Calibration, Optimality and Financial Mathematics

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Abstract


This thesis consists of a summary and five papers, dealing with financial applications of optimal stopping, optimal control and volatility.

In Paper I, we present a method to recover a time-independent piecewise constant volatility from a finite set of perpetual American put option prices.

In Paper II, we study the optimal liquidation problem under the assumption that the asset price follows a geometric Brownian motion with unknown drift, which takes one of two given values. The optimal strategy is to liquidate the first time the asset price falls below a monotonically increasing, continuous time-dependent boundary.

In Paper III, we investigate the optimal liquidation problem under the assumption that the asset price follows a jump-diffusion with unknown intensity, which takes one of two given values. The best liquidation strategy is to sell the asset the first time the jump process falls below or goes above a monotone time-dependent boundary.

Paper IV treats the optimal dividend problem in a model allowing for positive jumps of the underlying firm value. The optimal dividend strategy is of barrier type, i.e. to pay out all surplus above a certain level as dividends, and then pay nothing as long as the firm value is below this level.

Finally, in Paper V it is shown that a necessary and sufficient condition for the explosion of implied volatility near expiry in exponential Lévy models is the existence of jumps towards the strike price in the underlying process.

Keywords: perpetual put option, calibration of models, piecewise constant volatility, optimal liquidation of an asset, incomplete information, optimal stopping, jump-diffusion model, optimal distribution of dividends, singular stochastic control, implied volatility, exponential Lévy models, short-time asymptotic behavior.

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This thesis is based on the following papers, which are referred to in the text by their Roman numerals.


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1. Introduction

This thesis consists of an introduction and five papers. All five papers are devoted to the study of financial mathematics in continuous time.

Future stock prices are not deterministic and they are modeled by stochastic processes, which are based on for example Brownian motions or jump processes. The fair price of a financial derivative with the stock as underlying is not known by any law of nature, but by financial theory it is shown to be equal to an expected value of the discounted future price.

There is a connection between the expected value of a function of a stochastic process and parabolic differential equations. Therefore both stochastic methods and methods from parabolic differential equations are used in this thesis.

One of the most common investment problems is: when is the best time for the owner of a stock to sell her/his stock? To answer this question, one needs to know the future distribution of the underlying stock. To evaluate the future distribution, numerical estimates of the parameters in the underlying model are required. However, in order to have a good estimate of the drift of a stock, historical data of the stock prices for decades or even hundreds of years is needed. Optimal selling of an asset under incomplete information about the model parameters constitutes one of the main themes in this thesis, compare Paper II and III.

Another main theme of the thesis is the volatility of the underlying stock. The market practice is to quote option prices in terms of their implied volatilities, which can be calculated by inverting the Black-Scholes formula. As a dimensionless unit, implied volatility provides an easy comparison between options with different features. However, the implied volatility calculated from the market data is typically not a constant, which motivates the study of more elaborate models. Paper I deals with perpetual American put options with a piecewise constant volatility. Paper V studies small-time asymptotics of the implied volatility in exponential Lévy models.

Finally, Paper IV treats an optimal dividend problem in a model allowing for positive jumps of the underlying company value. This problem belongs to the area of stochastic control.
1.1 European options

A European option is the simplest type of option. A European call/put option is a contract that gives the holder the right, but not the obligation to buy/sell one share of the underlying stock at a prespecified price \( K \), known as strike price, at a prespecified time \( T \), known as maturity, from the underwriter of the option. Denote the stock price at time \( t \) by \( S(t) \). If the stock price \( S(T) \) is greater than the strike price \( K \) at maturity \( T \), then the holder of the European call option will make a profit \( S(T) - K \) by buying a share of the stock for \( K \) and selling it instantly for \( S(T) \). If \( S(T) \) is less than the strike \( K \), then the holder will simply do nothing. Thus the profit that the holder makes at maturity is \( (S(T) - K)^+ \) and \( s \mapsto (s - K)^+ \) is called the payoff function or the contract function of the European call option. Similarly, the payoff function for a European put option is \( s \mapsto (K - s)^+ \).

We will also discuss another type of option, American options, in detail later in Section 1.5 and 1.6. A central topic here is the valuation of the option price at times before maturity.

