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Market Making and Portfolio Liquidation under Uncertainty*

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Market making and optimal portfolio liquidation in the context of electronic limit order books are of considerably practical importance for high frequency (HF) market makers as well as more traditional brokerage firms supplying optimal execution services for clients. In general the two problems are based on probabilistic models defined on certain reference probability spaces. However, due to uncertainty in model parameters or in periods of extreme market turmoil, ambiguity concerning the correct underlying probability measure may appear and an assessment of model risk, as well as the uncertainty on the choice of the model itself, becomes important, as for a market maker or a trader attempting to liquidate large positions, the uncertainty may result in unexpected consequences due to severe mispricing. This paper focuses on the market making and the optimal liquidation problems using limit orders, accounting for model risk or uncertainty. Both are formulated as stochastic optimal control problems, with the controls being the spreads, relative to a reference price, at which orders are placed. The models consider uncertainty in both the drift and volatility of the underlying reference price, for the study of the effect of the uncertainty on the behavior of the market maker, accounting also for inventory restriction, as well as on the optimal liquidation using limit orders.

Keywords: High frequency trading; Market making; Optimal execution; Stochastic control; Hamilton-Jacobi-Bellman equation.

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1. Introduction

1.1. *High Frequency Trading*

Modern information technology allows traders in equity markets to process information, including order submissions and cancelations, at high frequency and high speed. The traders/firms who utilize the new technology for intraday trading for their own accounts are generally called high frequency traders (HFTs) and HFTs are now major players in equity markets [12, 14, 37–39, 42, 43, 51]. Accordingly, by the US financial regulator (SEC) [59], high frequency trading is defined as: 1) The use of extraordinarily high-speed and sophisticated computer programs for generating, routing and executing orders; 2) Use of co-location services and individual data feeds offered by exchanges and MTFs to minimize network and other types of latencies; 3) Very short timeframes for establishing and liquidating positions; 4) Submission of numerous orders that are canceled shortly after submission; 5) Ending the trading day in as close to a flat position as possible.

In general high frequency trading can be specific to certain traders or funds or be embedded in larger strategies or management of client orders. A recognized fact is that HFTs apply several different trading strategies [12, 38, 42, 43, 51]. A general and rough classification of members of a stock exchange is into the classes HFTs, non-HFTs, and hybrid firms that engage in both HFT and trading for clients. A more refined classification renders a distinction between HF market makers and opportunistic HFTs, the latter often being arbitrageurs and directional traders.

Today HF market makers on many exchanges make up a substantial part of the total HFT activity [38, 39, 42, 43, 51]. For example, a recent descriptive study of members at NasdaqOMX in Stockholm shows that during two distinct periods of trading in 2011 and 2012, market makers within the group of HFTs represent 60-70% of the trading volume and that more than 80% of the HFT limit order submissions originate from market makers [39]. These findings are in line with previous research and confirm that market making is a core part of the HFT business and, as such, it is of interest to further understand the principle on which HF market makers base their business.

This paper focuses on the market making problem in the context of high frequency trading as well as the optimal portfolio liquidation problem using limit orders, both of which can be addressed essentially using the same modeling framework. The emergence of electronic trading as a major mechanism for trading financial assets makes the study of the order book central to understanding the mechanisms of price formation, market making, and optimal portfolio liquidation [18, 20]. In the standard price-time priority order-driven markets, buy and sell orders are matched continuously subject to price and time priority, given that there exists sufficient liquidity. The order book is the list of all buy and sell limit orders, with their corresponding prices and sizes, at a given instant of time. Essentially, three types of orders can be submitted: limit orders, market orders, cancelations of previous limit orders. Limit orders are stored in the order book until they are either executed

against an incoming market order or they are canceled.

The limit order book is where HF market makers post their orders and implement their strategies. The increase in computer power has made it possible for HF market makers to deploy ever more complicated trading strategies to profit from changes in market conditions. A key to the success of strategies is the speed at which HF market makers can process information and news events to take trading decision. By the very definition, a characteristic of HF market makers is that the strategies are designed to hold close to no inventories over very short periods of time, from seconds to at most one day, to avoid exposure both to markets after the close and to avoid posting collateral overnight [1, 51].

HF market makers profit from posting limit orders on both sides of the order book turning positions over very quickly to make a very small margin per round trip transaction, the latter often referred to as earning the spread. The profit made by a HF market maker is then generated by repeating this procedure as many times as possible during each trading day. For HF market makers price anticipation and prediction concerning the order flow are important drivers of profit. Strategies that do not include in their limit orders a buffer to cover adverse selection costs, or that strategically post orders deeper in the book to avoid being picked off, may result in accumulated losses as a consequence of trading with other market participants who possess private or better information. In the long term, HF market makers who are not able to incorporate short-term price predictability in their optimal HF market making strategies, as well as account for adverse selection costs, are very likely to be driven out of the market.

To devise an optimal schedule of a large order, or optimal portfolio liquidation, participants may choose a mixture of market and limit orders. The mixture is influenced by the characteristics of the order flow, the volumes and shape of the order book, as well as the structure of transaction fees and rebates. Through an optimal scheduling, a trader tries to control the trading cost.

There is now a growing literature on modeling dynamic HF market-maker strategies based on optimal postings and cancelations of limit orders to maximize expected terminal wealth over a fixed horizon T whilst penalizing inventories [8, 15–17, 33, 35, 36], where the HF market makers are characterized as ultra-fast market players whose trading horizon T is at most one trading day. All limit orders are canceled an instant later if not filled and inventories are optimally managed and set to zero at T . A starting point for much of these developments is the early work on optimal postings by a securities dealer developed in [40] and more recently modified in [8] to model market making in limit order books. The main characteristic of this model is that it does not explicitly consider the limit order book but instead models liquidity statistically, which is an advantage compared to limit order book models when it comes to mathematical tractability. The model in [8, 40] has been further complemented by explicitly accounting for inventory constraints [15, 33]. More sophisticated models have also been proposed including for example richer dynamics of market orders, impact on the limit order book, adverse selection effects and pre-

dictability, to deal with high frequency market making [17]. Further generalizations include market orders and limit orders at best as well as next to best bid and ask together with stochastic spreads [36]. These models, all dealing with high frequency trading and market making, in the end rely either on first-order approximations or on numerical approximations for the associated partial differential equations, while one particular stylized model manages to provide simple expressions for the optimal quotes of the market maker [33].

The order book is also where a strategy for optimal portfolio liquidation using limit orders is implemented. The purpose of devising an optimal schedule of a large order, buy or sell, is to control the trading cost by balancing a trade-off between market impact and market risk. On the one hand, market impact demands trading to be slow. On the other hand, the presence of market risk favors faster trading. An optimal schedule of a large order may involve the use of market orders and limit orders in combination, as well as a routing of the orders to different exchanges including dark-pools [13, 19, 28, 29, 32, 34, 46, 47]. During the continuous auction process implemented by most electronic trading pools, market participants send their orders to a queuing system where a first-in-first-out queue stands at each possible price. If a buy (sell) order reaches a queue of sell (buy) orders, a transaction occurs. Orders filling queues are said to be passive or liquidity-providing orders and are often less expensive compared to the more aggressive market orders which are referred to as liquidity taking orders. In practice, the issue of optimally posting orders in the order book and optimal liquidation of a portfolio have become more and more important. Since the pioneering work on optimal scheduling of large orders [3, 11], the original approaches have been recently generalized in several directions [4, 26, 27, 30, 44, 48, 49, 58]. While a few papers attempt to consider the problem at the level of the interactions with the order book [2, 53, 55], most of them focus on the dynamics initiated by aggressive orders hitting a resilient order book and ignore trading with passive orders. In fact, optimal liquidation with limit orders included has only been studied very recently [10, 19, 32, 34]. In this paper we consider trading algorithms which are at the most passive end of the spectrum and we focus on algorithms which use only passive limit orders placed in the limit order book. In this case, a trading algorithm has to find the places in the order book where it will post orders as well as the sizes of the orders to be placed.

1.2. *Risk and Uncertainty*

A HF market maker faces several types of risk, one of which is model risk or uncertainty, the uncertainty being linked to the uncertainty of the choice of the model itself and in this paper we restrict our attention to uncertainty in the drift and volatility of the underlying midprice process. Naturally, uncertainty in this sense may lead to mispricing resulting in severe consequences for the market maker.

In the standard framework of mathematical finance and, more generally, in decision theory, the starting point is the specification of a stochastic model: a set of

future scenarios and a probability measure P on these outcomes. Meanwhile, there are circumstances, especially in financial decision making, where the decision maker or risk manager is not able to attribute a precise probability to future outcomes. This is called uncertainty, and is different from risk as in the case of risk a unique probability measure on future outcomes is specified [45]. More precisely, in the case of uncertainty there exists ambiguity as a market maker faces several possible specifications $P^1, P^2, \dots$ for the probabilities on future outcomes and it is shown that aversion to ambiguity plays an important role in decision making [21, 23]. In the literature, decision under ambiguity has been explored on, for example, its axiomatic foundations [23, 31], its implications for the behavior of security prices [24, 56], and coherent risk measures and their extensions [5, 25, 41].

Uncertainty in the sense described above is basically always a reality due to the model risk induced in any model through parameter estimation. Furthermore, this uncertainty may become even more relevant during extreme market conditions, the latter often accompanied by a lack of liquidity and a lack of trading, under which a sense of ambiguity concerning the correct underlying probability measure becomes critical. A perhaps even more relevant, and definitely much more frequent type of market behavior where uncertainty enters, is the so-called phenomenon of “liquidity evaporation” (reduction in market depth) before large economic announcements (company earnings, unemployment statistics, Fed funds rate). Due to significant uncertainty in the levels of prices and volatility around such announcements many HFTs prefer to shut down their strategies and as a result there is a dramatic drop in market depth. Actual examples of this phenomenon can be found at [52].

