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Analysis of the IRB asset correlation coefficient with an application to a credit portfolio

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ABSTRACT

Today banks play a central role in the global economy. Amongst other things, they contribute to the economic development of a country by lending money to companies. Doing their business banks expose them selves to several types of financial risks classifiable as follows: market risk, credit risk, operational risk and liquidity risk. In this paper we will mainly focus on the credit risk, namely the risk that a counterpart's reliability downgrades and the risk that a counterpart goes default.

We know how banks are dependent on each other in the actual economic system. It suffices to remember the devastating impact of Lehman Brother's bankruptcy on the global economy to understand why it is crucial to regulate banks' activities. The Basel committee initiated in 1988, tries to set rules to avoid deviant practices of banks and thereby tries to guaranty a healthier business fabric for the future.

Through this study we propose an analysis of the asset correlation coefficient such as described in Basel's IRB approach, which obliges banks to allocate a minimal capital to cover unexpected losses coming from borrowers who cannot honour their commitments. We compare the different methods used to compute this correlation coefficient. Finally, we apply it on Nordea's credit portfolio and try to determine an estimation of this parameter from the bank's datasets.

Key words: Basel framework, IRB, RWA, asset correlation coefficient, probabiliy of default, systematic risk, credit risk, VaR, Vasicek model, curve fitting, MLE, bootstrap.

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Part I
Introduction

1 Short presentation of the Basel framework

The Basel framework was initiated in 1988 by the central banks of 13 members of OECD. The set of rules resulting from Basel agreements is basically aiming at regulating the credit activity of banks for a safer global economic system. Since its creation, The Basel framework has been updated to take into account the major risks that weighed upon the latest financial crises.
As of year 2013, three versions of Basel framework have been released. By way of introduction, I am going to describe succinctly the main measures adopted by Basel committee to regulate banks activity. Then, I will to present Nordea which is a major bank in Northern Europe. Finally I will explain what Nordea may benefit from analysing and optimising its way to estimate the asset correlation parameter according to Basel IRB approach.

1.1 Basel I and the Cook ratio

In 1988, the first Basel accord introduced a new concept for capital requirement with the so-called Cook Ratio. This prudential ratio was set arbitrarily to 8%, meaning that the minimal sum of capital and subordinated debt securities that banks are asked to allocate to secure its credit activity, should reach at least 8% of the total amount of credit lines issued to its customers. The remaining 92% could be covered by deposit, loans etc.
As a first step in credit risk regulation, the Basel I agreement was far from perfect. This had the effect to decrease its relevance. Indeed in practice, the Basel I methodology showed imperfections especially in its way of computing the level of reliability (or quality) of borrowers, which was very far from reflecting reality. Amongst other things, it did not take into account the risks supported by financial institutions coming from derivatives products written off balance. An amendment to correct this failure was released in 1996 where a new capital ratio to cover off balance contracts was introduced. Despite its lack of consistency the Basel I accord had been put into effect for a relatively long period until Basel II came.

1.2 Basel II and the Mc Donough ratio

In 2004, the Basel committee started studying on a new version of the regulatory framework and released in 2005 the well-known accord Basel II which is still in use in 2013. In this second variant, an emphasis was brought on the credit risk measurement, where the reliability of counterparts was taken into account in a more accurate way. This could be done through the introduction of the IRB Internal Rating Based approach. Basically this approach introduces a new company rating system, which is self-supported by the banks. The Basel II framework is based on three pillars as follows.

1.2.1 Pillar one: the capital requirements

Here as a complement of the credit risk, two new elements were introduced: the operational risk and the market risk. The Cook ratio is replaced by the new Mc Donough ratio: 8% on a weighted combination of total operational risk, market risk and credit risk. The proposed formula is as following:

$$C > 0.08 \quad (0.85\ CR + 0.10\ OR + 0.05\ MR)$$

where
C denotes the minimal capital required by the financial institution,
CR denotes the total Credit Risk supported by the financial institution,
OR denotes the total Operational Risk,
MR denotes the total Market Risk.

It is worth mentioning that Basel II proposes different techniques to compute CR, OR and MR. Namely, with respect to assets considered and some eligibility requirements, banks may compute their own total credit risk by choosing amongst: the Standard approach, the Foundation IRB (FIRB) or the Advanced IRB. Similarly, regarding operational risk, three methods are proposed: the STA for Standard Approach, the internal measurement approach or the BIA for basic indicator approach. Finally, the market risk is computed using a Value at Risk approach: either by a general VaR or by a specific VaR.

1.2.2 Pillar two: the capital management regulation

The second pillar specifies how regulators, like central banks, may act to assure better risk management in a global scale as well as better transparency and better reliability for credit actors. Here, a larger set of risks is considered like the systemic risk, the concentration risk, the legal risk etc. More concretely, this pillar allows authorities to test the statistical methods used by banks to assess their needs for capital (back testing), to investigate the reliability of the data used, or to ask for actors to provide them with a stress test report, focusing on the validity of capital requirements under some economic downturn context (stress testing).

1.2.3 Pillar three: the market discipline

The third pillar contributes to guarantee a certain level of transparency regarding the financial reports provided by banks to the public. Financial actors are submitted to a set of rules promoting reporting standardisation. As a consequence, it becomes easier to read, to assess and to compare documentation regarding risk assessment processes, risk exposure and capital adequacy of the different credit providers.

1.3 Basel III and the LCR and NSFR ratios

The Basel III accord comes up in 2010 as a response to the failures observed under the "sub-prime crisis" and is still in calibration period. At this time i.e. 2007, it turned out that bank's balance and off balance had grown excessively, for instance by the purchase of a significant amount of derived products. Meanwhile both the quality and the level of the capital allocated to face economic downturns were degrading. Moreover, many financial institutions figured out that they did not have sufficient liquidities to contend a liquidity crisis. In this context all the prerequisites for a lasting economic depression was unified.
First, the banking system had to face a strong increase in company defaults due to the depreciation of their assets returns, especially coming from asset backed securities like "sub-primes". The mortgages essentially of type real estate became worthless implying uncertainty about banks' solvency, which in addition to interdependency between banks originated a crisis of trust and of liquidity.
Hence, it appeared obvious that the regulatory authorities had to think about a new version of

Basel II to complete the framework.

Basel III differs from its previous version in many points. In the next paragraph, we are going to summarize the main changes that have been brought in response to the latest crisis.

1.3.1 The solvency ratio

Changes in solvency ratio are done to upgrade the quality of elements taken into account in the computation of the economic capital of banks.
In the definition of capital according to Basel III, Common Equity Tier 1 replaces Core Tier 1, Tier 2 is replaced by Lower Tier 1 capital and Tier 3 Capital becomes Tier 2. Tier 3 disapears. This means that in Basel III, assets of better quality are used to compute banks' capital needs. The minimum ratio of total capital to RWAs is upgraded from 8% to 10.5/%.

1.3.2 The liquidity ratio

a) The Liquidity Coverage Ratio (LCR)

This new ratio is meant to show how a bank can resist a liquidity crunch during a thirty-days period without requiring refunding from central banks. In other words, it aims at granting that banks own enough quality assets, providing them with high level of liquidity even under crisis periods.

$$LCR = \frac{LA}{NetCFO}, \qquad \text{(LCR must be higher than 1)}$$

where,
LA denotes Liquid Assets defined as cash, deposits in central banks and non-risked bonds i.e. sovereign bonds, bond issued by companies rated at least AA-,
NetCFO denotes Net Cash Flows Out defined as incomes from credits issued minus outcomes or losses coming from a stress situation.

b) The Net Stable Funding Ratio (NSFR)

This is a middle-term ratio aiming at granting that banks do not fund their business mainly by means of short term resources. It makes sure that they own enough stable resources to refund its long-term assets (where "stable" means resources having an initial maturity longer than one year). NSFR implies that the liabilities constituting the stable resources must be weighted according to the type of product and to the counterpart involved. In a similar way the needs for funding have to take into account a weighting of assets relative to their liquidity level.

1.3.3 The leverage ratio

Under the latest financial crisis it turned out that banks tended to be more underfunded than overfunded in terms of quality capital. In fact we observed that the price of some asset classes fell faster than expected. The leverage ratio is intended to upgrade the part of quality capital (Capital Tier One) and in the same time contributes to reducing the risk of failures propagation in the whole economic system (systemic risk). It is expressed as following:

$$LR = \frac{TA}{T1C}$$

where,
LR denotes the leverage ratio,
TA denotes the total assets,
T1C denotes Capital Tier 1 assets [1]

1.3.4 The capital conservation buffer: 2.5%

The conservation buffer is a new prudential ratio set at 2,5% of the Risk-Weighted Assets[2] (RWA) of the credit provider. This ratio is applied on Core Tier One capital as a complement to the mandatory 4,5% already required. This means that the percentage of Core Tier One capital actually required to cover the credit activity of a bank reaches at least 7% of the RWA (plus a seasonality rate lying between 0% and 2.5%).

1.3.5 The countercyclical buffer: 0 to 2.5%

This new buffer is intended to damp the effects relative to the cyclical behaviour of the economy. One can observe that the predisposition of banks to issue credits to customers is better under certain periods. But an excess of credit growth may lead to the bulid-up of system-wide risk. The buffer aims at slowing down banking activity when it overheats and in the opposite case, it encourages issuing credits under economic depressions. Concretely the buffer may add up to 2.5% of capital requirements consisting of common equity or equivalent assets. This has for immediate effect to increase the cost of credit and to decrease the demand.

1.3.6 The market risk

In order to take extremely rare events into consideration, Basel III introduced three new components in the assessment of Market risks. Namely, the "stressed VaR", the Incremental Risk Charge (IRC) and the Comprehensive Risk Measure (CRM).

As of 2013, all these modifications are not yet in use. Banks are asked to implement them progressively during a transition period, which began January the 1st of 2013 and will end 2019. The authorities have released a schedule to organise the transition between the Basel II and the Basel III framework.