1.2 The Black-Scholes market and arbitrage pricing

An arbitrage opportunity is the possibility of making a positive amount of money out of nothing with zero risk. We consider the classical Black-Scholes model for the financial market which consists of two assets, compare papers [5] and [17]. The first asset is a bank account denoted by \( B \) and it satisfies

\[
\frac{dB(t)}{B(t)} = rB(t)dt,
\]

where \( r \geq 0 \) is the constant risk free interest rate. The second asset is a stock and the stock price \( S \) is modeled by the stochastic differential equation

\[
dS(t) = \mu S(t)dt + \sigma S(t)dW_t, \quad (1.1)
\]

where \( \mu \) is the rate of return of the stock price, \( \sigma \) is the constant volatility and \( W \) is a standard Brownian motion.

An option with the stock as underlying can be replicated by a portfolio of stocks and bonds. Regardless of what happens on the financial market, the price of the option will be exactly the same as the value of the portfolio at the expiration date. Thus, there is no difference between holding an option and holding the portfolio from a financial point of view. To avoid arbitrage, the value of the option must equal the unique value of the replicating portfolio at any time before the expiration date.

Let \( g \) be the payoff function of the European option. The unique option price which does not introduce arbitrage to the market is given by

\[
C(s, t) = e^{-r(T-t)}E^Q_{s,t}[g(S(T))], \quad (1.2)
\]
where the subindices indicate that the process $S$ is started at $s$ at time $t$. Here the superindex $Q$ denotes that the expected value is taken under the risk neutral measure $Q$, which is equivalent to the objective measure $P$ and the discounted price process $e^{-ru}S(u)$ is a martingale under $Q$. Under the risk-neutral measure, the stock price has the dynamics

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q_t,$$

(1.3)

where $W^Q$ is a standard Brownian motion under $Q$. The volatilities in (1.1) and (1.3) are the same and the only difference between the two stochastic differential equations is the drift. Notice that the drift $\mu$ does not play a part in option pricing. The solutions to (1.1) and (1.3) are geometric Brownian motions and the explicit solution of (1.3) is

$$S(u) = S(t) \exp\{(r - \sigma^2/2)(u - t) + \sigma(W^Q_u - W^Q_t)\}.$$

The expected value of a function of a diffusion process is also the solution to a parabolic partial differential equation by the Feynman-Kac representation theorems. The price $C$ of the European call option satisfies the Black-Scholes equation

$$C_t + \frac{1}{2}\sigma^2 s^2 C_{ss} + rsC_s - rC = 0.$$

By a change of variable, the Black-Scholes equation transforms into the heat equation, which can be solved easily. One can also use the known distribution of $S$ under $Q$ and the representation (1.2) to calculate the solution. The price of the European option is given by the classical Black-Scholes formula

$$C^\text{BS}(t, T, s, K, \sigma) := sN\left(\frac{\ln(s/K) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}}\right) - Ke^{-r(T-t)}N\left(\frac{\ln(s/K) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}}\right),$$

(1.4)

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.

### 1.3 Implied volatility

If all other parameters are fixed in the Black-Scholes formula (1.4), then the option price $C^\text{BS}$ is a strictly increasing function of the volatility $\sigma$. Thus the Black-Scholes implied volatility can be defined as $\hat{\sigma}(t, K) : (0, \infty) \times (0, \infty) \to [0, \infty)$ such that

$$C(t, s) = C^\text{BS}(t, T, s, K, \hat{\sigma}(t, K)).$$
where \( C \) is a given market price of a call option. Notice that for a given option price \( C \), there is no closed form solution for the implied volatility.

The implied volatility computed from the market data is usually not a constant. It is typically observed that the implied volatility is relatively big for options that are deep in-the-money or out-of-the-money and relatively small for at-the-money options. The volatility can be modeled as a stochastic process, which is known as stochastic volatility. The volatility can also be modeled as a function of time and stock price, which is known as local volatility. Note that local volatility is a special case of stochastic volatility. In Paper I we assume that the volatility is piecewise constant.

Despite the assumption of constant volatility, the Black-Scholes formula is still widely used in the financial industry. However, it is not used to model stock prices, instead, it is used as a tool to translate options prices into implied volatilities. It is a market practice to quote the option prices in terms of their corresponding implied volatilities, since the dimensionless units provide a better comparison between options with different features.