In the situation discussed above it could be rational for the market maker to respond by posting limit orders more based on an emphasis on the worst-case scenarios than on what would be suggested by an assumption on Savage expected utility [57]. The model of this paper can help explain the phenomena discussed as we in this paper capture one aspect of such uncertainty through an ambiguous underlying asset-returns distribution. As outlined below, our model explores how uncertainty may increase the bid-ask spread and hence reduce liquidity, as well as the relation between inventory penalization and bid-ask spread. This analysis is also applied to the optimal liquidation problem.

2. Model

A realistic model for the market making problem, as well as the problem of optimal portfolio liquidation, should account for realistic order book dynamics and price dynamics including richer dynamics of market and limit orders, cancellations, impact on the limit order book, impact on the price dynamics, adverse selection effects etcetera. Such models can be devised and to some extent calibrated, see [17] in particular, but by accounting for all these aspects analytic tractability is easily lost and one has to rely either on first order approximations or on numerical approximations for the associated partial differential equations. In this paper our

focus is on the market making problem, as well as the problem of optimal portfolio liquidation using limit orders, and a distinct feature of our paper is that we consider these problems under model risk or uncertainty. In particular, we want to exemplify the effect of this uncertainty on the behavior of the market maker, accounting also for inventory restrictions, as well as for the optimal liquidation using limit orders. Naturally these problems can be formulated in any of the models suggested in the papers mentioned in the previous section, resulting in a more or less analytically tractable paper and presentation. Based on this we here have deliberately chosen to stay in the more classical settings of [8, 40], see also [33], since many of the effects of uncertainty can already be derived in these papers. In particular, using an ansatz similar to [33, 34], we are able to essentially explicitly solve the worst-case stochastic control problem formulated below.

To outline the details of our set-up we here consider the one-dimensional setting of one order book. Assume that the midprice $S = \{S_t\}_{t \in [0, T]}$ is described by an Itô-diffusion process

$$dS_t(\omega) = \mu_t(\omega)dt + \sigma_t(\omega)dW_t(\omega), t \in [0, T], \omega \in \Omega, \quad (2.1)$$

where $W = \{W_t\}_{t \in [0, T]}$ is a standard Wiener process on $\mathbb{R}$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$. Here $\mu = \{\mu_t\}_{t \in [0, T]}$ and $\sigma = \{\sigma_t\}_{t \in [0, T]}$ are $\mathcal{F}_t$ -adapted processes satisfying sufficient conditions which ensure the existence of a strong solution to (2.1). To study the situation where market participants consider the model defined by (2.1) as uncertain, model risk or uncertainty is incorporated into the model by assuming that for almost all $\omega \in \Omega$,

$$-\bar{\mu} \leq \mu_t(\omega) \leq \bar{\mu}, \quad \underline{\sigma} \leq \sigma_t(\omega) \leq \bar{\sigma}, \quad \forall t \in [0, T], \quad (2.2)$$

where $0 < \bar{\mu} < \infty$ and $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$. In particular, in the following a market participant captures model risk or uncertainty by using solely $\bar{\mu}$, $\underline{\sigma}$, and $\bar{\sigma}$, in connection with (2.1), to make decisions.

In the case of market making, we consider a market maker who is only posting limit orders and who is allowed to control the ask and bid quotes, denoted by $p^+ = \{p_t^+\}_{t \in [0, T]}$ and $p^- = \{p_t^-\}_{t \in [0, T]}$, by continuously posting limit orders on both sides of the book. The distance from the midprice is determined by the $\mathcal{F}_t$ -adapted controls $\delta^+ = \{\delta_t^+\}_{t \in [0, T]}$ and $\delta^- = \{\delta_t^-\}_{t \in [0, T]}$, which are the associated spreads defined by $\delta_t^+ = p_t^+ - S_t$ and $\delta_t^- = S_t - p_t^-$, $t \in [0, T]$. In addition, let $\{Q_t\}_{t \in [0, T]}$, $Q_t \in \mathbb{Z}$ denote the inventory, which varies with the execution of limit orders placed by the market maker, and let $\{X_t\}_{t \in [0, T]}$, $X_t \in \mathbb{R}$ be the cash process, which also varies as the market maker buys or sells the asset. We assume that the dynamics of Q_t and X_t are governed by

$$\begin{aligned} dQ_t &= dN_t^- - dN_t^+, \\ dX_t &= [S_t + \delta_t^+]dN_t^+ - [S_t - \delta_t^-]dN_t^-, \end{aligned}$$

where $N^\pm = \{N_t^\pm\}_{t \in [0, T]}$ are two independent Poisson processes with intensities $\lambda^\pm(\delta^\pm)$, which are non-increasing functions determining the fill rates. The wealth

or PNL of the market maker, at t , is then given by

$$PNL_t = X_t + Q_t S_t.$$

Following [33, 34] we also add a bound, denoted by $Q > 0$, on the inventory the market maker is allowed to hold. In particular, we assume that a market maker with inventory Q will never post a bid quote and symmetrically that a market maker with inventory $-Q$, i.e. the market maker holds a short position of Q shares in the underlying stock, will never post an ask quote. This is a realistic restriction which imposes a risk limit on the inventory and this limit allows us, along the lines of [33, 34] to rigorously solve the stochastic control problem to be outlined.

Given a utility function $\psi(t, s, x, q)$, the value function associated with the market making problem under model risk or uncertainty is

$$v(t, s, x, q) = \sup_{\delta^+, \delta^-} \inf_{(\mu, \sigma) \in \mathcal{U}} E_{t, s, x, q}[\psi(S_T, X_T, Q_T)], \quad (2.3)$$

where $\mathcal{U}$ denotes the set of all $\mathcal{F}_t$ -adapted processes (μ, σ) satisfying (2.2), $t \in [0, T]$ is the current time, $s = S_t$ is the current midprice, $x = X_t$ is the current cash value, and $q = Q_t$ is the current inventory level. Furthermore, as discussed above, we impose the risk limit $q \in \{-Q, -Q+1, \dots, Q-1, Q\}$. The model hence accounts not only for different choices of utility functions ψ and inventory levels, but in particular also for model risk or uncertainty in the underlying model for the midprice S . In (2.3), the market maker's aversion to risk and to accumulating a large inventory, in absolute values and at $t = T$, is captured by the utility function ψ . The aversion to ambiguity, i.e. model uncertainty, is captured by the presence of the infimum with respect to $(\mu, \sigma) \in \mathcal{U}$. This worst-case approach, analogous to those for option pricing [6, 7, 22], distinguishes model risk or uncertainty from risk since the former is treated by taking the infimum/supremum over models whereas the latter by averaging over scenarios within a given model. Compared with model-averaging procedures, this worst-case approach is more conservative and robust, and requires less input from the market maker.

In the case of optimal portfolio liquidation using only limit orders, consider a trader who wants to liquidate a portfolio, consisting of q units of a stock at $t \in [0, T]$, during the time period $[t, T]$ by continuously posting limit orders on the ask side of the order book at the price level $S_t + \delta_t^+$, where $\delta^+ = \{\delta_t^+\}$ is assumed to be a nonnegative $\mathcal{F}_t$ -adapted control process. Let $\{\tilde{Q}_t\}$ denote the remaining stocks to be sold and $\{\tilde{X}_t\}$ the cash received so far. Then

$$\begin{aligned} d\tilde{Q}_t &= -dN_t^+, \\ d\tilde{X}_t &= [S_t + \delta_t^+]dN_t^+, \end{aligned}$$

where $N^+ = \{N_t^+\}$ is a jump process counting the number of stocks sold. Given a utility function $\psi(t, s, x, q)$, in this case the value function associated with the optimal liquidation problem under model risk or uncertainty is

$$\tilde{v}(t, s, x, q) = \sup_{\delta^+} \inf_{(\mu, \sigma) \in \mathcal{U}} E_{t, s, x, q}[\tilde{\psi}(S_T, \tilde{X}_T, \tilde{Q}_T)]. \quad (2.4)$$

The aversion to ambiguity, i.e. model uncertainty, is captured by the presence of the infimum with respect to $(\mu, \sigma) \in \mathcal{U}$.

In Section 3, the Hamilton-Jacobi-Bellman equations associated with (2.3) and (2.4) are first derived. They are then solved under further assumptions. In particular, the fill rates are assumed to be given by

$$\lambda^\pm(\delta^\pm) = Ae^{-\rho\delta^\pm}, \text{ where } A > 0 \text{ and } \rho > 0. \quad (2.5)$$

3. Result

3.1. *Hamilton-Jacobi-Bellman Equations*

To reduce the control problems under consideration to systems of ordinary differential equations, as in [33, 34], utility functions defined by CARA (Constant Absolute Risk Aversion) are used. Specifically, for market making, we assume that

$$\psi(s, x, q) = -\exp(-\gamma(x + qs - \eta|q|)), \quad (3.1)$$

where $\gamma, \eta \geq 0$, with γ measuring risk aversion and η representing a penalty on the inventory remaining at T . Note that the utility function in (3.1) does not penalize the inventory along its path, instead only the size of inventory at the terminal time T is penalized. In the reality of high frequency trading large inventories are penalized continuously as most HFTs tend to try to keep a modest inventory not only at the terminal date but throughout the trading day. In our context, a consequence of the fact that we only penalize inventory at the terminal time T is that the model should be used mainly for short time intervals, small T , iteratively and incrementally during the day. An attempt to develop a model which would penalize the inventory along its path, and over longer time intervals, may attempt to account for this by allowing the utility function to also depend on the average over time of the inventory held. However, such a specification renders a more involved stochastic control problem and we here stick with the modeling framework in [33, 34], which may be considered a stylized and slightly rough approximation of reality, but which however gives, for short time intervals, a realistic model for high frequency market making.

