Numerical simulations of the Cordilleran Ice Sheet (CIS)
Implementing a new module to the ice code ARCTIC-TARAH

Asako Fujisaki
Abstract

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Paleoglacialogy is an area of research where numerical reconstruction is actively used to understand the process of ice sheet growth and decay throughout past glacial periods, and numerical simulation of ice dynamics is a tool to reproduce the behaviors of the ice sheet and shelf under given conditions. ARCTIC-TARAH, designed by the Bolin Centre for Climate Research for simulating ice sheet dynamics, is developed based on the Pennsylvania State University Ice sheet model (PSUI). We arrange ARCTIC-TARAH for the simulation of the Cordilleran Ice Sheet (CIS), which periodically appears in the northwestern corner over North America during glacial periods, and simulate for 30,000 years in order for the researchers at the Bolin Centre to be able to perform real CIS paleo-simulations in the future.

Based on the Last Glacial Maximum (LGM), the Mid-Holocene (MH), and the present time (PT) climate data, several different experiments are run for arbitrary 10,000-30,000 years. We discuss the potential problems of the CIS simulation and suggest further improvements on the model. The major issues encountered are (1) the basal topography, (2) grounding line treatments, and (3) the climate setting. Both the steep and jagged surface of a mountain range and the basal topography near the edge of a continental slope require close attention for numerical stability. Because the topography in some areas of the coastline is very steep and the width of the continental slope is narrow, there is a large amount of ice mass that flows out into the ocean from land. The transition zone between the ice sheet and shelf may approach the continental slope in a relatively short time. This large volume flux across the grounding line on the steep seabed overwhelms the stability of the model. In addition, the model uses a PDD (positive-degree-day) method to approximate the budget of ice accumulation and ablation. We suggest either implementing a better PDD method or coupling it with a climate model to capture the coastal and continental climate characteristics, as well as the local extreme climate in the mountains and along the coast.
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1 Introduction

On Earth, there are currently two ice sheets, the Greenland Ice Sheet, and the Antarctic Ice Sheet. During past glacial periods, there were however other ice sheets that covered regions that today are ice-free, such as large parts of Canada, Siberia, northern Europe, and southern South America [1]. An ice sheet is composed of densely packed snow over hundreds and thousands of years and is a mass of glaciers and ice caps covering terrain greater than 50,000km$^2$ [2]. The presence of ice sheets contributes largely to the dynamics of the hydrosphere and the atmosphere, and thus, the ice sheet is one of the most significant players in controlling global climate in the long term.

Ice sheets may influence the following contributors to climate change: salinity, atmospheric composition, ocean circulation, and global heat transfer. During the last glacial maximum (LGM), the period between 26,000 to 19,000 years ago, the global ice volume was 52 million cubic kilometers greater, which is equivalent to 18 Greenland ice sheets today [3, 4], and the global sea level was, on average, about 121 meters lower than the current level [5]. Today, roughly 69 percent of total fresh water on Earth is in the form of the Greenland and the Antarctic ice sheets [6], and approximately 99 percent of terrestrial ice lies over these two regions [7]. It is likewise estimated that the sea level would rise as much as 70 meters if all ice sheets melted completely [8]. These simple numbers demonstrate that the ice sheets are important agents impacting the global hydrosphere, and thereby, global climate and its change. For instance, the release of freshwater from the ice sheets into the ocean will alter its salinity. During LGM, because the global freshwater budget was largely contained in ice sheets, ocean salinity was higher than it is today. Higher salinity reduces the solubility of carbon dioxide and releases it into the atmosphere from the ocean, which accelerates the temperature rise. Salinity also affects ocean circulation, so-called thermohaline circulation, which is driven by gradients of temperature and salinity. A water parcel differing in density from a neighboring one becomes more or less buoyant and is replaced by the adjacent parcel, creating horizontal and vertical stratification of ocean water. Today the dominant driving force in thermohaline circulation is temperature, whereas the salinity gradient was more relevant during LGM [9]. Similar to ocean water circulation, atmospheric circulation is largely driven by the temperature gradient. Ice sheets generally have high albedo, the ratio of reflected radiation over the total radiation reaching Earth’s surface, which maintains the polar regions at a low temperature. The area and the thickness of ice sheets are tightly associated with the change in neighboring surface temperatures. The greater the difference in temperature between the poles and the equator, the more dynamic the atmospheric heat transfer is; therefore, the presence of ice sheets alters the pattern of global energy transfer.
Although the change in present day Antarctica and Greenland ice masses, along with their effect on climate, have been studied extensively in the last couple of decades, the changes that ice sheets of the past underwent still remain unknown. There exist some climate proxy data and geological traces to estimate the presence of ice mass at certain locations and within a certain time range, but in many cases they are not sufficient enough to accurately understand the details of ice sheet dynamics. The more we understand about the behavior of past ice sheets and their contributions to past climate, the more accurately we can predict future climate with respect to change in present ice mass.

1.1 Ice sheet model

Numerical simulation is an important tool in investigating the behavior of a system that is difficult to examine with laboratory experiments for reasons such as cost, time, and scale, and also to predict past/future events. Numerical ice sheet modeling is only half a century old, but the models have been improved over time.

The scaled Shallow Ice Approximation (SIA) is a well-known simplification of full-Stokes equations applied to many ice models [10, 11], assuming the horizontal extent of the ice sheet is significantly larger than its vertical extent and relatively flat topography. This approximation is, however, not appropriate near ice margins on land and when approaching the grounding line, where the inland ice detaches from the ground and begins floating on the ocean (fig. 1). SIA is by construction not applicable to ice shelves. The ice shelf is a floating ice mass on the ocean attached to the ice sheet. Although a "shallowness" assumption of the ice shelf is the same as that of the ice sheet, the ice shelf deforms by stretching, whereas shearing deformation dominates the ice sheet flow (fig. 41). The ice shelf dynamics in a region that is far enough from an ice sheet-shelf transition are solved using an approximation called the Shallow Shelf Approximation (SSA) [10]. Figure 1 shows a schematic diagram of an ice sheet, shelf, and their transition zone.

ARCTIC-TARAH is a numerical ice sheet model and is a modified version of the Pennsylvania State University 3D Ice sheet-shelf hybrid model (PSUI), originally developed for simulations of Antarctic glaciations by D. Pollard and R.M. DeConto [13]. ARCTIC-TARAH is in use at the Bolin Centre for Climate Research and has been successfully applied to the paleoglacial simulation of the the Eurasian North [14] and to simulations of the interaction between ice dynamics and glacial hydraulics [15]. Here, we amend the model to be applicable to glaciations of the North American Continent, in particular, the Northwest.

The intent of this project is to apply the ARCTIC-TARAH model to the Cordilleran Ice Sheet (CIS) in the western part of North America and to observe possible growth of ice mass from an arbitrary ice-free condition and decay of the ice sheet from cold to warm periods. The CIS was a major ice sheet covering the
Figure 1: Cross-section of the ice sheet and shelf [12]. We note that there is no sediment contributions in the model unlike on the figure, e.g., all ice bottoms are only touching either bedrock or sea water.

Figure 2: Velocity profile in $x$ direction, modified from [12]. Surface and basal velocities in $y$ direction ($v_s, v_b$) are similar to ($u_s, u_b$). The vertical gradient of the horizontal velocity ($\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}$) for ice shelves is approximately zero. The basal sliding velocity ($\tilde{u}_b, \tilde{v}_b$), discussed in eq. (43) is the basal velocity ($u_b, v_b$) without the small gliding velocity at the base.

western part of the Continental Divide of the Americas during LGM. The eastern part of the CIS merged with the Laurentide Ice Sheet (LIS), which covered the majority of Canada and the northern United States during the same time period. Numerical simulations of LIS dynamics have been focused on relatively more than
that of the CIS because the steep topography of the CIS often violates the flat-base assumption of the SIA. The ice flow on the steep slopes, particularly with the mild annual temperature and high precipitation, is relatively dynamic, and the model must be capable of treating a large influx and efflux of ice in order to correspond with mass conservation correctly [16, 17]. CIS modeling requires a higher-order model for solving the Stokes equation, and some numerical treatments and grid refinements appropriate to the complex local topography [18, 16].

1.2 Goals of the project

This thesis project is a pilot study for the numerical simulation of the CIS. Having said that, it is not meant to be a realistic paleoclimate ice dynamic simulation. The main focus of the project is to configure the ARCTIC-TARAH model for the CIS. This is done by assigning suitable climate and topographical inputs, and by setting up appropriate numerical treatments for future studies. Another aim of the project is to create an interface for converting model outputs from netCDF format into the raster data format readable by ArcGIS, a mapping analysis software. GIS tools, including the ArcGIS software, are popular among collaborating geologists in the Department of Physical Geography and Quaternary Geology and the Bolin Centre for Climate Research at Stockholm University. It is a common barrier within a multidisciplinary team that the software and data formats used by one team are not familiar to the others. The GIS interface created as part of this thesis is a very useful tool that can be utilized by research teams to analyze and visualize future model output.

2 Theory

In this section, we introduce the basic theories and concepts underlying ice sheet dynamics, and then we present how they are implemented in the ARCTIC-TARAH model.

Although the time scale of ice sheet movement ranges from years to millennia, the ice sheets are continuously in motion, moving down along hills and streams to the bottom of valleys and oceans. Ice sheet dynamics is a study of motion and deformation patterns of a large ice mass driven by gravity. The pattern of motion is mainly governed by basal topography, type of ice-bed (e.g. soft, hard bed or stream), and atmospheric and geothermal heat. The motion is largely influenced by what underlies the ice. A rough surface disturbs the ice flow at the base, and the ice sheet erodes soft beds as it moves and changes the basal topography. For simplicity, we assume a homogeneous bed whose properties are characterized by its frictional response to ice sliding over it. The atmosphere contributes to addition
(snowfall) and loss (melting) of ice mass on the ice and bare ground surface. The
temperature at the ice base is often warmer than those in the interior part of the
ice mass due to the geothermal flux, with heat being generated by basal friction
and internal shearing. Because the melting point of ice is dependent on pressure,
meaning that it is not exactly 0°C or 273.15°K, the melting point deep in the
ice, known as the pressure melting point $T_{pm}$, is lower than that at the standard
atmosphere, 1 atm or 101,325 Pa [19]. These factors induce melting of ice at the
bottom and produce a thin water layer between the ice mass and bed which allows
the ice sheet to slide faster than a frozen-base ice sheet.

All quantities above (mass, velocity, temperature, etc.) closely interact with
each other to describe the overall motion of ice. In the model, they are iteratively
solved in each grid cell for every time step by fundamental equations derived from
the basic laws of physics. All symbols and variables used in this paper are listed
in tables 5 and 6 in the appendix.

2.1 Conservation Equations

The equations that govern the dynamics of nearly any materials including ice
are based on conservation principles. Ice thickness, velocity, and temperature are
computed from the basic mass, momentum, and energy balance and constitutive
equations. The equation is

**Mass balance**  This is also known as a continuity equation which states that the
total mass of an isolated system remains constant over time in spite of any change
in physical or chemical properties.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{U} = 0,$$

(1)

where $\rho$ is density, $t$ is time, and $\mathbf{U}$ is a velocity field $\mathbf{U}(t, x, y, z)$. ($\nabla \cdot$) denotes a divergence operator. For an incompressible material which we often assume in ice
modeling, the equation can be simplified as

$$\nabla \cdot \mathbf{U} = 0,$$

(2)

assuming uniform density $\rho$, and its rate of change over time is 0 ($\frac{\partial \rho}{\partial t} = 0$).

**Momentum balance**  Momentum $P$ is a product of mass and velocity and is also
conserved in the isolated system. The conservation of momentum is represented
by the Navier-Stokes equation:

$$\rho \frac{D \mathbf{U}}{Dt} = \nabla \cdot \sigma + \mathbf{F},$$

(3)
where \( \sigma \) is a stress tensor in three dimensions, and \( F \) is a body force per unit volume, e.g., a product of density \( \rho \) and gravity \( g \). \( \frac{D}{Dt} \) is a material derivative defined as

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U \cdot \nabla.
\]

(4)

The stress tensor is symmetric due to the conservation of angular momentum \( (\sigma = \sigma^T) \) and can be decomposed into a deviatoric part \( \hat{\sigma} \), and the hydrostatic pressure \( p \)

\[
\sigma = -pI + \hat{\sigma},
\]

(5)

where \( I \) is an identity matrix. In 3-by-3 matrix form eq. (5) is

\[
\sigma = \begin{pmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{pmatrix} = -\begin{pmatrix}
p & 0 & 0 \\
0 & p & 0 \\
0 & 0 & p
\end{pmatrix} + \begin{pmatrix}
\hat{\sigma}_{xx} & \hat{\sigma}_{xy} & \hat{\sigma}_{xz} \\
\hat{\sigma}_{yx} & \hat{\sigma}_{yy} & \hat{\sigma}_{yz} \\
\hat{\sigma}_{zx} & \hat{\sigma}_{zy} & \hat{\sigma}_{zz}
\end{pmatrix}.
\]

(6)

or

\[
\hat{\sigma} = \begin{pmatrix}
\sigma_{xx} + p & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} + p & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz} + p
\end{pmatrix}.
\]

(7)

The divergence of the stress tensor in eq. (3) is then

\[
\nabla \cdot \sigma = -\nabla p + \nabla \cdot \hat{\sigma},
\]

(8)

where \( \nabla \) denotes a gradient of a scalar field.

Using the relationship from eq. (8), eq. (3) becomes zero on the right hand side,

\[
-\nabla p + \nabla \cdot \hat{\sigma} + \rho g = 0,
\]

(9)

by assuming slow motion and no force of inertia from acceleration yield, \( \rho \frac{\partial U}{\partial t} + U \cdot \nabla U \approx 0 \), and by setting \( F = \rho g \).

