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Pricing the American Option Using Itô's Formula and Optimal Stopping Theory

Jonas Bergström

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Department of Mathematics
Uppsala University

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JONAS BERGSTRÖM

Abstract. In this thesis the goal is to arrive at results concerning the value of American options and a formula for the perpetual American put option. For the stochastic dynamics of the underlying asset I look at two cases. The first is the standard Black-Scholes model and the second allows for the asset to jump to zero i.e default. To achieve the goals stated above the first couple of sections introduces some basic concepts in probability such as processes and information. Before introducing Itô's formula this paper contains a not too rigorous introduction of stochastic differential equations and stochastic integration. Then the Black-Scholes model is introduced followed by a section about optimal stopping theory in order to arrive at the American option.

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1. Introduction

This section contains the definition of one of the most important building blocks in continuous probability namely Wiener process. This is followed by definitions of information and martingales.

Definition 1.1. *A Wiener process, $W = \{W_t; t \geq 0\}$, starting from $W_0 = 0$ is a continuous time stochastic process taking values in $\mathbb{R}$ s.t*

- *W has independent increments i.e $W_v - W_u$ and $W_t - W_s$ are independent whenever $u \leq v \leq s \leq t$*
- *$W_{s+t} - W_s \sim N(0, t)$*

Definition 1.2. *The symbol $\mathcal{F}_t^W$ denotes the information generated by the process $W = \{W_s; s \geq 0\}$ for $s \in [0, t]$. If Y is a stochastic process s.t $Y_t \in \mathcal{F}_t^W$, then Y is said to be adapted to the filtration $\mathcal{F}_t^W$. This simply means that Y can be observed at time t.*

For example let $Y_s := e^{-rs} W_s$. Then $Y_t \in \mathcal{F}_t^W$. This is because, given the trajectory of W between 0 and t, the value of Y can be determined.
If we define $Y_s := e^{-rs} W_{s+\epsilon}, \epsilon > 0$ then $Y_t \notin \mathcal{F}_t^W$ since $W_{t+\epsilon}$ exists in the "future" beyond our information at time t.

Definition 1.3. (Martingale) *The process $X_t \in \mathcal{F}_t^W$ is called a Martingale if*

- $E[|X_t|] < \infty$
- $E[X_t|\mathcal{F}_s^W] = E[X_s], \forall s \leq t$

$X_t \in \mathcal{F}_t^W$ *is called a* *Submartingale* *if* $E[X_t|\mathcal{F}_s^W] \geq E[X_s], \forall s \leq t$.
$X_t \in \mathcal{F}_t^W$ *is called a* *Supermartingale* *if* $E[X_t|\mathcal{F}_s^W] \leq E[X_s], \forall s \leq t$.

2. Stochastic Differential Equations

Let X_t be a stochastic process that resembles the value of an asset. What is a reasonable way to mathematically construct price evolutions in continuous time? That is, what can be said about $dX_t = X_{t+dt} - X_t$?

One can assume that X should change proportionally with the increment of time, dt. Driven by the assets fundamental values and market expectation, dt, should be amplified by a deterministic function of the asset. Call this function $u(X_t)$. Thus one arrives at

$$dX_t = u(X_t)dt.$$

To make this model more realistic one also adds a non-deterministic term $dW_t = W_{t+dt} - dW \sim N(0, dt)$ that is amplified by a deterministic function σ that depends on the same variables as u. The result is what is called a stochastic differential equation (SDE) that describes the local dynamics of a stochastic process in continuous time

$$\begin{cases} dX_t = u(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x \end{cases}$$

were the solution to this system is

$$X_t = x + \int_0^t u(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s$$

3. stochastic integrals

This section is devoted to give a good interpretation of

$$\int_0^t f(s)\, dW_s.$$

If one assumes that f is a simple function over $[0, t]$, meaning that $[0, t]$ can be split in smaller intervals were f is equal to some constant on respective intervals, then one can formulate the stochastic integral as

$$\int_0^t f(s)\, dW_s = \sum_{k=0}^{n-1} f(t_k)(W_{t_{k+1}} - W_{t_k})$$

where $0 = t_0 < ... < t_n = t$.