For optimal liquidation, assume that

$$\tilde{\psi}(s, x, q) = -\exp(-\gamma(x + qs - \eta q)), \quad (3.2)$$

where $\gamma, \eta \geq 0$. Here η can be interpreted as the cost per share incurred to liquidate the remaining inventory at T . Clearly, ψ and $\tilde{\psi}$ coincide for $q \geq 0$. Using the two utility functions, the optimization problems defined in (2.3) and (2.4) then become

$$\begin{aligned} & \sup_{\delta^+, \delta^-} \inf_{(\mu, \sigma) \in \mathcal{U}} E_{t,s,x,q} [-\exp(-\gamma(X_T + Q_T S_T - \eta|Q_T|))], \\ & \sup_{\delta^+} \inf_{(\mu, \sigma) \in \mathcal{U}} E_{t,s,x,q} [-\exp(-\gamma(\tilde{X}_T + \tilde{Q}_T S_T - \eta\tilde{Q}_T))]. \end{aligned} \quad (3.3)$$

Let

$$\mathcal{L} = \mathcal{L}^{\mu, \sigma} = \mu_t \partial_s + \frac{1}{2}(\sigma_t)^2 \partial_{ss}$$

denote the operator associated with the dynamics defined by (2.1). The Hamilton-Jacobi-Bellman equation [54] associated with the market making problem in (3.3) becomes

$$\sup_{\delta^+, \delta^-} \inf_{(\mu, \sigma) \in \mathcal{U}} \left[\partial_t v(t, s, x, q) + \mathcal{L}^{\mu, \sigma} v(t, s, x, q) + \lambda^+(\delta^+) [v(t, s, x + (s + \delta^+), q - 1) - v(t, s, x, q)] + \lambda^-(\delta^-) [v(t, s, x - (s - \delta^-), q + 1) - v(t, s, x, q)] \right] = 0, \quad (3.4)$$

for $(t, s) \in (0, T) \times \mathbb{R}$ and $(x, q) \in \mathbb{R} \times \{-Q, -Q+1, \dots, Q-1, Q\}$, with the terminal condition

$$v(T, s, x, q) = \psi(s, x, q).$$

Similarly, the Hamilton-Jacobi-Bellman equation [54] associated with the optimal liquidation problem in (3.3) becomes

$$\sup_{\delta^+} \inf_{(\mu, \sigma) \in \mathcal{U}} \left[\partial_t \tilde{v}(t, s, x, q) + \mathcal{L}^{\mu, \sigma} \tilde{v}(t, s, x, q) + \lambda^+(\delta^+) [\tilde{v}(t, s, x + (s + \delta^+), q - 1) - \tilde{v}(t, s, x, q)] \right] = 0 \quad (3.5)$$

for $(t, s) \in (0, T) \times \mathbb{R}$ and $(x, q) \in \mathbb{R} \times \{0, \dots, Q-1, Q\}$, with the terminal condition

$$\tilde{v}(T, s, x, q) = \tilde{\psi}(s, x, q).$$

In particular, the problems in (3.4)-(3.5) are easily seen to decouple into two independent optimization problems. Indeed, we first note that for fixed (δ^+, δ^-) and δ^+ , the step of initially taking the infimum with respect to $(\mu, \sigma) \in \mathcal{U}$ is equivalent to solving the optimization problem

$$\inf_{(\mu, \sigma) \in \mathcal{U}} \mathcal{L}^{\mu, \sigma} v = \inf_{(\mu, \sigma) \in \mathcal{U}} [\mu_t \partial_s v + \frac{1}{2} (\sigma_t)^2 \partial_{ss}^2 v],$$

which results in two separate problems,

$$\inf_{\mu \in [-\bar{\mu}, \bar{\mu}]} [\mu_t \partial_s v], \quad \inf_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} [(\sigma_t)^2 \partial_{ss}^2 v]. \quad (3.6)$$

Define

$$F(z) = \begin{cases} \underline{\sigma}, & z > 0, \\ \bar{\sigma}, & z \leq 0. \end{cases}$$

Apparently, the optimal controls in (3.6) are

$$\begin{aligned} \mu^* &= \mu^*(t, s, x, q) = -\bar{\mu} \operatorname{sgn}(\partial_s v(t, s, x, q)), \\ \sigma^* &= \sigma^*(t, s, x, q) = F(\partial_{ss}^2 v). \end{aligned} \quad (3.7)$$

Let

$$H^*(t, p, r) = -\bar{\mu}|p| + \frac{1}{2} (F(r))^2 r.$$

Then

$$\mathcal{L}^{\mu^*, \sigma^*} v(t, s, x, q) = H^*(t, \partial_s v(t, s, x, q), \partial_{ss}^2 v(t, s, x, q))$$

and the Hamilton-Jacobi-Bellman equation in (3.4) associated with the market making problem becomes

$$\begin{aligned} & \partial_t v(t, s, x, q) + H^*(t, \partial_s v(t, s, x, q), \partial_{ss}^2 v(t, s, x, q)) \\ & + \sup_{\delta^+} \lambda^+(\delta^+) [v(t, s, x + (s + \delta^+), q - 1) - v(t, s, x, q)] \\ & + \sup_{\delta^-} \lambda^-(\delta^-) [v(t, s, x - (s - \delta^-), q + 1) - v(t, s, x, q)] = 0, \end{aligned} \quad (3.8)$$

for $(t, s) \in (0, T) \times \mathbb{R}$ and $(x, q) \in \mathbb{R} \times \{-Q, -Q+1, \dots, Q-1, Q\}$, with the terminal condition

$$v(T, s, x, q) = \psi(s, x, q).$$

Similarly, for the optimal liquidation problem,

$$\begin{aligned} & \partial_t \tilde{v}(t, s, x, q) + H^*(t, \partial_s \tilde{v}(t, s, x, q), \partial_{ss}^2 \tilde{v}(t, s, x, q)) \\ & + \sup_{\delta^+} \lambda^+(\delta^+) [\tilde{v}(t, s, x + (s + \delta^+), q - 1) - \tilde{v}(t, s, x, q)] = 0, \end{aligned}$$

for $(t, s) \in (0, T) \times \mathbb{R}$ and $(x, q) \in \mathbb{R} \times \{0, \dots, Q-1, Q\}$, with the terminal condition

$$\tilde{v}(T, s, x, q) = \tilde{\psi}(s, x, q).$$

The next step is then to solve the optimization problems and to verify that the solutions constructed through the Hamilton-Jacobi-Bellman equations are indeed the solutions to the original optimization problems.

3.2. Market Making

Assume fill rates as in (2.5). Then, by (3.8) the Hamilton-Jacobi-Bellman equation associated with the market making problem becomes

$$\begin{aligned} & \partial_t v(t, s, x, q) + H^*(t, \partial_s v(t, s, x, q), \partial_{ss}^2 v(t, s, x, q)) \\ & + \sup_{\delta^+} A e^{-\rho \delta^+} [v(t, s, x + (s + \delta^+), q - 1) - v(t, s, x, q)] \\ & + \sup_{\delta^-} A e^{-\rho \delta^-} [v(t, s, x - (s - \delta^-), q + 1) - v(t, s, x, q)] = 0, \end{aligned} \quad (3.9)$$

$$v(T, s, x, q) = -\exp(-\gamma(x + qs - \eta|q|)).$$

Lemma 3.1. *For the market making problem, there exists a solution to the associated Hamilton-Jacobi-Bellman equation in (3.4).*

Proof. Indeed, by the above we see that it suffices to establish the existence of a solution to the associated Hamilton-Jacobi-Bellman equation in (3.9). To construct a solution to (3.9) we first, as in [33], consider an ansatz

$$v(t, s, x, q) = -\exp(-\gamma(x + qs - \eta|q|)) v_q(t)^{-\gamma/\rho},$$

where $v_q \in C^1(0, T)$, $q \in \{-Q, -Q + 1, \dots, Q - 1, Q\}$, are to be determined. With this ansatz, clearly,

$$\partial_t v = -\frac{\gamma}{\rho} \frac{v'_q}{v_q} v, \quad \partial_s v = -\gamma q v, \quad \partial_{ss}^2 v = (\gamma q)^2 v,$$

and

$$\begin{aligned} H^*(t, \partial_s v(t, s, x, q), \partial_{ss}^2 v(t, s, x, q)) \\ = -\gamma \bar{\mu} |qv| + \frac{1}{2} (F((\gamma q)^2 v))^2 (\gamma q)^2 v. \end{aligned}$$

Next, we apply the ansatz to the problems

$$\begin{aligned} (i) \quad \sup_{\delta^+} A e^{-\rho \delta^+} [v(t, s, x + (s + \delta^+), q - 1) - v(t, s, x, q)], \\ (ii) \quad \sup_{\delta^-} A e^{-\rho \delta^-} [v(t, s, x - (s - \delta^-), q + 1) - v(t, s, x, q)]. \end{aligned}$$

For problem (i) we have

$$\begin{aligned} & \sup_{\delta^+} A e^{-\rho \delta^+} [v(t, s, x + (s + \delta^+), q - 1) - v(t, s, x, q)] \\ &= \sup_{\delta^+} A e^{-\rho \delta^+} \left[\exp(-\gamma(x + qs - \eta|q|)) v_q(t)^{-\gamma/\rho} \right. \\ & \quad \left. - \exp(-\gamma(x + (s + \delta^+) + (q - 1)s - \eta|q - 1|)) v_{q-1}(t)^{-\gamma/\rho} \right] \\ &= \sup_{\delta^+} A e^{-\rho \delta^+} \left[\exp(-\gamma(x + qs - \eta|q|)) v_q(t)^{-\gamma/\rho} \right. \\ & \quad \left. - \exp(-\gamma(x + qs + \delta^+ - \eta|q - 1|)) v_{q-1}(t)^{-\gamma/\rho} \right] \\ &= v \sup_{\delta^+} A e^{-\rho \delta^+} \left[\exp(-\gamma(\delta^+ + \eta(|q| - |q - 1|))) \left(\frac{v_q(t)}{v_{q-1}(t)} \right)^{\gamma/\rho} - 1 \right]. \end{aligned}$$