The relationship between shear stress and strain rate of solid ice motion is non-linear, so called non-Newtonian flow. The viscosity increases with decreased shear stress, and such effect is called shear thinning or pseudoplastic. The viscosity is non-linearly dependent on temperature as well, and an ice mass is prone to deform as temperature increases. This non-linear constitutive relation for ice, called Glen’s flow law, describes the relationship among the stress, the strain rate \( \dot{\epsilon} \), and the ice temperature \( T \) as

\[
\dot{\epsilon} = A(T)\sigma_i^{n-1}\hat{\sigma},
\]

(10)

where \( A(T) \) is an Arrhenius temperature dependent coefficient, \( n \) is an ice rheological exponent, usually between 2 and 4, and \( \sigma_i \) is the second deviatoric stress.
invariant. The definition of the strain rate tensor in relation to the velocity gradient is
\[
\dot{\epsilon} = \frac{1}{2} (\nabla U + (\nabla U)^T) = \begin{pmatrix}
\frac{1}{2} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \\
\frac{1}{2} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\frac{1}{2} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}
\end{pmatrix}.
\] (11)

The second deviatoric stress invariant \(\sigma_{II}\) is written as
\[
\sigma_{II}^2 = \sum_i \sum_j \frac{1}{2} \hat{\sigma}_{ij} \hat{\sigma}_{ij},
\] (12)
where \(i\) and \(j\) are indeces of the matrix. The Arrhenius temperature dependent coefficient is given by
\[
A(T) = E A_0 \exp \left( \frac{Q}{R} \left( \frac{1}{T_{pm}} - \frac{1}{T} \right) \right),
\] (13)
where \(E\) is an enhancement flow factor, \(A_0\) is a flow law coefficient, \(Q\) is an activation energy, \(R\) is a gas constant, and \(T_{pm}\) is the ice melting point temperature, decreasing by depth from the ice surface, and \(T\) is the temperature of ice [20]. The enhancement flow factor is set 10 and 3 for ice sheets and shelves, respectively, in our model.

**Energy balance** The total energy equation, balanced with the sum of internal energy and kinetic energy, is expressed as
\[
\rho \frac{D(cT)}{Dt} = \nabla \cdot (k \nabla T) + \sigma : \dot{\epsilon},
\] (14)
where \(c\) is a specific heat capacity, \(k\) is a heat conductivity, and \((\sigma : \dot{\epsilon})\) is a double-dot product of the stress and strain rate tensors, \(\sum_i \sum_j (\sigma_{ij} \dot{\epsilon}_{ji})\). Hence, the energy conservation equation using the definition of material derivative eq. (4) is simplified as
\[
\rho c \left( \frac{\partial T}{\partial t} + U \cdot \nabla T \right) = k \nabla^2 T + \sigma : \dot{\epsilon},
\] (15)
by assuming specific heat capacity of ice \(c\) and heat conductivity \(k\) are constant in time and space. Boundary conditions for the base and free surface of ice are described in section 3.2.1.

In summary, this is a set of the conservation equations derived from eqs. (1), (3) and (14) for incompressible non-Newtonian ice flow:
Mass: \( \nabla \cdot \mathbf{U} = 0 \)

Momentum: \(- \nabla p + \nabla \cdot \mathbf{\sigma} + \rho g = 0 \) \hspace{1cm} (16)

Energy: \( \rho c \left( \frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla T \right) = k \nabla^2 T + \mathbf{\sigma} : \dot{\mathbf{\varepsilon}}. \)

In addition to taking the general mechanical characteristics of ice into consideration, the non-linear momentum equation can be further simplified based on the Shallow Ice Approximation (SIA) and the Shallow Shelf Approximation (SSA). As the result of reduction of the stress components, the internal energy term \((\mathbf{\sigma} : \dot{\mathbf{\varepsilon}})\) in the energy equation, and upper and lower boundary conditions can be also simplified.

### 2.2 Shallow Ice and Shallow Shelf Approximations

The mathematical descriptions of all ice sheet models are based on the principle of conservation equations above. It is, however, computationally expensive to solve for all stress components in the momentum equation throughout the large 3-dimensional domain over tens to hundreds of thousands of years. Although many ice models today use higher-order approximations for accuracy, these are three well-known approximations: hydrostatic approximation, Shallow Ice Approximation (SIA), and Shallow Shelf Approximation (SSA). ARCTIC-TARAH is implemented based on hydrostatic approximation, and a hybrid of SIA and SSA. We will give a brief overview of these fundamental zeroth-order approximations.

#### 2.2.1 Hydrostatic approximation

The hydrostatic approximation is applicable to both ice sheets and shelves and eliminates two components in the stress tensor in the vertical momentum balance equation. Since \( \mathbf{g} \) in eq. (16) is a scalar field downward in \( z \) direction, \( \mathbf{g} = (0, 0, -g)^T \), the momentum equation in component form is expressed as

\[
\begin{align*}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0 \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= \rho g,
\end{align*}
\hspace{1cm} (17)
\]
and this complete form of the equation is often called the full-Stokes equation. Shear stress $\sigma_{zx}$ and $\sigma_{zy}$ are more than two orders of magnitude smaller in comparison to $\sigma_{zz}$ [10], and therefore, the vertical gradient of $\sigma_{zz}$ is assumed to be only balanced by the gravity force.

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0
\]

\[
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0
\]

\[
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g.
\]

(18)

The longitudinal stress gradient in $z$ direction can be integrated as

\[
\sigma_{zz} = -\rho g (h_s - z)
\]

(19)

where $h_s$ is ice surface altitude, and $z$ is the vertical axis of the Cartesian coordinate system. The upper condition at the ice surface is 0 ($z=h_s$), and the lower boundary condition at the ice base $-\rho gh$ ($z=h_b=h_s-h$) where $h_b$ is the basal elevation and $h$ is ice thickness: $h=h_s-h_b$.

2.2.2 Shallow Ice Approximation (SIA)

The concept of SIA was initially introduced by Fowler and Larsen in 1978. Derivations of SIA and SSA require a scaling analysis of full-Stokes equation and perturbation expansion, and the detail is well-described in [21, 22, 23]. Briefly speaking, by assuming a small depth-to-width aspect ratio,

\[
\frac{H}{L} \ll 1,
\]

(20)

and relatively flat bed topography, the deformation of the large shallow ice mass is only driven by the shear stresses in the horizontal plane, $\sigma_{xz}$ and $\sigma_{yz}$ [10]. The shear stress in the vertical planes ($\sigma_{xy}$, $\sigma_{yz}$) and the longitudinal stresses deviators ($\hat{\sigma}_{xx}$, $\hat{\sigma}_{yy}$, $\hat{\sigma}_{zz}$) are negligible, so that the normal stresses ($\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{zz}$) are equal to negative pressure, $-p$, from the relationship in eq. (5); therefore, the momentum
equation can be further modified as

\[-\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0,\]
\[\frac{\partial \sigma_{yx}}{\partial x} - \frac{\partial p}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0,\]
\[\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} - \frac{\partial p}{\partial z} = \rho g.\]  

(21)

Since \(\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p\) and \(\sigma_{zz}\) is balanced by the hydrostatic pressure as shown in eq. (19), an expression for pressure is

\[p = \rho g (h_s - z),\]  

(22)

and therefore, \(\frac{\partial \sigma_{xz}}{\partial z}\) and \(\frac{\partial \sigma_{yz}}{\partial z}\) in eq. (21) are equal to

\[\frac{\partial \sigma_{xz}}{\partial z} = \frac{\partial p}{\partial x} = \rho g \frac{\partial h_s}{\partial x}, \quad \frac{\partial \sigma_{yz}}{\partial z} = \frac{\partial p}{\partial y} = \rho g \frac{\partial h_s}{\partial y}.\]  

(23)

Now, compared to the full-Stokes equation in eq. (17), the stress components in the momentum equation are largely simplified to

\[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho g \frac{\partial h_s}{\partial x},\]
\[\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \rho g \frac{\partial h_s}{\partial y},\]  

(24)
\[\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g.\]

This approximation is only appropriate at the interior part of ice sheets up to about 10km away from the ice terrestrial margin and the boundary between ice sheets and shelves, known as the grounding line (fig. 1) [10].

2.2.3 Shallow Shelf Approximation (SSA)

Ice shelves are thick ice masses that float on the ocean and are an extension of the land ice. The continuous motion of ice sheets toward the coastline pushes out the ice shelf farther away to the ocean, and the outer edge of the floating ice mass, the calving front, eventually breaks off from the ice shelf. It is known as ice calving and is a type of ice ablation.
The SSA is applicable to the interior part of ice shelves, approximately between 10km away from the grounding line and calving front [10]. The major difference from ice sheet motion is that there is almost no basal shear stress induced by the ocean in contact with the ice base (fig. 41). The horizontal velocity profile of the ice shelf far enough from the grounding line is vertically uniform [10].

\[ \frac{\partial u}{\partial z} \approx 0, \quad \frac{\partial v}{\partial z} \approx 0. \]  
\[ (25) \]

The very small basal drag also implies that the vertical shear stresses \( (\sigma_{xz}, \sigma_{yz}) \) on the ice surface and base are assumed to be equal. Thus, in addition to the simplification from the hydrostatic approximation eq. (18), the vertical gradient of the vertical shear stress, \( \frac{\partial \sigma_{xz}}{\partial z} \) and \( \frac{\partial \sigma_{yz}}{\partial z} \), can also be eliminated from the momentum balance equation. Higher-order shelf approximations, however, often maintain these vertical shear terms [24].

\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \]
\[ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0 \]
\[ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g \]  
\[ (26) \]

Unlike the ice sheet motion, the ice shelf deforms rather by horizontal stretching than shearing.

### 2.3 Ice Dynamics

In the following subsections we will introduce equations for ice temperature, ice velocity, and ice thickness implemented in the ARCTIC-TARAH model.

#### 2.3.1 Ice temperature

This is the modified energy eq. (16) for the ice temperature \( T(x,y,\zeta,t) \) in the model [25]:

\[ \frac{\partial T}{\partial t} = -u \frac{\partial T}{\partial x} - v \frac{\partial T}{\partial y} - w \frac{\partial T}{\partial \zeta} + \frac{k_i}{\rho_i c_i h^2} \left( \frac{\partial^2 T}{\partial \zeta^2} \right) + \frac{Q_i}{\rho_i c_i} \]  
\[ (27) \]

where \( T \) is the absolute temperature, \( \zeta \) is a scaled vertical coordinate \( \frac{h-z}{h} \) (see fig. 4) ranging from 0 (ice surface) to 1 (ice bottom), \( h \) is ice thickness, \( u \) and \( v \) are
horizontal velocities, $w'$ is a scaled vertical velocity $\frac{d\xi}{dt}$, $\rho_i$ is the density of ice, $c_i$ is the specific heat capacity of ice, $k_i$ is the thermal conductivity of ice, and $Q_i$ is the internal shear heating which is equivalent to $\sigma : \dot{\varepsilon}$ in the energy conservation eq. (16).

The reason why the scaled $\xi$ coordinate is employed instead of the ordinary $z$ coordinate is to avoid the upper and lower boundary errors [26]. Imposing the boundary conditions on the surface and the bottom of the ice which are not horizontal to the vertical grid is numerically problematic with the finite difference method. The upper and lower boundary conditions are assigned normal to the surface and the base of the ice, and it is difficult to follow each surface of structured 3D Cartesian grid cells at the vertical domain and re-impose the boundary conditions onto the grid surface. Figure 3 compares the uniformly spaced $z$ coordinate system and the scaled $\xi$ coordinate system in 2D. In the $z$ coordinate system, the real boundary condition between the ice and the atmosphere/bedrock (black-headed triangle arrows) and the boundary imposed on the domain (white-headed arrows) must be nearly identical to avoid the boundary errors. It requires additional treatments such as finer grids and more accurate higher-order difference methods near the upper and lower boundaries. Moreover, the number of grids within the vertical domain increases or decreases as ice thickness varies by time and space. The scaled $\xi$ coordinate, in contrast, has a constant number of vertical layers, and the ice surface and base are transformed to be horizontal to the vertical axis so that the boundary between the ice and the atmosphere/bedrock is always at the edge of the domain. While it adds extra steps to transform 3D equations back and forth, the scaled and transformed coordinate system is widely applied to many ice models for accuracy [20].

Because of the shallowness of the ice mass, the horizontal heat transfer is negligible relative to the vertical heat diffusion [27]. Hence, only $xz$, $yz$, $zx$, and $zy$ components of the stress and strain rate are taken into account as the following [28]:

$$Q_i = \sigma : \dot{\varepsilon} = \sigma_{xx}\dot{\varepsilon}_{xx} + \sigma_{yx}\dot{\varepsilon}_{yx} + \sigma_{zx}\dot{\varepsilon}_{zx} + \sigma_{zy}\dot{\varepsilon}_{zy}. \quad (28)$$

From the SIA and SSA, the horizontal gradient of the vertical velocity is small in comparison to the vertical gradient of the horizontal velocity [29, 30], $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \gg \frac{\partial w}{\partial z}$, and therefore, eq. (11) becomes

$$\dot{\varepsilon} = \begin{pmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{xy} & \dot{\varepsilon}_{xz} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_{yy} & \dot{\varepsilon}_{yz} \\ \dot{\varepsilon}_{zx} & \dot{\varepsilon}_{zy} & \dot{\varepsilon}_{zz} \end{pmatrix} = \left( \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \frac{1}{2} \frac{\partial u}{\partial z} \right). \quad (29)$$

Using the symmetry of the stress tensor and the relations in eqs. (6) and (29), the
internal shear heating equation is rewritten as

\[ Q_i = \sigma_{zx} \frac{\partial u}{\partial z} + \sigma_{zy} \frac{\partial v}{\partial z}. \]  

(30)

2.3.2 Ice velocity

There are two kinds of ice velocity equations integrated in the model. The first one is three-dimensional, calculated in each grid cell for the ice temperature equation. The second one is a SIA-SSA hybrid two-dimensional velocity, which is unique and proposed by Pollard and DeConto (2006).
3D ice velocity: $U(x, y, \zeta, t)$ The horizontal ice velocity $(u, v)$ in the advective part of eq. (27) is an addition of the basal velocity $(u_b, v_b)$ and the internal shear ice velocity $(u_i, v_i)$,

$$u = u_b - 2(\rho_i g)^n |\nabla h_s|^{n-1} \frac{\partial h_s}{\partial x} \int_{h_b}^{z} A(T)(h_s - z')^n \, dz'$$  \hspace{1cm} (31a)

$$v = v_b - 2(\rho_i g)^n |\nabla h_s|^{n-1} \frac{\partial h_s}{\partial y} \int_{h_b}^{z} A(T)(h_s - z')^n \, dz'$$  \hspace{1cm} (31b)

or using $\zeta$ instead of $z$

$$u = u_b - 2(\rho_i g)^n |\nabla h_s|^{n-1} \frac{\partial h_s}{\partial x} \int_{\zeta}^{1} h^{n-1} A(T)(\zeta')^n \, d\zeta'$$  \hspace{1cm} (32a)

$$v = v_b - 2(\rho_i g)^n |\nabla h_s|^{n-1} \frac{\partial h_s}{\partial y} \int_{\zeta}^{1} h^{n-1} A(T)(\zeta')^n \, d\zeta'$$  \hspace{1cm} (32b)

where $n$ is the ice rheological coefficient, $h_b$ is the bedrock altitude, $h_s$ is the ice surface altitude: $h_s = h_b + h$ for ice sheets and $h_s = S + h(1 - \frac{\rho_i}{\rho_w})$ for ice shelves ($S$ is sea level, $\rho_w$ is density of water, and $\rho_i$ is density of ice), and $A(T)$ is the ice rheological coefficient as discussed in section 2.1 [20, 25]. These equations are derived by integrating a combination of SIA, Glen’s flow law, and the second deviatoric stress invariant from eqs. (10), (12) and (24) [30].

