^{n+1}$ we mean the intersection of an Euclidean neighborhood U of (x_0, t_0) with $\mathbb{R}^n \times (-\infty, t_0]$. In the following a local extremum is to be understood with respect to parabolic neighborhoods.

Definition 6.1 *Let $D \subset \mathbb{R}^{n+1}$ be a bounded open set. A function $u \in C(D)$ is said to be a viscosity supersolution to (6.1) in D if, for given $\phi \in C^2(D)$, we have*

$$F(D^2\phi(x_0, t_0), D\phi(x_0, t_0), x_0, t_0) - \partial_t \phi(x_0, t_0) \leq 0$$

whenever $u - \phi$ has a local minimum at (x_0, t_0) . A function $u \in C(D)$ is said to be a viscosity subsolution to (6.1) in D if, for given $\phi \in C^2(D)$, we have

$$F(D^2\phi(x_0, t_0), D\phi(x_0, t_0), x_0, t_0) - \partial_t \phi(x_0, t_0) \geq 0$$

whenever $u - \phi$ has a local maximum at (x_0, t_0) . If u is both a viscosity supersolution and a viscosity subsolution to (6.1) in D , then u is said to be a viscosity solution to (6.1) in D .

Recall that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain if there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial\Omega$ and $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighborhood of $\partial\Omega$ and such that, for each i ,

$$\begin{aligned} \Omega \cap B(x_i, r_i) &= \{y = (y', y_n) \in \mathbb{R}^n : y_n > \phi_i(y')\} \cap B(x_i, r_i), \\ \partial\Omega \cap B(x_i, r_i) &= \{y = (y', y_n) \in \mathbb{R}^n : y_n = \phi_i(y')\} \cap B(x_i, r_i), \end{aligned} \quad (6.3)$$

in an appropriate coordinate system and for a Lipschitz function $\phi_i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$. The Lipschitz constants of Ω are defined to be $M = \max_i \|\nabla \phi_i\|_\infty$, $r_0 := \min_i r_i$ and we will often refer to Ω as a Lipschitz domain with parameters M and r_0 . If Ω is a Lipschitz domain with parameters M and r_0 , then there exists, for any $x_0 \in \partial\Omega$, $0 < r < r_0$, a point $A_r(x_0) \in \Omega$, such that

$$M^{-1}r < d(x_0, A_r(x_0)) < r, \quad \text{and} \quad d(A_r(x_0), \partial\Omega) \geq M^{-1}r.$$

We let $\Omega_T = \Omega \times (0, T)$, $T > 0$, and in following we let $A_r(x_0, t_0) = (A_r(x_0), t_0)$ whenever $(x_0, t_0) \in S_T := \partial\Omega \times (0, T)$ and $0 < r < r_0$. The complete proof of the following theorem is given in [N1].

Theorem 6.2 *Let $F : \mathcal{M}_n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy (6.2) for some (λ, Λ, η) . Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with parameters M, r_0 and $T > 0$. Let u be a non-negative viscosity solution to (6.1) in Ω_T vanishing continuously on S_T . Let $0 < \delta \ll \min\{r_0/2, \sqrt{T}\}$ be a fixed constant. Suppose $(x_0, t_0) \in S_T$ such that $\delta^2 < t_0 < T - \delta^2$ and assume that $r < \delta/2$. Then there exists $c = c(\lambda, \Lambda, \eta, M, \text{diam}(\Omega), T, \delta)$, $1 \leq c < \infty$, such that*

$$u(x, t) \leq cu(A_r(x_0, t_0))$$

whenever $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$.

Note that Theorem 6.2 is proved in [BG] under the addition assumption that Ω is a $C^{1,1}$ -domain. The latter implies that Ω satisfies a uniform inner and outer ball condition based on which one can prove, using a barrier argument, see [BG], that any non-negative function in the class $\mathcal{S}(\lambda, \Lambda, \eta)$ introduced in Definition 2.1 in [N1], decays linearly at the lateral boundary. Note, following [W1] and using standard notation that if $D \subset \mathbb{R}^{n+1}$ is bounded open set, then $\mathcal{S}(\lambda, \Lambda, \eta) = \mathcal{S}(\lambda, \Lambda, \eta, D)$ is the set of all functions u which are continuous in D and which simultaneously are viscosity subsolutions to the equation $\mathcal{H}_{\lambda, \Lambda, \eta}^+ u = 0$ and viscosity supersolutions to the equation $\mathcal{H}_{\lambda, \Lambda, \eta}^- u = 0$ where $\mathcal{H}_{\lambda, \Lambda, \eta}^+$ and $\mathcal{H}_{\lambda, \Lambda, \eta}^-$ are uniformly parabolic operator defined in terms of the maximal and minimal Pucci extremal operators. The barrier argument in [BG] results in a comparison principle for non-negative functions in the class $\mathcal{S}(\lambda, \Lambda, \eta)$, vanishing on the lateral boundary, based on which the authors in [BG] are able to conclude Theorem 6.2 in the case Ω is a $C^{1,1}$ -domain. Since Theorem 6.2 is stated under much weaker geometric assumptions a different route of proof is needed and we claim that Theorem 6.2 can be proved using only fundamental principles like comparison principles, the Harnack inequality and Hölder decay estimates at the lateral boundary. Furthermore, our proof relies on the following simple decay estimate at the bottom of cylinders. Let $(x_0, t_0) \in \mathbb{R}^{n+1}$, $r > 0$, and let $C_r^+(x_0, t_0) = B(x_0, r) \times (t_0, t_0 + r^2)$. Let F be as in the statement of Theorem 6.2. Then there exist constants $c = c(\lambda, \Lambda, \eta)$, $1 \leq c < \infty$, and $\alpha = \alpha(\lambda, \Lambda, \eta)$, $0 < \alpha < 1$, such that the following is true. Assume that u is a viscosity solution to (6.1) in $C_{2r}^+(x_0, t_0)$, that u is continuous on the closure of $C_{2r}^+(x_0, t_0)$ and that $u = 0$ on $\partial_p C_{2r}^+(x_0, t_0) \cap \{(x, t) : t = t_0\}$. Then

$$|u(x, t)| \leq c \left(\frac{|x| + |t|^{1/2}}{r} \right)^\alpha \sup_{C_{2r}^+(x_0, t_0)} |u| \quad (6.4)$$

whenever $(x, t) \in C_r^+(x_0, t_0)$. Note that this estimate follows from elementary barrier type arguments and we refer to section 2.4 in [W2]. Furthermore, we claim that Theorem 6.2 is simply a consequence of the comparison principle, the solvability of the Dirichlet problem in cylinders of the form $C_r(x, t)$, the Harnack inequality, Hölder decay estimates at the lateral boundary, and the estimate in (6.4). In fact, we also claim that Theorem 6.2 remains true also under the assumption that $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0 in the sense of [JK], and that it probably also extends to the setting of the time-dependent $\text{Lip}(1, 1/2)$ -domains considered in [N] and in the more general setting of parabolic NTA-domains considered in [HLN]. However, for simplicity we here stay with the same setting as in [N1] and with the formulation in Theorem 6.2.

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