The optimal choice of δ^+ is

$$\delta_*^+(t) = \frac{1}{\gamma} \log(1 + \gamma/\rho) + \frac{1}{\rho} \log \frac{v_q(t)}{v_{q-1}(t)} - \eta(|q| - |q - 1|)$$

and

$$\begin{aligned} & \sup_{\delta^+} A e^{-\rho \delta^+} [v(t, s, x + (s + \delta^+), q - 1) - v(t, s, x, q)] \\ &= -v A e^{-\rho \delta_*^+} \frac{\gamma}{\gamma + \rho} \\ &= -v \frac{A \gamma}{\gamma + \rho} \frac{v_{q-1}(t)}{v_q(t)} (1 + \gamma/\rho)^{-\frac{\rho}{\gamma}} e^{\eta \rho (|q| - |q - 1|)}. \end{aligned}$$

Similarly, for problem (ii),

$$\begin{aligned}
& \sup_{\delta^-} A e^{-\rho \delta^-} [v(t, s, x - (s - \delta^-), q + 1) - v(t, s, x, q)] \\
&= \sup_{\delta^-} A e^{-\rho \delta^-} \left[\exp(-\gamma(x + qs - \eta|q|)) v_q(t)^{-\gamma/\rho} \right. \\
&\quad \left. - \exp(-\gamma(x - (s - \delta^-) + (q + 1)s - \eta|q + 1|)) v_{q+1}(t)^{-\gamma/\rho} \right] \\
&= \sup_{\delta^-} A e^{-\rho \delta^-} \left[\exp(-\gamma(x + qs - \eta|q|)) v_q(t)^{-\gamma/\rho} \right. \\
&\quad \left. - \exp(-\gamma(x + qs + \delta^- - \eta|q + 1|)) v_{q+1}(t)^{-\gamma/\rho} \right] \\
&= v \sup_{\delta^-} A e^{-\rho \delta^-} \left[\exp(-\gamma(\delta^- + \eta(|q| - |q + 1|))) \left(\frac{v_q(t)}{v_{q+1}(t)} \right)^{\gamma/\rho} - 1 \right].
\end{aligned}$$

The optimal choice of δ^- is

$$\delta_*^-(t) = \frac{1}{\gamma} \log(1 + \gamma/\rho) + \frac{1}{\rho} \log \frac{v_q(t)}{v_{q+1}(t)} - \eta(|q| - |q + 1|)$$

and

$$\begin{aligned}
& \sup_{\delta^-} A e^{-\rho \delta^-} [v(t, s, x - (s - \delta^-), q + 1) - v(t, s, x, q)] \\
&= -v A e^{-\rho \delta_*^-} \frac{\gamma}{\gamma + \rho} \\
&= -v \frac{A \gamma}{\gamma + \rho} \frac{v_{q+1}(t)}{v_q(t)} (1 + \gamma/\rho)^{-\frac{\rho}{\gamma}} e^{\eta \rho(|q| - |q+1|)}.
\end{aligned}$$

Hence, with the ansatz, (3.9) becomes

$$\begin{aligned}
& -\frac{\gamma}{\rho} \frac{v'_q(t)}{v_q(t)} v(t) - \gamma \bar{\mu} |qv(t)| + \frac{1}{2} (F((\gamma q)^2 v(t)))^2 (\gamma q)^2 v(t) \\
& - \frac{\gamma A}{\rho + \gamma} (1 + \gamma/\rho)^{-\frac{\rho}{\gamma}} \frac{v_{q-1}(t)}{v_q(t)} e^{\eta \rho(|q| - |q-1|)} v(t) \\
& - \frac{\gamma A}{\rho + \gamma} (1 + \gamma/\rho)^{-\frac{\rho}{\gamma}} \frac{v_{q+1}(t)}{v_q(t)} e^{\eta \rho(|q| - |q+1|)} v(t) = 0, \\
& v_q(T) = 1.
\end{aligned}$$

Substituting $F((\gamma q)^2 v(t)) = F(-v_q(t)^{-\gamma/\rho})$ into the equation gives

$$\begin{aligned}
v'_q(t) &= \rho \bar{\mu} |qv_q(t)| + \frac{1}{2} (F(-v_q(t)^{-\gamma/\rho}))^2 \gamma q^2 \rho v_q(t) \\
&\quad - A (1 + \gamma/\rho)^{-(1+\frac{\rho}{\gamma})} v_{q+1}(t) e^{\eta \rho(|q| - |q+1|)} \\
&\quad - A (1 + \gamma/\rho)^{-(1+\frac{\rho}{\gamma})} v_{q-1}(t) e^{\eta \rho(|q| - |q-1|)}, \\
v_q(T) &= 1.
\end{aligned} \tag{3.10}$$

Note that the presence of the absolute values $|qv_q|$ and the factor $(F(-v_q(t)^{-\gamma/\rho}))^2$ complicate the problem relative to the one in [33]. To proceed, recall that we only allow the inventory q to take values in a finite range $\{-Q, -Q + 1, \dots, Q - 1, Q\}$

and we now consider an auxiliary system with a positive sign assumption on v_q . For $q \in \{-Q+1, \dots, Q-1\}$,

$$\begin{aligned} u'_q(t) &= \rho\bar{\mu}|q|u_q(t) + \frac{1}{2}\bar{\sigma}^2\gamma q^2\rho u_q(t) \\ &\quad - A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}u_{q+1}(t)e^{\eta\rho(|q|-|q+1|)} \\ &\quad - A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}u_{q-1}(t)e^{\eta\rho(|q|-|q-1|)}, \\ u_q(T) &= 1, \end{aligned} \quad (3.11)$$

for $q = Q$,

$$\begin{aligned} u'_q(t) &= \rho\bar{\mu}|q|u_q(t) + \frac{1}{2}\bar{\sigma}^2\gamma q^2\rho u_q(t) \\ &\quad - A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}u_{q-1}(t)e^{\eta\rho(|q|-|q-1|)}, \\ u_q(T) &= 1, \end{aligned}$$

and for $q = -Q$,

$$\begin{aligned} u'_q(t) &= \rho\bar{\mu}|q|u_q(t) + \frac{1}{2}\bar{\sigma}^2\gamma q^2\rho u_q(t) \\ &\quad - A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}u_{q+1}(t)e^{\eta\rho(|q|-|q+1|)}, \\ u_q(T) &= 1. \end{aligned} \quad (3.12)$$

Let $U(t)$ be the $(2Q+1) \times 1$ -dimensional vector having $u_q(t)$ as the element on row $q \in \{Q, Q-1, \dots, 0, \dots, -Q+1, -Q\}$ and $E = \{E_{i,j}\}$ be a $(2Q+1) \times (2Q+1)$ -dimensional matrix with zero elements except the following. For $q \in \{Q-1, \dots, 1\}$,

$$\begin{aligned} E_{q,q-1} &= -A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}e^{\eta\rho}, \\ E_{q,q} &= \rho\bar{\mu}q + \frac{1}{2}\bar{\sigma}^2\gamma q^2\rho, \\ E_{q,q+1} &= -A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}e^{-\eta\rho} \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} E_{-q,-q-1} &= -A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}e^{-\eta\rho}, \\ E_{-q,-q} &= \rho\bar{\mu}q + \frac{1}{2}\bar{\sigma}^2\gamma q^2\rho, \\ E_{-q,-q+1} &= -A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}e^{\eta\rho}. \end{aligned}$$

In addition,

$$\begin{aligned} E_{Q,Q} &= \rho\bar{\mu}Q + \frac{1}{2}\bar{\sigma}^2\gamma Q^2\rho, \\ E_{Q,Q-1} &= -A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}e^{\eta\rho}, \\ E_{-Q,-Q} &= \rho\bar{\mu}Q + \frac{1}{2}\bar{\sigma}^2\gamma Q^2\rho, \\ E_{-Q,-Q+1} &= -A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}e^{\eta\rho}, \end{aligned}$$

and

$$\begin{aligned} E_{0,-1} &= -A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}e^{-\eta\rho}, \\ E_{0,0} &= 0, \\ E_{0,1} &= -A(1+\gamma/\rho)^{-(1+\frac{\rho}{\gamma})}e^{-\eta\rho}. \end{aligned} \quad (3.14)$$

With the constant coefficient matrix E defined in (3.13)-(3.14), the auxiliary system (3.11)-(3.12) can be rewritten as a system of first-order ordinary differential equations

$$U'(t) = EU(t), \quad U(T) = \bar{\mathbf{1}}_{2Q+1}, \quad (3.15)$$

where $\bar{\mathbf{1}}_{2Q+1}$ denotes the $(2Q+1)$ -dimensional vector having all entries identical to 1. A solution to (3.15) can be easily constructed as

$$U(t) = e^{-E(T-t)} \bar{\mathbf{1}}_{2Q+1}. \quad (3.16)$$

Next using Lemma 3.2 stated and proved below we can conclude that all components of $U(t)$ are nonnegative and hence they are solutions to the original system. This proves the existence of a solution to the Hamilton-Jacobi-Bellman equation in (3.9) and hence the proof of the lemma is complete. $\square$

Lemma 3.2. *Let $Q \in \mathbb{Z}_+$ and $T > 0$. Then there exist nonnegative constants $\alpha_1, \alpha_2, \alpha_3$ such that*

$$u_q(t) \geq e^{-(\alpha_1 Q^2 + \alpha_2 Q + \alpha_3)(T-t)}, \quad \forall q \in \{-Q, \dots, Q\}, \quad \forall t \in [0, T]. \quad (3.17)$$

In particular, $u_q(t)$ is nonnegative for all $q \in \{-Q, \dots, Q\}$ and $t \in [0, T]$.