The vertical velocity $w' = \frac{d\zeta}{dt}$ is

$$w' = \frac{1}{h} \left( M_b + \frac{\partial h}{\partial t} (1 - \zeta) + \frac{\partial u_h}{\partial x} + \frac{\partial v_h}{\partial y} \right) ,$$  \hspace{1cm} (33)

where $M_b$ is the melting/freezing rate at the base of the ice mass (basal mass balance) [20], and $u_h$ and $v_h$ are defined as

$$u_h = u_h \left( \frac{\int_{\zeta}^{1} \int_{\zeta}^{1} h^{n-1} A(T)(\zeta')^n \, d\zeta'' \, d\zeta'}{\int_{\zeta}^{1} \int_{\zeta}^{1} h^{n-1} A(T)(\zeta')^n \, d\zeta'' \, d\zeta'} \right)$$  \hspace{1cm} (34a)

$$v_h = v_h \left( \frac{\int_{\zeta}^{1} \int_{\zeta}^{1} h^{n-1} A(T)(\zeta')^n \, d\zeta'' \, d\zeta'}{\int_{\zeta}^{1} \int_{\zeta}^{1} h^{n-1} A(T)(\zeta')^n \, d\zeta'' \, d\zeta'} \right) .$$  \hspace{1cm} (34b)

At the lower boundary $\zeta = 1$ ($z = h_b$), the fraction of the double integrals is 0:

$$w' = \frac{1}{h} (M_b),$$  \hspace{1cm} (35)

and at the upper boundary $\zeta = 0$ ($z = h_s$), the double integral in the denominator is equal to that in the numerator:

$$w' = \frac{1}{h} \left( M_b + \frac{\partial h}{\partial t} + u_x \frac{\partial h}{\partial x} + v_y \frac{\partial h}{\partial y} \right) .$$  \hspace{1cm} (36)
The wavy underline term is equivalent to \((M_s - M_b)\) where \(M_s\) is the surface mass balance in eq. (53); this is discussed in detail in section 2.3.3, and thus, the vertical velocity at the ice surface is simply expressed by the mass balance term as

\[
w' = \frac{1}{h} (M_s).
\]  

(37)

Figure 4: Schematic diagram of \(\zeta\)-coordinate, and vertical discretization in the model. \(\zeta\)-coordinate points downward (the opposite of \(z\)-coordinate), defined as \(\zeta = \frac{h - z}{h}\); 0 at the top and 1 at the bottom of the ice. Unlike \(x\)- and \(y\)-coordinates, \(\Delta \zeta\) is not uniform and the grid size is finer near the upper and lower boundaries.

2D vertical-mean ice velocity: \(\bar{U}(x, y, t)\) Since the vertical profile of the horizontal velocity for ice sheets differs from that of ice shelves due to the difference in their basal shear stresses (fig. 41), the ice velocity equation in the model is a combination of the SSA and SIA equations [13]. Firstly, the location of a sheet-shelf transition zone, grounding line, and the out flux of ice across the grounding line \(q_g\) need to be determined. The grounding line flux is parameterized by a technique suggested by Schoof [31].

Schoof’s grounding line treatment At the grounding line, ice area flux \(q_g\) from the ice mass on land into the floating ice on the ocean is calculated as

\[
q_g = \left( \frac{\bar{A}(\rho_i g)^{n+1}(1 - \rho_i/\rho_w)^n}{4^n C} \right) \frac{\frac{1}{n+1}}{h_{\text{ms}}+1} \left( \frac{\tau}{\tau_f} \right)^n \frac{\frac{m_{\text{ms}}+n+3}{n+1}}{h_g^{m_{\text{ms}}+n+3}},
\]  

(38)
where $\rho_w$ and $\rho_i$ are density of water and ice, $\bar{A}$ is the vertical mean of the Arrhenius temperature dependent coefficient in eq. (13): 

$$\bar{A} = \frac{1}{h} \int A \, dz,$$

$C$ is the basal sliding coefficient between bed and ice, $n$ and $m_s$ are ice rheological and basal sliding exponents 3 and 2, respectively [31]. $\tau$ is the longitudinal stresses ($\tau_{xx}$, $\tau_{yy}$) downstream of the grounding line, and $\tau_f$ is the free stress, defined as $\frac{1}{2}\rho_i gh(1 - \frac{\rho_i}{\rho_w})$ [32]. The ice velocity at the grounding line is

$$u_g = \frac{q_g}{h_g}.$$  \hspace{1cm} (39)

Beyond the grounding line, the rate of change in horizontal velocities over the depth is zero from the shallow shelf approximation (SSA), and thus, $u_g$ is the same as the velocity of ice shelves (fig. 41).

**Hybrid of SIA and SSA** The vertical-mean horizontal velocities $\bar{U}(x,y,t)$ for both ice sheets and shelves are a combination of the scaled shearing SIA and stretching SSA equations [25]. $\bar{U}$ is the sum of the basal velocity and the vertical-mean internal velocity in each ice column ($\bar{U} = U_b + \bar{U}_i$). In order to derive the accurate internal velocity, a better approximation of the vertical shear stress ($\sigma_{zx}$, $\sigma_{zy}$) is required. In the zeroth-order shallow ice approximation, the shear stress is often solved only by the hydrostatic driving force ($-\rho_i gh \frac{\partial h}{\partial x}$, $-\rho_i gh \frac{\partial h}{\partial y}$) as in section 2.2.2, but horizontal stretching equations below for $\bar{u}(x,y)$ and $\bar{v}(x,y)$ with additional basal shear stresses are incorporated in the model [13]. These equations are basically derived from the SSA eq. (26) in terms of velocity components with an extra basal shear term added,

$$\frac{\partial}{\partial x} \left[ \frac{2\mu h}{A} \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[ \frac{\mu h}{A} \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \right] = \rho_i g h \frac{\partial h_s}{\partial x} + \frac{f_g}{C_n} |u_b^2 + v_b^2|^{\frac{1-m}{2m}} u_b $$ \hspace{1cm} (40a)

$$\frac{\partial}{\partial y} \left[ \frac{2\mu h}{A} \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) \right] + \frac{\partial}{\partial x} \left[ \frac{\mu h}{A} \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \right] = \rho_i g h \frac{\partial h_s}{\partial y} + \frac{f_g}{C_n} |u_b^2 + v_b^2|^{\frac{1-m}{2m}} v_b $$ \hspace{1cm} (40b)

where $\mu$ is the strain rate-dependent shear viscosity:

$$\mu = \frac{1}{2} (\dot{\epsilon}_t^2)^{\frac{1-n}{2m}}. $$ \hspace{1cm} (41)

$f_g$ is a floating ice flag from 0 to 0.5: when the ice is on land as the ice sheet, the flag $f_g$ is 0.5, and when ice is floating on ocean as the ice shelf, $f_g$ ranges from 0 to 0.5. $f_g$ is parameterized as a function of the water column height as

$$f_g = 0.5 \max \left[ 0, 1 - \frac{h_w}{s_{dev}} \right]. $$ \hspace{1cm} (42)
where \( h_w \) is the height of the water column, and \( s_{dev} \) is the standard deviation of observed bed elevations, which is fixed at 100m in our model. In short, \( f_g \) allows to add extra basal stress for the ice shelf whose water column underneath is less than 100m. Remember that the SSA equation in eq. (26) is only applicable to the interior part of the ice shelf. This \( f_g \) flag is responsible for smoothing a change in the basal shear stress in the sheet-shelf transition zone by adding the extra stress term whose effect diminishes with distance from the grounding line.

The basal sliding relation, initially developed by Weertman in 1957 and often referred to as Weertman’s sliding relation, is used for basal velocities for the sheet flow \( u_b \) and \( v_b \) (fig. 41),

\[
\tilde{u}_b = C|\sigma_{bx}|^{m-1}\tilde{\sigma}_b, \quad \tilde{v}_b = C|\sigma_{by}|^{m-1}\tilde{\sigma}_b,
\]

where \( \sigma_{bx} \) and \( \sigma_{by} \) are the basal shear stress in \( x \) and \( y \) directions, and \( \tilde{\sigma}_b \) is the basal stress [33].

The effective strain rate \( \dot{\epsilon}_e \), also known as the second invariant of strain rate, in eq. (41) for \( \mu \) is approximated as

\[
\dot{\epsilon}_e^2 \approx \left( \frac{\partial \tilde{u}}{\partial x} \right)^2 + \left( \frac{\partial \tilde{v}}{\partial y} \right)^2 + \frac{1}{4} \left( \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} \right)^2 + \frac{1}{4} \left( \frac{\partial \tilde{u}_i}{\partial z} \right)^2 + \frac{1}{4} \left( \frac{\partial \tilde{v}_i}{\partial z} \right)^2,
\]

or equivalently

\[
\dot{\epsilon}_e^2 = \dot{\epsilon}_{xx}^2 + \dot{\epsilon}_{yy}^2 + \dot{\epsilon}_{zz}^2 = 0,
\]

based on the definition of the second invariant tensor, and using the following relations:

\[
\dot{\epsilon}_{xx} = \frac{\partial \tilde{u}_i}{\partial x}, \quad \dot{\epsilon}_{yy} = \frac{\partial \tilde{v}_i}{\partial y}, \quad \dot{\epsilon}_{xy} = \frac{1}{2} \left( \frac{\partial \tilde{u}_i}{\partial y} + \frac{\partial \tilde{v}_i}{\partial x} \right), \quad \dot{\epsilon}_{xz} = \frac{1}{2} \frac{\partial \tilde{u}_i}{\partial z}, \quad \dot{\epsilon}_{yz} = \frac{1}{2} \frac{\partial \tilde{v}_i}{\partial z},
\]

where eq. (46) is from the mass conservation eq. (16), and eq. (47) is from the modified strain rate tensor (eq. (29)).

The internal shear equations (SIA) solving for the internal ice velocities \( u_i \) and \( v_i \) contain six shear stress components due to the symmetric nature of the stress tensor,

\[
\frac{\partial u_i}{\partial z} = 2A\sigma_{xz}(\sigma_{xx}^2 + \sigma_{yz}^2 + \dot{\sigma}_{xx}^2 + \dot{\sigma}_{yy}^2 + \sigma_{xy}^2 + \dot{\sigma}_{xx}\dot{\sigma}_{yy})^{\frac{m-1}{2}}, \quad \frac{\partial v_i}{\partial z} = 2A\sigma_{yz}(\sigma_{xx}^2 + \sigma_{yz}^2 + \dot{\sigma}_{xx}^2 + \dot{\sigma}_{yy}^2 + \sigma_{xy}^2 + \dot{\sigma}_{xx}\dot{\sigma}_{yy})^{\frac{m-1}{2}}.
\]
The term with dashed underlines is
\[
\hat{\sigma}_{xx}^2 + \hat{\sigma}_{yy}^2 + \sigma_{xy}^2 + \hat{\sigma}_{xx}^2 \hat{\sigma}_{yy}^2 = \left( \frac{2\mu}{A^2} \right)^2 \left[ \left( \frac{\partial \bar{u}}{\partial x} \right)^2 + \left( \frac{\partial \bar{v}}{\partial y} \right)^2 + \frac{\partial \bar{u}}{\partial x} \frac{\partial \bar{v}}{\partial y} + \frac{1}{4} \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right)^2 \right].
\] (49)

These stress and velocity relations are held by Glen’s flow law (eq. (10)) and the modified strain rate tensor (eq. (29)). Using the left-hand side LHS\(_x\) and LHS\(_y\) of eqs. (40a) and (40b), the vertical shear stresses are estimated as
\[
\sigma_{zx} = -\left( \rho_i gh \frac{\partial h_s}{\partial x} - \text{LHS}_x \right) \left( \frac{h_s - z}{h} \right),
\] (50a)
\[
\sigma_{zy} = -\left( \rho_i gh \frac{\partial h_s}{\partial y} - \text{LHS}_y \right) \left( \frac{h_s - z}{h} \right).
\] (50b)

Because the boxed additional basal stress term in eqs. (40a) and (40b) is 0 when the water column underneath the ice shelf is \(h_w \geq 100\), the left-hand side \(\text{LHS}_x = \rho_i gh \frac{\partial h_s}{\partial x}\) or \(\text{LHS}_y = \rho_i gh \frac{\partial h_s}{\partial y}\), so \(\sigma_{zx}\) and \(\sigma_{zy}\) are zero as shown in eq. (26). For the shelf flow with the water column \(0 \leq h_w < 100\) and the sheet flow, the shear stress is balanced with the basal friction multiplied by the scaled vertical coordinate \(\zeta (\equiv \frac{h_s - z}{h})\).

In the end of the ice velocity routine, the vertical integration of the internal velocity \(\bar{U}_i\) yields
\[
\bar{U}_i = \frac{1}{h} \int_0^1 \bar{U}_i \, d\zeta.
\] (51)

This depth-averaged internal velocity is added to the basal velocity \(U_b\) so that the vertical-mean velocity \(\bar{U}\) is carried to the next ice thickness calculation \(\bar{U} = \bar{U}_i + U_b\).