For a non-simple f one creates a sequence of simple functions, f_n with certain properties s.t

$$\int_0^t f(s)\,dW_s = \lim_{n\to\infty}\int_0^t f_n(s)dW_s.$$

Proposition 3.1. *For any process f, with conditions*

- $E[f^2] < \infty$
- *$f(\tau)$ is adapted to $\mathcal{F}_\tau^W$*

then

$$E[\int_s^t f(\tau)dW_\tau|\mathcal{F}_s^W] = 0$$

Proof. In this proof one assumes that f is simple because the full proof is outside of the scope of this thesis. From the law of iterated expectations it is true, for $s < t$, that

$$E[E[\int_s^t f(\tau)dW_\tau|\mathcal{F}_t^W]|\mathcal{F}_s^W] = E[\int_s^t f(\tau)dW_\tau|\mathcal{F}_s^W]$$

Now looking at the left-hand side inside the first expectation it follows that

$$E[\int_s^t f(\tau)dW_\tau|\mathcal{F}_t^W] = \sum_{k=0}^{n-1} E[f(\tau_k)(W_{\tau_{k+1}} - W_{\tau_k})|\mathcal{F}_t^W]$$

where $s = \tau_0 < ... < \tau_n = t$.

Since $f(\tau_k)$ depends on the value of the process W from s to τ_k (which are all given by $\mathcal{F}_t^W$, for $\forall k$ s.t $0 \le k \le n$). Because of independent increments $W_{\tau_{k+1}} - W_{\tau_k}$ does not depend on the interval $[s, \tau_k]$ and hence is independent of $f(\tau_k)$. Then it follows that

$$E[f(\tau_k)(W_{\tau_{k+1}} - W_{\tau_k})|\mathcal{F}_t^W] = E[f(\tau_k)|\mathcal{F}_t^W]E[W_{\tau_{k+1}} - W_{\tau_k}|\mathcal{F}_t^W] = 0$$

$$\Rightarrow E[\int_s^t f(\tau)dW_\tau|\mathcal{F}_t^W] = 0 \Rightarrow E[\int_s^t f(\tau)dW_\tau|\mathcal{F}_s^W] = E[0|\mathcal{F}_s^W] = 0$$

□

Theorem 3.2. *Assume that*

$$dX_t = u(X_t)dt + \sigma(X_t)dW_t.$$

If $u = 0$ $\mathbb{P}$-a.s $\forall t \Rightarrow X_t$ is a martingale.

Proof. $dX_t = u(X_t)dt + \sigma(X_t)dW_t$ have the solution

$$X_t = X_s + \int_s^t u(X_\tau)\,d\tau + \int_s^t \sigma(X_\tau)dW_\tau, s \le t.$$

Taking expected value yields

$$\begin{aligned} E[X_t|\mathcal{F}_s^W] &= E[X_s|\mathcal{F}_s^W] + E[\int_s^t u(X_\tau)\,d\tau|\mathcal{F}_s^W] + E[\int_s^t \sigma(X_\tau)dW_\tau|\mathcal{F}_s^W] \\ &= X_s + E[\int_s^t u(X_\tau)\,d\tau|\mathcal{F}_s^W] \\ &= X_s \end{aligned}$$

□

4. Itô's Formula

Itô's formula helps to give a description of the local dynamics of a stochastic process that is a function of an underlying process with a given stochastic differential.