[1] Let us remind that Capital Tier 1 in Basel III consists of :
- Common Equity Tier 1: 4.5% + 2.5% of conservative buffer
- Hybrids Assets (Trust-preferred securities, some convertible bonds).

[2] Each credit issued by a bank is registered as an asset in its portfolio. But in reality these assets are not homogeneous in term of risk. The Risk-Weighted Asset is a measure of the exposition of a bank issuing credits obtained by applying a weighting according to the intrinsic riskiness of each asset. Collateralised assets weights less than non-collateralised ones. The Risk-Weighted Asset amount determines the amount of capital needed by the bank according to Basel framework.

2 Short presentation of Nordea

Nordea Bank AB is the largest financial services group in Northern Europe. The bank is engaged in corporate merchant banking as well as retail banking and private banking. Since its creation in 1820, Nordea has been growing through a number of successive mergers and acquisitions as those ones that took place in the latest 1990s, involving financial institutions like Nordbanken, Merita Bank, Unibank and Kreditkassen. Today the bank owns more than 30 subsidiaries and employs about 33400 persons. Based in Stockholm, Nordea bank is mainly operating in Scandinavia, in the Baltic countries, in Poland and in Russia. However, the bank proposes also corporate banking solutions in other countries like Germany, UK, Singapore, China or the USA, as well as asset management services in Luxembourg, Belgium, France, Switzerland and Spain.
As of December 2012, Nordea counts around 10.4 millions private customers and another 600000 corporate clients. Nordea is managing a total of 218 bnEUR in assets and generated, during the financial year 2012, a total operating income of 10,236 mEUR (+7.7%) yielding an operating profit of 4,117 mEUR (+16.1%) and a net profit of 3,126 mEUR (+18.7%).
Nordea shares are quoted on NASDAQ OMX Nordic Exchange in Stockholm, Copenhagen and Helsinki. The main owners of the bank are: Sampo Group (21,4%), Nordea Fonden (3,9%), Swedbank Robur Funds (3.3%) and Alecta (2%)[3].
The main competitors of Nordea are SEB, Handelsbanken, Swedbank, Danske Bank and DnB.

In order to compare the main competitors in Northern Europe let's present some key figures coming from the financial reports 2012. We present it in the following table:

Bank	Nordea	Handelsbanken	SEB	Swedbank	DNB	Danske Bank
Total Assets	667	286.5	294.5	221.5	294	453
Total Loans	346	212.5	165.5	159	169	227
Operating Income	10	7	4.5	4.5	5.5	6
Operating Profit	4.1	2.1	1.7	2.2	2.3	1.1
Net Profit	3.1	1.7	1.5	1.7	1.8	0.6
Tot Loan/Tot Assets	52%	74%	56%	72%	57%	50%
Return on Assets ROA	0.46%	0.59%	0.51%	0.77%	0.61%	0.13%

Some financial figures to compare the main banks acting in Northern Europe (in millions EUR)

Looking at these figures, it appears that Nordea is first in total assets owned, in total loans issued, and yields the best operating profit and net profit in 2012. However, Nordea's Return On Assets[4] (ROA) doesn't lie in the top of the list, meaning that some competitors use their assets in a better way to generate profits.
The ratio Total Loan / Total Assets gives an idea of the weight of the credit activity in each bank. One may notice that Nordea, SEB, DnB and Danske Bank have a ratio approaching 50% meaning that the half of the assets owned by these banks are coming from retail banking and in consequence the other half should come from corporate & investment banking activities or from asset management activities.Handelsbanken and Swedbank whose ratio is lying around 75% are probably more focused on retail banking activities which reduces their exposure to risky activities linked to financial markets. It is worth mentioning that the two highest ROAs are coming from banks focusing on retail banking businesses.

Now, if we focus on the credit portfolio of Nordea in 2012, one can read from the latest report that the amount of loans issued to the public increased by 3% in comparison to 2011, to reach a total amount of 346 bnEUR, which is roughly speaking the half (52%) of the total asset owned by Nordea bank AB (677 bnEUR in 2012). Hence, it is easy to understand how much changes

[3]Figures as of September 2013
[4]Return On Assets = Net Profits/Total Assets.

in the Basel framework on capital requirements may impact the business of the group.
As presented earlier in the introduction, we saw that Basel framework leaves to banks a certain level of freedom in determining their own needs for capital. Therefore, it sounds rational that Nordea wants to be able to measure its credit risks with the highest accuracy. This is first of all, a way to respect the rules set by Basel accords, but also an analysis, which could contribute to avoiding losses of opportunities by allocating potential capital surplus to profitable investments.

In this context, we are going to analyse a parameter called the correlation coefficient ρ of the Risk-Weighted exposure formulae proposed by Basel's IRB approach. This correlation coefficient determines how the asset values of the borrowers depend on each other and indicates how the asset values of the borrowers depend on the systematic risk factor denoted by S. The level of ρ is fixed by the regulation, nevertheless some empirical measurements have already been carried out to determine a more realistic value of this correlation coefficient ρ with respect to some particuliar credit portfolios.
After analysing each of these methods we are going to apply them to Nordea's credit portfolio to derive estimations of this parameter by asset class and by country.
Reaching a high accuracy in parameter estimation would allow us to have a better idea of the level of economic capital required to cover unexpected losses inherent in the credit business of Nordea.

Part II

The asset correlation coefficient in the IRB approach

Basically, the IRB approach proposed by the Basel framework to compute the regulatory capital requirements of banks, comes from a credit portfolio model called the Asymptotic Risk Factor (ASRF) model, which in turn is derived from the Merton model.
We are going to explain briefly what these models assume and how they are constructed.

1 The Asymptotic Single Risk Factor approach (ASRF)

1.1 The Merton model

In this model, default events occur when the value of a firms asset becomes lower than the level of its liabilities at a certain maturity T. In 1974, Robert C. Merton proposed a method to evaluate the credit risk supported by a firm by taking equity return as a proxy for the assets return of the company observed.

Based on the assumption that the dynamics of a firms equity follows a Geometric Brownian Motion (GBM), one can model the evolution of the equity and compare it to the total level of the liabilities in the firms balance sheet after a given period.
Some additional assumptions are needed in Merton model, namely that there are no bankruptcy charges when default occurs, and that debt and equity are single and liquid assets.

$$\begin{cases} dV_t = \mu_V \, V_t \, dt + \sigma_V \, V_t \, dS_t \\ V_0 = V_0 \end{cases} \tag{1}$$

where:
V_t denotes company's equity value,
σ_V denotes firm's equity volatility,
μ_V denotes the return rate of the firm's equity,
dSt denotes the standard Wiener process associated to the systematic risk. $\mathcal{N}$(0,t).

Two events can occur at maturity t=T:

1. The equity level is lower than the liabilities level. In this case the company is in bankruptcy the shareholders lose their investments, whereas the creditors share the remaining equity value.
2. The equity level is higher than the liabilities level. In this case, the firm can reimburse the debt and shareholders may decide how to use the surplus of equity.

In other words, according to Merton's model, equity return can be considered as a vanilla option. The shareholders are long a call option on equitys value with strike the level of the debt and maturity T. The premium is the value invested by the shareholders in the firm, whereas the debtors are short a put option on equitys value with same strike and same maturity. The premium is the debt plus an interest rate.

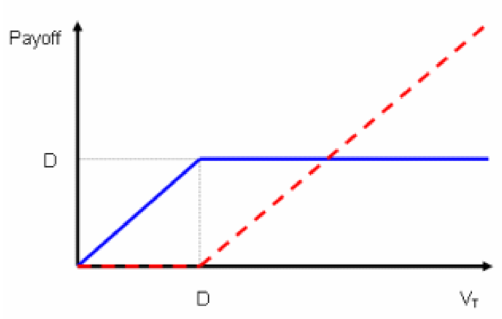

Figure 1: Payoff of shareholders / debtors with respect to the asset value V_T

Here,
V_T is the total value of the firms assets,
D is the level of total liabilities,
In red line: the payoff from the shareholders point of view.
In blue line : the payoff from debtors point of view.

The GBM (Equation 1) has for solution (see Arbitrage Theory in Continuous Time by T.Björk, Chapter 5):

$$V_t = V_0 \exp\left\{\left(\mu_V - \frac{1}{2}\sigma_V\right)t + \sigma_V S_t\right\} \tag{2}$$

1.2 The Vasicek model

The Vasicek formula is basically coming from Merton model in the sense that under some assumptions mentioned earlier, the assets of borrowing firms can be modelled by companies equity as a stochastic differential equation (cf. Equation 1). However, the difference between Merton model and Vasicek model is that in Vasiscek's Model, instead of taking the liabilities level to infer the PD of a firm, here the PD is given and we infer the debt level from the PD. The advantage of doing this way is that the levels of both asset return and liabilities disapear from the formula, which simplifies our model.
As an example, we are going to derive the probability of default of a single firm taking into account the systematic risk and the idiosyncratic risk. From the Merton model we can write the value of equity as following:

$$dV_t = \mu_V \, V_t \, dt + \sigma_V \, V_t \, dS_t + \beta_V V_t \, dB_t \tag{3}$$

where,
V_t denotes company's equity value at time t,
σ_V denotes the sensitivity of the firm to systematic risk,
μ_V denotes the return rate of the firm's equity,
β_V denotes the sensibility of the firm to idiosyncratic risk,

dB_t denotes the standard Wiener process associated to the idiosyncratic risk. $\mathcal{N}$(0,dt),
dS_t denotes the standard Wiener process associated to the systematic risk. $\mathcal{N}$(0,dt).