### 2.3.3 Ice thickness

The ice thickness \(h\) is derived from taking the vertical integral of the mass conservation eq. (16) using the Leibniz integral rule.
\[
\frac{\partial h}{\partial t} = -\nabla \cdot (\bar{U} h) + M_s - M_b,
\] (52)

where \(t\) is time, \(M_s\) is the surface mass balance: positive for accumulation and negative for ablation, and \(M_b\) is the basal melting. They are the net mass balance at the ice surface and base, including events such as calving at the ice shelf front, and refreezing/melting at the bottom of the floating ice in contact with sea water. The vertical-mean velocity field \(\bar{U}\) here is only in \(x\)- and \(y\)-planes (\(\bar{U}(x, y, t)\)), and the ice thickness equation is rewritten as
\[
\frac{\partial h}{\partial t} = -\bar{u}_x \frac{\partial h}{\partial x} - \bar{v}_y \frac{\partial h}{\partial y} + M_s - M_b,
\] (53)
where $\bar{u}_x$, $\bar{v}_y$ is a derivative of the vertical-mean horizontal velocity with respect to $x$, $y$, respectively. The advective term $-\bar{u}_x \partial h / \partial x - \bar{v}_y \partial h / \partial y$ can be also represented as the horizontal ice flux divergence $-\nabla \cdot \mathbf{q}$, namely the rate of change in ice thickness is balanced by a sum of ice flux in the horizontal directions since $M_s - M_b$ is ice mass flux in the vertical direction.

Ice surface elevation $h_s$ for the ice sheet where $\rho_w (S - h_b) < \rho_i h$ is

$$h_s = h + h_b, \quad (54)$$

and that of the ice shelf where $\rho_w (S - h_b) > \rho_i h$ is

$$h_s = S + h (1 - \frac{\rho_i}{\rho_w}), \quad (55)$$

where $S$ is the sea level, $\rho_w$ is the density of water, and $h_b$ is the bed elevation (fig. 5).

Figure 5: The relationship among ice thickness $h$, basal elevation $h_b$, water column height $h_w$, ice surface altitude $h_s$, and sea level $S$. One-sided solid arrows indicate elevations which can be positive or negative, whereas double-sided arrows are thickness, thus always non-negative.
3 Model Description

3.1 Model setup

We execute the model on a server with 16 GB RAM, 2 TB hard disk, and two Quad Core Intel Xeon Processor X5460 64-bit CPUs running at 3.16 GHz. The operating system is Gentoo Linux. The code is written in Fortran 77, and the output is in netCFD format.

3.1.1 Time range and domain

The model is run for 10,000 or 30,000 years and is forced through cold and warm periods. The time length is 0.5 years for stability, but the model output is recorded every 1000 years.

During the former glacial periods, the Cordilleran Ice Sheet was located in the western part of North America [34]. The model domain is defined from 66°00N down to 46°00N and from 152°00W to 112°00W, and the topography is generated from GEODAS Grid Translator from the National Oceanic and Atmospheric Administration (NOAA) based on ETOPO1 1-minute Global Relief grid database, shown in fig. 6 [35]. The grid size is 4 minutes by 4 minutes with 300 and 600 grid points along latitude and longitude lines, respectively. Unlike the uniform grid size in latitude $\Delta y=7413$ m, the length of the grid in longitude $\Delta x$ depends on its latitude; the grid size $\Delta x$ is approximately 3020 m at the northern boundary and 5150 m at the southern boundary. There are ten layers in the vertical coordinate. The grid size in a vertical column is a fraction of ice thickness, and the grids near the surface and the bottom of the ice are finer than those in the core ice column (fig. 4). We assume there is no deformable sediment layer beneath the ice mass, i.e., an ice sheet is only in contact with hard bedrock; and thus, the basal domain is fixed at all times, and there is no isostatic adjustment.

Topographic smoothing As mentioned earlier in section 1.2, the reason that modelling the Cordilleran Ice Sheet (CIS) is challenging in comparison to other ice sheet modelling is because of the flat-topography assumption in the Shallow Ice Approximation (SIA). The model faces a steepness issue. In a region where the basal topography $h_b$ is higher than -200 m (the bed elevation within the domain varies from -5817 m to 5524 m), there are 38 points whose slopes from one grid point to the next are greater than and equal to 20 degrees, and the steepest slope is 30.6 degrees with 1989 m difference in elevation. The application of the SIA to a steep and rugged topography generally introduces spurious mass at a grid boundary if upstream cells receive positive mass balance, which violates the mass conservation [17]. Explosions of ice mass (extreme ice thickness) are indeed observed, partic-
Figure 6: 4’×4’ basal elevation from 66°00N to 46°00N and from 152°00W to 112°00W [35].

ularly at grids where bed height of the surrounding four grid points are higher, similar to an inverted square pyramid.

There are several ways to resolve this kind of numerical instability, such as refining the grid, using other numerical methods (finite element method, finite volume method etc.), and so on. The simplest method which we apply to our experiments is to smooth the bed elevation, but still preserve the original landscape morphology as much as possible. Only grid points located higher than the bed elevation -200m ($h_b$>-200m) with a threshold slope angle of 11° are subject to interpolation. Grid points which have slope angles greater than the threshold angle in relation to their neighboring points are smoothed by taking either 9 or 7-point averages both forward and backward in space. 1.32% out of 600×300 grid points are treated (fig. 7), and the maximum slope angle is reduced to 17.7° (fig. 8).
Figure 7: Difference plot of the original and smoothed topography. The smoothing treatment is only applied to the area where $h_b > -200m$ in light blue. There are especially many small dots (indications of the rugged topography) along the western side of the mountain range and at the lower right corner of the domain. Mountain peaks near the coastline of Alaska are relaxed. (black line: coast line)
Figure 8: Details of smoothing: original on the left, and the difference between the original and smoothed surface on the right. Examples of rugged topography (top), and extremely steep topography (bottom).
3.1.2 Input data

There are three sets of annual mean surface air temperature and annual total precipitation data: the COLD climate based on the last glacial maximum (LGM: 21,000 years before present), the WARM climate based on the mid-Holocene (MH: 6,000 years before present), and the PT climate based on the average of year 1980, 1990 and 2000. The first two sets of data are obtained from the Palaeoclimate Modelling Intercomparison Project Phase 2 (PMIP2) [36], and the present data are provided by the North American Regional Reanalysis [37]. All sources of input data are summarized on table 1. Monthly surface temperature and precipitation data are extracted and then are converted to the annual mean, and spline interpolation is applied to fit them to the corresponding grid cells in the model domain (fig. 10).

For the first experiment, no variations of temperature or precipitation in time and space are given, and the simulation is run for 10,000 years. Secondly, the surface temperature and precipitation are set constant throughout the 30,000-year simulations using COLD, WARM, and PT temperature and precipitation data sets. The third experiment is to provide the model with time-dependent temperature based on the global average temperature fluctuation data [38]. Because there is no 30,000-year local or regional temperature cycle data available to the public, we simply linear-interpolate the global data and fit it to COLD, WARM and PT data to observe deglaciation after the cold period (fig. 11). No time-dependence in precipitation is given. All simulations start with an arbitrary ice free condition. The time and spacial dependency on the temperature and precipitation in each experiment is summarized in table 2.

<table>
<thead>
<tr>
<th>Data</th>
<th>Name</th>
<th>Year</th>
<th>Database</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_b$</td>
<td>ETOPO1</td>
<td>'99-'01</td>
<td>GEODAS Grid Translator</td>
<td>66'-46'N,152'-112'W</td>
</tr>
<tr>
<td>$T_a$ &amp; $Pr$</td>
<td>COLD</td>
<td>21k BP</td>
<td>PMIP 2</td>
<td>monthly mean $T_a/Pr$</td>
</tr>
<tr>
<td></td>
<td>WARM</td>
<td>6k BP</td>
<td>PMIP 2</td>
<td>monthly mean $T_a/Pr$</td>
</tr>
<tr>
<td></td>
<td>PT</td>
<td>'80,'90,'00</td>
<td>NARR</td>
<td>averaged annual mean $T_a$</td>
</tr>
<tr>
<td>$\Delta T_a$</td>
<td>-</td>
<td>30k-0 BP</td>
<td>WDC PALEO</td>
<td>global mean $T_a$</td>
</tr>
</tbody>
</table>

Table 1: List of input data sources.

3.2 Numerical implementation

All partial differential equations, including ones not introduced in section 2.3, are approximated by finite difference methods in ARCTIC-TARAH. There are two important subroutines for our project, icetherm and icedyn where all equations discussed in detail in section 2.3 are implemented.
### Table 2: Summary of input data, and parameters for each experiment. (ind: independent, dep: dependent)

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Surface Air Temperature</th>
<th>Annual Precipitation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Space</td>
<td>Time</td>
</tr>
<tr>
<td>Experiment 1</td>
<td>ind</td>
<td>ind</td>
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<tr>
<td>Experiment 2</td>
<td>dep</td>
<td>ind</td>
</tr>
<tr>
<td>Experiment 3</td>
<td>dep</td>
<td>dep</td>
</tr>
</tbody>
</table>

#### 3.2.1 subroutine icetherm

The ice temperature $T(x, y, \zeta, t)$ in each ice column is solved in the subroutine icetherm in the model.

The ice velocities ($u, v, w'$) in the advective part of eq. (27) are solved based on eqs. (32a), (32b) and (33) which are independent from the vertically averaged velocities derived from SIA and SSA equations, ($\bar{u}, \bar{v}$). There are two options to calculate the advective term explicitly, $-(\mathbf{U} \cdot \nabla)T$. The following are the methods expressed in 1D:

**First order backward finite difference:**

$$T_{j}^{n+1} = T_{j}^{n} - u \frac{\Delta t}{\Delta x} (T_{j}^{n} - T_{j-1}^{n}), \quad (56)$$

**Second order backward finite difference:**

$$T_{j}^{n+1} = T_{j}^{n} - u \frac{\Delta t}{2\Delta x} (3T_{j}^{n} - 4T_{j-1}^{n} + T_{j-2}^{n}), \quad (57)$$

where $\Delta t$ and $\Delta x$ are a step size in time and space. We use the 1st order method for robustness and computational cost, although the order of accuracy is only first order, both in time and space. The 2nd order method is second order accurate in space and first order accurate in time; however, in order to apply this method, the accuracy of the basal topography must be sacrificed to some degree.

The vertical heat diffusion and source term $\frac{k_{i}}{\rho_{c_{i}} h^{2}} \frac{\partial^{2} T}{\partial \zeta^{2}} + \frac{Q_{i}}{\rho_{c_{i}}} \frac{\Delta t}{\eta}$ is solved implicitly as

**Second order central finite difference:**

$$T_{j}^{n} = T_{j}^{n-1} + \frac{k_{i}}{\rho_{c_{i}} h^{2}} \frac{\Delta t}{(\Delta \zeta)^{2}} (T_{j+1}^{n} - 2T_{j}^{n} + T_{j-1}^{n}) + \frac{\Delta t}{\rho_{c_{i}} \eta} Q_{i}, \quad (58)$$
where $\Delta \zeta$ is a grid length in $z$-direction. It is second and first order accurate in space and time, respectively, and using $\gamma$ and $\eta$ in eq. (58), a tridiagonal matrix is arranged as following:

\[
\begin{pmatrix}
1 + 2\gamma & -\gamma & 0 \\
-\gamma & 1 + 2\gamma & -\gamma \\
& \ddots & \ddots \\
0 & \cdots & 1 + 2\gamma & -\gamma & 1 + 2\gamma \\
\end{pmatrix}
\begin{pmatrix}
T_1 \\
T_2 \\
\vdots \\
T_{n-1} \\
T_n \\
\end{pmatrix}
= \begin{pmatrix}
T_0 + \frac{k_\eta}{(\Delta \zeta)^2} T_{n-1} \\
T_1 \\
\vdots \\
T_{n-1} + \eta Q_i \\
T_n + \eta(Q_g + Q_b) \\
\end{pmatrix}.
\]

(59)

It is an 11-by-11 matrix which consists of ten ice layers, as well as one bedrock layer for ice sheets or one water layer for ice shelves.

The upper and lower boundary conditions are imposed at the first and last elements on the right-hand side of the vector. If the ice exists, the upper boundary condition of the ice temperature $T(x, y, 0, t)$ is set to ice surface temperature, $T_i$, which is a function of annual mean surface air temperature, $T_a$ (fig. 9) [39].

\[
T_i = \begin{cases}
T_m & \text{if } (T_m - T_a) \leq -(T_z - T_m), \\
\frac{T_a(\pi - \beta) - (T_z - T_m)\sin\beta + \beta T_m}{\pi} & \text{if } -(T_z - T_m) < (T_m - T_a) < (T_z - T_m) \\
T_a & \text{if } (T_z - T_m) \leq (T_m - T_a)
\end{cases}
\]

(60)

where $T_m$ is a melting point temperature, $T_z$ is 288.06°K, and $\beta$ is defined as

\[
\beta = \arccos \left( \frac{T_m - T_a}{T_z - T_m} \right).
\]

(61)

The temperature at the bottom of the ice sheet $T(x, y, 1, t)$ is a sum of the vertical conductive heat flux $\left( \frac{k_i}{\Delta \zeta} \right) \frac{\partial T}{\partial \zeta}$ from the bedrock top, which is equal to a constant geothermal flux $Q_g$, and the basal shear heating $Q_b = \sigma_b \cdot U_b$ where $\sigma_b$ is basal stress and $U_b$ is horizontal basal velocities in $x$ and $y$ planes. If the ice bottom temperature exceeds a basal pressure melting point $T_{pm}$, then it is set to $T_{pm}$, and the excess heat contributes to melting the basal ice. Whereas the lower boundary condition for the ice shelf is set equal to $T_{pm}$ everywhere [25]. If there is no ice, the ice temperatures throughout the vertical column are set equal to the surface air temperature $T_a$. Figure 12 illustrates a flow chart describing the choice of the upper and lower boundary conditions.