Theorem 4.1. (Itô's Formula) *Assume that the stochastic process* $X = \{X_t; t \geq 0\}$ *has the differential* $dX_t = u(X_t)dt + \sigma(X_t)dW_t$ *where* u,$\sigma \in \mathcal{F}_t^W$. *Define a new stochastic process* $f(t, X_t)$, *where* f *is assumed to be smooth. Given the multiplication table*

$$\begin{cases} (dt)^2 = 0 \\ dt \cdot dW_t = 0 \\ (dW_t)^2 = dt \end{cases}$$

the stochastic differential for f *becomes*

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 \tag{1}$$

or equivalently

$$df = (\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2})dt + \sigma\frac{\partial f}{\partial x}dW_t$$

Proof. To prove that (1) holds for all t we look at the taylor expansion around the fixed point (t, x).

$$f(t+h, x+k) = f(t, x)+h\frac{\partial f}{\partial t}(t, x)+k\frac{\partial f}{\partial x}(t, x)+\frac{1}{2}(h^2\frac{\partial^2 f}{\partial t^2}+2hk\frac{\partial^2 f}{\partial t\partial x}+k^2\frac{\partial^2 f}{\partial t^2})+I$$

$$I = \frac{1}{3!}(h\frac{\partial}{\partial t} + k\frac{\partial}{\partial x})^3 f(t + sh, x + sk)$$

where $0 < s < 1$. Now let $h \to dt$ and $k \to dX_t$ to get

$$f(t+dt, x+dX_t) = f(t, x)+\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial x}dX_t+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2+\frac{\partial^2 f}{\partial t\partial x}dt{\cdot}dX_t+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2+I$$

Since $(dt)^2 = 0$, $dt \cdot dX_t = dt(udt + \sigma dW_t) = 0$ and the fact that $(\frac{\partial}{\partial t}dt + \frac{\partial}{\partial x}dX_t)^3 = 0$ it follows that

$$df = f(t+dt, x+dX_t) - f(t,x) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

Now we obtain

$$(dX_t)^2 = u^2(dt)^2 + 2u\sigma dt \cdot dW_t + \sigma^2(dW_t)^2 = \sigma^2 dt$$

$$\Rightarrow df = (\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2})dt + \sigma\frac{\partial f}{\partial x}dW_t$$

□

Theorem 4.2. (Feynman-Kač) *Assume that F solves the boundary value problem*

$$\begin{cases} \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} - rF = 0 \\ F(T,s) = \Phi(s). \end{cases}$$

Also assume that $E[|\sigma(X_s)\frac{\partial F}{\partial x}(s,X_s)e^{-rs}|^2] < \infty$ and $\sigma(X_t)\frac{\partial F}{\partial x}e^{-rt}$ is adapted to $\mathcal{F}_t^W$.
Assume that X_s is the solution to

$$\begin{cases} dX_s = u(X_s)ds + \sigma(X_s)dW_s \\ X_t = x \end{cases}$$

Then it follows that

$$F(t,x) = e^{-r(T-t)}E_{t.x}[\Phi(X_T)]$$

where

$$E_{t,x}[\ .\ ] = E[\ .\ |X_t = x].$$

Proof. To prove this define a new stochastic process

$$f(s,X_s) = e^{-rs}F(s,X_s)$$

and use Itô's formula.

$$df = e^{-rt}(-rF + \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2})dt + \sigma e^{-rt}\frac{\partial F}{\partial x}dW_t$$

where

$$u = u(X_t), \sigma = \sigma(X_t), F = F(t,X_t)$$

etc. The solution to this SDE is

$$f(T,X_T) = f(t,X_t) + \int_t^T e^{-rs}(-rF + \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2})ds + \int_t^T \sigma e^{-rs}\frac{\partial F}{\partial x}\,dW_s$$

where the integrands are evaluated at (s,X_s).

