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Existence of pathwise unique Langevin processes on polytopes with perfect reflection at the boundary

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Abstract

Exploiting an explicit projection from the real line into an interval, we prove existence and pathwise uniqueness of one-dimensional Langevin processes confined to an interval with perfect reflection at the boundary. This result is subsequently generalized to multi-dimensional Langevin processes confined to box domains or general polytopes.

Keywords: Langevin process, integrated Brownian motion, Lagrangian stochastic model, reflecting boundary

2000 MSC: 60J50, 60J65, 60G15

1. Introduction

We study a multi-dimensional Langevin process describing the time evolution of a process (X, U) , where X denotes the position and U the velocity of a particle. To properly define the Langevin process studied in this article, we first introduce some notation. Given an integer $d \geq 1$, let $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the standard inner product and norm, respectively, in the Euclidean d -space $\mathbb{R}^d$. Given $T > 0$, let $\mathcal{C}([0, T], \mathbb{R}^d)$ denote the class of continuous functions on $[0, T]$ with values in $\mathbb{R}^d$ and let $\mathbb{D}([0, T], \mathbb{R}^d)$ denote the set of càdlàg functions on $[0, T]$ with values in $\mathbb{R}^d$, i.e. functions which are right continuous with left limits. Let $D \subset \mathbb{R}^d$ be an open polytope, let W be a d -dimensional standard Wiener process and let $(X_0, U_0) \in D \times \mathbb{R}^d$. The Langevin process $(X, U) \in \mathcal{C}([0, T], \overline{D}) \times \mathbb{D}([0, T], \mathbb{R}^d)$ studied here is defined as

$$\begin{aligned} X_t &= X_0 + \int_0^t U_s ds, \\ U_t &= U_0 + W_t - 2 \sum_{0 \leq s \leq t} \langle U_{s-}, n_D(X_s) \rangle n_D(X_s) \mathbb{I}_{\{X_s \in \partial D\}}, \end{aligned} \quad (1.1)$$

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for all $t \in [0, T]$. Here ∂D is the boundary of D , $n_D(x)$ is the outward unit normal vector of D at $x \in \partial D$ and $\mathbb{I}_{\{X_s \in \partial D\}}$ is equal to 1 whenever $X_s \in \partial D$ and 0 otherwise. The definition of n_D at the edges and corners of D will be further discussed in Section 3. The rightmost term in the velocity equation is constructed so that the confinement is maintained. The behaviour of the process at the boundary corresponds to perfect reflection in the sense that the incident and reflected angles are identical and the norm of the velocity remains unchanged whenever the particle hits the boundary.

A first study of the free Langevin process was conducted by McKean [9], who derived the explicit density of the process and proved that the paths of the one-dimensional free Langevin process, almost surely, never cross $(0, 0)$. More detailed investigations of the successive hitting times of the free Langevin process have been carried out by Lachal [8]. The confined Langevin process has been studied for a variety of different boundary conditions and we here mention the work of Bertoin [2] regarding absorbing boundaries and that of Jacob [6],[7] regarding elastic boundaries. In recent years there has been a breakthrough in the study of the nonlinear Langevin process that arises when a density dependent drift term is added to the velocity equation in (1.1). Bossy, Jabir and Talay [5] proved well-posedness of the free nonlinear Langevin process and later extended their results to the case when the position process is confined to a half space [4]. Well-posedness of the nonlinear Langevin process confined to bounded $\mathcal{C}^3$ -domains is currently being developed by Bossy and Jabir using methods from the field of partial differential equations [3]. The study of the confined Langevin process with perfect reflection is motivated by its application to Lagrangian stochastic models in computational fluid dynamics. In particular, Langevin processes are used in probability density function (PDF) methods for simulation of turbulent flows. For the general theory of PDF methods in turbulence, we refer the reader to the monograph by Pope [10] and for an application of the well-posedness results by Bossy et al., the reader is referred to [1].

In this article we prove existence in law of a pathwise unique Langevin process confined to a polytope, as defined in (1.1). To this end, we first consider a one-dimensional Langevin process and construct an explicit projection from the unbounded position-velocity space $\mathbb{R} \times \mathbb{R}$ into the semibounded position-velocity domain $[0, a] \times \mathbb{R}$, for some positive real number a . Our projection lends a few ideas from, and exhibits some similarity with, the projection from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}_+ \times \mathbb{R}$ appearing in Proposition 3.1 in [4]. We subsequently use the projection onto $[0, a] \times \mathbb{R}$ to deduce the existence and pathwise uniqueness of the confined Langevin process from the corresponding properties of the free Langevin process in the entire space. This result is then generalized to, in turn, box domains of the form $D = (0, a_1) \times \dots \times (0, a_d)$ and the considerably more complex case of general polytopes. Note that our results are disjoint to the results by Bossy and Jabir [3] on $\mathcal{C}^3$ -domains, as we consider domains with edges and corners. To put our result into the context of the articles mentioned above, we note that the box domains considered in this article are the natural ones to use in the downscaling problems studied in [1]. The extension of our results to nonlinear Langevin processes is work in progress.

The remainder of this article is organized as follows. In Section 2 we state and prove the existence and pathwise uniqueness result for a one-dimensional Langevin process confined to a bounded interval $[0, a]$. In Section 3 we extend the one-dimensional result from Section 2 to multi-dimensional box domains and finally, in Section 4, we arrive at the existence and uniqueness result for polytopes.