**Arrhenius temperature dependent coefficient** The Arrhenius temperature dependent coefficient $A(T)$ is determined by eq. (13), and the calculation is
very straightforward since the function is only dependent on the ice temperature. Hence, two different sets of constants for activation energy $Q$, and flow law coefficient $A_0$ are applied; the one is when the ice temperature $T$ is greater than and equal to $266.65\, ^\circ K$ ($-6.5\, ^\circ C$), and another is when $T$ is less than that (eq. (62)). We use the following constants for $Q$ and $A_0$:

$$
A_0 = \begin{cases} 
2.00 \times 10^{-16} Pa^{-3} yr^{-1} & \text{if } T(i,j,k) \geq 266.65\, ^\circ K \\
1.66 \times 10^{-16} Pa^{-3} yr^{-1} & \text{if } T(i,j,k) < 266.65\, ^\circ K.
\end{cases}
$$

$$
Q = \begin{cases} 
95.45 kJ/mol & \text{if } T(i,j,k) \geq 266.65\, ^\circ K \\
78.20 kJ/mol & \text{if } T(i,j,k) < 266.65\, ^\circ K.
\end{cases}
$$

(62)

**Sea level** The sea level calculation is based on benthic $\delta^{18}O$ data for the last 5.32 million years, recorded from 57 different sites around the world [40]. The time increment of the $\delta^{18}O$ data is 1000 years, and they are linearly interpolated by two nearest data points to the model time resolution ($\Delta t=0.5$) and converted to the sea level.

### 3.2.2 subroutine icedyn

The ice velocity and ice thickness calculations are implemented in the subroutine icedyn. There are two Picard iteration loops in icedyn: an inner loop called A-loop and an outer loop called C-loop. The outer C-loop performs most of non-shelf calculations such as the grounding line treatment eq. (38), SIA eqs. (48a) and (48b), and ice thickness eq. (52). After identifying the location of the grounding line in the C-loop, the simulation enters the A-loop, which operates ice shelf related calculations such as SSA eqs. (40a) and (40b). Threshold values are assigned to the target solution in each loop ($u_b$ and $v_b$ for the A-loop, and $h$ for the C-loop) such that the iterative process continues until the difference between the current
and previous target solutions converges to less than 0.001. The maximum number of iterations, however, is set to 10 or 15 per loop, depending on the experiment.

**SSA equations: A-loop** The horizontal stretching eqs. (40a) and (40b) are solved by a successive-over-relaxation (SOR) method which is often used for a very large sparse system of partial differential equations [41], combined with a checkerboard scheme. For a system of linear equations $Ax = b$ where $A$ is a square matrix with $n$ rows and columns, the SOR is an iterative method to approximate a solution $x$ by computing a combination of the newly updated solution $x^k$ and the solution from the previous time step $x^{k-1}$ [42] as

$$\hat{x}_i = \frac{1}{A_{ii}}(b_i - \sum_{j=1}^{i-1} A_{ij} x_j^k - \sum_{j=i+1}^{n} A_{ij} x_j^{k-1}),$$

$$x_i^k = (1 - \omega)x_{i-1}^{k-1} + \omega \hat{x}_i,$$  

where $\omega$ is a constant relaxation parameter, which is set to 1.2 in our case. If $\omega = 1$, the SOR is equivalent to the Gauss-Seidel method.

The checkerboard scheme is iterated as shown in fig. 13. The ice velocity at a point $U_{i,j}$ (white cell) is approximated by four immediate neighbors (red cells): $U_{i+1,j}$, $U_{i-1,j}$, $U_{i,j+1}$, and $U_{i,j-1}$. Instead of performing an update on the neighbor cell $U_{i+1,j}$ (red cell) at the next step, the checkerboard scheme completes the calculation over all white cells and then iterates over the red cells whose solutions only depend on newly updated solutions of the white cells. This combination of SOR and checkerboard schemes speeds up the convergence rate [43].

**Ice thickness equation: C-loop** After evaluating the internal shear eqs. (48a) and (48b) using the solutions from eqs. (49) and (50) time implicitly, the ice thickness eq. (53) is solved by an alternating-direction-implicit (ADI) method. "Alternating direction" refers to splitting two-dimensional problems into two one-dimensional problems that alternatively calculates the numerical solution in 1D strips in the $x$ direction, and then the solution in the 1D strips in $y$ direction [44]. In addition, the time stepping is a two-step process; $h_{ij}^{n+1}$ is evaluated from $h_{ij}^{n+1/2}$ and $h_{ij}^n$. This 1D two-time-step problem is formulated in a tridiagonal matrix and solved by a tridiagonal solver. Again, this C-loop (the whole ice sheet and shelf calculations) is repeated until $h$ converges.

Flowcharts of the subroutine *icetherm* and *icedyn* are shown in figs. 16 and 17. The ice rheological exponent $n$, and the basal sliding exponent $m_s$ in equations in the chart are replaced with their constant (3, 2) values for simplicity.
3.2.3 Climate system

The ice melting rate for each grid point is obtained from a positive degree-days (PDD) method. It is a statistical relationship between the annual ice melting rate and the sum of days whose daily mean surface air temperature is above freezing point $0^\circ C$ or $237.15^\circ K$ [45]. In the current setting of the model, PDD is only a function of the annual mean surface air temperature $T_a$ as follows:

$$
PDD = \begin{cases} 
365(T_a - T_m) & (T_m - T_a) \leq (T_z - T_m) \\
(T_a - T_m)\beta + (T_z - T_m)\sin\beta \frac{365}{\pi} & -(T_z - T_m) < (T_m - T_a) < (T_z - T_m) \\
\frac{1}{365} & (T_z - T_m) \leq (T_m - T_a) 
\end{cases}
$$

where $T_a$ is the annual mean surface air temperature, $\beta$ is defined in eq. (61), and $T_z$ is a constant $288.06^\circ K$ in our ARCTIC-TARAH model. The melting rate (MR) equation is based on Szafraniec’s study [45],

$$
MR = \begin{cases} 
\frac{1}{1000}(6.75PDD - 64.15) & PDD \geq \frac{64.15}{6.75} \\
0.005PDD & \text{otherwise}
\end{cases}
$$

Figure 14 (bottom) represents a relationship between $T_a$ and MR. The top figure shows the fraction of snowfall and rainfall at a given annual surface air temperature $T_a$. If $T_a$ is $261^\circ K$ (-12$^\circ C$) and the total precipitation $Pr$ is 2000mm/yr, the snow budget is 1600mm/yr, and the rain budget is 400mm/yr.

The annual precipitation is given as an input, and the annual snowfall is determined by a simple ratio in the following equation:

$$\frac{\text{Snowfall}}{\text{Precipitation}} = \begin{cases} 
0 & (T_m - T_a) \leq -(T_z - T_m) \\
1 - \frac{\beta}{\pi} & -(T_z - T_m) < (T_m - T_a) < (T_z - T_m) \\
1 & (T_z - T_m) \leq (T_m - T_a)
\end{cases}
$$

The rainfall is an inverse of the snowfall. Figure 15 is an example of how the climate setting contributes to the ice accumulation and ablation with the PT dataset. The lower figures are total rainfall (left) and snowfall (right). The one on the upper right corner represents the melting rate (MR). The difference from the snowfall to the melting rate is the total budget of ice from the climate which is shown on the upper left corner. If the total budget is negative, there is no net ice accumulation from the snowfall, and yet the horizontal input of ice from neighboring grid cells due to ice sheet/shelf flows is possible.
Figure 10: Air surface temperatures $T_a$ in °K (left), and total annual precipitations $P_r$ in mm/yr (right) for COLD (top), WARM (middle), and PT (bottom). The main reason why $P_r$ for the COLD and WARM climate lack a spatial variation is that the COLD and WARM data sets are interpolated from data with only $8 \times 15$ grid resolution within our domain [36], whereas the original PT data have $1197 \times 2397$ grids in our domain and are interpolated down to our $300 \times 600$ resolution [37]. Because of the high resolution of the original PT input data, the local extreme precipitation is captured unlike in the COLD and WARM datasets.
Figure 11: Experiment 3: The minimum (left), and the maximum (right) air surface temperature distributions (°K) for COLD, WARM, and PT scenarios [36, 37]. The coldest period in the last 30,000 years is at 19.6kyrBP, and the warmest is at 2.1kyrBP (fig. 26) [38], and therefore, we enforce the coldest climate at 10.4kyr and the warmest climate at 27.9kyr from the start in Experiment 3.
Presence of ice

\[ T(x, y, 0, t) = T_i(x, y, t) \]

\[ T(x, y, 0, t) = T_a(x, y, t) \]

Figure 12a: Ice surface temperature: upper boundary condition \( T(x, y, 0, t) \)

\[ T(x, y, 1, t) = T_{pm}(x, y, t) \]

\[ T < T_{pm} \]

\[ T(x, y, 1, t) = T_{pm}(x, y, t) \]

\[ T \geq T_{pm} \]

\[ T(x, y, 1, t) = T_a(x, y, t) \]

Figure 12b: Ice bottom temperature: lower boundary condition \( T(x, y, 1, t) \)

Figure 13: Checkerboard scheme. Circles and squares represent data from previous \((k - 1)\) and current time \((k)\), respectively. The calculation is performed only on the white cells, using the previous time dataset of the four surrounding red cells (left). After updating all white cells to the current time data, the remaining red cells are solved using the neighboring white cells at the current time (right).
Figure 14: Top: The relationship between the surface air temperature and the fraction of snow/rain over annual precipitation. At 0°C (273.15°K), 50% of the total precipitation $P_r$ is counted as snowfall, and another 50% is as rainfall. If the $T_a$ is below -15°C (258.15°K), no rainfall is considered. Bottom: The upper limit of ice melting per year for a given surface air temperature (a combination of eqs. (65) and (66)). The MR, however, is dependent on the summer temperature as well as the annual mean temperature (discussed in section 4.4.1). Approximately 10m/yr of melting at 0°C (273.15°K) might be an overestimate at a certain location, but considering that the summer mean temperature reaches above 10°C (283.15°K), the potential annual MR is not far from the reality [46].
Figure 15: Examples of PT climate data: the net budget from accumulation and ablation (upper left), the ablation budget (upper right), the amount of rain out of total precipitation (lower left), and the amount of snow, e.g., accumulation budget (lower left), based on the data provided by the North American Regional Reanalysis [37]. The ice accumulates when the total budget is positive or when the sum of the surface snow accumulation and the horizontal ice volume flux is greater than the melting budget.
subroutine icetherm

Horizontal velocities
\[ u = u_b - 2(\rho_i g)^3 |\nabla h_s| \frac{2\partial h_s}{\partial x} \int_{\zeta}^{1} h^2 A(T)(\zeta')^3 \, d\zeta' \]
\[ v = v_b - 2(\rho_i g)^3 |\nabla h_s| \frac{2\partial h_s}{\partial y} \int_{\zeta}^{1} h^2 A(T)(\zeta')^3 \, d\zeta' \]

Internal shear heating
\[ Q_i = \frac{\partial u}{\partial z} \sigma_{xx} + \frac{\partial v}{\partial z} \sigma_{xy} \]

Vertical velocity
\[ u_h = u_h \left( \frac{\int_{\zeta}^{1} \int_{\zeta}^{1} h^2 A(T)(\zeta'')^3 \, d\zeta'' \, d\zeta'}{\int_{\zeta}^{1} \int_{\zeta}^{1} h^2 A(T)(\zeta'')^3 \, d\zeta'' \, d\zeta'} \right) \]
\[ v_h = v_h \left( \frac{\int_{\zeta}^{1} \int_{\zeta}^{1} h^2 A(T)(\zeta'')^3 \, d\zeta'' \, d\zeta'}{\int_{\zeta}^{1} \int_{\zeta}^{1} h^2 A(T)(\zeta'')^3 \, d\zeta'' \, d\zeta'} \right) \]
\[ w' = \frac{d\zeta}{dt} = \frac{1}{h} \left( M_b + \frac{\partial h}{\partial t}(1 - \zeta) + \frac{\partial u_h}{\partial x} + \frac{\partial v_h}{\partial y} \right) \]

Adveective term in the heat equation
\[ \frac{\partial T_{adv}}{\partial t} = -u \frac{\partial T}{\partial x} - v \frac{\partial T}{\partial y} - w' \frac{\partial T}{\partial \zeta} \]

Complete vertical heat diffusion
\[ \frac{\partial T}{\partial t} = T_{adv} + \frac{k_i}{\rho_i c_i} \frac{\partial^2 T}{\partial \zeta^2} + \frac{Q_i}{\rho_i c_i} \]

Figure 16: subroutine icetherm
subroutine icedyn

sheet/shelf ID

Grounding line treatment

\[ U_b = \frac{1}{h_g} \sqrt{\frac{A(\rho_i g)^3 (1 - \rho_i / \rho_w)^3}{64C}} \]

C-loop

Horizontal stretching equations

\[ \begin{align*}
\text{LHS}_x &= \rho_i g h \frac{\partial h}{\partial x} + f_g \frac{C}{h_g^3/3} |u_b|^2 + v_b^2 |u_b| \\
\text{LHS}_y &= \rho_i g h \frac{\partial h}{\partial y} + f_g \frac{C}{h_g^3/3} |u_b|^2 + v_b^2 |v_b|
\end{align*} \]

Basal stress

\[ \tilde{\sigma} = C |U_b| \bar{U}_b \]

Only ice sheet

Elboop

Figure 17: subroutine icedyn
4 Results

Three types of experiment are carried out with the following settings:

1. Uniform temperature and precipitation in space and time (10,000-year run).

2. Temperature and precipitation variations in space, but not in time (30,000-year run).

3. Temperature variation in space and time, and precipitation variation only in space (30,000-year run).

The data sources and time and spatial dependencies are summarized in tables 1 and 2. The variations in Experiment 2 and 3 are given by COLD, WARM, and PT datasets (fig. 10), and the temperature varies by time based on the global average temperature fluctuation data [38].

First order v.s. second order backward finite differences The advective term in the ice temperature eq. (27) is either solved by the first or second order backward finite differences (FD) as described in section 3.2.1. We apply the second order method first, and if it does not work, we employ the more robust first order method. When the second order method fails, the model typically experiences extremely low temperatures.