If a model of the underlying stock price is given, there is very little chance to obtain an explicit expression of the implied volatility. Instead, asymptotic behavior of the implied volatility in a given underlying model is studied, for example, small strike, big strike, small maturity and large maturity.

1.4 Lévy process and option pricing

A Lévy process is a stochastic process with independent, stationary increments. The most famous examples of Lévy processes are the Brownian motion and the Poisson process. A Lévy process is characterized by a triplet \((\gamma, \sigma, \nu)\), where \( \gamma \) is the drift, \( \sigma \) is the volatility of the Gaussian component and \( \nu \) is the Lévy measure controlling the behavior of the jumps and satisfies

\[
\int_{\mathbb{R}} (x^2 + 1) \nu(dx) < \infty.
\]

A Lévy process can be decomposed into three independent parts, which are a linear drift, a Brownian motion and a pure jump process. If \( \nu(R) < \infty \), which means that almost all paths of the Lévy process have finite number of jumps on every compact interval, then the Lévy process is said to have finite activity. Thus the jump part of the Lévy process is just a compound Poisson process. A Lévy process is said to have infinite activity if \( \nu(R) = \infty \). In that case the Lévy process has infinitely many small jumps and finitely many big jumps. A Lévy process is called spectrally negative when there are no positive jumps, i.e. \( \nu(0, \infty) = 0 \). Similarly, the Lévy process is called spectrally positive when there are no negative jumps, i.e. \( \nu(-\infty, 0) = 0 \).

Let \( X \) be a Lévy process. Then we call \( e^X \) an exponential Lévy process. Exponential Lévy processes are used to model stock prices and they generalize
the Black-Scholes model by adding jumps and preserving the independent and stationary property.

To price European options, using the martingale condition we assume that

\[
\gamma = -\frac{\sigma^2}{2} - \int_R (e^y - 1 - y1_{|y|\leq 1}) v(dy) + r, \\
\int_{|y|>1} e^y v(dy) < \infty.
\]

Given an equivalent martingale measure \(Q\), the European option price in an exponential Lévy model is given by

\[
C(t, s) = e^{-r(T-t)} E^Q [ (se^{r(T-t)} + X_T - X_t - K)^+] ,
\]

(1.5)

If we assume that \(\sigma > 0\), then \(C\) is a smooth function on \(s\) and it satisfies the following parabolic integro-differential equation

\[
C_t + rsC_s + \frac{1}{2} \sigma^2 s^2 C_{ss} + \int [C(t, se^y) - C(t, s) - s(e^y - 1)C_s] v(dy) - rC = 0,
\]

compare [6]. Notice that if the jumps do not exist, this equation is reduced to the Black-Scholes equation.

1.5 American options and optimal stopping

Compared to European options, which can only be exercised at a fixed time, American options give the holder more freedom to choose when to exercise. The holder of an American option has the right, but not the obligation to exercise the option at any time before the expiration time \(T\). Therefore, at any time before \(T\) the holder needs to decide if the option should be exercised immediately or later. Let \(g\) be the payoff function. The owner will receive the amount \(g(S(t))\) at time \(t\), if the option is exercised at time \(t\). Since the option can at least be exercised at the maturity \(T\), the price of an American option is never less than the corresponding price of the European option. The holder can always choose to exercise the option immediately, so the value of the option is no less than the payoff right now.

The arbitrage free price of the American option at time \(t\) with contract function \(g\) is given by

\[
V(s, t) = \sup_{t \leq \tau \leq T} E_{s,t}^Q [ e^{-r(T-t)} g(S(\tau))] ,
\]

(1.6)

compare [3] and [11]. Here the supremum is taken over all times \(\tau\) that are stopping times with respect to the filtration generated by the stochastic process \(S\). Intuitively speaking, considering only stopping times means that one can only use historical stock prices but not future prices to decide whether to
exercise or to wait. Studying the American options includes two parts. First, to compute the value function $V$ defined in (1.6). Second, to determine the optimal exercising strategy.

For American call options, in which the payoff function is $g(s) = (s - K)^+$, because of the submartingale property of $e^{-rt}(S(u) - K)^+$ under the risk neutral measure $Q$, the best exercising time is always the expiration time. Thus the price of an American call equals the price of the corresponding European call, which can be calculated by the Black-Scholes formula.