2. Langevin processes confined to intervals

As a first step we consider, in this section, a one-dimensional special case. We construct and investigate a projection from $\mathbb{R}^2$ into $[0, a] \times \mathbb{R}$, for some positive constant a . Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ be a given filtered probability space endowed with a standard one-dimensional Wiener process $W^{1,f}$ and let $(X_0^1, U_0^1) \in (0, a) \times \mathbb{R}$. The free one-dimensional Langevin process $(X_t^{1,f}, U_t^{1,f})$ is defined as

$$X_t^{1,f} = X_0^1 + \int_0^t U_s^{1,f} ds, \quad U_t^{1,f} = U_0^1 + W_t^{1,f}, \quad (2.2)$$

for all $t \in [0, T]$. By means of the projection described below, we will show existence and pathwise uniqueness for the one-dimensional Langevin process $(X_t^{1,c}, U_t^{1,c})$ confined to the interval $[0, a]$, which is defined, for all $t \in [0, T]$, as

$$\begin{aligned} X_t^{1,c} &= X_0^1 + \int_0^t U_s^{1,c} ds, \\ U_t^{1,c} &= U_0^1 + W_t^{1,c} - 2 \sum_{0 < s \leq t} U_s^{1,c} \mathbb{I}_{\{X_s^{1,c} \in \{0, a\}\}}. \end{aligned} \quad (2.3)$$

Define the continuous function $F_a : \mathbb{R} \rightarrow [0, a]$ as

$$F_a(x) = \sum_{k \in \mathbb{Z}} |x - 2ka| \mathbb{I}_{\{x \in [(2k-1)a, (2k+1)a)\}}, \quad (2.4)$$

so that F_a is a periodic extension of the absolute value function on $[-a, a]$ to the entire real line. Denote by Z_a the set of integer multiples of a ,

$$Z_a = \{ka : k \in \mathbb{Z}\}, \quad (2.5)$$

and define the function $S_a : \mathbb{R} \setminus Z_a \rightarrow \{-1, 1\}$ as

$$S_a(x) = \begin{cases} 1, & x \in (2ka, (2k+1)a), \text{ for some } k \in \mathbb{Z}, \\ -1, & x \in ((2k-1)a, 2ka), \text{ for some } k \in \mathbb{Z}. \end{cases} \quad (2.6)$$

Note that $F'_a(x) = S_a(x)$ on $\mathbb{R} \setminus Z_a$ and also that the projection used in [4] is a limit case of our projection obtained by letting $a \rightarrow \infty$. Next, we want to define the stochastic process $S_t^{1,f}$ as the càdlàg version of $S_a(X_t^{1,f})$. In order to assert that $S_t^{1,f}$ is well-defined, we need information regarding the hitting times of $X_t^{1,f}$ on the set Z_a .

Lemma 2.1. *The sequence of hitting times $\{\tau_n^1\}_{n \geq 0}$, defined as $\tau_0^1 = 0$ and*

$$\tau_n^1 = \inf\{t > \tau_{n-1}^1 : X_t^{1,f} \in Z_a\}, \quad (2.7)$$

for $n \geq 1$, is well-defined and grows to infinity.

Proof. McKean [9] showed that if $(X_0^1, U_0^1) \neq (0, 0)$, then $\mathbb{Q}$ -almost surely, the paths of $(X_t^{1,f}, U_t^{1,f})$ never cross $(0, 0)$. Trivially, this statement holds also with $(0, 0)$ replaced by $(y, 0)$, for any fixed $y \in \mathbb{R}$. Since $(X_0^1, U_0^1) \neq (ka, 0)$ for all $k \in \mathbb{Z}$, it follows that, $\mathbb{Q}$ -almost surely, the paths of $(X_t^{1,f}, U_t^{1,f})$ never cross $(ka, 0)$, for any $k \in \mathbb{Z}$. Hence

$$\begin{aligned} & \mathbb{Q}\left(\exists t \geq 0 : (X_t^{1,f}, U_t^{1,f}) \in Z_a \times \{0\}\right) \\ &= \sum_{k \in \mathbb{Z}} \mathbb{Q}\left(\exists t \geq 0 : (X_t^{1,f}, U_t^{1,f}) = (ka, 0)\right) = 0. \end{aligned} \quad (2.8)$$

As $X_t^{1,f}$, $\mathbb{Q}$ -almost surely, always hits Z_a with nonzero velocity, the sequence of hitting times $\{\tau_n^1\}_{n \geq 0}$ is well-defined.

To show that $\{\tau_n^1\}_{n \geq 0}$ grows to infinity, we prove that, for any $T > 0$, only finitely many elements of the set $\{t \geq 0 : X_t^{1,f} \in Z_a\}$ are found in $[0, T]$. If this were not the case, then at least one of the following two situations would occur.

1. There exists an integer k and a countably infinite set $\tau^{1,k} \subset [0, T]$ such that $X_t^{1,f} = ka$ for all $t \in \tau^{1,k}$.
2. There exists a countably infinite subset $K \subset \mathbb{Z}$ such that, for all $k \in K$, there exists at least one $t \in [0, T]$ such that $X_t^{1,f} = ka$.