4.1 Experiment 1: Uniform temperature and precipitation in space and time

The intent of this experiment is to verify the stability of the model on the smoothed topography throughout the entire domain. The temperature and precipitation are fixed at 263.15K (-10°C) and 2850mm/yr, respectively, which provide a sufficient amount of ice, theoretically 36.5mm of net ice accumulation per year if the topography is completely flat, and there is no sliding or deformation of the ice. The result is shown in figs. 18 and 19.

The first order backward FD is applied to this experiment. The maximum number of iterations in the A-loop and the C-loop is 10 (fig. 17), and the number of C-loop iterations reaches the maximum in 650 years from the start. As soon as ice shelves form on the ocean, approximately after 1300 years, the A-loop requires 10 iterations, and therefore, the overall convergence rate is not very optimal.
Figure 18: Experiment 1: Ice thickness $h$ (left) and ice surface altitude $h_s$ (right) after a 10,000-year run, depicting classic ice dynamics where ice mass flows from mountains and hills to valleys and streams.

Figure 19: Experiment 1: Ice surface velocity $U_s$ (left) and ice bottom velocity $U_b$ (right) in logarithmic scale after a 10,000-year run. $U_s$ and $U_b$ are the Euclidean norm of horizontal velocities, $U_s = \sqrt{u_s^2 + v_s^2}$ and $U_b = \sqrt{u_b^2 + v_b^2}$. Ice slides and deforms relatively fast near the coastline.

4.2 Experiment 2: Temperature and precipitation variations in space

Spacial variations in temperature and precipitation are given when it has been confirmed that any particular grid point within the domain is capable of numerically
resisting ice accumulation/ablation and ice influx/eﬄux. Although the second order backward FD is a successful method to use in WARM and PT case studies, the COLD scenario experiences numerical instability in the vertical temperature diffusion (table 3).

<table>
<thead>
<tr>
<th>Method</th>
<th>COLD</th>
<th>WARM</th>
<th>PT</th>
</tr>
</thead>
<tbody>
<tr>
<td>second order backward FD</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>first order backward FD</td>
<td>✓</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 3: Experiment 2: Finite difference methods for the vertical temperature diffusion equation. The symbols indicate the method failed (×), was successful (√), or was not performed (-).

As in Experiment 1, the maximum number of iterations in the A-loop and C-loop is set to 10. The ice mass, if any, continues to grow until the end of the simulation, unless it flows to the area where the surface air temperature is high enough to melt it, or it is calved at the edge of ice shelves. The rate of calving is defined constant at 4m/yr.

4.2.1 COLD climate

The model with the COLD climate scenario suffers from numerical instability both in ice temperature and velocity. As described earlier, the stability of the ice temperature is improved by increasing the robustness of the difference method in exchange for its accuracy and by selecting a 1-year time step. Although a 1-year time step simulation provides reasonable numerical solutions for ice thickness and temperature, we detect persistent oscillations in the ice velocity solutions in the middle of the ice mass. The complexity of hybrid SIA-SSA velocity calculations requires even a smaller time interval for stability. It is the reason for applying a 0.5-year increment for a 30,000-year simulation. Compared to the 1-year simulation, the oscillations are drastically reduced, but some minor oscillations are still observed sometime during the simulation (fig. 20). They produce a locally jagged ice surface because the numerical solution of the horizontal velocity directly influences that of ice thickness (eq. (53)). Figure 21 illustrates the longitudinal cross-section of the ice surface near the oscillations ($\Delta t = 1$).

The total ice volume increases up to 15,000 years from the beginning, and stabilizes beyond that despite the continuous input of snow (figs. 22 and 24). Ice flows out through the boundary or to the region where the surface air temperature is high enough to melt it. No ice shelf is produced even with the COLD scenario. The possible reasons are (1) the coarse original input dataset (8×15 grid resolution), and (2) the PDD method which does not well-represent the difference between the
Figure 20: Experiment 2 - COLD: Oscillations in the ice surface velocity solutions after the 15,000-year simulations with different time intervals, represented in a log_{10} scale (142°W-112°W). The oscillations appear in the middle of the ice mass with $\Delta t = 1$ and stay there until the end of the simulation (left). We still observe a few minor oscillations with the $\Delta t = 0.5$ model result, but they disappear rather quickly and arise again at different locations (right). Fast flowing lines in red at the ice front on the right do not represent the reality well since they are more than in order of 3-4 (1-10km/yr). However, these numerical solutions match with existing deep and narrow valleys, and are rather steady (the ice flows steadily fast) throughout the simulation, unlike the instability observed in the $\Delta t=1$ result. The yellow part extending to the interior between 52° to 60°N like a haze is also likely to be small oscillations.

oceanic climate and the continental climate. This problem is discussed in detail in section 4.4.1.

4.2.2 WARM climate

There is no ice growth with the WARM temperature and precipitation data which are based on the mid-Holocene period [36]. It is known to be warmer than today, and this null result is as expected.
Figure 21: Experiment 2 - COLD: Cross-section of ice at 58°N after a 15,000-year run with the constant COLD setting (132°W-112°W). The thick black line indicates the basal topography, and blue lines are ice accumulation at 1000 year intervals. The oscillations in ice velocity solutions affect the smoothness of the ice surface.

Figure 22: COLD: the change in total ice volume (blue) and maximum ice thickness (green)

4.2.3 PT climate

Unlike COLD and WARM climate datasets, the PT data contain local extreme climates along the coast and mountains. Although no large-scale ice sheets or shelves exists in North America today, there are still some glaciers and icefields. We indeed observe some ice growth in the solution where they occur around Denali National Park and Preserve and Wrangell-St. Elias National Park and Preserve (fig. 25). The ice thickness and volume reach equilibrium in 2000 years, relatively early in the simulation.
4.3 Experiment 3: Temperature variation in space and time, and precipitation variation in space

Validating the stability of the model using the constant COLD, WARM, and PT datasets, we introduce temperature variations to each scenario at every time step, $\Delta t = 0.5$. The variation is based on the global average temperature fluctuation data in the last 3 million years [38]. We simplify the fluctuation data of the last 30,000 years to a piecewise linear function and calibrate each COLD, WARM, and PT temperature data (fig. 26). The surface temperature is at its minimum at 10,400 years, and maximum at 27,900 years from the beginning of the simulation. The temperature distributions at these time steps (the coldest and warmest) are shown in fig. 11. Unlike the constant temperature case study, we apply the first order backward FD method in space on the vertical temperature diffusion equation for all COLD, WARM, and PT scenarios (table 3). The second order method fails approximately at 9900 (COLD), 8100 (WARM), and 13,400 (PT) years from the beginning of the simulation.

<table>
<thead>
<tr>
<th></th>
<th>COLD</th>
<th>WARM</th>
<th>PT</th>
</tr>
</thead>
<tbody>
<tr>
<td>second order backward FD</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>first order backward FD</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
</tr>
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</table>

Table 4: Experiment 3: Finite difference methods for the vertical temperature diffusion equation. The symbols indicate the method failed ($\times$), or was successful ($\checkmark$).

Considering the issue of the convergence rate from Experiment 1 and 2, we increase the maximum number of iterations for ice velocity and thickness calculations in the A-loop and the C-loop to 15. Another reason is that, from comparing the temperature distribution maps (fig. 11) with the COLD ice growth in Experiment 2 (fig. 24), it is expected that the majority of land and coastlines will be
Figure 24: Experiment 2 - COLD: Ice thickness $h$ (left) and ice surface altitude $h_s$ (right) from three different time slices: 1000, 10,000, and 30,000 years.
Figure 25: Experiment 2 - PT: Ice thickness $h$ (left), ice surface velocity $U_s$ (upper right), and ice bottom velocity $U_b$ (lower right) in a logarithmic scale. The first color bar is for ice thickness, and the second one is for the velocity.

largely covered in ice.

### 4.3.1 COLD climate

The difference between the original COLD climate temperature (9000 years) and the lowest temperature at 10,400 years is only 0.875°C, and the surface air temperature is lower than that of the original COLD temperature for only 2700 years within the 30,000-year simulation. Hence, the total ice volume and ice thickness during the coldest period are within the range of Experiment 2 (fig. 22). The ice front hardly reaches the coast.

### 4.3.2 WARM climate

Ice covers the majority of the land surface, and ice shelves extend from the Alaskan coast and occupy most of the Canadian coastline during the cold period. This requires an enormous amount of computational power, and once the ice shelf appears, the CPU time drastically increases. The model eventually stops around 9500 years due to a singular tridiagonal matrix operation for the ice thickness.
**Figure 26:** Surface air temperature variations for COLD (blue) and WARM/PT (red) data. The mean global temperature change curve in the last 30,000 years (dotted lines) is simplified as a piecewise linear function (colored lines). Colored circles represent the original COLD, WARM, and PT datasets because the originals are based on the Last Glacial Maximum (21,000 years BP (before present)), Mid-Holocene (6000 years BP), and the average of year 1980, 1990 and 2000 (0 years BP), respectively [36, 37]. For example, in the case of the COLD scenario, the simulation starts at +1.25°C and ends at +15.02°C from the original COLD dataset. The surface temperature at 9000 years is equivalent to the original COLD temperature because it is based on the Last Glacial Maximum model result from [36]. Since there is only 0.005°C difference between the data at 30,000 (PT) and 24,000 (WARM) years from the start in the global average temperature data, we eliminate the WARM calibration line to simplify the figure and let the WARM and PT lines be equal in the figure.

Equation (division by zero). The residual of solutions from the current and previous iterations (the difference between the solutions at the 15th and 14th iterations, \(\max |h_{ij}^{15th} - h_{ij}^{14th}|\)) starts to exceed a thousand around 9000 years, and when the model is aborted, the residual reaches 6500.

In order to determine whether or not the instability is directly relevant to the size of \(\Delta t\) relative to the grid size \(\Delta x\), \(\Delta y\), and \(\Delta \zeta\), \(\Delta t\) is further reduced to 0.25 years. Except for the size of time step, the setting of the following additional experiment is identical to Experiment 3 (WARM).

**WARM climate with a smaller time step:** \(\Delta t = 0.25\). Although the experiment succeeds with the smaller time step, the total run time of this experiment is 35 days and, at worst, it takes 5 minutes to move to the next time step despite the fact that the last half of the simulation (15,000-30,000 year period) consumes only 4 days out of the total (fig. 27). By comparing the ice area and computing time, we can easily discover that the numerical solutions at the ice shelf do not converge.
well. Once the ice shelf completely disappears, the simulation runs as fast as the PT and COLD models; the number of iterations in the C-loop (fig. 17) is almost immediately halved, and the solutions are stabilized and well-converge with several iterations approximately in 500 years after the absence of the ice shelf. We do not observe a significant improvement of the convergence rate, and therefore, the convergence problem will not be efficiently solved by further discretization of a time length.

4.3.3 PT climate

The ice starts to grow in the northern half of the domain and gradually fills up the south, particularly on the mountain tops, and ice shelves extend to nearshore areas during the coldest period. The majority of the ice disappears after 20,000 years, and a few glaciers remain as observed in Experiment 2 at the end of the simulation.

First order v.s. second order backward finite difference  As in Experiment 2, both the first order and second order backward finite difference methods are applied to the temperature diffusion equation, and the second order method is unstable for all case studies in Experiment 3. The timing at which the simulation is aborted varies: approximately at 9100 (COLD), 8000 (WARM), and 13,400 (PT) years from the beginning of the simulation, and the results from the first order and second order methods are compared in fig. 28. The order of accuracy of the temperature diffusion equation does not, evidently, affect the accumulation and advection of ice sheets, but ice shelves extend farther with the first order method (c.f., the WARM results in fig. 28). The ice shelf calculations are more sensitive to the accuracy of the ice temperature than that of ice sheets. Relative differences between the ice area solutions with the first order \( S_1 \) and the second order method \( S_2 \) \( \left( \frac{S_1 - S_2}{S_2} \right) \) lies between ±0.0015 for ice sheets, whereas the relative differences between solutions for ice shelves differ from -0.25 to 0.1 (figs. 29 and 30). In comparison to the ice volume result (fig. 29), the total floating ice area result (fig. 30) with the first order method only differs slightly from that of the second order method. This means that the accuracy of the temperature equation is sensitive to the vertical extent of the shelf rather than its horizontal extent. This makes sense since the vertically averaged ice shelf velocity (horizontal extent) is independent of the temperature equation. However, the lower boundary condition of the ice shelf is simply a function of ice thickness or a pressure melting point \( T_{pm} \), unlike that of the ice sheet (fig. 12). Inaccuracy of the ice shelf temperature near the base relative to the pressure melting point may cause the basal melting and refreezing which unnecessarily adds and subtracts the ice shelf thickness, which destabilizes the ice volume.
Figure 27: Comparison of computing time, the ice area, and the change in the surface air temperature. The presence of ice shelves enormously impedes the process rather than that of ice sheets. Even a very small amount of ice shelves requires 5 hours to compute per 200-year simulation.
Figure 28: Comparison between first order (left) and second order (right) finite difference methods for solving the vertical heat diffusion equation. COLD (top) is at 9000, WARM (middle) is at 8000, and PT (bottom) is at 13,000 years from the beginning of the simulation; they are the record before the second order FD model is aborted due to instability. There is no obvious difference in ice thickness and the total ice volume on the ice sheet, but the accuracy of the method is sensitive to the ice shelf calculation, observed in the WARM cases.
Figure 29: Total ice volume for ice sheet (grounded ice: top), and ice shelf (floating ice: bottom). There is no significant difference in the ice sheet volume between the use of the first order method and the second order method, solving for the vertical heat diffusion equation, but the increase of ice shelf volume is slightly unstable (WARM\textsuperscript{0.5}-1st) with the first order method.
Figure 30: Total ice area for ice sheet (grounded ice: top) and ice shelf (floating ice: bottom). Although the ice area of the ice shelf is not as accurate as that of the ice sheet, its relative error is very small, compared to the error in the ice volume (fig. 29).
Figure 31: Experiment 3 - COLD: Ice thickness $h$ in meters ($\Delta t = 0.5$). Even with the COLD scenario, the ice does not reach to the ocean, and the possible reason is discussed in section 4.4.1.
Figure 32: Experiment 3 - WARM: Ice thickness $h$ in meters ($\Delta t = 0.25$).
Figure 33: Experiment 3 - PT: Ice thickness $h$ in meters ($\Delta t = 0.5$).
4.4 Analysis

4.4.1 PDD method

As mentioned in section 3.2.3, the seasonal temperature cycle at the given annual surface air temperature is the same regardless of the local climate regimes. Figure 34 is an example, describing how the seasonal temperature variability affects a computation of PDD under the same $T_a$.