Proof. There may be several ways to prove that for $q \in \{-Q, \dots, Q\}$ and $t \in [0, T]$, $u_q(t)$ is nonnegative. The proof below by contradiction is along the lines of [33]. Indeed, let $a_4 > 0$ be an additional constant to be chosen. If (3.17) is not true, then for $a_4 > 0$ fixed, there exists $\epsilon > 0$ such that

$$\min_{t \in [0, T], |q| \leq Q} e^{-2\alpha_4(T-t)} (u_q(t) - e^{-(\alpha_1 Q^2 + \alpha_2 Q + \alpha_3)(T-t)}) + \epsilon(T-t) < 0. \quad (3.18)$$

Assume that t^* and q^* , with $|q^*| < Q$, realize the minimum in (3.18). Then

$$\begin{aligned} & 2\alpha_4 e^{-2\alpha_4(T-t^*)} (u_{q^*}(t^*) - e^{-(\alpha_1 Q^2 + \alpha_2 Q + \alpha_3)(T-t^*)}) \\ & + e^{-2\alpha_4(T-t^*)} (u'_{q^*}(t^*) - (\alpha_1 Q^2 + \alpha_2 Q + \alpha_3) e^{-(\alpha_1 Q^2 + \alpha_2 Q + \alpha_3)(T-t^*)}) \geq \epsilon. \end{aligned}$$

That is,

$$\begin{aligned} & 2\alpha_4 u_{q^*}(t^*) + u'_{q^*}(t^*) \\ & - (\alpha_1 Q^2 + \alpha_2 Q + \alpha_3 + 2\alpha_4) e^{-(\alpha_1 Q^2 + \alpha_2 Q + \alpha_3)(T-t^*)} \geq \epsilon e^{2\alpha_4(T-t^*)}. \end{aligned}$$

Substituting the equation for u'_{q^*} gives

$$\begin{aligned} & 2\alpha_4 u_{q^*}(t^*) + \rho \bar{\mu} |q^*| u_{q^*}(t^*) + \frac{1}{2} \bar{\sigma}^2 \gamma (q^*)^2 \rho u_{q^*}(t^*) \\ & - A(1 + \gamma/\rho)^{-(1+\frac{\rho}{\gamma})} u_{q^*+1}(t^*) e^{\eta \rho(|q^*| - |q^*+1|)} \\ & - A(1 + \gamma/\rho)^{-(1+\frac{\rho}{\gamma})} u_{q^*-1}(t^*) e^{\eta \rho(|q^*| - |q^*-1|)} \\ & - (\alpha_1 Q^2 + \alpha_2 Q + \alpha_3 + 2\alpha_4) e^{-(\alpha_1 Q^2 + \alpha_2 Q + \alpha_3)(T-t^*)} \geq \epsilon e^{2\alpha_4(T-t^*)}. \end{aligned} \quad (3.19)$$

Now let

$$\begin{aligned}\alpha_5 &= A(1 + \gamma/\rho)^{-(1+\frac{\rho}{\gamma})} e^{-\eta\rho}, \\ \alpha_4 &= \alpha_5 + \bar{\mu}\rho Q\end{aligned}$$

and

$$\begin{aligned}B_1 &= 2\bar{\mu}\rho Q + \rho\bar{\mu}|q^*|, \\ B_2 &= A(1 + \gamma/\rho)^{-(1+\frac{\rho}{\gamma})} u_{q^*+1}(t^*) e^{\eta\rho(|q^*| - |q^*+1|)} \\ &\quad + A(1 + \gamma/\rho)^{-(1+\frac{\rho}{\gamma})} u_{q^*-1}(t^*) e^{\eta\rho(|q^*| - |q^*-1|)} - 2\alpha_5 u_{q^*}(t^*).\end{aligned}$$

By construction, $B_1 \geq 0$ and $B_2 \geq 0$. (3.19) can then be rewritten as

$$\begin{aligned}&\frac{1}{2}\bar{\sigma}^2\gamma(q^*)^2\rho u_{q^*}(t^*) + B_1 u_{q^*}(t^*) - B_2 \\ &- (\alpha_1 Q^2 + \alpha_2 Q + \alpha_3 + 2\alpha_4) e^{-(\alpha_1 Q^2 + \alpha_2 Q + \alpha_3)(T-t^*)} \geq \epsilon e^{2\alpha_4(T-t^*)}.\end{aligned}$$

That is,

$$\begin{aligned}&\frac{1}{2}\bar{\sigma}^2\gamma(q^*)^2\rho(u_{q^*}(t^*) - e^{-(\alpha_1 Q^2 + \alpha_2 Q + \alpha_3)(T-t^*)}) \\ &+ B_1(u_{q^*}(t^*) - e^{-(\alpha_1 Q^2 + \alpha_2 Q + \alpha_3)(T-t^*)}) - B_2 \\ &- B_3 e^{-(\alpha_1 Q^2 + \alpha_2 Q + \alpha_3)(T-t^*)} \geq \epsilon e^{2\alpha_4(T-t^*)},\end{aligned}\tag{3.20}$$

where

$$B_3 = \alpha_1 Q^2 + \alpha_2 Q + \alpha_3 + 2\alpha_4 - \frac{1}{2}\bar{\sigma}^2\gamma(q^*)^2\rho - B_1.$$

This shows that there exist nonnegative constants $\alpha_1, \alpha_2, \alpha_3$ such that all terms on the left-hand side in (3.20) are negative, which contradicts to the original statement. Hence (3.17) holds in the case $|q^*| < Q$. The case $|q^*| = Q$ can be treated similarly and it is omitted here. This completes the proof. $\square$

Using Lemma 3.1 and Lemma 3.2 above we conclude that

$$\begin{aligned}U(t, s, x, q) &= -\exp(-\gamma(x + qs - \eta|q|)) u_q(t)^{-\gamma/\rho}, \\ U(t) &= e^{-E(T-t)} \bar{\mathbf{1}}_{2Q+1},\end{aligned}\tag{3.21}$$

is a solution to the associated Hamilton-Jacobi-Bellman equation in (3.4). Here, for $q \in \{-Q, -Q+1, \dots, 0, \dots, Q-1, Q\}$, $\{u_q\}$ are the components of $U(t)$ given in (3.16). In particular, the function in (3.21), for $q \in \{-Q, -Q+1, \dots, 0, \dots, Q-1, Q\}$, is the candidate for being the value function for the original optimization problem associated to the market making problem defined in (3.3). Furthermore, the candidates for the optimal controls are

$$\begin{aligned}\mu^* &= -\bar{\mu} \operatorname{sgn}(q), \quad \sigma^* = \bar{\sigma}, \\ \delta_*^+(t) &= \frac{1}{\gamma} \log(1 + \gamma/\rho) + \frac{1}{\rho} \log \frac{u_q(t)}{u_{q-1}(t)} - \eta(|q| - |q-1|), \quad q \neq -Q, \\ \delta_*^-(t) &= \frac{1}{\gamma} \log(1 + \gamma/\rho) + \frac{1}{\rho} \log \frac{u_q(t)}{u_{q+1}(t)} - \eta(|q| - |q+1|), \quad q \neq Q.\end{aligned}\tag{3.22}$$

Here δ_*^+ and δ_*^- depend on $\bar{\mu}$ and $\bar{\sigma}$, through E and $\{u_q\}$. For μ^* and σ^* , there are natural interpretations. Indeed, if the inventory is positive, $q > 0$, the worst-case scenario is that the asset price goes down and if negative, $q < 0$, the worst-case scenario is that the price goes up. In all cases, a market maker has to handle the highest possible volatility $\bar{\sigma}$.

Remark 3.1. We emphasize that when interpreting our results the reader should keep in mind that in our model the market maker is only allowed to post limit orders and that it is in the context of this model that the highest possible volatility $\bar{\sigma}$ represents the worst case volatility. Considering more flexible market makers, e.g market makers who submit both limit as well as market orders, the profitability of more aggressive HFT strategies may increase with volatility, see [9].