It is worth mentioning that μ_V, σ_V and β_V are estimated constants and that dSt and dBt are independent Wiener processes.
As in Equation 2 the solution to this stochastic differential equation can be rewritten as:

$$V_t = V_0 \exp\left\{\mu_V\, t - \frac{t}{2}(\sigma_V^2 + \beta_V^2) + \begin{bmatrix} \sigma_V & \beta_V \end{bmatrix} \begin{bmatrix} dS_t \\ dB_t \end{bmatrix}\right\} \tag{4}$$

Taking a time horizon of one year, we can write t=1 and thus S_1 and B_1 follow a $\mathcal{N}$(0,1) distribution. The equation above simplifies like:

$$V_1 = V_0 \exp\left\{\mu_V - \frac{1}{2}(\sigma_V^2 + \beta_V^2) + \begin{bmatrix} \sigma_V & \beta_V \end{bmatrix} \begin{bmatrix} S_1 \\ B_1 \end{bmatrix}\right\} \tag{5}$$

To model default events we introduce a new random variable D, which is assumed to be Bernoulli distributed: $\mathcal{B}(PD)$

$$D = \begin{cases} 0\, with\, probability\, 1 - PD \\ 1\, with\, probability\, PD \end{cases}$$

Here, D=1 means that default event has occurred. In other words, after a period of one year we observe that the level of V_1 stands below the level of liabilities (L). Thus, we can write:

$$PD = P\left(V_0 \exp\left\{\mu_V - \frac{1}{2}(\sigma_V^2 + \beta_V^2) + \begin{bmatrix} \sigma_V \beta_V \end{bmatrix} \begin{bmatrix} S_1 \\ B_1 \end{bmatrix}\right\} < L\right)$$
$$PD = P\left(\mu_V - \frac{1}{2}(\sigma_V^2 + \beta_V^2) + \begin{bmatrix} \sigma_V & \beta_V \end{bmatrix} \begin{bmatrix} S_1 \\ B_1 \end{bmatrix} < \ln \frac{L}{V_0}\right)$$
$$PD = P\left(\begin{bmatrix} \sigma_V & \beta_V \end{bmatrix} \begin{bmatrix} S_1 \\ B_1 \end{bmatrix} < \ln \frac{L}{V_0} - \mu_V + \frac{1}{2}(\sigma_V^2 + \beta_V^2)\right)$$

We know that B_1 and S_1 are mutually independent and $\mathcal{N}$(0,1) distributed. Furthermore, $\sigma_V S_1$ and $\beta_V B_1$ are respectively $\mathcal{N}(0, \sigma_V^2)$ and $\mathcal{N}(0, \beta_V^2)$, from which it is easy to find that $\sigma_V S_1 + \beta_V B_1$ is $\mathcal{N}(0, \sigma_V^2 + \beta_V^2)$ distributed.
Now we can standardize the random variable in the right hand side of the inequality and we get that:

$$PD = P\left(\frac{\sigma_V S_1 + \beta_V B_1}{\sqrt{\sigma_V^2 + \beta_V^2}} < \frac{\ln \frac{L}{V_0} - \mu_V + \frac{1}{2}(\sigma_V^2 + \beta_V^2)}{\sqrt{\sigma_V^2 + \beta_V^2}}\right) \tag{6}$$
$$PD = \Phi\left(\frac{\ln \frac{L}{V_0} - \mu_V + \frac{1}{2}(\sigma_V^2 + \beta_V^2)}{\sqrt{\sigma_V^2 + \beta_V^2}}\right) \tag{7}$$

Now let $\rho_V = \dfrac{\sigma_V^2}{\sigma_V^2 + \beta_V^2}$ meaning that ρ_V is the proportion of systematic risk.

From Equation 6 we get that,

$$\begin{aligned}
\frac{\sigma_V\, S_1 + \beta_V\, B_1}{\sqrt{\sigma_V^2 + \beta_V^2}} &= \frac{\sigma_V\, S_1}{\sqrt{\sigma_V^2 + \beta_V^2}} + \frac{\beta_V\, B_1}{\sqrt{\sigma_V^2 + \beta_V^2}}, \\
\frac{\beta_V^2}{\sigma_V^2 + \beta_V^2} &= 1 - \rho_V, \\
\frac{\sigma_V\, S_1 + \beta_V\, B_1}{\sqrt{\sigma_V^2 + \beta_V^2}} &= \sqrt{\rho_V}\, S_1 + \sqrt{1-\rho_V}\, B_1.
\end{aligned}$$

and can rewrite equation 6 as follows,

$$PD = P\left(\sqrt{\rho_V}\, S_1 + \sqrt{1-\rho_V}\, B_1 < \frac{\ln \frac{L}{V_0} - \mu_V + \frac{1}{2}\,(\sigma_V^2 + \beta_V^2)}{\sqrt{\sigma_V^2 + \beta_V^2}}\right) \tag{8}$$

which is equivalent to,

$$\Phi^{-1}(PD) = \left(\frac{\ln \frac{L}{V_0} - \mu_V + \frac{1}{2}\,(\sigma_V^2 + \beta_V^2)}{\sqrt{\sigma_V^2 + \beta_V^2}}\right). \tag{9}$$

To go further, this means that:

$$D = \begin{cases} 0\, if\, \sqrt{\rho_V}\, S_1 + \sqrt{1-\rho_V}\, B_1 > \Phi^{-1}(PD) \\ 1\, if \sqrt{\rho_V}\, S_1 + \sqrt{1-\rho_V}\, B_1 \le \Phi^{-1}(PD) \end{cases}.$$

If we have an estimation of the probability of default of the firm given the systematic risk factor (representing the state of economy), mathematically written: P(D=1|S_1=y) then we can compute:

$$\begin{aligned}
P(D=1|S_1=y) &= P\left(\sqrt{\rho_V}\, S_1 + \sqrt{1-\rho_V}\, B_1 \le \Phi^{-1}(PD)\,|\, S_1 = y\right), \qquad (10)\\
P(D=1|S_1=y) &= P\left(\sqrt{\rho_V}\, y + \sqrt{1-\rho_V}\, B_1 \le \Phi^{-1}(PD)\right), \\
P(D=1|S_1=y) &= P\left(B_1 \le \frac{\Phi^{-1}(PD) - \sqrt{\rho_V}\, y}{\sqrt{1-\rho_V}}\right), \\
P(D=1|S_1=y) &= P\left(B_1 \le \frac{\Phi^{-1}(PD) - \sqrt{\rho_V}\, y}{\sqrt{1-\rho_V}}\right).
\end{aligned}$$

Finally, because we know that B_1 is $\mathcal{N}(0,1)$ distributed we conclude easily that:

$$P(D=1|S_1=-y) = \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho_V}\, y}{\sqrt{1-\rho_V}}\right) \tag{11}$$

This formula is also called the conditional probability of default (CPD) of a firm.

1.3 The Asymptotic Risk Factor approach (ASRF)

The ASRF approach is used by the Basel framework to compute the capital needed to prevent a bank from bankruptcy under a one year period, with a probability of more than $q = 0.999$.
If Φ is the cumulative standard normal distribution, Φ^{-1} the inverse standard normal distribution, q the level of confidence required by the bank, ρ_V the correlation between returns on the assets of borrowers in the portfolio, then C_V represents the proportion of capital required by the bank and C_V is given by:

$$C_V = \Phi \left\{ \frac{\Phi^{-1}(PD) + \sqrt{\rho_V}\, \Phi^{-1}(q)}{\sqrt{1-\rho_V}} \right\} \tag{12}$$

One can notice that the CPD equation (11) becomes equation (12) if we simply let $\Phi^{-1}(q) = -y$. We can interpret this by saying that -y represents the number y of standard deviations away from the mean of a $\mathcal{N}(0,1)$ to the left of the distribution.

This formula has been derived making additional assumptions:

1. The portfolio is sufficiently fine grained (the number of assets N $\geq$ 1000) so that the idiosyncratic risk is diversified away. Only the systematic risk remains, this is the reason why we call it a single factor model.
2. Firm's asset returns are correlated to the common systematic factor S, which is N(0,1) distributed.
3. The Loss Given Default (LGD) is assumed equal to one for each exposure, meaning that when a loan defaults there is no recovery possibility.
4. The loan generates no cash flows.

From practice, banks know that in reality the LGDs, PDs, and asset correlations are not equal from a company to another. Hence, if we want to compute the exact percentage of capital C_p needed per unit of exposure, we let LGD_i denote the Loss Given Default of firm i, w_i denote the exposure weight of each asset i in the portfolio (i.e. $w_i = \frac{EAD_i}{\sum_{i=1}^{N} EAD_i}$), and according to Gordy (2003) and Pykhtin and Dev (2002) we get:

$$C_p = \sum_{i=1}^{N} (w_i\, LGD_i) \times \Phi \left\{ \frac{\Phi^{-1}(PD_i) + \sqrt{\rho_i}\, \Phi^{-1}(q)}{\sqrt{1-\rho_i}} \right\} \tag{13}$$

The advantage of this expression, apart from its simplicity, is that one can figure out that the capital required to add a single loan to any large portfolio depends only on the obligor's attributes and not on the properties of the portfolio it is added to. This is called the portfolio invariance and it constitutes a real advantage from the Basel framework point of view, in the sense that all credit institutions are handled equitably regardless of the composition of their own portfolio.
It is worth mentioning that this portfolio invariance property comes from the assumptions made earlier, namely the high granularity of the portfolio giving the asymptotic property and the assumption on the single systematic risk factor model.
If idiosyncratic risk was not taken away by high diversification, then adding a single loan to a portfolio would impact the capital needs differently in accordance to the portfolio composition, because the covariance of the asset with the portfolios would tend to differ.

To go further Schönbucher (2002a) and Wehrspohn (2002) investigated what Equation (13) would become in the case of homogeneous portfolios. In the one factor model, by dividing a whole portfolio of credits in fine grained homogeneous clusters where all assets of a same cluster have the same PD, the same LGD, the same EAD, the same correlation ρ and the same expiry date, they came up to the statement that the percentage of capital needed to cover the whole portfolio with only $q = (1 - \alpha)$ of default probability becomes:

$$C_P = \sum_{k=1}^{N} EAD_k \times LGD_k \, \Phi \left\{ \frac{\Phi^{-1}(PD_k) + \sqrt{\rho_k}\, \Phi^{-1}(\alpha)}{\sqrt{1-\rho_k}} \right\}. \tag{14}$$

Here,
N is the number of fine grained homogeneous sub-portfolios,
α is the level of confidence required[5].
PD_k is the Probability of Default of the sub-portfolio k,
EAD_k is the Exposure At Default of the sub-portfolio k,
LGD_k is the Loss Given Default rate of the sub-portfolio k.
ρ_k is the correlation coefficient between assets in the sub-portfolio k.