Suppose that the value function $V(s, t)$ is known. If $V(s, t) > g(s)$, which means that the value of the option at time $t$ is greater than the income of exercising the option at $t$, then to end the option at $t$ is suboptimal. An important result in the optimal stopping theory says that if the payoff is continuous and $\sup_{t \leq u \leq T} g(S(u))$ is integrable, then the supremum in (1.6) is obtained at the stopping time

$$\tau^* = \inf\{u \geq t : V(S(u), u) = g(S(u))\},$$

compare for example Appendix D in [12]. This implies that the optimal exercise strategy is to do nothing if the value of the option is strictly greater than the payoff and to exercise the first time the option value equals the payoff.

Therefore it is natural to introduce the continuation region $\mathcal{C}$ and the stopping region $\mathcal{D}$

$$\mathcal{C} := \{(s, t) : V(s, t) > g(s)\} \quad \mathcal{D} := \{(s, t) : V(s, t) = g(s)\}.$$

The optimal stopping time is then given by

$$\tau^* = \inf\{u \geq t : (S(u), u) \notin \mathcal{C}\}.$$

Recall that the statements above are based on the assumption that the value function is known. In the next section we will investigate the value function.

1.6 American put options and free boundary problems

A useful method to study the value function $V$ is the free boundary approach, compare [13], [16] and [18]. Other ways of characterization of $V$ are, for example, using excessive functions or in terms of variational inequalities, see [7], [10] and [4]. By free boundary problem we means that there exists a boundary, which is to be determined while solving the problem, that divides the whole time-space domain into continuation region and stopping region. In the stopping region, the value function simply equals the payoff function. In the continuation region, the value function satisfies a parabolic differential equation with certain boundary conditions.
1.6.1 American put options with finite horizon

To illustrate, we take the example of American put options with finite horizon in the Black-Scholes model, where the volatility is a constant. There exists a time dependent function $b(t)$ such that the continuation region is given by

$$
\mathcal{C} := \{(s,t) \in (0,\infty) \times [0, T) : s > b(t)\}.
$$

The stopping region is

$$
\mathcal{D} := \{(s,t) \in (0,\infty) \times [0, T) : s \leq b(t)\}
$$

together with the points $(T,s)$ for all $s \geq 0$. The optimal strategy is thus to exercise the option the first time the stock price falls below the boundary.

Moreover, the value function $(t,s) \mapsto V(t,s)$ is continuous on $[0,T] \times (0,\infty)$ and $C^{1,2}$ on the continuation region $\mathcal{C}$. For a fixed $t$, the value function $V(t,s)$ is decreasing and convex in $s$. The optimal stopping boundary $b(t)$ is continuous and increasing in $t$ and $b(T-) = K$. Those statements and the proofs are usually referred to as the standard arguments in optimal stopping, compare [19].

The value function $V$ together with the free boundary $b(t)$ satisfies the free boundary problem

$$
\begin{cases}
V_t + \frac{1}{2}\sigma^2 s^2 V_{ss} + rsV_s - rV = 0 & \text{for } s > b(t) \\
V = (K-s)^+ & \text{for } s < b(t) \\
V_s = -1 & \text{for } s = b(t) \\
V = (K-s)^+ & \text{for } t = T,
\end{cases}
$$

see [12]. For a fixed $t$, the value function $V$ is not only continuous in $s$ but also continuously differentiable at the level $b(t)$, which is known as smooth fit condition. Moreover, the optimal stopping boundary $b(t)$ is the unique solution to an integral equation, which is known as the free boundary equation, see [19].

In paper II, we carry on an analysis of an optimal stopping problem and provide the corresponding integral equation. The procedure is similar to that for American put options discussed above.

Optimal stopping problems usually do not have explicit solutions. In the next section, we study the perpetual American put option, which has explicit solution and is a classical example in optimal stopping.