![Figure 34: Different PDDs with the same annual surface air temperature $T_a$. The temperature is converted to the Celsius instead of in the Kelvin in order to present the idea of "positive-degree" clearly. PDDs are approximately 775 °C·day on the left and 88 °C·day on the right case. Using the eq. (66), the melt rates are 5.17m/yr and 0.53m/yr, respectively.](image)

The seasonal temperature fluctuation varies by location with many environmental factors, and the climate in the Pacific Northwest especially differs between west (coastal climate) and east (continental climate) of the Coast Mountains [47, 48]. Generally, a smaller amplitude of the seasonal cycle is observed on the west side of the mountains due to humid air mass from the ocean, whereas the temperature variation is relatively larger on the east side (fig. 35 - Data source: the Environment Canada [49], the NOAA [50], and the Western Regional Climate Center [51]).

These phenomena sensitively influence the ice margin of the Cordillean Ice Sheet (CIS). The geological traces indicate that the southern edge of the ice sheet extended approximately down to 48°N almost parallelly [34, 52]. The western CIS
margin expanded around the Yukon-Alaskan border as in fig. 31, but in addition, a tail of ice mass evolved along the Alaska Range and a part of the Aleutian Range during LGM [53]. Our simulation result with the COLD temperature and precipitation, however, shows that the ice margin does not fully reach the state of Alaska, western Washington, nor the British Columbia Coast (fig. 31). This is likely due to the inaccuracy of our current PDD estimation (eq. (65)). Although the spacial variation of annual mean surface air temperature $T_a$ is nearly parallel to the coast line in Canada, the interior part of the southern domain is as warm as the coast area, and south central Alaska is as cold as Yukon during the summer (fig. 36). A melting-point-contour line in the figure better fits the historical outline of CIS (upper left of fig. 36).

Figure 35: Comparison of the modern seasonal temperature cycles from west (red) and east (blue) of the Coast Mountains. The annual average temperature is Calgary, AB$^{[1]}$=$4.1^\circ$C, Annex Creek, AK$^{[2]}$=$4.2^\circ$C, Yellowknife, NT$^{[1]}$=$-4.6^\circ$C, and Mankomen Lake, AK$^{[3]}$=$-4.5^\circ$C. Calgary and Yellowknife belong to the continental climate (blue lines), and Annex Creek is located beside Taku Inlet and has the typical oceanic climate characteristics (red line). Mankomen Lake rather belongs to a transitional climate zone between the oceanic and continental climates (red line), but it sufficiently illustrates that the temperature cycles considerably differ by climate zone. Although the annual surface air temperatures are high at present such that there are no glaciers at these locations, these seasonal variations have a significant impact on the overall ice distribution during the cold period. Data Source: the Environment Canada (1971-2000)$^{[1]}$, the NOAA (1981-2010)$^{[2]}$, and the Western Regional Climate Center (1965-1983, 2002-2008)$^{[3]}$. 

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Figure 36: Annual mean surface air temperature $T_a$ in °K (left) and July mean surface air temperature $T_j$ (right). Extra contour lines in $T_a$ figures on the left indicate where July mean temperature is at the melting point (three small area on the upper left corner for PT). In the COLD case, the model underestimates ice accumulation in the Southcentral (Anchorage area), Interior (Fairbanks area), and Inside Passage (Juneau area) regional districts, and overestimate the extension of ice mass in the contiguous United States. The bottom figures indicate that with the better PDD method, more glaciers and ice caps may appear around the Alaskan Coast, especially at the Chugach Mountains near the Prince William Sound, AK in the PT result.

The PDD values, currently computed only from the annual mean temperature, generally underestimate the amount of ice melting on the east side of the Coast Mountains and overestimate the melting near the coast. For example, by lowering the temperature peak $T_j$ from 10°C to 5°C on the left in fig. 34, and by keeping
the right figure unchanged and the same annual mean temperature $T_a$ for both figures, e.g., $(T_a, T_j)_{\text{left}} = (-10^{\circ}\text{C}, 5^{\circ}\text{C})$ and $(T_a, T_j)_{\text{right}} = (-10^{\circ}\text{C}, 2^{\circ}\text{C})$, PDDs on the left and right are 313$^{\circ}\text{C}$-day and 88$^{\circ}\text{C}$-day. With only 3$^{\circ}$C difference in the July mean temperature, but still with the same annual mean temperature, their annual melt rates differs from 2.05m/yr to 0.53m/yr, respectively. The annual net ice accumulation depends on the total precipitation as well, but the contribution of seasonal temperature cycle, particularly during the summer, is not negligible.

The best solution is to incorporate a climate model, and calculate the amount of ice accumulation and melting at each grid cell since the original resolution of the COLD and WARM input data is only $8 \times 15$ within our domain and is interpolated to fit our $300 \times 600$ grids (the original PT input data have $1197 \times 2397$ grids in our domain). It requires, however, an enormous amount of additional computational power. Instead, it can be improved by integrating the July mean surface air temperature into the PDD function as follows [54, 55]:

$$PDD = \frac{1}{T_{\text{dev}} \sqrt{2\pi}} \int_0^a \int_0^{\infty} T_d \exp \left( -\frac{(T_d - T_{\text{ac}}(t))^2}{2T_{\text{dev}}^2} \right) dT_d, \quad (68)$$

where $a$ is 1 year, $T_{\text{dev}}$ is a standard deviation of the surface air temperature, $T_d$ is the daily surface air temperature, and $T_{\text{ac}}$ is the annual surface temperature cycle,

$$T_{\text{ac}}(t) = T_a + (T_j - T_a) \cos \frac{2\pi t}{a}. \quad (69)$$

It requires implementation of an efficient method to solve a double integral, but it is simple and certainly a more accurate estimate of PDD, compared to eq. (65). An analytical solution for these equations is well-described by Calov and Greve [56].

### 4.4.2 Convergence rate

The results provide us with a reasonable outline of the CIS build-up and decay, but the solution in at least one grid point in the domain does not converge well. At every time step, we only extract the worst A-loop and C-loop residuals of solutions from the current and previous iterations within $300 \times 600$ grid cells, and therefore, we are not yet certain whether a particular grid cell or several grid cells in different locations are causing the problem. Since the WARM and PT simulations take more than 10 days, it is time consuming to verify what exact grid cells suffer from numerical instability, despite it technically being an easy task.

**C-loop convergence** We guesstimate from the test results, however, that the initial cause of the bad C-loop convergence is somewhere near the grounding line.
Since Experiment 1 (constant temperature and precipitation throughout the domain) also experiences bad convergence from the beginning of the simulation (around 700 years) in spite of a relatively small amount of ice load in comparison to Experiment 2 and 3, the cause is more likely to be the existence of ice rather than the amount of ice at the particular locations. The C-loop convergence rate exceeds 1.0: COLD-never, WARM-after 300 years, and PT-after 5400 years. By extracting the regions where ice is never built for COLD, and ice is newly accumulated between 0 and 1000 years for WARM and between 5000 and 6000 years for PT, we realize that those locations are neither steep nor rugged, but mostly along the grounding line (fig. 37). In addition, ice shelves disappear around 13,700 years in the case of PT. The C-loop convergence does not recover immediately, but once the ice front is away from the coastline about 16,000 years, the solution starts to converge well. It is very likely that the source of bad convergence in the C-loop is caused by the grounding line treatment, although whether it happens along the entire grounding line or at the grounding line with a certain unique topography remains unknown.

**A-loop convergence** The A-loop converges very fast until the ice shelf appears. The number of A-loop iterations is largely either 1 or a set maximum (10 for Experiment 1 and 2, and 15 for Experiment 3), and the solutions of the horizontal stretching eqs. (40a) and (40b) poorly converge, most likely, as soon as the ice shelf arises. A bad convergence rate and unstable solutions from the A-loop extremely hinder the efficiency of the simulation. Because the A-loop is nested in the C-loop, 15 iterations in each loop is $15 \times 15$ iterations total just to calculate ice velocity and thickness for one grid cell. It is not clear why the solutions from the horizontal stretching equations do not converge well despite the fact that successive-over-relaxation (SOR) is known to be a fast converging method [57].

One possible reason could be the topography near the grounding line. The continental shelf along the west coast of North America is narrow, and the continental slope is steep [58]. The extra basal shear term in eqs. (40a) and (40b) is a shelf-sheet transition term, but when the sea level is at the edge of the continental shelf, $h_w$ next to the grounding line is large enough that $f_g$ from eq. (42) jumps from 0.5 to 0. In fact, this happens at most of the grounding lines, particularly during the coldest period when the sea level is low (fig. 38). $f_g$ serves as an "on-off switch" rather than a "smoothing term" at the ice sheet-shelf transition zone, and therefore, the solution near the grounding line may not be smooth (fig. 39). One of unique features of ARCTIC-TARAH is the hybrid SIA and SSA equations. If $f_g$ is not truly involved in eqs. (40a), (40b) and (50) near the grounding line, these equations are almost identical to zeroth-order SIA and SSA immediately beyond the grounding line (section 2.2). There are no notable issues with the convergence
rate [59], applying the same model to the Eurasian North where the continental shelf is very wide [58]. For these reasons, one of the sources that overwhelms the A-loop convergence rate could be from the horizontal stretching eqs. (40a) and (40b) at the unstable ice sheet-shelf grounding zone, particularly at the grounding line creeping on the steep continental slope.

### 4.4.3 Steep topography

As described in the section 3.1.1, the grid size used in the model is 4’×4’ in latitude and longitude. The range of length scales in longitude $\Delta x$ from 3020m at the north boundary to 5150m at the south boundary, and the grid size in latitude
Δy is uniform, 7413m from the west to east boundaries. There are 755 points in latitude and 420 points in longitude where the difference between two neighboring elevations is greater than 1000m, and the maximum differences are 3955m and 1899m in latitude and longitude, respectively.

As shown in fig. 20, the numerical solution of the ice velocity at deep and narrow valleys is not reasonably solved because of a large influx of ice from the surrounding grid cells. This kind of complex geometry requires a fine spatial resolution in order to derive an accurate solution [16]. It is, however, not possible for us to refine the grid size because of the computational memory limit. Since nearly a half of the domain is occupied by ocean and the northeast corner of the domain is relatively flat, an adaptive Cartesian mesh generation technique may provide efficiency in the memory usage to allow further refinement. In addition to the mountain ranges, a close attention to grid size at the continental slope and grounding lines are required to ensure the efficiency and stability. The grounding line on fig. 38 reaches lower than -2600m, and there are only a few grid cells within this narrow V-shape ice bottom.

There are, however, extremely steep locations such as cliffs that violate the assumption of the SIA no matter how much the grid is refined. Alternatively, Egholm (2011) and Jarosch (2012) successfully added higher order terms to strain
rate components and a diffusion term, respectively, for their steep mountain glacier simulations [18, 17].

4.5 GIS interface

The output data format of the model is in netCDF (Network Common Data Format), commonly used in earth science. We employ Panoply as a quick netCDF data viewer, and MATLAB for post-processing the data, such as creating plots and movies by converting .nc files into .mat files. In addition, netCDF data are converted to raster layers and tables which enable further spatial analyses by ArcMap. The conversion requires one of ArcToolboxes called Multidimensional Tool and some additional environmental settings. All processes are scripted in Python, which can be executed in the Python window on ArcMap by loading the code. Some work examples by ArcMap and a Python code are located in the Appendix section 7.2. We recommend ArcScene if an advanced three-dimensional display environment is preferred.
5 Discussion

In ARCTIC-TARAH, the numerical solutions of partial differential equations (PDEs) are approximated by the finite difference method (FDM). Although this method is widely understood and relatively easy to implement, it is relatively a difficult tool for handling problems with complex geometries in multi-dimensions [60]. Both the steepness of the Canadian Rockies and their jaggedness create instability in the model. We force the model to work by smoothing the basal topography, increasing the number of iterations for convergence, and reducing the time increment by a half year over 30,000-year runs.

One solution in the long-term could be to re-implement the model with either the finite element method (FEM) or the finite volume method (FVM) for a more flexible grid structure. The implementation of grounding line calculations should be reconsidered, the better climate forcing using the existing surface air temperature and precipitation data is required to capture the seasonal climate characteristics in the Pacific Northwest coastal, mountain, and continental regions, and the refinement of grids around the sheet-shelf transition zone may enhance the stability of the model. Although this would be much more difficult to implement than FDM, it would be a worthwhile effort, considering the accuracy and efficiency of the simulation, as well as the flexibility that would allow for the model to be applied to regions with various topographies.

6 Acknowledgments

This thesis is dedicated to the memory of my mother who passed away during the completion of this project and who had wished to see me complete a master’s programme in Computational Science the most.

I would like to thank my supervisor Nina Kirchner at Stockholm Universitet and my reviewer Per Lötstedt at Uppsala Universitet for assisting with the work of my thesis project over an extended period of time. I also want to show my appreciation for Per Wahlund for advising me on Fortran, Rickard Pettersson for the ArcGIS help, Philipp Hancke for the system support, and my friend Marina Dunaravich for proofreading this thesis.
References


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[59] Nina Kirchner. personal communication. Stockholm University / the Bolin Centre for Climate Research.