In other words if a portfolio is constituted of homogeneous sub-portfolios, then the value of regulatory capital to cover the entire portfolio is just the sum of the amounts required to cover each sub-portfolios.
There are many ways to create homogeneous sub-portfolios. The first idea would be to sort the assets by sector/industry and by country. But other techniques more accurate may be used like K-means clustering, hierarchical clustering, the expectation maximization algorithm, or by principal component analysis.
To respect the granularity requirement it is said that each cluster should be made of at least 1000 assets.

In fact, in practice everybody knows that it is hardly possible to fulfil all the conditions made in assumptions to model credit risk. For example creating homogeneous and fine grained portfolios could sound quite unrealistic in reality. Furthermore, it does not sound reasonable to affirm that every exposure in a portfolio represents a similar risk of defaulting.
Although the ASRF approach is leading to some imperfections in the estimation of the capital requirements, it presents some advantages in a regulatory point of view: it is relatively easy to implement, it yields the same capital charge when adding an asset independently on the structure of the initial portfolio, and it yields sufficiently good estimations of credit risk.

Using multi-factor models would increase the accuracy of estimations but would be harder to implement because it requires a correlation matrix between each asset, where each element has to be estimated. When the number of factors becomes big, the gain in accuracy made by using a multi-factor model faces a dimensionality challenge, which makes the implementation effort substantially greater.
Some commercial credit risk models like "Credit Metrics", "Creditrisk+" and "Moody's-KMV" propose a multi-factor approach to forecast credit risk with better accuracy.

[5]In IRB approach α is arbitrarily chosen as 0.999, which is a very conservative value

2 Basel's IRB formula

2.1 The Risk Weight Asset Formula (RWA)

In their lending business banks are able to forecast the average level of losses that they should experience over a specific time period. This is called the Expected Losses (EL) and financial institutions cover these potential losses by write-offs, provisions etc.
Losses above the expected level are called the unexpected losses (UL), they may occur at any time and with any intensity. Through the IRB approach, Basel II provides banks with a formula aiming at covering these unexpected losses.
Based on our explanation of the ASRF approach, we can clearly understand how this formula has been build up.

If LGD denotes the losses given default (in %),
PD the probability of default (in %),
$\sqrt{\rho}$ the correlation coefficient between assets and the systematic factor,
MA the maturity adjustment,
SF the scaling factor and finally,
MCR the Minimum Capital Required,
then the risk weight (in %) RW is given by:

$$RWA = \left\{ \underbrace{LGD\,\Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}\,\Phi^{-1}(0.999)}{\sqrt{1-\rho}}\right)}_{\text{Conditional expected losses or VAR}} - \underbrace{LGD\,PD}_{\text{Expected losses}} \right\} \times MA \times SF \times MCR \times EAD \tag{15}$$

where,
MA $= \dfrac{1+(M-2.5)\,b}{1-1.5\,b}$, with M the time to maturity, b $= \big(0.11852 - 0.05478 \times \ln(PD)\big)^2$,
SF = 1.06,
MCR = 12.5.

The Maturity Adjustment MA is only used when the portfolio consists of corporate assets having different maturities.
It can be explained on the one hand because long-term credits are always considered riskier than short-term credits, and on the other hand because maturity affects more firms having a lower probability of default.

The Scaling Factor SF is just a coeffcient used by the regulation to make the formula more or less conservative. Now the scaling factor is equal to 1.06.

The MCR is the Minimum Capital Required. MCR times 12.5 gives the RWA.

2.2 The parameters of the RWA formula

2.2.1 PD: The Probability of Default

This parameter is estimated by the internal rating system of the bank. PDs are supposed to reflect expected default rates under normal business conditions. Basically, there are two types of models used to determine PDs: Accounting based models and market based models.

2.2.2 LGD: The Losses Given Default

The LGD is the percentage of exposure that a bank may loose if the client goes default. Under economic downturn, the losses given default are likely to be higher than under growth periods. Banks are estimating their own LGDs for each client.

2.2.3 EAD: The Exposure At Default

The EAD is an estimate of the amount of money that a bank is likely to loose if the customer goes default. The advanced IRB approach allows banks to estimate their own EADs for each client. Basiccally, the EAD consists of two part: the amount currently drawn and the credit conversion factor (CCF).

2.3 Definition of the asset correlation coefficient

The asset correlation ρ in the ASRF model gives a measure of how the asset values of the borrowers depend on each other. Alternatively it may be seen as the degree of dependence between the assets of a credit portfolios and the general state of economy[6], which is specified by the systematic risk factor S (See Equation 10).
This parameter is impossible to observe directly and is quiet hard to estimate due to a lack of historical data.

In the IRB approach, the Basel framework sets the correlation coefficient to different values depending on the asset class we are dealing with. Every bank are using these levels of correlation in their RWA calculations despite knowing that these values are arbitrary chosen and do not necessarily match the levels that should be applied for their own credit portfolios.
The formulas proposed by the Basel II framework are shown here after:

2.3.1 Large Corporates and Institutions (LCI)

For sovereign, banks and large corporates[7] exposures:

$$\rho_{LCI} = 0.24 - 0.12 \times \frac{1 - e^{-50.PD}}{1 - e^{-50}}$$

So according to IRB approach, ρ_{LCI} is bounded like following : $0.12 \leq \rho_{LCI} \leq 0.24$

2.3.2 Small and Medium Enterprises (SME)

[6]It is noteworthy to recall that if ρ is the correlation coefficient between assets, then the correlation coefficient between assets and the economic context is given by $\sqrt{\rho}$.

[7]In IRB context, large corporates are defined as firms generating a yearly turnover greater than 50 millions EUR

- For SMEs generating a yearly turnover lying between 5 millions EUR and 50 millions EUR: (5 millions < T ≤ 50 millions EUR)

$$\rho_{ME} = \rho_{LCI} - 0.04 \times \left(1 - \frac{T-5}{45}\right)$$

So according to IRB approach, ρ_{ME} is bounded like following : $0.08 \leq \rho_{ME} \leq 0.24$

- For SME generating a yearly turnover smaller than 5 millions EUR: (T ≤ 5 millions EUR)

$$\rho_{SE} = 0.20 - 0.12 \times \frac{1 - e^{-50.PD}}{1 - e^{-50}}$$

So according to IRB approach, ρ_{SE} is bounded like following : $0.08 \leq \rho_{SE} \leq 0.20$

2.3.3 Retail (Households)

- For Residential Mortgages:

$$\rho_R = 0.15$$

- For qualifying revolving:

$$\rho_R = 0.04$$

- For other retail exposures:

$$\rho_R = 0.16 - 0.13 \times \frac{1 - e^{-35.PD}}{1 - e^{-35}}$$

These formulas are in fact coming from earlier empirical studies conducted on the correlation coefficient between assets, which showed firstly that the its value seems to increase with decreasing PD (see Figure 2), meaning that asset return correlations are higher between healthy firms, and secondly that the correlation coefficient tends to increase with the firm size, meaning that large corporates are more sensitive to the global economic context than small ones (Dietsch and Petey, 2004; Dllmann and Scheule, 2003).

2.4 Relation between the asset correlation and the RWA formula

Now, knowing that the RWA formula is a function of the LGD, the PD and of the correlation coefficient ρ $\big($ RWA=f(LGD, PD, ρ) $\big)$ we want to show the effect of changes in correlation coefficient on the RWA for a given LGD of 10%. The results are shown here after:

It is interesting to observe how RWA levels may differ with respect to the asset class considered. Figure 3 shows merely that the RWA formula is very sensitive to the correlation coefficient value and based on this remark, one may easily understand that banks are interested in estimating more accurate values of this coefficient with respect to the real composition of their credit portfolio.