Theorem 3.1. *Let U be defined as in (3.21). Then $U(t, s, x, q)$ is the value function for the original optimization problem associated to the market making problem defined in (3.3)((2.3)). Furthermore, the associated optimal controls are given in (3.22).*

Proof. Let $\hat{\delta} := (\hat{\delta}^+, \hat{\delta}^-)$ and $(\hat{\mu}, \hat{\sigma}) \in \mathcal{U}$ be the admissible control processes and let $t \in [0, T]$. Consider the following processes for $\tau \in [t, T]$:

$$\begin{aligned} dS_\tau^{t,s} &= \hat{\mu}_\tau d\tau + \hat{\sigma}_\tau dW_\tau, S_t^{t,s} = s, \\ dX_\tau^{t,x,\hat{\delta}} &= (S_\tau + \hat{\delta}_\tau^+) dN_\tau^+ - (S_\tau - \hat{\delta}_\tau^-) dN_\tau^-, X_t^{t,x,\hat{\delta}} = x, \\ dQ_\tau^{t,q,\hat{\delta}} &= dN_\tau^+ - dN_\tau^-, Q_t^{t,q,\hat{\delta}} = q, \end{aligned} \quad (3.23)$$

where N^- and N^+ are point processes with intensities $\{\lambda_\tau^-\}$ and $\{\lambda_\tau^+\}$, respectively. Here $\lambda_\tau^- = Ae^{-k\hat{\delta}_\tau^-} \mathbf{1}_{\{Q_{\tau-} < Q\}}$ and $\lambda_\tau^+ = Ae^{-k\hat{\delta}_\tau^+} \mathbf{1}_{\{Q_{\tau-} > -Q\}}$. Since $\hat{\delta}^+ \geq 0$ and $\hat{\delta}^- \geq 0$ the intensities are bounded. Define

$$t_n = T \wedge \{\tau > t, |S_\tau - s| \geq n \text{ or } |N_\tau^+ - N_t^+| \geq n \text{ or } |N_\tau^- - N_t^-| \geq n\}, n \in \mathbb{N}.$$

Since U is smooth we see, simply using Itô's formula, that

$$\begin{aligned} U(t_n, X_{t_n}^{t,x,\hat{\delta}}, Q_{t_n}^{t,q,\hat{\delta}}, S_{t_n}^{t,s}) &= U(t, x, q, s) \\ &+ \int_t^{t_n} (\partial_\tau U(\tau, X_\tau^{t,x,\hat{\delta}}, Q_\tau^{t,q,\hat{\delta}}, S_\tau^{t,s}) + \mathcal{L}^{\hat{\mu}, \hat{\sigma}} U(\tau, X_\tau^{t,x,\hat{\delta}}, Q_\tau^{t,q,\hat{\delta}}, S_\tau^{t,s})) d\tau \\ &+ \int_t^{t_n} (U(\tau, X_{\tau-}^{t,x,\hat{\delta}} + S_\tau^{t,s} + \hat{\delta}_\tau^+, Q_{\tau-}^{t,q,\hat{\delta}} - 1, S_\tau^{t,s}) - U(\tau, X_{\tau-}^{t,x,\hat{\delta}}, Q_{\tau-}^{t,q,\hat{\delta}}, S_\tau^{t,s})) \lambda_\tau^+ d\tau \\ &+ \int_t^{t_n} (U(\tau, X_{\tau-}^{t,x,\hat{\delta}} - S_\tau^{t,s} + \hat{\delta}_\tau^-, Q_{\tau-}^{t,q,\hat{\delta}} + 1, S_\tau^{t,s}) - U(\tau, X_{\tau-}^{t,x,\hat{\delta}}, Q_{\tau-}^{t,q,\hat{\delta}}, S_\tau^{t,s})) \lambda_\tau^- d\tau \\ &+ \int_t^{t_n} \hat{\sigma}_\tau \partial_s U(\tau, X_\tau^{t,x,\hat{\delta}}, Q_\tau^{t,q,\hat{\delta}}, S_\tau^{t,s}) dW_\tau \\ &+ \int_t^{t_n} (U(\tau, X_{\tau-}^{t,x,\hat{\delta}} + S_\tau^{t,s} + \hat{\delta}_\tau^+, Q_{\tau-}^{t,q,\hat{\delta}} - 1, S_\tau^{t,s}) - U(\tau, X_{\tau-}^{t,x,\hat{\delta}}, Q_{\tau-}^{t,q,\hat{\delta}}, S_\tau^{t,s})) dM_\tau^+ \end{aligned}$$

$$+ \int_t^{t_n} (U(\tau, X_{\tau^-}^{t,x,\hat{\delta}} - S_{\tau}^{t,s} + \hat{\delta}_{\tau}^-, Q_{\tau^-}^{t,q,\hat{\delta}} + 1, S_{\tau}^{t,s}) - U(\tau, X_{\tau^-}^{t,x,\hat{\delta}}, Q_{\tau^-}^{t,q,\hat{\delta}}, S_{\tau}^{t,s})) dM_{\tau}^-,$$

where M^+ and M^- are the compensated processes associated respectively with N^+ and N^- for the intensities processes $\{\lambda_{\tau}^+\}$ and $\{\lambda_{\tau}^-\}$. Since each U is continuous and positive on a compact set, it has a positive lower bound and $u_q^{-\frac{\gamma}{k}}$ is bounded along the trajectory, independently of the trajectory. In addition, $\hat{\delta}^+$ and $\hat{\delta}^-$ are bounded from below. Hence, all terms in the previous stochastic integrals are bounded and, by the definition of t_n , the local martingales are in fact martingales. Taking expectation both sides gives

$$\begin{aligned} & \mathbb{E}[U(t_n, X_{t_n^-}^{t,x,\hat{\delta}}, Q_{t_n^-}^{t,q,\hat{\delta}}, S_{t_n}^{t,s})] = U(t, x, q, s) \\ & + \mathbb{E} \left[\int_t^{t_n} (\partial_{\tau} U(\tau, X_{\tau}^{t,x,\hat{\delta}}, Q_{\tau}^{t,q,\hat{\delta}}, S_{\tau}^{t,s}) + \mathcal{L}^{\hat{\mu}, \hat{\sigma}} U(\tau, X_{\tau}^{t,x,\hat{\delta}}, Q_{\tau}^{t,q,\hat{\delta}}, S_{\tau}^{t,s})) d\tau \right] \\ & + \mathbb{E} \left[\int_t^{t_n} (U(\tau, X_{\tau^-}^{t,x,\hat{\delta}} + S_{\tau}^{t,s} + \hat{\delta}_{\tau}^+, Q_{\tau^-}^{t,q,\hat{\delta}} - 1, S_{\tau}^{t,s}) - U(\tau, X_{\tau^-}^{t,x,\hat{\delta}}, Q_{\tau^-}^{t,q,\hat{\delta}}, S_{\tau}^{t,s})) \lambda_{\tau}^+ d\tau \right] \\ & + \mathbb{E} \left[\int_t^{t_n} (U(\tau, X_{\tau^-}^{t,x,\hat{\delta}} - S_{\tau}^{t,s} + \hat{\delta}_{\tau}^-, Q_{\tau^-}^{t,q,\hat{\delta}} + 1, S_{\tau}^{t,s}) - U(\tau, X_{\tau^-}^{t,x,\hat{\delta}}, Q_{\tau^-}^{t,q,\hat{\delta}}, S_{\tau}^{t,s})) \lambda_{\tau}^- d\tau \right]. \end{aligned}$$

Next, by essentially repeating the corresponding argument in [33] we can conclude that

$$\lim_{n \rightarrow \infty} \mathbb{E}[U(t_n, X_{t_n^-}^{t,x,\hat{\delta}}, Q_{t_n^-}^{t,q,\hat{\delta}}, S_{t_n}^{t,s})] = \mathbb{E}[U(T, X_T^{t,x,\hat{\delta}}, Q_T^{t,q,\hat{\delta}}, S_T^{t,s})].$$

In particular, letting $n \rightarrow \infty$ we see that the equality

$$\begin{aligned} & \mathbb{E}[U(T, X_T^{t,x,\hat{\delta}}, Q_T^{t,q,\hat{\delta}}, S_T^{t,s})] = U(t, x, q, s) \\ & + \mathbb{E} \left[\int_t^T (\partial_{\tau} U(\tau, X_{\tau}^{t,x,\hat{\delta}}, Q_{\tau}^{t,q,\hat{\delta}}, S_{\tau}^{t,s}) + \mathcal{L}^{\hat{\mu}, \hat{\sigma}} U(\tau, X_{\tau}^{t,x,\hat{\delta}}, Q_{\tau}^{t,q,\hat{\delta}}, S_{\tau}^{t,s})) d\tau \right] \\ & + \mathbb{E} \left[\int_t^T (U(\tau, X_{\tau^-}^{t,x,\hat{\delta}} + S_{\tau}^{t,s} + \hat{\delta}_{\tau}^+, Q_{\tau^-}^{t,q,\hat{\delta}} - 1, S_{\tau}^{t,s}) - U(\tau, X_{\tau^-}^{t,x,\hat{\delta}}, Q_{\tau^-}^{t,q,\hat{\delta}}, S_{\tau}^{t,s})) \lambda_{\tau}^+ d\tau \right] \\ & + \mathbb{E} \left[\int_t^T (U(\tau, X_{\tau^-}^{t,x,\hat{\delta}} - S_{\tau}^{t,s} + \hat{\delta}_{\tau}^-, Q_{\tau^-}^{t,q,\hat{\delta}} + 1, S_{\tau}^{t,s}) - U(\tau, X_{\tau^-}^{t,x,\hat{\delta}}, Q_{\tau^-}^{t,q,\hat{\delta}}, S_{\tau}^{t,s})) \lambda_{\tau}^- d\tau \right], \end{aligned}$$

holds whenever the triple $(S_{\tau}^{t,s}, X_{\tau^-}^{t,x,\hat{\delta}}, Q_{\tau^-}^{t,q,\hat{\delta}})$ satisfies the system in (3.23) for some admissible control processes $\hat{\delta} = (\hat{\delta}^+, \hat{\delta}^-)$, $(\hat{\mu}, \hat{\sigma}) \in \mathcal{U}$. In particular, applying the controls in (3.22) we see that

$$\mathbb{E}[U(T, X_T^{t,x,\delta_*}, Q_T^{t,q,\delta_*}, S_T^{t,s,\mu^*,\sigma^*})] = U(t, x, q, s),$$

where $\delta_* = (\delta_*^+, \delta_*^-)$ and S_T^{t,s,μ^*,σ^*} is the process for the asset based on (μ^*, σ^*) . Using that U solves the Hamilton-Jacobi-Bellman equation in (3.9) we also deduce, using the above, that

$$\begin{aligned} & \mathbb{E}[U(T, X_T^{t,x,\hat{\delta}}, Q_T^{t,q,\hat{\delta}}, S_T^{t,s})] \leq U(t, x, q, s) \\ & + \mathbb{E} \left[\int_t^T (\mathcal{L}^{\hat{\mu}, \hat{\sigma}} U(\tau, X_{\tau}^{t,x,\hat{\delta}}, Q_{\tau}^{t,q,\hat{\delta}}, S_{\tau}^{t,s}) - \mathcal{L}^{\mu^*, \sigma^*} U(\tau, X_{\tau}^{t,x,\hat{\delta}}, Q_{\tau}^{t,q,\hat{\delta}}, S_{\tau}^{t,s})) d\tau \right]. \end{aligned}$$