Since F solves the PDE above it follows that

$$f(T,X_T) = f(t,X_t) + \int_t^T \sigma e^{-rs}\frac{\partial F}{\partial x}\,dW_s \Leftrightarrow e^{-rT}F(T,X_T) = e^{-rt}F(t,x) + \int_t^T \sigma e^{-rs}\frac{\partial F}{\partial x}\,dW_s$$

$$\Leftrightarrow F(t,x) = e^{-r(T-t)}F(T,X_T) - \int_t^T \sigma e^{-r(s-t)}\frac{\partial F}{\partial x}\,dW_s$$

Taking expected values yields

$$F(t,x) = e^{-r(T-t)}E[\Phi(X_T)]$$

□

5. Asset Dynamics in the Black-Scholes model

Within the framework of the Black-Scholes(BS) model there exist two assets: A stock and a bond.

5.1. **Bond.** Bonds have the following dynamics in the BS-model:

$$dB_t = rB_t dt$$

Note that this is equivalent to

$$\frac{dB_t}{dt} = rB_t \Leftrightarrow B_t = B_0 e^{rt}$$

and that the local rate of return is equal to:

$$\frac{dB_t}{B_t \cdot dt} = r$$

5.2. **Stock.** Stocks are said to have the following dynamics in the BS-model:

$$dX_t = uX_t dt + \sigma X_t dW_t.$$

Here the constants u and σ are, respectively, the local mean of return and the volatility of X_t. In contrast to a bond the local rate of return on a stock is stochastic:

$$\frac{dX_t}{X_t \cdot dt} = u + \sigma\frac{dW_t}{dt}$$

Theorem 5.1. *The solution to the equation*

$$\begin{cases} dX_t = rX_t dt + \sigma X_t dWt \\ X_0 = x_0 \end{cases}$$

is given by $X_t = x_0 \exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t)$ *and in addition we have* $E[X_t] = x_0 e^{rt}$

Note: Here the drift term of X_t is r rather than u. This is because of so called "risk neutral valuation" which is explained in theorem 6.2.

proof
Use Itôs formula on $f(t, W_t) = x_0 \exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t)$.

$$f(t,x) = x_0 \exp((r - \frac{1}{2}\sigma^2)t + \sigma x)$$

$$\frac{\partial f}{\partial t} = (r - \frac{1}{2}\sigma^2)f(t,x), \frac{\partial f}{\partial x} = \sigma f(t,x), \frac{\partial^2 f}{\partial x^2} = \sigma^2 f(t,x)$$

$$\Rightarrow df = f(t,x)((r - \frac{1}{2}\sigma^2)dt + \sigma dW_t + \frac{1}{2}\sigma^2 (dW_t)^2)$$

From the multiplication table in theorem (4) it follows that

$$df = f(t,x)(rdt + \sigma dW_t) \Leftrightarrow dX_t = rX_t dt + \sigma X_t dW_t$$

To prove the second claim rewrite the expression above as

$$X_t = x_0 + \int_0^t rX_\tau \, d\tau + \int_0^t \sigma X_\tau \, dW_\tau$$

Now take expected values to get

$$E[X_t] = x_0 + r\int_0^t E[X_\tau]\, d\tau$$

$$m(s) := E[X_s] \Rightarrow m(t) = x_0 + r\int_0^t m(\tau)\, d\tau$$

$$\Rightarrow \frac{dm}{dt} = rm(t) \Rightarrow m(t) = e^{rt}m(0) = e^{rt}x_0$$

6. Options and Option Pricing

An option is a contract derived from some underlying asset which gives the holder of the contract the right (but not the obligation) to buy or sell the underlying asset for a determined amount, called the exercise price. The option is called European if the contract can only be used at a specific time in the future, called maturity. If the option can be exercised at any time between today and maturity then the option is called American. The right to buy is called a call option and the right to sell is called a put option.

Definition 6.1. *A contingent claim, χ, is a random variable that is adapted to $\mathcal{F}_T^X$ where T is called the maturity, i.e at time T the payoff of χ can be determined by looking at process $X = \{X_t; t \geq 0\}$.*
A claim is said to be "simple" if $\chi = \Phi(X_T)$, where Φ is called a "contract-function". The call option and the put option are two simple contingent claims who has contract functions $\max\{x - K, 0\}$ *and* $\max\{K - x, 0\}$ *respectively, where K is the strike price.*

To price an option one first assumes that the market where financial assets are traded is complete, meaning that every payoff structure of a contract can be replicated using bonds and stocks.