The first case cannot occur since, as was first noted by McKean [9], $X_t^{1,f} = 0$ occurs, $\mathbb{Q}$ -almost surely, only finitely many times on compact time intervals. Trivially, this statement holds for $X_t^{1,f} = ka$ as well and the set $\tau^{1,k}$ cannot be countably infinite for any k . The second case cannot occur since the paths of a Brownian motion are, $\mathbb{Q}$ -almost surely, bounded on finite time intervals. Hence, $\mathbb{Q}$ -almost surely, $X_t^{1,f}$ cannot have visited countably infinitely many of the elements in Z_a during the time interval $[0, T]$. This proves that, $\mathbb{Q}$ -almost surely, the set $\{t \in [0, T] : X_t^{1,f} \in Z_a\}$ is finite for all $T > 0$ and, as a consequence, $\{\tau_n^1\}_{n \geq 0}$ must grow to infinity. $\square$

It follows from Lemma 2.1 that, $\mathbb{Q}$ -almost surely, $X_t^{1,f}$ hits the set Z_a , on which S_a was undefined, a finite number of times during each finite time interval $[0, T]$. This implies that the càdlàg version of $S_a(X_t^{1,f})$ is a well-defined stochastic process which we denote by $S_t^{1,f}$.

Theorem 2.2. *There exists a one-dimensional $\mathcal{F}_t$ -adapted Wiener process $W^{1,c}$, such that*

$$X_t^{1,c} = F_a(X_t^{1,f}), \quad U_t^{1,c} = S_t^{1,f} U_t^{1,f}, \quad (2.9)$$

is the pathwise unique solution to (2.3) for all $t \in [0, T]$. The sequence of hitting times $\{\tilde{\tau}_n^1\}_{n \geq 0}$, defined as $\tilde{\tau}_0^1 = 0$ and

$$\tilde{\tau}_n^1 = \inf\{t > \tilde{\tau}_{n-1}^1 : X_t^{1,c} \in \{0, a\}\}, \quad (2.10)$$

for $n > 1$, is well-defined and grows to infinity.

Proof. First note that (2.9) and the definition of F_a imply the identity

$$\{t \geq 0 : X_t^{1,f} = Z_a\} = \{t \geq 0 : X_t^{1,c} \in \{0, a\}\}, \quad (2.11)$$

from which it follows that $\{\tau_n^1\}_{n \geq 0} = \{\tilde{\tau}_n^1\}_{n \geq 0}$. The well-posedness and growth to infinity of $\{\tilde{\tau}_n^1\}_{n \geq 0}$ thus follows immediately from Lemma 2.1. By construction, $S_t^{1,f}$ is a pure jump process and since both $S_t^{1,f}$ and $U_t^{1,f}$ are càdlàg $\mathcal{F}_t$ -adapted processes, so is $U_t^{1,c}$. The integration by parts formula implies

$$\begin{aligned} U_t^{1,c} &= S_t^{1,f} U_t^{1,f} = U_0^1 + \int_0^t S_{s-}^{1,f} dU_s^{1,f} + \sum_{0 < s \leq t} U_s^{1,f} \Delta S_s^{1,f} \\ &= U_0^1 + W_t^{1,c} + \sum_{0 < s \leq t} \underbrace{U_{s-}^{1,f} S_{s-}^{1,f}}_{=U_{s-}^{1,c}} \frac{\Delta S_s^{1,f}}{S_{s-}^{1,f}}, \end{aligned} \quad (2.12)$$

where the last equality follows by the Lévy characterization of a Wiener process. It is easy to see that $W_t^{1,c}$ must be $\mathcal{F}_t$ -adapted. Moreover, $S_s^{1,f}$ jumps if $X_s^{1,f} \in Z_a$, which, by (2.11) is exactly when $X_s^{1,c} \in \{0, a\}$. If $S_{s-}^{1,f} = 1$, then $\Delta S_s^{1,f} = -2$ and if $S_{s-}^{1,f} = -1$, then $\Delta S_s^{1,f} = 2$, so altogether we get

$$U_t^{1,c} = U_0^1 + W_t^{1,c} - 2 \sum_{0 < s \leq t} U_{s-}^{1,c} \mathbb{I}_{\{X_s^{1,c} \in \{0, a\}\}}, \quad (2.13)$$

which is the second equation in (2.3). Next, using that $F_a'(x) = S_a(x)$ on $\mathbb{R} \setminus Z_a$ and the fact that the set $\{t \geq 0 : X_t^{1,f} \in Z_a\}$ is countable, we get

$$\begin{aligned} X_t^{1,c} &= F_a(X_t^{1,f}) = F_a(X_0^{1,f}) + \int_0^t S_a(X_s^{1,f}) dX_s^{1,f} = X_0^1 + \int_0^t S_s^{1,f} U_s^{1,f} ds \\ &= X_0^1 + \int_0^t U_s^{1,c} ds, \end{aligned} \quad (2.14)$$

which is the first equation in (2.3). Thus $\mathbb{Q}$ -almost surely, $X^{1,c} = F_a(X^{1,f})$ and $U^{1,c} = S^{1,f} U^{1,f}$. To conclude, pathwise uniqueness follows by induction exactly as in Proposition 3.1 in [4] using the properties of $\{\tilde{\tau}_n^1\}_{n \geq 0}$ and the fact $X_0^1 \in (0, a)$. We omit the details. $\square$

3. Langevin processes confined to box domains

In this section we consider box domains of the form $D = (0, a_1) \times \cdots \times (0, a_d)$, for some choice of positive real numbers $a_1, \dots, a_d$. We prove existence and pathwise uniqueness of a multi-dimensional Langevin process confined to a box domain by considering a simple extension of the results in Section 2. For every point $x \in \partial D$, we define

$$I_L(x) = \{i \in \{1, \dots, d\} : x_i = 0\}, \quad I_U(x) = \{i \in \{1, \dots, d\} : x_i = a_i\}, \quad (3.15)$$

Let e_i , $i \in \{1, \dots, d\}$, be the i^{th} unit coordinate vector. On the faces of the boundary, we define the outward unit normal vector of D at $x \in \partial D$ as

$$n_D(x) = \begin{cases} e_i, & \text{if } I_L(x) = \emptyset \text{ and } I_U(x) = \{i\}, \\ -e_i, & \text{if } I_L(x) = \{i\} \text{ and } I_U(x) = \emptyset. \end{cases} \quad (3.16)$$

Due to Remark 3.5 below, we may let $n_D(x)$ be undefined at the edges and corners of the boundary. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a given filtered probability space endowed with a standard d -dimensional Wiener process W^f and let $(X_0, U_0) \in D \times \mathbb{R}^d$. Define the free d -dimensional Langevin process (X_t^f, U_t^f) as

$$X_t^f = X_0 + \int_0^t U_s^f ds, \quad U_t^f = U_0 + W_t^f, \quad (3.17)$$

for all $t \in [0, T]$.