In particular, we deduce that

$$\sup_{\delta^+, \delta^-} \inf_{(\mu, \sigma) \in \mathcal{U}} E_{t,s,x,q} [\psi(S_T, X_T, Q_T)] \leq U(t, x, q, s)$$

with equality when the controls (δ_*^+, δ_*^-) and (μ^*, σ^*) are applied. Hence $U(t, s, x, q)$ is the value function for the original optimization problem associated to the market making problem defined in (3.3)((2.3)) and the proof of the lemma is complete. $\square$

3.3. Portfolio Liquidation

For the optimal portfolio liquidation problem, to construct a solution to

$$\begin{aligned} & \partial_t v(t, s, x, q) + H^*(t, \partial_s v(t, s, x, q), \partial_{ss}^2 v(t, s, x, q)) \\ & + \sup_{\delta^+} A e^{-\rho \delta^+} [v(t, s, x + (s + \delta^+), q - 1) - v(t, s, x, q)] = 0, \\ & v(T, s, x, q) = -\exp(-\gamma(x + qs - \eta q)), \end{aligned}$$

where $q \in \{0, \dots, Q - 1, Q\}$, consider an ansatz

$$\tilde{v}(t, s, x, q) = -\exp(-\gamma(x + qs - \eta q)) \tilde{v}_q(t)^{-\gamma/\rho},$$

where $v_q \in C^1(0, T)$, $q \in \{0, \dots, Q - 1, Q\}$, are to be determined. As in the case of market making, the problem is reduced to find $\{\tilde{v}_q\}$ such that

$$\begin{aligned} \tilde{v}_q'(t) &= \rho \bar{\mu} |q \tilde{v}_q(t)| + \frac{1}{2} (F(-\tilde{v}_q(t)^{-\gamma/\rho}))^2 \gamma q^2 \rho \tilde{v}_q(t) \\ &\quad - A(1 + \gamma/\rho)^{-(1+\frac{\rho}{\gamma})} \tilde{v}_{q+1}(t) e^{\eta \rho(|q| - |q+1|)}, \\ \tilde{v}_q(T) &= 1. \end{aligned}$$

To proceed, first consider an auxiliary system with a sign assumption on $\tilde{v}_q$:

$$\begin{aligned} \tilde{u}_q'(t) &= \rho \bar{\mu} q \tilde{u}_q(t) + \frac{1}{2} \bar{\sigma}^2 \gamma q^2 \rho \tilde{u}_q(t) - A(1 + \gamma/\rho)^{-(1+\frac{\rho}{\gamma})} \tilde{u}_{q+1}(t) e^{\eta \rho(|q| - |q+1|)}, \\ \tilde{u}_q(T) &= 1, \end{aligned}$$

where $q \geq 0$. In this case the candidates for the optimal controls are

$$\begin{aligned} \mu^* &= -\bar{\mu}, \quad \sigma^* = \bar{\sigma}, \\ \delta_*^+(t) &= \frac{1}{\gamma} \log(1 + \gamma/\rho) + \frac{1}{\rho} \log \frac{v_q(t)}{v_{q-1}(t)} - \eta(|q| - |q-1|). \end{aligned} \quad (3.24)$$

In particular, for a selling position, the worst-case scenario is that the asset price goes down with a speed determined by $\mu^* = -\bar{\mu}$, under the highest possible volatility $\bar{\sigma}$. In addition, with the same argument as in the case of the market making problem, it can be concluded that the solution to the Hamilton-Jacobi-Bellman equation associated with the optimal liquidation problem is

$$\begin{aligned} \tilde{U}(t, s, x, q) &= -\exp(-\gamma(x + qs - \eta q)) \tilde{u}_q(t)^{-\gamma/\rho}, \\ \tilde{U}(t) &= e^{-\tilde{E}(T-t)} \mathbf{1}_{Q+1}, \end{aligned} \quad (3.25)$$

where $\tilde{E}$ is a $(Q+1) \times (Q+1)$ -dimensional submatrix of E defined by $q = 0, \dots, Q$, i.e. $\tilde{E} = \{E_{i,j}\}$, $i, j \in \{Q, Q-1, \dots, 0\}$. The following statement holds.

Theorem 3.2. *Let $\tilde{U}$ be defined as in (3.25). Then $\tilde{U}(t, s, x, q)$ is the value function to the original optimization associated to the optimal liquidation problem defined in (3.3)((2.4)). In particular, in the optimal liquidation problem the optimal controls are given in (3.24).*

Theorem 3.2 can be proved in a similar way as Theorem 3.1. We omit further details.

4. Numerical Illustration

By Theorem 3.1, in the market making problem the optimal controls $\delta_*^+(t)$ and $\delta_*^-(t)$ are given by (3.22). In this section we show, numerically and for the market making problem, that the model, and the solution of the problem formulated and solved in the previous sections, produce results which are in line with intuition. Recall, see Theorem 3.1, that the value function for the original optimization problem associated to the market making problem defined in (3.3)((2.3)) is the function $U(t, s, x, q)$ defined in (3.21). Furthermore, the associated optimal controls are given in (3.22). In particular,

$$\begin{aligned}\delta_*^+(t) &= \delta_*^+(\gamma, \eta, \rho, A, T, q, \bar{\mu}, \bar{\sigma}, t), \\ \delta_*^-(t) &= \delta_*^-(\gamma, \eta, \rho, A, T, q, \bar{\mu}, \bar{\sigma}, t),\end{aligned}$$

and we recall that $\lambda^\pm(\delta^\pm) = Ae^{-\rho\delta^\pm}$, with $A > 0$ and $\rho > 0$.

To fix the model we have to specify γ , η , defining the utility function, and ρ , A , defining the fill rates. Furthermore, we have to fix T . Concerning T , γ , η , ρ , and A , we in our numerical illustration let

$$T = 60 \text{ s}, \quad \gamma = 0.01 \text{ Tick}^{-1}, \quad \eta = 0.1 \text{ Tick}, \quad \rho = 0.3 \text{ Tick}^{-1}, \quad A = 0.9 \text{ s}^{-1} \quad (4.1)$$

as our baseline setting. Note that this implies that we only consider the model in the range from 0-60 s and that we in the baseline setting also have defined a value for the parameter η , measuring the level of penalization to inventory remaining at T . Though we will illustrate the effect of different values of η , the above gives at hand that $q, \bar{\mu}, \bar{\sigma}, t$ are the degrees of freedom which remain to be explored. Concerning t we let

$$t = 57 \text{ s, i.e. } T - t = 3 \text{ s.} \quad (4.2)$$

The choice $T - t = 3$ in (4.2) may be realistic for a high frequency market maker applying the model, incrementally, over and over again on very short time intervals. Furthermore, this is different compared to [33] where the authors instead focus on asymptotic expansions for spreads as $T \rightarrow \infty$ something which seems less relevant in the context of high frequency trading. However, we note that the values for γ , ρ and A in (4.1) are similar to those used in [33].

Based on the above we in the following consider $\delta_*^\pm = \delta_*^\pm(t)$ as functions of $q, \bar{\mu}, \bar{\sigma}$. Recall that q is the initial level of inventory held by the market maker at t . Starting from q , as time evolves the actual inventory of the market maker during the time interval $[t, T]$ will then be random. In the following we first discuss $\delta_*^\pm$, for q fixed, as functions of $\bar{\mu}, \bar{\sigma}$. Subsequently, we fix either $\bar{\mu}$ or $\bar{\sigma}$ and consider $\delta_*^\pm$ as functions of q and the variable, among $\bar{\mu}$ and $\bar{\sigma}$, which is not fixed.

Figure 1 shows the spreads, $\delta_*^\pm$, posted by the market maker, for q and $\bar{\sigma}$ fixed, as a function of the uncertainty in the drift. We have here put $q = 25$, i.e. currently the market maker has a positive inventory, and $\bar{\sigma} = 0.3 \text{ Tick } s^{-1/2}$. In particular, based on these choices we consider the functions $\bar{\mu} \rightarrow \delta_*^\pm(\gamma, \eta, \rho, A, T, q, \bar{\mu}, \bar{\sigma}, t)$ for $\bar{\mu}$ in the range $[0, 1] \text{ Tick } s^{-1}$. In Figure 1 we see that δ_*^+ , the ask spread, and δ_*^- , the bid spread, are monotonically decreasing and monotonically increasing, respectively, as functions of $\bar{\mu}$. Furthermore, $\delta_*^-(\bar{\mu}) \geq \delta_*^+(\bar{\mu})$. In particular, the model produces results in line with intuition: since $q > 0$ the market maker is eager to reduce inventory and as a result posts ask prices closer to the mid price to increase the probability of getting a hit, while posting bid prices further away from the mid price to decrease the probability of adding to the inventory. Figure 2 and Figure 3 show the same thing as Figure 1 but for $\eta = 0$ and $\eta = 1$, respectively. In the case $\eta = 0$ there is no explicit penalization on inventory in the utility function at the terminal and as a consequence the bid and ask come closer. In the case $\eta = 1$, Figure 3, there is a higher penalization of the remaining inventory at T and as a result the ask is pushed even lower and the bid is pushed even higher. In fact, in this case the ask spread may even become negative for large uncertainty in the drift and in this case we have put $\delta_*^+ = 0$.