1.6.2 Perpetual American put options

A perpetual American put option is an American put option with expiration time infinity, i.e. $T = \infty$. This means that the holder can exercise the option at any time in the future and the time left to maturity is always infinity. Thus the
option price does not depend on the current time. The arbitrage free price of the American perpetual put option is given by

\[ V(s) = \sup_{\tau} E_s^Q [e^{-r\tau}(K - S_\tau)]. \tag{1.8} \]

Suppose that the stock price follows the Black-Scholes model. Using the result from the last section, we see that there exists a constant \( b \), which is the optimal stopping boundary, such that the stopping time

\[ \tau_b = \inf \{ t \geq 0 : S_t \leq b \} \]

attains the supremum in (1.8). The free boundary problem in the previous section is reduced into the following problem

\[
\begin{aligned}
\frac{1}{2} \sigma^2 s^2 V_{ss} + rsV_s - rV &= 0 \quad \text{for } s > b \\
V &= (K - s)^+ \quad \text{for } s < b \\
V_s &= -1 \quad \text{for } s = b,
\end{aligned}
\tag{1.9}
\]

compare [16] and [12]. Solving the free boundary problem (1.9) gives the explicit expression for the option price

\[
V(s) = \begin{cases} 
(K - b)(s/b)^{2r/\sigma^2} & \text{for } s > b \\
K - s & \text{for } s \leq b
\end{cases}
\tag{1.10}
\]

where the optimal stopping boundary is given by

\[ b = \frac{K}{1 + \sigma^2/(2r)}. \]

It is easy to check that the value function \( V \) is \( C^2 \) on \((0,b) \cup (b,\infty)\), \( C^1 \) at \( b \) and convex. The constant optimal stopping boundary \( b \) is less than the time dependent optimal stopping boundary \( b(t) \) for all \( t \). When \( t \to -\infty \), we have that \( b(t) \) converges to \( b \).

If an option price in the continuation region is given, then it is straightforward to calculate the volatility which reproduces the option price by (1.10). This is referred to as the inverse problem and is similar to the procedure of finding the implied volatility by inverting the Black-Scholes formula. In Paper I we study the inverse problem for the perpetual American put option with piecewise constant volatility.

### 1.7 Optimal selling under incomplete information

Optimal selling/ Optimal purchasing of an asset is one of the most common investment problems. A typical problem is, for example, when is the best time to buy or sell stocks, funds, real estate, gold and so on. Suppose that an agent
owns an asset and the task is to determine when to sell this asset in order to maximize his wealth. The asset price is modeled by a stochastic process. If all the parameters of the model are known, then the expected rate of return of the asset can be calculated and the optimal strategy can be made based on that.

However, in practice, the assumption of complete information is unrealistic. The volatility of the underlying is possible to estimate by observing the path of the asset price for a short period of time. However, the drift could be rather difficult to estimate with a high precision. Let’s take a simple example. Assume that the stock price follows

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

and $S_0 = s$ with unknown drift $\mu$. To estimate $\mu$, we observe the price trajectory with time length $u$, which is to be determined. The estimate of the drift $\mu$ is

$$\hat{\mu} = \frac{\ln(S_t/s)}{t} + \frac{\sigma^2}{2}.$$

The 95% confidence interval is $(\hat{\mu} - 1.96\sigma/\sqrt{t}, \hat{\mu} + 1.96\sigma/\sqrt{t})$. Even if the volatility is small $\sigma = 0.1$, one needs approximately 100 years of data of stock prices in order to have a rather tight confidence interval $(\hat{\mu} - 0.02, \hat{\mu} + 0.02)$.

An optimal selling problem with incomplete information is formulated as

$$V(s) = \sup_{0 \leq \tau \leq T} E_s[e^{-\tau r}S_\tau],$$

where the supremum is taken over stopping times with respect to the filtration generated by the asset price process. Suppose that we do not have complete information of the parameters of the model for $S$.

Although in the beginning, the agent is not sure about the parameters in the underlying model, he can observe the price fluctuations and estimate the parameter based on the observations, then make the decisions. For example, if the price increases very fast, which indicates that the expected rate of return is big, then the agent would keep this asset as long as possible. If the price drops drastically, which indicates that the rate of return is negative, then the agent would sell the asset immediately. Thus, we guess there exists a boundary that separates the selling region from the keeping region.

The general procedure of treating optimal selling problems with incomplete information is to reduce it into a one-dimensional optimal stopping problem with complete information. Then the problem can be studied using the standard analysis in Section 1.5 and 1.6, compare Paper II and III.
1.8 Optimal dividend problem

The optimal dividend problem, which belongs to the area of stochastic control, investigates the strategy to maximize the expected value of the discounted dividends distributed to the shareholders of a company.