Define the continuous function $F_{(a_1, \dots, a_d)} : \mathbb{R}^d \rightarrow \overline{D}$, where $\overline{D}$ denotes the closure of D , as

$$F_{(a_1, \dots, a_d)}(x_1, \dots, x_d) = (F_{a_1}(x_1), \dots, F_{a_d}(x_d)), \quad (3.18)$$

with F_{a_i} as defined in (2.4). Define the function $S_{(a_1, \dots, a_d)} : (\mathbb{R} \setminus Z_{a_1}) \times \dots \times (\mathbb{R} \setminus Z_{a_d}) \rightarrow \{-1, 1\} \times \dots \times \{-1, 1\}$ as

$$S_{(a_1, \dots, a_d)}(x_1, \dots, x_d) = (S_{a_1}(x_1), \dots, S_{a_d}(x_d)), \quad (3.19)$$

with S_{a_i} as defined in (2.6). Recall that $F'_{a_i}(x_i) = S_{a_i}(x_i)$ on $\mathbb{R} \setminus Z_{a_i}$, for all $1 \leq i \leq d$. The following lemma follows easily from Lemma 2.1.

Lemma 3.3. *The sequence of hitting times $\{\tau_n\}_{n \geq 0}$, defined as $\tau_0 = 0$ and*

$$\tau_n = \inf\{t > \tau_{n-1} : (X_t^f)_i \in Z_{a_i} \text{ for some } 1 \leq i \leq d\}, \quad (3.20)$$

for $n \geq 1$, is well-defined and grows to infinity.

Lemma 3.3, whose proof we have omitted, asserts that the càdlàg version of $S_{(a_1, \dots, a_d)}(X_t^f)$ is a well-defined stochastic process which we denote by S_t^f .

Theorem 3.4. *There exists a d -dimensional $\mathcal{F}_t$ -adapted Wiener process W^c , such that*

$$X_t^c = F_{(a_1, \dots, a_d)}(X_t^f), \quad (U_t^c)_i = (S_t^f)_i (U_t^f)_i, \text{ whenever } i \in \{1, \dots, d\}, \quad (3.21)$$

is the pathwise unique solution to (1.1) in $D = (0, a_1) \times \dots \times (0, a_d)$, for all $t \in [0, T]$. The sequence of hitting times $\{\tilde{\tau}_n\}_{n \geq 0}$ defined as $\tilde{\tau}_0 = 0$ and

$$\tilde{\tau}_n = \inf\{t > \tilde{\tau}_{n-1} : X_t^c \in \partial D\}, \quad (3.22)$$

for $n > 1$, is well-defined and grows to infinity.

Proof. By construction, the properties of $\{\tilde{\tau}_n\}_{n \geq 0}$ follow directly from the properties of $\{\tau_n\}_{n \geq 0}$ given in Lemma 3.3. Next, in analogy with (2.12)-(2.13), we obtain, by the integration by parts formula,

$$(U_t^c)_i = (U_0)_i + (W_t^c)_i - 2 \sum_{0 < s \leq t} (U_{s-}^c)_i \mathbb{I}_{\{(X_s^c)_i \in \{0, a_i\}\}}. \quad (3.23)$$

By the construction of the projection and Remark 3.5 below, it is clear that, $\mathbb{P}$ -almost surely, for all s such that $X_s^c \in \partial D$, we have $(X_s^c)_i \in \{0, a_i\}$ for exactly one $i \in \{1, \dots, d\}$. The outward unit normal vector $n_D(X_s^c)$ is then parallel to e_i for this i and hence

$$(U_{s-}^c)_i = \langle U_{s-}^c, n_D(X_s^c) \rangle n_D(X_s^c)_i, \quad (3.24)$$

whereas, for all other indices $j \in \{1, \dots, d\}$, we have $\mathbb{I}_{\{(X_s^c)_j \in \{0, a_j\}\}} = 0$ and $(n_D(X_s^c))_j = 0$. This proves that U_t^c , as defined in (3.21), solves the second equation in (1.1).