### 7 Appendix

#### 7.1 Symbol tables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x, y$</td>
<td>horizontal coordinates</td>
<td>$m$</td>
</tr>
<tr>
<td>$z, \zeta$</td>
<td>vertical coordinate, scaled vertical coordinate $0 \leq \zeta = \frac{h - z}{h} \leq 1$</td>
<td>-</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
<td>$yr$</td>
</tr>
<tr>
<td>$U$</td>
<td>total velocity field $U(x, y, \zeta, t)$ for each velocity component $u, v, w'$</td>
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<td>$\bar{U}$</td>
<td>vertical averaged velocity field $\bar{U}(x, y, t)$ for horizontal velocity $\bar{u}, \bar{v}$</td>
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</tr>
<tr>
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<td>basal velocity field $U(x, y, t)$ for horizontal velocity $u_b, v_b$</td>
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<td>surface velocity field $U(x, y, t)$ for horizontal velocity $u_s, v_s$</td>
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</tr>
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<td>internal velocity field $U_i(x, y, t)$ for horizontal velocity $u_i, v_i$</td>
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<td>$dx, dy$</td>
<td>lengths of grids in $x$ and $y$ directions</td>
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</tr>
<tr>
<td>$F$</td>
<td>body force</td>
<td>$N/m^3$</td>
</tr>
<tr>
<td>$g$</td>
<td>gravitational field</td>
<td>$m/s^2$</td>
</tr>
<tr>
<td>$I$</td>
<td>identity matrix</td>
<td>-</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>stress tensor</td>
<td>$N/m^2$</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>deviatoric stress tensor</td>
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<tr>
<td>$\sigma_{11}$</td>
<td>second invariant of stress tensor</td>
<td>$N/m^2$</td>
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<tr>
<td>$\sigma_{bx}, \sigma_{by}$</td>
<td>basal shear stress in $x$ and $y$</td>
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<td>$\hat{\sigma}_b$</td>
<td>basal stress</td>
<td>$N/m^2$</td>
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<td>$\dot{\epsilon}$</td>
<td>strain rate tensor</td>
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<td>$\dot{\epsilon}_e$</td>
<td>effective strain rate</td>
<td>$yr^{-1}$</td>
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<td>$h$</td>
<td>ice thickness</td>
<td>$m$</td>
</tr>
<tr>
<td>$h_g$</td>
<td>ice thickness at grounding line</td>
<td>$m$</td>
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<tr>
<td>$h_b$</td>
<td>bedrock elevation</td>
<td>$m$</td>
</tr>
<tr>
<td>$h_s$</td>
<td>ice surface altitude; $h_s = h_b + h$</td>
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<tr>
<td>$h_w$</td>
<td>water column thickness</td>
<td>$m$</td>
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<td>$S$</td>
<td>sea level</td>
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<tr>
<td>$\rho$</td>
<td>density</td>
<td>$kg/m^3$</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure</td>
<td>$Pa$</td>
</tr>
<tr>
<td>$c$</td>
<td>specific heat capacity</td>
<td>$J/^\circ K$</td>
</tr>
<tr>
<td>$k$</td>
<td>heat conductivity</td>
<td>$W/m/^\circ K$</td>
</tr>
<tr>
<td>$A$</td>
<td>Arrhenius temperature dependent coefficient</td>
<td>$/yr/Pa^3$</td>
</tr>
<tr>
<td>$\bar{A}$</td>
<td>vertically mean ice rheological coefficient</td>
<td>$/yr/Pa^3$</td>
</tr>
<tr>
<td>$q_g$</td>
<td>ice area flux to ocean at a grounding line; $q_g = u_g h_g$</td>
<td>$m^3/yr$</td>
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<tr>
<td>$Q_i$</td>
<td>internal shear heating</td>
<td>$N/m^2/yr$</td>
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<tr>
<td>$M_b$</td>
<td>basal mass balance</td>
<td>$kg/m^3$</td>
</tr>
<tr>
<td>$M_s$</td>
<td>surface mass balance</td>
<td>$kg/m^3$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>longitudinal stresses downstream of the grounding line $(\tau_{xx}, \tau_{yy})$</td>
<td>$N/m^2$</td>
</tr>
<tr>
<td>$\tau_f$</td>
<td>free stress at the grounding line, $\frac{1}{2} \rho_i g h_g (1 - \frac{\rho_i}{\rho_w})$</td>
<td>$N/m^2$</td>
</tr>
<tr>
<td>$\sigma_f$</td>
<td>free stress: $\frac{1}{2} \rho_i g h_g (1 - \frac{\rho_i}{\rho_w})$</td>
<td>-</td>
</tr>
<tr>
<td>$u, v$</td>
<td>horizontal velocities in $x$ and $y$ directions</td>
<td>$m/yr$</td>
</tr>
<tr>
<td>$w, w'$</td>
<td>vertical velocity and scaled vertical velocity: $\frac{\partial \zeta}{\partial y}$</td>
<td>$m/yr$</td>
</tr>
<tr>
<td>$\bar{u}, \bar{v}$</td>
<td>total velocities in $x$ and $y$ directions: $\bar{u} = \bar{u}_i + u_b$</td>
<td>$m/yr$</td>
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<tr>
<td>$\bar{u}_i, \bar{v}_i$</td>
<td>averaged internal velocities in $x$ and $y$ directions</td>
<td>$m/yr$</td>
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<tr>
<td>$u_b, v_b$</td>
<td>basal velocities in $x$ and $y$ directions</td>
<td>$m/yr$</td>
</tr>
</tbody>
</table>
$u_b, v_b$  | basal sliding velocities in x and y directions | m/yr  
$u_g$  | ice velocity at the grounding line | m/yr  
$\mu$  | strain rate-dependent shear viscosity | Pa · yr  
$f_g$  | floating ice flag $0 \leq f_g \leq 0.5$ | -  
$T$  | ice temperature | °K  
$T_i$  | ice surface temperature | °K  
$T_a$  | annual mean surface air temperature | °K  
$T_{pm}$  | pressure melting point: $T_{pm} = T_m - 8.66 \times 10^{-4} h\zeta$ | °K  
$T_j$  | July mean surface air temperature | °K  
$T_d$  | daily mean surface air temperature | °K  
$T_{ac}$  | annual mean surface air temperature cycle | °K  
$T_{dev}$  | standard deviation of surface air temperature | °K  
$\beta$  | $\arccos\left(\frac{T_m - T_a}{T_z - T_m}\right)$ | -  
$Pr$  | annual total precipitation | m/yr  
$PDD$  | positive degree-days | day °C  
$MR$  | annual melting rate | m³/yr  

Table 5: List of symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_i$</td>
<td>density of ice</td>
<td>910 kg/m³</td>
</tr>
<tr>
<td>$\rho_w$</td>
<td>density of water</td>
<td>1000 kg/m³</td>
</tr>
<tr>
<td>$c_i$</td>
<td>specific heat capacity of ice</td>
<td>2009 J/°K</td>
</tr>
<tr>
<td>$k_i$</td>
<td>heat conductivity of ice</td>
<td>662,560 W/m/°K</td>
</tr>
<tr>
<td>$A_0$</td>
<td>ice rheological coefficient</td>
<td>$2.0 \times 10^{-16}$ or $1.66 \times 10^{-16}$ yr⁻³ Pa⁻³</td>
</tr>
<tr>
<td>$Q$</td>
<td>activation energy</td>
<td>$9.525 \times 10^4$ or $7.820 \times 10^4$ J/mol</td>
</tr>
<tr>
<td>$R$</td>
<td>gas constant</td>
<td>8.314 J/mol</td>
</tr>
<tr>
<td>$E$</td>
<td>enhancement factor</td>
<td>10 (sheet), 3 (shelf)</td>
</tr>
<tr>
<td>$T_m$</td>
<td>melting temperature of ice</td>
<td>273.15°K</td>
</tr>
<tr>
<td>$g$</td>
<td>gravitational force</td>
<td>9.81 m/s²</td>
</tr>
<tr>
<td>$n$</td>
<td>ice rheological exponent</td>
<td>3</td>
</tr>
<tr>
<td>$m_s$</td>
<td>basal sliding exponent</td>
<td>2</td>
</tr>
<tr>
<td>$C$</td>
<td>basal sliding coefficient between bed and ice</td>
<td>$10^{-12}$ m/yr/Pa²</td>
</tr>
<tr>
<td>$Q_g$</td>
<td>geothermal heat flux</td>
<td>0.08 W/yr/m²</td>
</tr>
<tr>
<td>$s_{dev}$</td>
<td>the standard deviation of the observed bed elevations</td>
<td>100 m</td>
</tr>
<tr>
<td>$T_z$</td>
<td>threshold temperature for PDD and precipitation</td>
<td>288.06°K</td>
</tr>
<tr>
<td>$\pi$</td>
<td>ratio of a circle’s circumference to its diameter</td>
<td>3.14</td>
</tr>
<tr>
<td>$a$</td>
<td>year cycle</td>
<td>1 yr</td>
</tr>
</tbody>
</table>

Table 6: List of constants
7.2 GIS interface supplement

Figure 40: ArcMap must have a Multidimensional Tool extension to convert a netCDF file to raster files. One-by-one conversions are available from ArcToolbox.

Figure 41: An example of Experiment 3 COLD scenario at 2000 years from the start with a Python code on the left.
Figure 42: Basal topography on ArcMap.

Figure 43: Ice thickness at 10000 years from the start (Experiment 3 COLD scenario).

Figure 44: Ice surface altitude at 10000 years from the start (Experiment 3 COLD scenario).
To convert a netCDF file to raster layers and a table.

You must have a "Multidimensional Tools" in the ArcToolbox.
Make sure it's turned on (> Customize > Extensions... on the desktop bar)
Edit the directory "fort_92_nc = xxx" below *# Local variables:* to where a .nc
file is located.
You may not be able to execute "Define Projection", but it seems an ArcGIS bug
that appears when you convert netCDF to raster. Comment out the code if it
doesn't work.

Change the transparency of "bed_elevation" to 15% and locate it right above
"hillshade_base". Try not to display 0 on "ice_thickness" and other files
by checking the box "Display Background Value: 0" in the properties.

You can see each time series by right-clicking properties and selecting a
*netCDF* tab. Choose the time series from the drop-down menu.

---

# netCDF_raster_table.py
# To convert a netCDF file to raster layers and a table.

# Local variables:
fort_92_nc = "D:\fort.92.nc"
longitude = "longit"
latitude = "latit"
h = "h"
hw = "hw"
hb = "hb"
hs = "hs"
budgall = "budgall"
budgrain = "budgrain"
budgsnow = "budgsnow"
budgmelt = "budgmelt"

import arcpy
import os
from arcpy import env
from arcpy.sa import *
import sys

print "Start extracting from netCDF".format(sys.version)

# Local variables:
fort_92_nc = "D:\fort.92.nc"

longitude = "longit"
latitude = "latit"
h = "h"
hw = "hw"
hb = "hb"
hs = "hs"
budgall = "budgall"
budgrain = "budgrain"
budgsnow = "budgsnow"
budgmelt = "budgmelt"
table_list = "toti;tota;totig;totag;totif;totaf"
## utop = "utop"
## vtop = "vtop"
## ubot = "ubot"
## vbot = "vbot"
## ua = "ua"
## va = "va"
##
# Process : NetCDF -> Raster
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, h, longitude, latitude, "ice_thickness", "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, hw, longitude, latitude, "water_height", "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, hb, longitude, latitude, "bed_elevation", "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, hs, longitude, latitude, "icesurf_altitude", "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, budgall, longitude, latitude, "total_ice_budget", "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, budgrain, longitude, latitude, "rain_budget", "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, budgsnow, longitude, latitude, "snow_budget", "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, budgmelt, longitude, latitude, "melt_budget", "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, "utop", longitude, latitude, utop, "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, "vtop", longitude, latitude, vtop, "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, "ubot", longitude, latitude, ubot, "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, "vbot", longitude, latitude, vbot, "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, "ua", longitude, latitude, ua, "", "", "BY_VALUE")
arcpy.MakeNetCDFRasterLayer_md(fort_92_nc, "va", longitude, latitude, va, "", "", "BY_VALUE")
print "finish raster"

# Process : NetCDF -> Table
arcpy.MakeNetCDFTableView_md(fort_92_nc, table_list, "ice_table", "time", "", "BY_VALUE")
arcpy.MakeNetCDFTableView_md(fort_92_nc, "sealev", "sea_level", "time", "", "BY_VALUE")
print "finish table"

# Process: Define Projection
arcpy.DefineProjection_management("ice_thickness", "GEOGCS["GCS_North_American_1983",DATUM["D_North_American_1983",\Spheroid["GRS_1980",6378137.0,298.2572221010],PRIMEM["Greenwich",0.0],\UNIT["Degree",0.0174532925199433]]")
arcpy.DefineProjection_management("water_height", "GEOGCS["GCS_North_American_1983",DATUM["D_North_American_1983",\Spheroid["GRS_1980",6378137.0,298.2572221010],PRIMEM["Greenwich",0.0],\UNIT["Degree",0.0174532925199433]]")
arcpy.DefineProjection_management("bed_elevation", "GEOGCS["GCS_North_American_1983",DATUM["D_North_American_1983",\Spheroid["GRS_1980",6378137.0,298.2572221010],PRIMEM["Greenwich",0.0],\UNIT["Degree",0.0174532925199433]]")
arcpy.DefineProjection_management("icesurf_altitude", "GEOGCS["GCS_North_American_1983",DATUM["D_North_American_1983",\Spheroid["GRS_1980",6378137.0,298.2572221010],PRIMEM["Greenwich",0.0],\UNIT["Degree",0.0174532925199433]]")
SPHEROID ['GRS_1980', 6378137.0, 298.257222101], PRIMEM ['Greenwich', 0.0],
UNIT ['Degree', 0.0174532925199433])

SPHEROID ['GRS_1980', 6378137.0, 298.257222101], PRIMEM ['Greenwich', 0.0],
UNIT ['Degree', 0.0174532925199433]]")

SPHEROID ['GRS_1980', 6378137.0, 298.257222101], PRIMEM ['Greenwich', 0.0],
UNIT ['Degree', 0.0174532925199433]]")

SPHEROID ['GRS_1980', 6378137.0, 298.257222101], PRIMEM ['Greenwich', 0.0],
UNIT ['Degree', 0.0174532925199433]]")

SPHEROID ['GRS_1980', 6378137.0, 298.257222101], PRIMEM ['Greenwich', 0.0],
UNIT ['Degree', 0.0174532925199433]]")

print "finish projection"

# Process: Hillshade
arcpy.HillShade_3d ("bed_elevation", "hillshade_base", "315", "45", "NO_SHADOWS", "1")

print "finish hillshade"