Optimal dividend problem is also known as optimal harvesting problem and it is used in areas other than the financial industry to determine the optimal harvesting policy for renewable resources such as fish stocks.

In the setting of the classical optimal dividend problem, the value of the underlying company is modeled as a linear Brownian motion. The problem is solved using methods from singular stochastic control theory, compare [2] and [22]. The value function can be found by the free boundary approach and it is $C^2$ on the boundary. Recall that the value function of the optimal stopping problem in Section 1.6 is only $C^1$ on the boundary. The optimal strategy is of the barrier type, which is to pay out all the money above a certain level as dividend, and then pay nothing as long as the value of the company is below this level. This barrier strategy can be proved to be the best strategy by a verification argument using Ito’s formula and martingale techniques.

Recent literature treats problems in models with negative jumps, see [1], [9], [14] and [15]. The application of such models is in the insurance industry and the negative jumps can be interpreted as insurance claims. In Paper IV we treat problems in models allowing for positive jumps, which can be interpreted as the income from inventions of a research based company. The optimal strategies for both cases are of the barrier type.
2. Summary of papers

In this chapter we give short summaries of each of the papers included in this thesis.

2.1 Paper I

Given a price for a European call option, the Black-Scholes implied volatility can be calculated by inverting the Black-Scholes formula, as explained in Section 1.3. In paper [8] the unique volatility, which recovers the perpetual American put option prices, is found under the assumption that a continuum of option prices is given. However, in practice, option prices are provided only for a finite set of strikes, so interpolation is needed and the volatility is very sensitive to how it is interpolated.

In this paper we propose an algorithm for calibration of a time-independent piecewise constant volatility to a discrete set of perpetual American put option prices. The connection between the option price and its related ordinary differential equation, which has explicit solutions in the case of piecewise constant volatility, plays a key role in the paper.

We assume that the stock price satisfies
\[ dX_t = rX_t dt + \sigma(X_t)X_t dW_t, \]
where \( \sigma(\cdot) \) is a positive function. Then the related ordinary differential equation is
\[ \frac{1}{2} \sigma^2(x) x^2 u_{xx} + rxu_x - ru = 0. \] (2.1)
The price of a perpetual American put option defined in (1.8) can be expressed in terms of the linearly independent solutions of (2.1).

First, we assume that the volatility \( \sigma(\cdot) \) is a given piecewise constant function. Then the solutions to (2.1) have explicit expressions. Thus the explicit formula of the option price can be derived.

Next, assume that a finite set of option prices are given and they are increasing and strictly convex in the strikes. We propose a method to compute a piecewise constant volatility which can reproduce those prices. The whole computation process can be decomposed into elementary calculations in many fixed disjoint intervals. In each interval, one needs to solve two nonlinear
equations with two unknown variables and it does not involve differentiation of option prices.

In fact, given a finite set of option prices, there are many volatilities that can reproduce those prices. The piecewise constant function is a natural candidate and the corresponding computation procedure is comparatively simpler than that for other volatility models.

2.2 Paper II

The drift of an asset could be difficult to estimate with a high precision from historical data, compare Section 1.7. In this paper, we study the optimal liquidation problem under the assumption that the asset price follows a geometric Brownian motion with unknown drift, which takes one of two given values. Clearly, this problem is trivial if the drift is known. In that case, one only needs to compare the expected rate of return of the asset price to the interest rate and the optimal selling time is either right now or the terminal time.

In our setting, the drift is not known from the beginning but we have an initial estimate for its distribution. As time goes by, one can observe the path of the asset price and then update one’s beliefs about the drift distribution. For example, if the price increases very fast, then it is more likely that the drift is big, so the asset should be kept as long as possible. If the price decreases quickly, then it is more likely that the drift is small, so we would like to sell it as soon as possible and deposit the money in the bank instead. In fact, this intuition is proved to be true later.

We first formulate an optimal selling problem with incomplete information. Then we apply filtering techniques and the Girsanov transformation to reduce it to a one-dimensional optimal stopping problem with an affine pay-off function with complete information. Next we study an optimal stopping problem, which can be easily converted to the reduced problem and is similar to the one for American put options, by standard methods from optimal stopping, compare Section 1.6.

We show that the optimal selling time is the first time the price falls below a time-dependent boundary. This boundary is monotonically increasing, continuous and satisfies a non-linear integral equation, which is similar to the free boundary equation for the American put option. We also study the problem of when to close a short position in the asset.