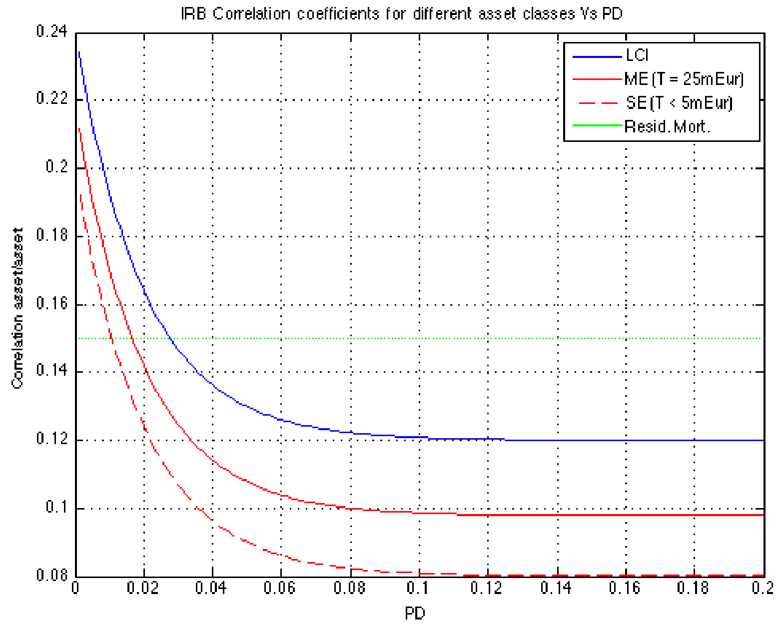

Figure 2: Levels of correlation coefficient with respect to PDs

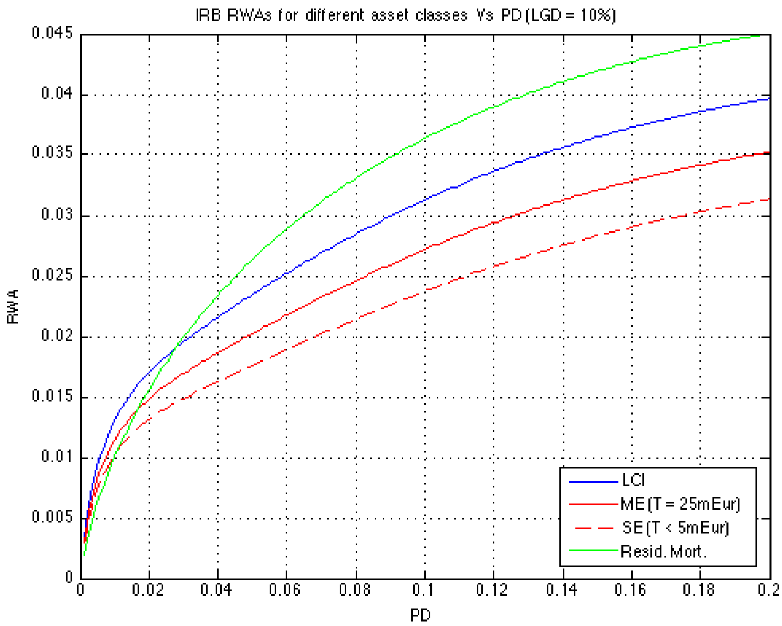

Figure 3: Levels of RWA with respect to PD

Since the correlation coefficient is a parameter which is not observable, the financial industry has been working on finding estimation techniques to assess their own correlation coefficient while respecting the financial rules imposed by the regulation.
The next section is devoted to the methodologies used in credit risk analysis. After presenting the main methods used by the practitioners to estimate the correlation parameter, we are going to apply them to some available datasets and see which levels of correlation are more likely to match Nordea's credit portfolio.

Part III

Application of empirical methods to estimate the asset correlation coefficient with respect to Noredas's credit portfolio

In this paper we are going to use three different methods. The first one called the Fitch method is about comparing the CVaR of two different distributions to extract an implicit correlation coefficient. The second is about maximizing the maximum likelihood function of our portfolio based on the ASRF approach and extracting the optimal argument for the parameter. Finally, based on bayesian inference we can find prior probability functions of the probability of default also called likelihood functions, and similarly to the second method we will optimize the function to get the most likely estimate of the correlation coefficient after introducing a time correlation matrix beween the different states of macroeconomy.

1 The Fitch Rating's method : Estimation of the implicit asset correlation coefficient

1.1 Description of the method

This method proposed by the rating agency Fitch Ratings is based on an empirical analysis of the Conditional VaR. Basically, we fit a Beta distribution to observed annualized loss rates time series in order to compute an estimated Conditional VAR at 99,9 % : $CVaR_e$. This $CVaR_e$ is in turn compared to the Conditional VaR derived by means of the IRB formula $CVaR_{irb}$. Equating the two CVaR levels yields an implicit estimated value of the correlation coefficient ρ_e.

A prerequisite before applying this method is to create several data set of annualized loss rates depending on the location and on the asset class from which the losses are observed (We know that the UL_{irb} values are different with respect to the asset class considered).
Furthermore, it is important to check the robustness of the time series. The frequency of the data as well as its size are determinant in the quality of the results. Finally, to avoid underestimating our implicit correlation coefficient it is worth mentioning that the data sets should include both periods of economic boom and periods of economic recession.

The beta distribution $\mathcal{B}(\hat{\alpha}, \hat{\beta})$ depends on two parameters alpha and beta, which are estimated by using a straight-forward method of moments calculation. Applying this method we get an expression for each estimated parameter:

If we let μ be the annualized loss rate sample mean and σ the standard deviation, we get that,

$$\hat{\alpha} = \mu\left(\frac{\mu\,(1-\mu)}{\sigma^2} - 1\right), \tag{16}$$

$$\hat{\beta} = (1-\mu)\left(\frac{\mu\,(1-\mu)}{\sigma^2} - 1\right). \tag{17}$$

After estimating $\hat{\alpha}$ and $\hat{\beta}$ we use the probability density of the beta distribution to compute its 99.9 % quantile, which is equivalent to our estimated conditional VaR : $CVaR_e$ (i.e. $UL_e + EL_e$).
For each country and each asset class according to the Basel II definition, we derive a yearly average PD denoted $\overline{PD}$ and we compute the associated conditional VaR at 99,9% : $CVaR_{irb}$. We compare them to their respective $CVaR_e$ obtained previously thanks to the beta fitted distribution. Equating both conditional VaR yields an implicit estimated ρ_e for each country and each asset class:

$$B(\hat{\alpha}, \hat{\beta}, 99.9\%) = CVaR_e \iff CVaR_{irb} = \Phi\left(\frac{\Phi^{-1}(\overline{PD}) + \sqrt{\rho}\,\Phi^{-1}(99.9\%)}{\sqrt{1-\rho}}\right) \tag{18}$$

$$\Rightarrow \Phi^{-1}(B(99.9\%, \hat{\alpha}, \hat{\beta})) = \frac{\Phi^{-1}(\overline{PD}) + \sqrt{\rho}\,\Phi^{-1}(99.9\%)}{\sqrt{1-\rho}} \tag{19}$$

For notation simplification let b= $\Phi^{-1}\big(B(\hat{\alpha}, \hat{\beta}, 99, 9\%)\big)$, p= $\Phi^{-1}(\overline{PD})$ and u= $\Phi^{-1}(99.9\%)$. We may rewrite equation (19) as:

$$b\sqrt{1-\rho} = p + u\sqrt{\rho} \tag{20}$$

$$\rho = \frac{(b^2+p^2)(b^2+u^2) - 2bp(bp + u\sqrt{(b^2+u^2-p^2)})}{(b^2+u^2)^2} \tag{21}$$

Equation (21) gives the value of the implicit assets correlation coefficient ρ.

1.2 Application on Nordea's portfolio

Due to the lack of data on losses rates we decided to apply this method using time series on observable default frequencies (ODF) also called ODF (annualised default frequencies) at Nordea. Two data sources are used:

1.2.1 Nordea dataset

Description of the data sets available:

- For Large corporates and institutions asset class:
•
- For SME asset class:
•
- For Retail asset class:
•

1.2.2 Hybrid dataset: Nordea and SCB

This data set is a mix of data coming from both Nordea and the Swedish statistics agency (SCB). The advantage is that it provides us with longer times series on ODF, however it doesn't sort the data by asset class.

Both of the dataset described here over are valid according to the Basel Committee (BIS 2006, part 2, paragraph 463) to apply the IRB approach for PD estimations. It is said that at least five years historical default data are required. Note that it is always preferable to use data coming from both bullish and bearish market trends to avoid underestimation and overestimation of the probability of default.

1.3 Results of the Fitch method

Based on our data sets and after implementation of the described method on SAS we got the following results for 2012:

ALL	ODF mean	ODF std d	$\hat{\alpha}$	$\hat{\beta}$	$\overline{PD_{12}}$	$\sqrt{\rho}$
Sweden	•	•	•	•%	•%	•%
Finland	•	•	•	•%	•%	•%
Norway	•	•	•	•%	•%	•%
Denmark	•	•	•	•%	•%	•%

Table 1: Implicit correlation asset/systematic factor for all asset classes, 2012

LCI	ODF mean	ODF std d	$\hat{\alpha}$	$\hat{\beta}$	$\overline{PD_{12}}$	$\sqrt{\rho}$	$\sqrt{\rho_{reg}}$
Sweden	•	•	•	•	•%	•%	•%
Finland	•	•	•	•	•%	•%	•%
Norway	•	•	•	•	•%	•%	•%
Denmark	•	•	•	•	•%	•%	•%

Table 2: Implicit correlation asset/systematic factor for LCI asset class, 2012

SME	ODF mean	ODF std d	$\hat{\alpha}$	$\hat{\beta}$	$\overline{PD_{12}}$	$\sqrt{\rho}$	$\sqrt{\rho_{reg}}$
Sweden	•	•	•	•	•%	•%	•%
Finland	•	•	•	•	•%	•%	•%
Norway	•	•	•	•	•%	•%	•%
Denmark	•	•	•	•	•%	•%	•%

Table 3: Implicit correlation asset/systematic factor for SME asset class, 2012

Retail	ODF mean	ODF std d	$\hat{\alpha}$	$\hat{\beta}$	$\overline{PD_{12}}$	$\sqrt{\rho}$	$\sqrt{\rho_{reg}}$
Sweden	•	•	•	•	•%	•%	•%
Finland	•	•	•	•	•%	•%	•%
Norway	•	•	•	•	•%	•%	•%
Denmark	•	•	•	•	•%	•%	•%

Table 4: Implicit correlation asset/systematic factor for Retail asset class, 2012

1.4 Analysis of the Fitch method

To analyse our correlation coefficient estimations we are going to give confidence intervals for each result. We use the so called "Bootstrap" methodology on each estimation to get a 95% confidenc interval.

1.4.1 Bootstrap algorithm

We are gooing to carry out the bootstrap method by means of a SAS macro based on the datasets prevously used. For each asset class, each country, and each estimation of PD (MLE or average ODF), we run the following algorithm:

1 We generate new time series of number of default a large number of times N[8]. Each sample has the same size[9] as the observed time serie (intital data) and is constructed by random draws with replacement from the empiric distribution F_n.

2 For each sample i (time series) we derive the maximum likelihood estimate of ρ denoted ρ_i^*, i=1,...,N. First with ML estimation of PD ($\widehat{PD}$) and then by using the yearly average PD derived from Nordea's dataset ($\overline{PD}$).

3 We put in order the ρ_i^* from the smallest to the biggest $\rho_{(1)}^* \leq \rho_{(2)}^* \leq \cdots \leq \rho_{(9999)}^* \leq \rho_{(10000)}^*$. Then it is possible to estimate both, the 0.025 and 0.975 quantiles of the distribution by picking respectively $\rho_{(250)}^*$ and $\rho_{(9750)}^*$, that is the 25^{th} smallest and the 25^{th} biggest values from ρ_1^*,...,ρ_N^*.