Figure 4 shows the spreads, $\delta_*^\pm$, posted by the market maker, for q and $\bar{\mu}$ fixed, as a function of the uncertainty in the volatility. Again $q = 25$ and we let $\bar{\mu} = 0.1 \text{ Tick } s^{-1}$. In particular, based on these choices we consider the functions $\bar{\sigma} \rightarrow \delta_*^\pm(\gamma, \eta, \rho, A, T, q, \bar{\mu}, \bar{\sigma}, t)$ for $\bar{\sigma}$ in the range $[0, 1] \text{ Tick } s^{-1/2}$. Figure 4 shows similar patterns as Figure 1: δ_*^+ and δ_*^- , are monotonically decreasing and monotonically increasing, respectively, as functions of $\bar{\sigma}$ and $\delta_*^-(\bar{\sigma}) \geq \delta_*^+(\bar{\sigma})$.

Finally, Figure 5 and Figure 6 show, respectively, for $q = 25$, the ask spread and the bid spread as functions of $\bar{\mu}$ and $\bar{\sigma}$. Figure 5 and Figure 6 give the three-dimensional pictures of the patterns outlined above. Put together, Figure 1 - Figure 6 indicate, for severe uncertainty in the direction of the stock price, that a market maker may be forced to post bets so far from the midprice, which in reality implies that the market maker may be exiting the market. Naturally a consequence of this is, assuming that market makers are supplying a substantial portion of the liquidity, as indicated in the introduction, that uncertainty may lead to a temporary reduction or evaporation of liquidity in the market.

In Figure 7 - Figure 12 we allow q to vary and we study the effect on $\delta_*^\pm$. To start with, in Figure 7 we consider a situation where the market maker is determined to post bets around the midprice with spread $\delta_* := \delta_*^+$, if $q > 0$, and $\delta_* := \delta_*^-$, if $q < 0$,

fixed to 1 Tick. Figure 7 then shows the optimal initial level of inventory q held by the market maker as a function of uncertainty in the drift $\bar{\mu}$. To construct the curves we consider, for δ_* defined as above and fixed to 1, the function $\bar{\mu} \rightarrow q(\bar{\mu})$ where $q(\bar{\mu})$ satisfies $\delta_*(\gamma, \eta, \rho, A, T, q(\bar{\mu}), \bar{\mu}, \bar{\sigma}, t) = \delta_*$ with γ, η, ρ, A, T as in (4.1) and $\bar{\sigma} = 0.3 \text{ Tick } s^{-1/2}$. We see that the optimal initial inventory held is decreasing as the uncertainty in the drift is increasing. In particular, being determined to post bets making a fixed spread through a market turbulence resulting in uncertainty in the drift forces a market maker to reduce the inventory accordingly. More uncertainty results in a lower inventory. In the same situation, Figure 8 show the same pattern but with the initial level of inventory q held by the market maker as a function of uncertainty in the volatility $\bar{\sigma}$.

Figure 9 and Figure 10 show, respectively, the ask spread and the bid spread as functions of $\bar{\mu}$ and q , and Figure 11 and Figure 12 show, respectively, the ask spread and the bid spread as functions of $\bar{\sigma}$ and q . Figure 9 confirms, for t and $\bar{\sigma}$ fixed, that

- (i) if the initial level of inventory q at t is fixed and positive, then δ_*^+ is decreasing with respect to the uncertainty in the drift $\bar{\mu}$;
- (ii) if the initial level of inventory q at t is fixed and negative, then δ_*^+ is increasing with respect to the uncertainty in the drift $\bar{\mu}$;
- (iii) if uncertainty in the drift $\bar{\mu}$ is fixed, the δ_*^+ is decreasing with respect to the initial level of inventory q at t .

Figure 10 confirms the same, but reverse pattern, for δ_*^- . Figure 11 confirms, for t and $\bar{\mu}$ fixed, that

- (i') if the initial level of inventory q at t is fixed and positive, then δ_*^+ is decreasing with respect to the uncertainty in the volatility $\bar{\sigma}$;
- (ii') if the initial level of inventory q at t is fixed and negative, then δ_*^+ is increasing with respect to the uncertainty in the volatility $\bar{\sigma}$;
- (iii') if uncertainty in the volatility $\bar{\sigma}$ is fixed, the δ_*^+ is decreasing with respect to the initial level of inventory q at t .

Again, Figure 12 confirms the same, but reverse pattern, for δ_*^- .

5. Conclusions and future research

This paper focuses on the market making problem and the optimal liquidation problem using limit orders, accounting for model risk or uncertainty. The idea of the paper is that whenever a sense of ambiguity concerning the correct underlying probability measure for a model becomes critical it would be rational for a market maker to respond by posting limit orders more based on an emphasis on the worst-case scenarios than on what would be suggested by an assumption on a Savage expected utility. For the market making problem and the optimal liquidation problem we formulate our models as (worst-case) stochastic optimal control problems, with the

controls being the spreads, relative to a reference price, at which orders are placed. The models consider uncertainty in both the drift and volatility of the underlying reference price. Using a particular ansatz we are able to solve the problems for the associated Hamilton-Jacobi-Bellman equations and we rigorously prove that the solutions constructed give the value functions to the original optimization problems. The effects on spreads and inventory are quantified and numerically illustrated. The model outlined, and the subsequent numerical illustration, are able to capture the rationale discussed and aspects of such uncertainty in the case of an ambiguous distribution of the underlying asset returns. In particular, we show how uncertainty influences the bid and ask spreads, optimal inventory levels held, and we discuss implication on potential reductions of liquidity.

Future research will be devoted to further studies of the market making problem and the optimal liquidation problem under uncertainty but in more general models compared to the models studied in this paper. In this paper we have deliberately chosen to stay in the more classical settings of [8, 40], see also [33], since, as we have shown, many interesting effects of uncertainty can already be derived in these settings. However, for further applicability it is important to allow for more general utility functions and to consider more flexible market makers, e.g. market makers who submit both limit as well as market orders. As stated in the introduction, a characteristic of HF market makers is that the strategies are designed to hold close to no inventories over very short periods of time, from seconds to at most one day, to avoid being exposed to market risk. From this perspective, in the model solved in this paper, a consequence of the fact that we only penalize inventory at the terminal time T is that the model should be used mainly for short time intervals, small T , iteratively and incrementally during the day. However, it is interesting to generalize this to a model applicable for longer time intervals and in which a large inventory is penalized whenever it occurs during the trading day, not only at the end. In practice this would imply an implementation of more or less hard risk limit for the inventory during the trading day. This kind of models can be formulated but the mathematical analysis in this case becomes considerably harder and the value functions and controls will have to be solved numerically. Concerning more flexible market makers and more flexible liquidation algorithms, we in this paper only consider situations where limit orders are submitted. In models allowing for both limit as well as market orders the worst case scenarios under uncertainty may be different compared to the situation considered in this paper. For instance and as discussed, considering more flexible strategies the profitability may very well increase with volatility, though, in our model, the highest volatility is a worst case scenario. Finally, in general any of the structures of our model, e.g. the fill rates, the underlying price process, can be refined in attempts to produce even more realistic models for the problems at hand and concerning the actual behaviors of HFTs.

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Fig. 1. Relation between Spreads Posted by a Market Maker and Uncertainty in Drift

Fig. 2. Relation between Spreads Posted by a Market Maker and Uncertainty in Drift

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Fig. 3. Relation between Spreads Posted by a Market Maker and Uncertainty in Drift

Fig. 4. Relation between Spreads Posted by a Market Maker and Uncertainty in Volatility

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Fig. 5. Relation between the Ask Spread Posted by a Market Maker and Uncertainty in Drift and Volatility

Fig. 6. Relation between the Bid Spread Posted by a Market Maker and Uncertainty in Drift and Volatility

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Fig. 7. Relation between the Inventory Level Held by a Market Maker and Uncertainty in Drift

Fig. 8. Relation between the Inventory Level Held by a Market Maker and Uncertainty in Volatility

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Relation between ask spread and uncertainty in drift & initial inventory ($T = 60$ s, $\gamma = 0.01$ Tick $^{-1}$, $\eta = 0.1$ Tick, $\rho = 0.3$ Tick $^{-1}$, $A = 0.9$ s $^{-1}$, $t = 57$ s, $\sigma = 0.3$ Tick s $^{-1/2}$)

Fig. 9. Relation between the Ask Spread Posted by a Market Maker, Initial Level of Inventory and Uncertainty in the Drift

Relation between bid spread and uncertainty in drift & initial inventory ($T = 60$ s, $\gamma = 0.01$ Tick $^{-1}$, $\eta = 0.1$ Tick, $\rho = 0.3$ Tick $^{-1}$, $A = 0.9$ s $^{-1}$, $t = 57$ s, $\sigma = 0.3$ Tick s $^{-1/2}$)

Fig. 10. Relation between the Bid Spread Posted by a Market Maker, Initial Level of Inventory and Uncertainty in the Drift

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Relation between ask spread and uncertainty in volatility & initial inventory ($T = 60$ s, $\gamma = 0.01$ Tick $^{-1}$, $\eta = 0.1$ Tick, $\rho = 0.3$ Tick $^{-1}$, $A = 0.9$ s $^{-1}$, $t = 57$ s, $\mu = 0.1$ Tick s $^{-1}$)

Fig. 11. Relation between the Ask Spread Posted by a Market Maker, Initial Level of Inventory and Uncertainty in Volatility

Relation between bid spread and uncertainty in volatility & initial inventory ($T = 60$ s, $\gamma = 0.01$ Tick $^{-1}$, $\eta = 0.1$ Tick, $\rho = 0.3$ Tick $^{-1}$, $A = 0.9$ s $^{-1}$, $t = 57$ s, $\mu = 0.1$ Tick s $^{-1}$)

Fig. 12. Relation between the Bid Spread Posted by a Market Maker, Initial Level of Inventory and Uncertainty in Volatility