2.3 Paper III

In this paper we study an optimal selling problem, which is similar to the one in Paper II. We investigate when is the best time to sell an asset under the assumption that the asset price follows a jump-diffusion with unknown jump
intensity, which takes one of two given values. First we formulate an optimal selling problem with incomplete information. Then by filtering techniques and equivalent measure transformations for jump process, the original problem is reduced into a one-dimensional optimal stopping problem with complete information. Then we study an optimal stopping problem, which can be easily converted to the reduced problem.

The best selling strategy is to sell the asset the first time the jump process falls below or goes above a time-dependent monotone boundary, which depends on the distribution of the jump size of the compound Poisson process. The optimal selling strategy depends only on the jump process and is independent from the Brownian motion and the drift. We show that this boundary is monotone and right-continuous.

Note that in Paper II, the information of the drift is mixed with the Brownian noise. However, in Paper III, the Brownian motion does not serve as noise.

2.4 Paper IV

The objective in this paper is to maximize the expected value of the discounted dividends paid to the share holders of a company until the ruin time. The value of the company is modeled by a jump-diffusion with only positive jumps. The upwards jumps can be interpreted as the net present value of future profit of inventions of a research based firm.

It can be shown that the solution to the optimal dividend problem for a jump diffusion is the limit of a sequence of stochastic control problems for a diffusion using a fixed point method in optimal stopping theory. Each problem in the sequence can be solved using standard approach for stochastic singular control of a diffusion. By this method, we can show the regularities of the value function and provide verification results. The fixed point method here is adapted from the corresponding one in optimal stopping theory and a key contribution of the paper is to give the technical details of applying this method in stochastic control.

For each problem in the sequence, the optimal dividend strategy is of barrier type. Thus the optimal dividend strategy for the original problem is of barrier type as well, i.e. to pay all the money above a certain level as dividends and pay nothing if the value is below this level.

The rate of convergence of the barriers and the value functions is exponential. We also provide a sensitivity analysis of the solution with respect to the parameters of the model.
2.5 Paper V

In this paper we study small time behavior of the Black-Scholes implied volatility in exponential Lévy models, compare Section 1.4.

As the time to maturity goes to zero, the short time implied volatility has two types of possible behaviors. It either goes to infinity or converges to the volatility of the Gaussian component of the underlying Lévy process. We show that a necessary and sufficient condition for the volatility explosion is the existence of jumps towards the strike price in the underlying process, i.e. the convergence happens if and only if the model belongs to the category of the three cases. Denote the strike price by $K$ and the stock price by $S$. (i) $S < K$ in spectrally negative exponential Lévy models; (ii) $S > K$ in spectrally positive exponential Lévy models; (iii) $S = K$, which is shown in [20], see also [21].

The general principle of the proof is to compare the lower and upper estimates of the option price to the Black-Scholes asymptotics near expiry. The lower and upper estimates, which have explicit formulas, can be found using the convexity property.
3. Sammanfattning på svenska (Summary in Swedish)


Det antas ofta att parametrarna för de underliggande modellerna är givna. För att få en bra uppskattning av driften kan det dock behövas en mycket lång tid för observationer av tillgångspriserna. Således utgör optimal stopptidpunkt med ofullständig information om modellens parametrar en av de viktigaste teman i denna avhandling. I artikel II och III studerar vi den optimala tidpunkten att sälja en tillgång under ofullständig information i två olika miljöer. I artikel II antas att tillgångens pris följer en geometrisk Brownsk rörelse med okänd drift, som tar en av två givna värden. I artikel III antar vi att tillgångens pris följer en diffusionsmodell med hopp och där hoppen har okänd intensitet som tar en av två givna värden. I båda artiklarna formulerar vi först ett optimalt försäljningsproblem med ofullständig information. Därefter tillämpar vi filtreringstekniker och måttransformation för att reducera det till ett endimensionellt optimalt stopptidpunktsproblem med en affin utbetalningsfunktion med fullständig information. Därefter kan vi behandla det reducerade
problemet med de vanliga metoderna för optimal stopptidpunkt. Vi visar att i båda fallen är den optima försäljningsstrategin att sälja tillgången första gången priset faller under en tidsberoende gräns.


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