4 The confidence interval is then given by $CI_\rho := [\rho_{250}^* \; ; \rho_{9750}^*]$.

1.4.2 Note on the error coming from the bootstrap method

The error coming from the boostrap method is of two types:

1 The first source of error comes from the fact that we approximate the population distribution F with an empiric distribution F_n.

2 The second source of error comes from the fact that instead of computing our estimate for our emipric distribution F_n in particular, we do it for N=10000 time series resampled from F_n.

It is worth mentionning that the first source of error can be decreased if we increase the size of our time series. Whereas the second one can be decreased by incresaing N the number of itreations. The results are shown in the next section.

[8]We choose to run N=10000 iterations to reach an acceptable accuracy in our confidence intervals

[9]in our case the samples have size 7 or 5 (for finland)

1.4.3 Confidence intervals results

Country	**CI 95%**	**2010**	**2011**	**2012**	**2013**
Sweden	Upper	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%
	Lower	•%	•%	•%	•%
	Std Dev.	•%	•%	•%	•%
Finland	Upper	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%
	Lower	•%	•%	•%	•%
	Std Dev.	•%	•%	•%	•%
Norway	Upper	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%
	Lower	•%	•%	•%	•%
	Std Dev.	•%	•%	•%	•%
Denmark	Upper	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%
	Lower	•%	•%	•%	•%
	Std Dev.	•%	•%	•%	•%

Table 5: Implicit Correlation Coefficient for mixed assets from Hybrid dataset

GRAPH

(a) Icc ALL and 95 % CI, SWEDEN

GRAPH

(b) Icc ALL and 95 % CI, FINLAND

GRAPH

(c) Icc ALL and 95 % CI, NORWAY

GRAPH

(d) Icc ALL and 95 % CI, DENMARK

Figure 4: Estimation of the implicit correlation coef. for ALL asset class

Country	**CI 95%**	**2010**	**2011**	**2012**	**2013**
Sweden	Upper	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%
	Lower	•%	•%	•%	•%
	Std Dev.	•%	•%	•%	•%
Finland	Upper	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%
	Lower	•%	•%	•%	•%
	Std Dev.	•%	•%	•%	•%
Norway	Upper	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%
	Lower	•%	•%	•%	•%
	Std Dev.	•%	•%	•%	•%
Denmark	Upper	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%
	Lower	•%	•%	•%	•%
	Std Dev.	•%	•%	•%	•%

Table 7: Implicit Correlation Coefficient for LCI asset class

GRAPH

(a) Icc LCI and 95 % CI, SWEDEN

GRAPH

(b) Icc LCI and 95 % CI, FINLAND

GRAPH

(c) Icc LCI and 95 % CI, NORWAY

GRAPH

(d) Icc LCI and 95 % CI, DENMARK

Figure 5: Estimation of the implicit correlation coef. for LCI asset class

Country	**CI 95%**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**
Sweden	Upper	•%	•%	•%	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%	•%	•%	•%
	Lower	•%	•%	•%	•%	•%	•%	•%
	stddev	•%	•%	•%	•%	•%	•%	•%
Finland	Upper	•%	•%	•%	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%	•%	•%	•%
	Lower	•%	•%	•%	•%	•%	•%	•%
	stddev	•%	•%	•%	•%	•%	•%	•%
Norway	Upper	•%	•%	•%	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%	•%	•%	•%
	Lower	•%	•%	•%	•%	•%	•%	•%
	stddev	•%	•%	•%	•%	•%	•%	•%
Denmark	Upper	•%	•%	•%	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%	•%	•%	•%
	Lower	•%	•%	•%	•%	•%	•%	•%
	stddev	•%	•%	•%	•%	•%	•%	•%

Table 8: Implicit Correlation Coefficient for SME asset class

GRAPH

(a) Icc SME and 95 % CI, SWEDEN

GRAPH

(b) Icc SME and 95 % CI, FINLAND

GRAPH

(c) Icc SME and 95 % CI, NORWAY

GRAPH

(d) Icc SME and 95 % CI, DENMARK

Figure 6: Estimation of the implicit correlation coef. for SME asset class

Country	**CI 95%**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**
Sweden	Upper	•%	•%	•%	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%	•%	•%	•%
	Lower	•%	•%	•%	•%	•%	•%	•%
	stddev	•%	•%	•%	•%	•%	•%	•%
Finland	Upper	•%	•%	•%	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%	•%	•%	•%
	Lower	•%	•%	•%	•%	•%	•%	•%
	stddev	•%	•%	•%	•%	•%	•%	•%
Norway	Upper	•%	•%	•%	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%	•%	•%	•%
	Lower	•%	•%	•%	•%	•%	•%	•%
	stddev	•%	•%	•%	•%	•%	•%	•%
Denmark	Upper	•%	•%	•%	•%	•%	•%	•%
	Icc_AS	•%	•%	•%	•%	•%	•%	•%
	Lower	•%	•%	•%	•%	•%	•%	•%
	stddev	•%	•%	•%	•%	•%	•%	•%

Table 9: Implicit Correlation Coefficient for Retail asset class

(a) Icc Retail and 95 % CI, SWEDEN

(b) Icc Retail and 95 % CI, FINLAND

(c) Icc Retail and 95 % CI, NORWAY

(d) Icc Retail and 95 % CI, DENMARK

Figure 7: Estimation of the implicit correlation coef. for Retail asset class

This method gives quiet low estimations of the correlation coefficient in comparison to those provided by the reglementation. Applying these values would significantly reduce the RWA. However, it is important to keep in mind all the assuptions we made in this model to come up to these results.
We made the assumption that our portfolios were well diversified (more than 1000 assets) so that only the systematic risk remains, the idiosycratic risk beeing diversified away. In addition, we said that our portfolios were homogeneous, which allowed us to use the average PD as value for the PD parameter. We assume that the state of the macroeconomy is an independant random variable normally distributed. We chosed to fit the distributions with a Beta distribution, and we used short time series to estimate the parameters of the beta distribution. These short time series do not contain data from a whole economic cycle expected for the mixed dataset for Sweden and Denmark where data from 1982 are recorded.
It is worth mentionning that according to this model the SMEs seems to be more correlated to the systematic factor than the LCIs are, which is contradictory to the observations made in other studies. Furthermore we may observe that longer times series tend to yield larger levels of correlation asset systematic factor: •% for Sweden and •% for Denmark. Although all these assumptions are source of error, the model can give us an idea of the levels for the correlation coefficient with respect to Nordea's credit portfolio.
The Bootstrap method reveals relatively narrow confidence intervals at 95% of the estimations and the measured standard deviations never exceed 4%, which means that the volatility of the results is relatively low with this method.

GRAPH

(a) Fitted beta distribution of ODF in Sweden

GRAPH

(b) Fitted beta distribution of ODF in Denmark

GRAPH

(c) Fitted beta distribution of ODF in Norway

GRAPH

(d) Fitted beta distribution of ODF in Finland

Figure 8: Graphical representation of the fitted ODF distributions for LCI asset class

2 The Vasicek MLE method

Maximum likelihood estimation is the most widely used and the most important method of estimation. Practitioners in credit risk use this method with an application to the Vasicek model to determine an estimation of the asset correlation parameter. In this section we are going to carry out this method with Nordea's dataset and the mixed (SCB and Nordea) dataset.

2.1 Description of the method

Recall that in the Vasicek model, the probability of observing default conditional to a systematic factor S is as show in Equation 11. This equation can be written in term of correlation asset/systematic risk factor ($\sqrt{\rho}$) as follows:

$$CP_t(S) = \Phi\left(\frac{\Phi^{-1}(P_t) - \sqrt{\rho}S}{\sqrt{1-\rho}}\right) \tag{22}$$

In this approach, we assume that given a systematic factor S, the default occurrences are independent with each other over time, i.e. knowing that D_t borrowers goes default doesn't give any information about the number of defaults D_{t+1} we are going to observe next year. Each year, the probability of observing D defaults over N (total number of exposures in the credit portfolio) independent random draws follows a binomial distribution with probability CP_t(S).
The goal is to find the estimator $\hat{\rho_s}$ which is maximizing the chances to get exactly the same combination i.e. D defaults over N exposures for each year.
To do that let's define the Likelihood function L(ρ_s,**x**) for our past observations of defaults **x**:

$$L(\rho_s, \mathbf{x}) = \prod_{t=1}^{T} \int_{-\infty}^{\infty} \binom{N_t}{D_t} CP_t(S)^{D_t}(1 - CP_t(S))^{N_t - D_t}\, \mathrm{d}\Phi S \tag{23}$$

Note that in Equation 23 we integrate over the factor S because S is not observable and in consequence we assume that S is normally distributed N(0,1). In term of dW, Equation 23 may be rewritten as follows:

$$L(\rho_s, \mathbf{x}) = \prod_{t=1}^{T} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}.\mathrm{e}^{-\frac{S^2}{2}} \binom{N_t}{D_t} CP_t(S)^{D_t}(1 - CP_t(S))^{N_t - D_t}\, \mathrm{d}S \tag{24}$$

Now, we want to maximize this Likelihood function. As usual we maximise the log-likelihood function derived straight forward from Equation 25:

$$lnL(\rho_s, \mathbf{x}) = \frac{T}{\sqrt{2\pi}}.\sum_{t=1}^{T} ln \int_{-\infty}^{\infty} \binom{N_t}{D_t} CP_t(S)^{D_t}(1 - CP_t(S))^{N_t - D_t}\, \mathrm{e}^{-\frac{S^2}{2}} \mathrm{d}S \tag{25}$$

which is equivalent to maximizing:

$$lnL(\rho_s, \mathbf{x}) = \sum_{t=1}^{T} ln \int_{-\infty}^{\infty} CP_t(S)^{D_t}(1 - CP_t(S))^{N_t - D_t}\, \mathrm{e}^{-\frac{S^2}{2}} \mathrm{d}S \tag{26}$$

The maximum likelihood estimators $\widehat{PD}$ and $\hat{\rho}$ are thus described as:

$$(\widehat{PD}, \hat{\rho}) = arg\, max\, \big(lnL(\rho_s, \mathbf{x})\big) \tag{27}$$

2.2 Application on Nordea's credit portfolio

As in the previous section, we apply this method using time series on observed default frequencies (ODF). Two data sources are used:

2.2.1 Nordea dataset

Description of the data sets available:

- For Large corporates and institutions asset class:
•
- For SME asset class:
•
- For Retail asset class:
•

2.2.2 Hybrid dataset: Nordea and SCB

This data set is a mix of data coming from both Nordea and the Swedish statistics agency (SCB). The advantage is that it provides us with longer times series on ODF, however it doesn't sort the data by asset class.