The proofs of pathwise uniqueness and the fact that X_t^c , as defined in (3.21), solves the first equation in (1.1) are analogous to the corresponding proofs in Theorem 2.2 and are thus omitted. $\square$

Remark 3.5. By the construction of the projection used in Theorem 3.4, we note that the confined d -dimensional Langevin process hits an edge or a corner of D if there exist $m, n \in \mathbb{N}$ and two distinct $i, j \in \{1, \dots, d\}$ such that

$$\tau_n^i = \tau_m^j, \quad (3.25)$$

where $\{\tau_n^i\}_{n \geq 0}$ is defined as in Lemma 2.1 for the confined one-dimensional Langevin process $((X_t^c)_i, (U_t^c)_i)$. The probability for this event is bounded from above by

$$\begin{aligned} & \frac{1}{2} \sum_{i, j \in \{1, \dots, d\}, i \neq j} \mathbb{P} \left(\exists n \in \mathbb{N} : (X_{\tau_n^i}^c)_j \in Z_{a_j} \right) \\ & \leq \frac{1}{2} \sum_{i, j \in \{1, \dots, d\}, i \neq j} \sum_{n=0}^{\infty} \int_0^{\infty} \mathbb{P}(\tau_n^i \in dt, (X_t^c)_j \in Z_{a_j}) \\ & = \frac{1}{2} \sum_{i, j \in \{1, \dots, d\}, i \neq j} \sum_{n=0}^{\infty} \int_0^{\infty} \mathbb{P}(\tau_n^i \in dt) \mathbb{P}((X_t^c)_j \in Z_{a_j}) = 0. \end{aligned} \quad (3.26)$$

The last step holds since the distribution for X_t^j is diffuse and Z_{a_j} is countable.

The crucial aspect of the box domains considered in this section is that they allow for the construction of a deterministic map of the locations of the free Langevin process onto those of the confined Langevin process. On more general domains, however, the location of the free Langevin process at a given time instant may correspond to a number of different possible locations for the confined Langevin process. To determine the correct location of the confined Langevin process, we need to know the trajectory of the free Langevin process up to that time. As a result, in more general domains, we cannot construct a

deterministic map analogous to the one defined in this section. Nevertheless, by constructing a sequence of maps, it is possible, as we shall see in the next section, to arrive at similar results for polytopes.

Applying the deterministic map used in this section, it is a straightforward exercise to derive the density of the confined Langevin equation on box domains. It is likely that the properties of this density can be used to prove existence of solutions to nonlinear confined Langevin processes on box domains following the approach of [4]. This is work in progress.

Corollary 3.6. *Let $g(t; y, v; x, u)$ be the probability density function of the one-dimensional free Langevin process starting from (y, v) . The probability density function of the confined Langevin process in $D = (0, a_1) \times \cdots \times (0, a_d)$ starting from (y, v) is given by*

$$\begin{aligned} & \Gamma(t, y, v; x, u) \\ &= \prod_{i=1}^d \sum_{k \in \mathbb{Z}} (g(t; y_i, v_i; x_i + 2ka_i, u_i) + g(t; y_i, v_i; 2ka_i - x_i, -u_i)). \end{aligned} \quad (3.27)$$

4. Langevin processes confined to polytopes

Denote, for any polytope $D \subset \mathbb{R}^d$, the finite set of open $(d-1)$ -dimensional facets of D by A_D . On each of the open facets $E \in A_D$, the outward unit vector is well-defined and constant, that is $n_D(x) := n_E$ for all $x \in E$. Moreover, we may define, for all facets $E \in A_D$, a reflection map $M_E : \mathbb{R}^d \rightarrow \mathbb{R}^d$ reflecting points in $\mathbb{R}^d$ orthogonally through the facet E . We shall prove existence and pathwise uniqueness of a Langevin process confined to a polytope by constructing a sequence of maps of the free Langevin process onto the confined Langevin process.

Let (X_t^f, U_t^f) be the free Langevin process as defined in (3.17). For each path of (X_t^f, U_t^f) , we define a sequence $\{\tau_n\}_{n \geq 0}$ of hitting times, a sequence $\{E_n\}_{n \geq 1}$ of $(d-1)$ -dimensional polytopes in $\mathbb{R}^d$ and a sequence $\{D_n\}_{n \geq 0}$ of d -dimensional polytopes in $\mathbb{R}^d$.

Definition 4.7. *Let $D_0 = D$, $\tau_0 = 0$ and define, for $n \geq 1$,*

$$\tau_n = \inf\{t > \tau_{n-1} : X_t^f \in \partial D_{n-1}\}. \quad (4.28)$$

Moreover, define, for $n \geq 1$, E_n as the unique element $E \in A_{D_{n-1}}$ such that $X_{\tau_n}^f \in E$ and

$$D_n = \{x \in \mathbb{R}^d : M_{E_n}(x) \in D_{n-1}\}. \quad (4.29)$$

Note that the elements of $A_{D_{n-1}}$ can be obtained from the elements of A_D by applying the linear map $M_{E_{n-1}} \circ \cdots \circ M_{E_1}$ on each element of A_D .

We shall prove that these three sequences are well-defined and unique for a given path of (X_t^f, U_t^f) . Moreover, the sequences can be used to construct a linear map from (X_t^f, U_t^f) onto (X_t^c, U_t^c) and to deduce the existence of a pathwise unique Langevin process confined to a polytope.

Clearly, the set $\partial_c D := \{x \in \partial D : x \notin E \text{ for all } E \in A_D\} \subset \partial D$ is $(d-2)$ -dimensional. Modifying the argument in Remark 3.5 slightly, it is straightforward to prove that, $\mathbb{P}$ -almost surely, X_t^f will not hit $\partial_c D$ and that, as a consequence, X_t^f must hit ∂D at a point belonging to one of the elements of A_D . Hence, E_1 is well-defined and, as a consequence, so is M_{E_1} and D_1 . This argument can be repeated to show that E_n and D_n are well-defined and uniquely determined by X_t^f as long as the result provided by the following lemma holds.