After importing the different datasets on SAS, we implement the method by means of a SAS command called "proc nlmixed" which is a solver for optimization problems. As PD is a parameter taking part in the log likelihood function (see Equation 27) we may either choose to handle it as a parameter to be estimated, thus when maximizing the log likelihood function the solver will yield the most likely estimates for PD and for ρ_s; or we may handle PD as a given value (i.e. the average PD of each ODF samples) and enter it as a constant in the solver to get only one most likely estimate : ρ_s.
The two ways have been implemented and the results are shown in the next section.

2.3 Results of the Vasicek MLE method

It is worth notifying that the output of this method is amongst other things an estimation of the correlation coefficient between assets and the systematic risk factor S.
Hence, to get the correlation coefficient asset/asset it suffices to rise the output to the power of two.
The reason why we get two tables for each asset class is that we applied the method first by ML-estimating PD and ρ_s, and then by taking the average of PD as a constant in the likelihood function so that only ρ_s remains as a parameter to be estimated. These two ways are implemented using the same datasets used in the Fitch Method.

2.3.1 All asset classes, SCB and Nordea dataset

Country	$\widehat{PD}$	$\rho_{(AA)}$	$\sqrt{\rho}$
Sweden	•%	•%	•%
Finland	•%	•%	•%
Norway	•%	•%	•%
Denmark	•%	•%	•%

MLE of $\widehat{PD}$ and $\hat{\rho}$

Country	$\overline{PD_{all}}$	$\rho_{(AA)}$	$\sqrt{\rho}$
Sweden	•%	•%	•%
Finland	•%	•%	•%
Norway	•%	•%	•%
Denmark	•%	•%	•%

MLE of $\hat{\rho}$ only

2.3.2 LCI, Nordea dataset only

Country	$\widehat{PD}$	$\sqrt{\rho_{irb}}$	$\sqrt{\rho}$
Sweden	•%	•%	•%
Finland	•%	•%	•%
Norway	•%	•%	•%
Denmark	•%	•%	•%

MLE of $\widehat{PD}$ and $\hat{\rho}$

Country	$\overline{PD_{lci}}$	$\sqrt{\rho_{irb}}$	$\sqrt{\rho}$
Sweden	•%	•%	•%
Finland	•%	•%	•%
Norway	•%	•%	•%
Denmark	•%	•%	•%

MLE of $\hat{\rho}$ only

2.3.3 SME, Nordea dataset only

Country	$\widehat{PD}$	$\sqrt{\rho_{irb}}$	$\sqrt{\rho}$
Sweden	•%	•%	•%
Finland	•%	•%	•%
Norway	•%	•%	•%
Denmark	•%	•%	•%

MLE of $\widehat{PD}$ and $\hat{\rho}$

Country	$\overline{PD_{sme}}$	$\sqrt{\rho_{irb}}$	$\sqrt{\rho}$
Sweden	•%	•%	•%
Finland	•%	•%	•%
Norway	•%	•%	•%
Denmark	•%	•%	•%

MLE of $\hat{\rho}$ only

2.3.4 Retail, Nordea dataset only

Country	$\widehat{PD}$	$\sqrt{\rho_{irb}}$	$\sqrt{\rho}$
Sweden	•%	•%	•%
Finland	•%	•%	•%
Norway	•%	•%	•%
Denmark	•%	•%	•%

MLE of $\widehat{PD}$ and $\hat{\rho}$

Country	$\overline{PD_{rtl}}$	$\sqrt{\rho_{irb}}$	$\sqrt{\rho}$
Sweden	•%	•%	•%
Finland	•%	•%	•%
Norway	•%	•%	•%
Denmark	•%	•%	•%

MLE of $\hat{\rho}$ only

2.4 Analysis and evaluation of the method

As we did it previously we use the bootstrap method to give 95% confidence intervals for our estimates[10].
In addition we give the standard deviation of the estimations obtained after 10000 iterations.

[10]See Section 1.4.1 and 1.4.2 to have a description of the method.

2.4.1 Confidence interval results

For mixed assets (SCB and Nordea datasets):

ALL MLE (rho,PD)	**SWE**	**FIN**	**NOR**	**DEN**
Upper (Q9750)	•%	•%	•%	•%
MLE of $\sqrt{\rho}$	•%	•%	•%	•%
Lower (Q250)	•%	•%	•%	•%
Mean rho BS	•%	•%	•%	•%
Std dev rho BS	•%	•%	•%	•%

ALL MLE rho, $\overline{PD}$	**SWE**	**FIN**	**NOR**	**DEN**
Upper (Q9750)	•%	•%	•%	•%
MLE of $\sqrt{\rho}$	•%	•%	•%	•%
Lower (Q250)	•%	•%	•%	•%
Mean rho BS	•%	•%	•%	•%
Std dev rho BS	•%	•%	•%	•%

Table 10: 95 % confidence interval of the ML-estimate of $\sqrt{\rho_{all}}$

For LCI asset class (Nordea dataset):

LCI MLE(rho,PD)	**SWE**	**FIN**	**NOR**	**DEN**
Upper (Q9750)	•%	•%	•%	•%
MLE of $\sqrt{\rho}$	•%	•%	•%	•%
Lower (Q250)	•%	•%	•%	•%
Mean rho BS	•%	•%	•%	•%
Std dev rho BS	•%	•%	•%	•%

LCI MLE rho, $\overline{PD}$	**SWE**	**FIN**	**NOR**	**DEN**
Upper (Q9750)	•%	•%	•%	•%
MLE of $\sqrt{\rho}$	•%	•%	•%	•%
Lower (Q250)	•%	•%	•%	•%
Mean rho BS	•%	•%	•%	•%
Std dev rho BS	•%	•%	•%	•%

Table 11: 95 % confidence interval of the ML-estimate of $\sqrt{\rho_{lci}}$

For SME asset class (Noreda dataset):

SME MLE(rho,PD)	**SWE**	**FIN**	**NOR**	**DEN**
Upper (Q9750)	•%	•%	•%	•%
MLE of $\sqrt{\rho}$	•%	•%	•%	•%
Lower (Q250)	•%	•%	•%	•%
Mean rho BS	•%	•%	•%	•%
Std dev rho BS	•%	•%	•%	•%

SME MLE rho, $\overline{PD}$	**SWE**	**FIN**	**NOR**	**DEN**
Upper (Q9750)	•%	•%	•%	•%
MLE of $\sqrt{\rho}$	•%	•%	•%	•%
Lower (Q250)	•%	•%	•%	•%
Mean rho BS	•%	•%	•%	•%
Std dev rho BS	•%	•%	•%	•%

Table 12: 95 % confidence interval of the ML-estimate of $\sqrt{\rho_{sme}}$

For Retail asset class (Noreda dataset):

HH MLE (rho,PD)	**SWE**	**FIN**	**NOR**	**DEN**
Upper (Q9750)	•%	•%	•%	•%
MLE of $\sqrt{\rho}$	•%	•%	•%	•%
Lower (Q250)	•%	•%	•%	•%
Mean rho BS	•%	•%	•%	•%
Std dev rho BS	•%	•%	•%	•%

HH MLE rho, $\overline{PD}$	**SWE**	**FIN**	**NOR**	**DEN**
Upper (Q9750)	•%	•%	•%	•%
MLE of $\sqrt{\rho}$	•%	•%	•%	•%
Lower (Q250)	•%	•%	•%	•%
Mean rho BS	•%	•%	•%	•%
Std dev rho BS	•%	•%	•%	•%

Table 13: 95 % confidence interval of the ML-estimate of $\sqrt{\rho_{hh}}$

3 The Vasicek MLE method with time correlation

In this section we are going to describe a final method based on Vacisek MLE method, which may be used to estimate the asset correlation coefficient ρ. The difference with the previous method (described in Section 2) is that we introduce a time correlation between each systematic factor (S_t, S_{t+1}). Indeed, it is rational to think that there should exist some dependence between the economic state of year t and year t+1. An other argument for introducing time correlation is that the exposures included in credit portfolios are often the same over many years. Thus, The conditional PD should be correlated over time.

3.1 Description of the method

To carry out this method we assume that we have at our disposal a times series of ODF including for each observed year t, the number of defaults D_t and the total number of exposures denoted N_t for at least five years.
Furthermore, we assume that the portfolio is well diversified with at least one thousand exposures so that the effect of the idiosyncratic factor can be negligible, and we say that the PD is homogeneous for the whole credit portfolio, meaning that at a certain time, all exposures have the same probability of going default.
As mentioned in the short introduction in addition to the dependence of the borrowers' default behaviour, we now assume that there is some degree of dependence between the systematic factors at different time points.