Lemma 4.8. *The sequence $\{\tau_n\}_{n \geq 0}$ of hitting times is well-defined and grows to infinity.*

Proof. We have already verified that X_t^f , $\mathbb{P}$ -almost surely, hits ∂D_{n-1} at an open facet. This implies that $\{\tau_n\}_{n \geq 0}$ is well-defined if X_t^f , $\mathbb{P}$ -almost surely, hits ∂D_{n-1} with non-tangential velocity, that is if

$$\langle U_{\tau_n}^f, n_{D_{n-1}}(X_{\tau_n}^f) \rangle \neq 0, \text{ for all } n. \quad (4.30)$$

The key tool in the proof of this result is the observation that, for all unit vectors $e \in \mathbb{R}^d$, the process $(\langle X_t^f, e \rangle, \langle U_t^f, e \rangle)$ is identical in law to a one-dimensional Langevin process. Indeed, this follows easily from definitions (2.2) and (3.17). We prove (4.30) by induction and introduce the notation $e_n := n_{D_{n-1}}(X_{\tau_n}^f)$, $x_{D_{n-1}} := \langle X_{\tau_n}^f, e_n \rangle$.

First, let $n = 1$. From the density of the one-dimensional Langevin process, it is easily deduced that, $\mathbb{P}$ -almost surely, $(\langle X_t^f, e_1 \rangle, \langle U_t^f, e_1 \rangle) \neq (x_{D_0}, 0)$ for any given $t > 0$. Since $X_0 \notin \partial D_0$ and the velocity of the Langevin process is finite, it is also clear that $\tau_1 > 0$. Hence, $(\langle X_{\tau_1}^f, e_1 \rangle, \langle U_{\tau_1}^f, e_1 \rangle) \neq (x_{D_0}, 0)$ and, as $\langle X_{\tau_1}^f, e_1 \rangle = x_{D_0}$ holds by definition, we conclude that (4.30) is valid for $n = 1$.

To prove (4.30) for $n \geq 2$, we assume that $\langle U_{\tau_{n-1}}^f, n_{D_{n-2}}(X_{\tau_{n-1}}^f) \rangle \neq 0$, which, by the definition of D_{n-1} , implies that $\langle U_{\tau_{n-1}}^f, n_{D_{n-1}}(X_{\tau_{n-1}}^f) \rangle \neq 0$. Note that the induction assumption asserts that τ_n and e_n are well-defined and that $\tau_n > \tau_{n-1}$. In [9] it was shown that a one-dimensional Langevin process almost surely does not hit $(0, 0)$ at any given positive time. The Markov property at time τ_{n-1} then asserts that, $\mathbb{P}$ -almost surely, $(\langle X_{\tau_{n-1}+t}^f, e_n \rangle, \langle U_{\tau_{n-1}+t}^f, e_n \rangle)$ does not hit $(x_{D_{n-1}}, 0)$ for any given $t > 0$. As $\tau_n > \tau_{n-1}$, we conclude that $(\langle X_{\tau_n}^f, e_n \rangle, \langle U_{\tau_n}^f, e_n \rangle) \neq (x_{D_{n-1}}, 0)$ and, as $\langle X_{\tau_n}^f, e_n \rangle = x_{D_{n-1}}$ holds by definition, (4.30) is valid also for $n \geq 2$. This completes the proof that $\{\tau_n\}_{n \geq 0}$ is well-defined.

To show that $\{\tau_n\}_{n \geq 0}$ grows to infinity, we prove that, for any $T > 0$, only finitely many τ_n are found in $[0, T]$. We assume that this is not the case, implying that at least one of the index sets I_1 , I_2 and I_3 , defined as

$$\begin{aligned} I_1 &= \{n \in \mathbb{N} : E_{n-1} = E_n, \tau_n \in [0, T]\}, \\ I_2 &= \{n \in \mathbb{N} : E_{n-1} \neq E_n, \partial E_{n-1} \cap \partial E_n \neq \emptyset, \tau_n \in [0, T]\}, \\ I_3 &= \{n \in \mathbb{N} : E_{n-1} \neq E_n, \partial E_{n-1} \cap \partial E_n = \emptyset, \tau_n \in [0, T]\}, \end{aligned} \quad (4.31)$$

respectively, is countably infinite. The number of $(d-1)$ -dimensional facets of a d -dimensional polytope is bounded, so the minimal distance between points belonging to two different, non-intersecting facets of the polytope must be bounded

from below by some positive constant r_0 . Hence, if I_3 is countably infinite, the free Langevin process must travel the positive distance r_0 infinitely many times during $[0, T]$. But, as mentioned in the proof of Lemma 2.1, the free Langevin process, $\mathbb{P}$ -almost surely, cannot travel an infinite distance in finite time.

We conclude that I_3 must be finite, which implies that, for some sufficiently large integer N_0 , all indices $n > N_0$ belong to either I_1 or I_2 . Hence, for all $n > N_0$, the intersection $\bar{E}_{n-1} \cap \bar{E}_n$ is nonempty and is, by construction of the reflection map, invariant under M_{E_n} and belongs to ∂D_n . Next, the intersection $\bar{E}_{n-1} \cap \bar{E}_n \cap \bar{E}_{n+1}$ is either nonempty, invariant under $M_{E_{n+1}} \circ M_{E_n}$ and belongs to ∂D_{n+1} or empty. In the first case, the intersection $\bar{E}_{n-1} \cap \dots \cap \bar{E}_{n+2}$ is either nonempty, invariant under $M_{E_{n+2}} \circ \dots \circ M_{E_n}$ and belongs to ∂D_{n+2} or empty. Continuing in this fashion, we see that if $\bigcap_{n \geq N_0} \bar{E}_n$ is empty, then there exists a positive integer m such that $\bar{E}_{n-1} \cap \dots \cap \bar{E}_{n+m} \in \partial D_{n+m}$ is nonempty, whereas $\bar{E}_{n-1} \cap \dots \cap \bar{E}_{n+m+1}$, with $\bar{E}_{n+m+1}$ being a facet of ∂D_{n+m} , is empty. As the number of $(d-1)$ -dimensional facets of a d -dimensional polytope is bounded, the Langevin process must then have travelled a positive distance r_1 , depending on the geometry of the polytope but not on m and n , during the time interval $[\tau_{n-1}, \tau_{n+m+1}]$. During the time interval $[0, T]$, the Langevin process can only travel the distance r_1 a finite number of times. Hence, there must exist an integer $N_1 \geq N_0$ such that $\bigcap_{n \geq N_1} \bar{E}_n$ is nonempty.

Being the intersection of $(d-1)$ -dimensional facets, the set $\bigcap_{n \geq N_1} \bar{E}_n$ is at most $(d-1)$ -dimensional. If $\bigcap_{n \geq N_1} \bar{E}_n$ is exactly $(d-1)$ -dimensional, then there exists a hyperplane E such that $\bigcap_{n \geq N_1} \bar{E}_n \subset E$ and the one-dimensional Langevin process $\langle X_t^f, n_E \rangle$ hits the point $\langle \bigcap_{n \geq N_1} \bar{E}_n, n_E \rangle$ countably infinitely many times during the time interval $[0, T]$. But, as was shown in Section 3.1 in [4], this situation, $\mathbb{P}$ -almost surely, cannot occur. If, on the other hand, the dimension of $\bigcap_{n \geq N_1} \bar{E}_n$ is less than $d-1$, then either does there exist an integer $N_2 > N_1$ such that $\bigcap_{n \geq N_2} \bar{E}_n$ is $(d-1)$ -dimensional or not. In the first case, the argument for $\bigcap_{n \geq N_1} \bar{E}_n$ above can be applied to $\bigcap_{n \geq N_2} \bar{E}_n$ and we arrive at the same contradiction. In the latter case, the Langevin process will continue to hit new facets of the polytope and as the angle between two different facets of the polytope is bounded from below by some constant, the Langevin process will travel an infinite distance in finite time unless X_t^f converges to the at most $(d-2)$ -dimensional set $\limsup_n \bar{E}_n$. But X_t^f , $\mathbb{P}$ -almost surely, cannot hit a $(d-2)$ -dimensional set and we have reached a contradiction. Hence, $\mathbb{P}$ -almost surely, only finitely many τ_n are found in $[0, T]$ and, as a consequence, $\{\tau_n\}_{n \geq 0}$ must grow to infinity. $\square$

We have shown that the three sequences $\{\tau_n\}_{n \geq 0}$, $\{E_n\}_{n \geq 1}$ and $\{D_n\}_{n \geq 0}$ are well-defined and unique for a given path of (X_t^f, U_t^f) . It remains to show that we can construct a pathwise unique solution to (1.1) by means of these three sequences and the pathwise unique free Langevin process. Before proving this result, we collect a number of observations. Any facet E of a polytope belongs to a hyperplane with equation $\langle x, n_E \rangle = c$ for some constant c . The

reflection map M_E is explicitly given by

$$M_E(x) = x - 2(\langle x, n_E \rangle - c) n_E, \text{ for all } x \in \mathbb{R}^d \quad (4.32)$$

Denote, for any differentiable map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, the Jacobian matrix of f by $J(f)$. Note also that if f is linear, then $J(f)$ is constant. In particular, the Jacobian matrix of the reflection map M_E is

$$J(M_E) = I - 2n_E n_E^T, \quad (4.33)$$

where I is the identity matrix, n_E is interpreted as a column vector and n_E^T denotes the transpose of n_E . Note that $J(M_E)$ is symmetric and orthogonal.

Theorem 4.9. *There exists a d -dimensional $\mathcal{F}_t$ -adapted Wiener process W^c , such that (X^c, U^c) defined as*

$$X_t^c = X_t^f, \quad U_t^c = U_t^f, \text{ for } \tau_0 \leq t < \tau_1, \quad (4.34)$$

and

$$X_t^c = M_{E_1} \circ \cdots \circ M_{E_n}(X_t^f), \quad U_t^c = J(M_{E_1}) \cdots J(M_{E_n}) U_t^f, \quad (4.35)$$

for $\tau_n \leq t < \tau_{n+1}$, is the pathwise unique solution to (1.1) for all $t \in [0, T]$. The sequence of hitting times $\{\tilde{\tau}_n\}_{n \geq 0}$, defined as $\tilde{\tau}_0 = 0$ and

$$\tilde{\tau}_n = \inf\{t > \tilde{\tau}_{n-1} : X_t^c \in \partial D\}, \quad (4.36)$$

for $n > 1$, is well-defined and grows to infinity.