Let PD denote the homogeneous probability of default of the credit portfolio.
(D_t,N_t,p_t) denotes the number of defaults observed year t, the total number of exposures observed year t and the observed default frequency of year t respectively. With $p_t = D_t \ / \ N_t$.
S_t denotes as usually the systematic factor of year t. As this parameter is not observable, we assume that it follows a standard normal distribution N(0,1). Now if we look at the realisation of a random path for S : $\{S_1, S_2, \ldots, S_T\}$ with T $\leq$ 5, we define the time correlation of the systematic factor by the covariance matrix of size T×T as :

$$\forall\, \theta \in [0,1), \quad C_\theta = \begin{bmatrix} 1 & \theta & \theta^2 & \cdots & \theta^{T-1} \\ \theta & 1 & \theta & \cdots & \theta^{T-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta^{T-2} & \cdots & \theta & 1 & \theta \\ \theta^{T-1} & \cdots & \theta^2 & \theta & 1 \end{bmatrix}$$

This covariance matrix says that the time correlation is exponentially decreasing with increasing time lag.
The asset correlation coefficient is as usually denoted ρ with $0 \leq \rho < 1$, and reflects the degree of dependence between each exposure. In other words this is a measure of sensibility of the portfolio to the systematic risk.
As in the previous methods we make use of the probability of default conditional on the systematic factor S_t at time t. CP_t is a function of PD, ρ, S and is denoted CP_t. Its formula is given by Equation 11 in section two of this paper.

Now, using Vasicek model, we may write the probability of observing D_1 defaults out of N_1 exposures at time t=1 and D_2 defaults out of N_2 exposures at time t=2, ... , D_T defaults out of N_T exposures at time T, given a realisation of systematic factors (S_1,S_2, ... ,S_T) as the joint probability function:

$$P\left[X_1 = D_1, X_2 = D_2, \cdots, X_T = D_T \mid S_1, S_2, \cdots, S_T\right] = \prod_{t=1}^{T} \binom{N_t}{D_t} CP_t^{D_t} (1 - CP_t)^{N_t - D_t} \quad (28)$$

Now, à priori we do not know the future realisations of the systematic factor $\{S_1, S_2, \cdots, S_T\}$. So let's take a look at the marginal density of the joint probability function:

P $[X_1 = D_1, X_2 = D_2, \cdots, X_T = D_T]$.

Recall that by definition for continuous distributions the marginal density function of X is given by the following formula:

$P_X(x) = \int_y P_{X,Y}(x, y)\ \mathrm{d}y = \int_y P_{X|Y}(x|y)\, P_Y(y)\ \mathrm{d}y$

As a consequence of it, if we marginalize out the variable S of our equation 28, we get that the marginal density of the joint distribution P $[X_1 = D_1, X_2 = D_2, \cdots, X_T = D_T]$ is as follows:

$$P\left[X_1 = D_1, X_2 = D_2, \cdots, X_T = D_T\right] = \int_{s_1}\int_{s_2} \cdots \int_{s_T} mNC_\theta(S_1, \cdots, S_T) \quad \prod_{t=1}^{T} \binom{N_t}{D_t} CP_t^{D_t} (1 - CP_t)^{N_t - D_t}\ \mathrm{d}(s_1, \cdots, s_T), \quad (29)$$

where mNC_θ denotes the multivariate normal density function with mean $\mu : [0, \cdots, 0]^{(1\times T)}$ and covariance matrix $C_\theta^{(T\times T)}$.

This function given in Equation 29 is nothing else than the multidimensional likelihood function of the joint probability $P\left[X_1 = D_1, X_2 = D_2, \cdots, X_T = D_T\right]$. Thus by maximizing it, we should be able to find out the most likely estimates for $\widehat{PD}$, $\hat{\rho}$ and $\hat{\theta}$:

$$(\widehat{PD}, \hat{\rho}, \hat{\theta}) = arg\,max \int_{s_1}\int_{s_2} \cdots \int_{s_T} mNC_\theta(S_1, \cdots, S_T) \quad \prod_{t=1}^{T} \binom{N_t}{D_t} CP_t^{D_t} (1 - CP_t)^{N_t - D_t}\ \mathrm{d}(s_1, \cdots, s_T) \quad (30)$$

In the next part we apply this method to Nordea's credit portfolio.

3.2 Application on Nordea's credit portfolio

The data set is the same as those used in the previous sections.

3.3 Issues encountered in the optimization problem

Implementing the optimization of Equation 30 on SAS presents computational issues. If we want to apply this method on Nordea's dataset, we realize quickly that we need to solve a high-dimentional integral with a non-linear integrand. This kind of computation is very demanding and could give unaccurate results due to computational instability. Thus, using proc nlmixed on SAS to solve this problem might jeopardize the outcome.

3.3.1 Computational stability of the problem

First we simplify the problem of Equation 30 to a bi-dimentional problem. The aim is to give a representation of the solution surface of the double integral described here under to get an idea about the computational stability of the problem.

$$(\hat{\rho}, \hat{\theta}) = arg\,max \int_{s_1}\int_{s_2} mNC_\theta(S_1, S_2) \quad \prod_{t=1}^{2} \binom{N_t}{D_t} CP_t^{D_t}(1 - CP_t)^{N_t - D_t} \ \mathrm{d}(s_1, s_2),$$

with C_θ the 2×2 covariance matrix.

Notice that here we do not ML-estimate PD. Instead we let the parameter PD take different constant values and we try to get ML-estimations of rho and theta.
If the surface shows some pic, then we can affirme that the two dimentional problem should be computationaly stable. On the contrary, if the surface is smooth and do not reveal any clear pic, we can already warn on stability issues to solve Equation 30. In the next section we propose an alternative method to bypass the unstability issues.

3.3.2 Alternative method to approximate the solution

1 Find a simplified expression for $P[D \mid S = y] = CP_t(S = y)$[11] by means of a Taylor expension. So that we get a simplified expression as $CP_t(S = y) = a_t + b_t \times S_t + rest$
a_t and b_t depend on P_t (the probability of default at time t) and ρ.

2 Replace the CP_t of Equation 29 with this the simplified expression obatined in 1 and let A be the product:

$A = \prod_{t=1}^{T} \binom{N_t}{D_t} \big(a_t + b_t S_t\big)^{D_t} \big(1 - (a_t + b_t S_t)\big)^{N_t - D_t}$

3 Observe that Equation 30 is nothing else than the the expected value of A. Write E[A] as a polynom:

E[A]= $\sum_{t=1}^{T} a_{k1,\cdots,kT}^{(P_t,\rho)} \times b_{k1,\cdots,kT}^{(P_t,\rho)} \ E\big[S_1^{k1} \times S_2^{k2} \times \cdots \times S_T^{kT}\big]$

with $E\big[S_1^{k1} \times S_2^{k2} \times \cdots \times S_T^{kT}\big]$ the k1, k2, ... kT moments of the multivariate normal distribution $N(0, C_\theta)$.

4 By applying points 1 - 3 we get an explicit function of the likelihood function given in Equation 30. The final step is to compute the value of this function with different values for the three parameters: θ, P_t and ρ.
For example we can take:

[11] the conditional probability of default given the systematic factor S=y (see Equation 22)

$\theta = [0 \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \quad 0.8 \quad 0.9];$
$P_t = [0 \quad 0.005 \quad 0.01 \quad 0.015 \quad 0.02 \quad 0.025 \quad 0.03 \quad 0.035 \quad 0.04 \quad 0.045 \quad 0.05];$
$\rho = [0 \quad 0.05 \quad 0.1 \quad 0.15 \quad 0.2 \quad 0.25 \quad 0.3 \quad 0.35 \quad 0.4 \quad 0.45];$

We retain the three values $\hat{\theta}, \hat{P}_t, \hat{\rho}$ which maximize the likelihood function as ML-estimates for Equation 30.

Part IV
Conclusion

This study aims at estimating the asset correlation coefficient of the RWA formula by using empirical methods. After describing the Basel II framework, we introduce the notion of economic capital and the RWA formula. This formula depends on four parameters, the PD, the LGD, the EAD and the correlation coefficient assets / systematic risk.
By default the regulation sets this correlation coefficient at some predetermined levels for each asset class and provides the same formula for all credit institutions. Risk analysts have been investigating some methods to estimate the asset correlation coefficient with respect to their own credit portfolios, having in mind that the levels obtained should be quiet different from the ones provided by the Basel II framework. In fact by using their own asset correlation estimates, it will be possible for banks to assess their credit risk more accurately and in turn to allocate the good level of capital as credit risk buffer.
We apply three of these methods on Nordea's credit portfolio and compute the estimates of this coefficient for each country and asset class. The first method, which was published by Fitch Ratings, proposes to fit the observed default frequencies observed by Nordea by a Beta distribution in order to compare the conditional VAR and in turn to get an implicit estimation of the parameter. The second one is based on the Vacisek single factor model and proposes a ML-estimation of the parameter. The third one is similar to the second method but introduces a time correlation between the systematic factor levels.
For each method we give a 95% confidence interval by means of the bootstrap method and we discuss the relevance of the estimates obtained.
All the methods are carried out thanks to SAS foundation and SAS enterprise guide. The implementation of the two first methods are relatively easy in comparison to the implementation of the third method, which is quiet complex. We tried to implement the last one with proc nlmixed but due to an uncertainty about proc nlmixed stability in such complex computation, we decided not to retain the results. Instead we discuss the main computational issues encountered, and try to give an alternative to approximate the estimates solving the problem.

References

1 Basel III: A Global Regulatory Framework for more Resilient Banks and Banking Systems (Dec 2010, Rev Jun 2011).

2 Basel III Correlations: An Empirical Analysis Reflecting the Financial Crisis (Nov 2011).

3 Investigating Implied Asset Correlation and Capital Requirements: Empirical Evidence from the Italian Banking System (2011).

4 Default Correlation: Empirical Evidence, Working Paper, Standard and Poors, Servigny and Renault (2002).

5 The Empirical Relationship Between Average Asset Correlation, Firm Probability of Default, and Asset Size, Lopez J.A. (2004).

6 Modeling and Estimating Default Correlations with the Asset Value Approach, Laffler Posch

7 Bayesian Estimation of Probabilities of Default for Low Default Portfolios, Dirk Tasche (2012).

8 Arbitrage Theory in Continuous Time, Thomas Björk (2009).

9 Stokastik: Sannolikhetsteori och statistikteori med tillmpningar, Sven Erik Alm & Tom Britton (2008).

10 The Little SAS Book: A Primer, Third Edition Lora D. Delwiche & Susan J. Slaughter.]