Proof. By construction, the properties of $\{\tilde{\tau}_n\}_{n \geq 0}$ follow immediately from the properties of $\{\tau_n\}_{n \geq 0}$ derived in Lemma 4.8. We will use induction to prove that (X^c, U^c) , as defined in (4.35), is a solution to (1.1). This is trivially true for $\tau_0 \leq t < \tau_1$ and we assume next that

$$X_t^c = M_{E_1} \circ \cdots \circ M_{E_{n-1}}(X_t^f), \quad U_t^c = J(M_{E_1}) \cdots J(M_{E_{n-1}}) U_t^f, \quad (4.37)$$

is a solution to (1.1) whenever $\tau_{n-1} \leq t < \tau_n$ for some arbitrary $n \geq 1$. Note that we adopt the notation that, for $n = 1$, $M_{E_1} \circ \cdots \circ M_{E_{n-1}} = I$ and $J(M_{E_1}) \cdots J(M_{E_{n-1}}) = 1$. By construction, $J(M_{E_1}) \cdots J(M_{E_n})$ is a càdlàg pure jump process whose value only changes at the discrete times τ_n . Hence, by the integration of parts formula, we obtain

$$\begin{aligned} U_t^c &= J(M_{E_1}) \cdots J(M_{E_n}) U_t^f = J(M_{E_1}) \cdots J(M_{E_{n-1}}) U_{\tau_n^-}^f \\ &\quad + J(M_{E_1}) \cdots J(M_{E_n}) U_{\tau_n}^f - J(M_{E_1}) \cdots J(M_{E_{n-1}}) U_{\tau_n^-}^f \\ &\quad + \int_{\tau_n}^t J(M_{E_1}) \cdots J(M_{E_n}) dU_s^f. \end{aligned} \quad (4.38)$$

By the induction assumption, the first term on the right hand side of (4.38) is

$$U_0 + W_{\tau_n}^c - 2 \sum_{0 < s < \tau_n} \langle U_{s-}^c, n_D(X_s^c) \rangle n_D(X_s^c) \mathbb{I}_{\{X_s^c \in \partial D\}}. \quad (4.39)$$

Regarding the mid terms on the right hand side of (4.38), we have

$$J(M_{E_n})U_{\tau_n}^f - U_{\tau_n}^f = (I - 2n_{E_n}n_{E_n}^T)U_{\tau_n}^f - U_{\tau_n}^f = -2\langle U_{\tau_n}^f, n_{E_n} \rangle n_{E_n}. \quad (4.40)$$

By construction $J(M_{E_1}) \cdots J(M_{E_{n-1}})n_{E_n} = n_D(X_{\tau_n}^c)$ and hence

$$\begin{aligned} & J(M_{E_1}) \cdots J(M_{E_n})U_{\tau_n}^f - J(M_{E_1}) \cdots J(M_{E_{n-1}})U_{\tau_n}^f \\ &= -2\langle U_{\tau_n}^f, n_{E_n} \rangle J(M_{E_1}) \cdots J(M_{E_{n-1}})n_{E_n} = -2\langle U_{\tau_n}^f, n_{E_n} \rangle n_D(X_{\tau_n}^c). \end{aligned} \quad (4.41)$$

Using the orthogonality of $J(M_{E_1}) \cdots J(M_{E_{n-1}})$, we may also conclude that

$$\langle U_{\tau_n}^f, n_{E_n} \rangle = \langle J(M_{E_1}) \cdots J(M_{E_{n-1}})U_{\tau_n}^f, n_D(X_{\tau_n}^c) \rangle = \langle U_{\tau_n}^c, n_D(X_{\tau_n}^c) \rangle, \quad (4.42)$$

which, combined with (4.41), proves that the mid terms on the right hand side of (4.38) equal $-2\langle U_{\tau_n}^c, n_D(X_{\tau_n}^c) \rangle n_D(X_{\tau_n}^c)$. Finally, due to the orthogonality of $J(M_{E_1}) \cdots J(M_{E_n})$ and the fact that $dU_s^f = dW_s^f$, it follows from the Lévy characterization of Wiener processes that the last term on the right hand side of (4.38) equals an increment $W_t^c - W_{\tau_n}^c$ of some Wiener process W^c . This completes the proof that U^c , as defined in (4.35), is a solution to the second equation in (1.1) for $\tau_n \leq t < \tau_{n+1}$.

Next, as the value of $J(M_{E_1}) \cdots J(M_{E_n})$ only changes at the discrete times τ_n , we obtain

$$\begin{aligned} X_t^c &= M_{E_1} \circ \cdots \circ M_{E_n}(X_t^f) = M_{E_1} \circ \cdots \circ M_{E_{n-1}}(X_{\tau_n}^f) \\ &\quad + M_{E_1} \circ \cdots \circ M_{E_n}(X_{\tau_n}^f) - M_{E_1} \circ \cdots \circ M_{E_{n-1}}(X_{\tau_n}^f) \\ &\quad + \int_{\tau_n}^t J(M_{E_1} \circ \cdots \circ M_{E_n})dX_s^f. \end{aligned} \quad (4.43)$$

By the induction assumption, the first term on the right hand side of (4.43) is equal to $X_0 + \int_0^{\tau_n} U_s^c ds$. Moreover, the mid terms on the right hand side of (4.43) vanish since, by construction, $X_{\tau_n}^f = X_{\tau_n}^f \in E_n$ and M_{E_n} is the identity map on E_n . Finally, since $M_{E_1} \circ \cdots \circ M_{E_n}$ is a composition of linear maps, it is straightforward to show that the Jacobian $J(M_{E_1} \circ \cdots \circ M_{E_n})$ can be written as $J(M_{E_1}) \cdots J(M_{E_n})$. Hence, the last term can be rewritten

$$\int_{\tau_n}^t J(M_{E_1} \circ \cdots \circ M_{E_n})dX_s^f = \int_{\tau_n}^t J(M_{E_1}) \cdots J(M_{E_n})U_s^f ds = \int_{\tau_n}^t U_s^c ds. \quad (4.44)$$

This completes the proof that X^c , as defined in (4.35), is a solution to the first equation in (1.1) for $\tau_n \leq t < \tau_{n+1}$. Finally, pathwise uniqueness follows by induction on n exactly as in Proposition 3.1 in [4] and we omit the details. $\square$

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