Another important assumption is the absence of arbitrage opportunities in the market. This assumption leads to the concept of a so called risk free measure when taking expected values of asset prices in the BS-model.

Theorem 6.2. *Let X_t resembles a stock price with dynamics*

$$dX_t = uX_t dt + \sigma X_t dW_t.$$

Assume that X_t is traded at a complete market that is free of arbitrage. If one takes expected values, X_t must have the following dynamics

$$dX_t = rX_t dt + \sigma X_t dW_t$$

in order to make the market free of arbitrage. This is called a risk free measure.

The theorem says that if, while taking an expected measure, the drift term is anything other then r there will exist an arbitrage opportunity.

Proof. $f(t, X(t)) := e^{-rt}X_t$ where X_t has the following dynamics when taking expected values

$$dX_t = \hat{u}X_t dt + \sigma X_t dW_t.$$

In this proof one can assume that $\hat{u} > r$.

By Itô's formula one arrives at

$$df = (\frac{\partial f}{\partial t} + \hat{u}X_t\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 f}{\partial x^2})dt + \sigma\frac{\partial f}{\partial x}dW_t$$

which has the solution

$$e^{-rt}X_t = e^{-r\tau}X_\tau + \int_\tau^t (\frac{\partial f}{\partial t} + \hat{u}X_s\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 X_s^2 \frac{\partial^2 f}{\partial x^2})ds + \int_\tau^t \sigma\frac{\partial f}{\partial x}dW_s$$

By replacing X_t by the deterministic function x it follows that

$$f(t, x) := e^{-rt}x, \frac{\partial f}{\partial t} = -re^{-rt}x, \frac{\partial f}{\partial x} = e^{-rt}, \frac{\partial^2 f}{\partial x^2} = 0.$$

Inserting this into the integrands in the expression above yields

$$e^{-rt}X_t = e^{-r\tau}X_\tau + \int_\tau^t (\hat{u} - r)xe^{-rs}\, ds + \int_\tau^t \sigma e^{-rs}dW_s$$

and by taking expectations it follows that

$$E[e^{-r(t-\tau)}X_t|\mathcal{F}_\tau] = X_\tau + (\hat{u} - r)\int_\tau^t xe^{-r(s-\tau)}\, ds > X_\tau.\forall\tau \leq t \tag{2}$$

The left hand side can be seen as the value of a contract at time τ that gives X_t at time t. The expression says that this contract is greater than X_τ even though they have the same payoff structure.

Now it exist an arbitrage opportunity in the market. By shortening the contract at time τ, one can immediately buy X_τ while still having capital at ones disposal. At time t one closes the short by selling the stock at value X_t and pay for the value of the contract at time t wich is X_t hence one makes a risk free profit.

If however we had the risk free measure i.e $\hat{u} = r$ then (2) becomes

$$E[e^{-r(t-\tau)}X_t|\mathcal{F}_\tau] = X_\tau$$

which is a correct pricing of such a contract. Hence no arbitrage opportunity exists.

□

Note: The last expression in the proof says that standing at time τ the value of a stock is equal to the discounted expected value of the stock given all information available at time τ. This is also in line with the so called efficient-market hypothesis.

7. The black-Scholes equation

To price a contingent claim one first makes a couple of financial and mathematical assumptions. First we assume that the market is free of arbitrage opportunities and that it is complete. Another financial assumption is that every portfolio consisting of bonds and stocks are so called self financed meaning that every new trades of acquisitions of assets must be financed by selling parts of the portfolio. No exogenous inflow or outflow of capital is allowed.

Here we are also dealing with simple contingent claims meaning that the contract only depends on the value of the asset at maturity, T i.e.
$\chi = \Phi(X_T)$.

Now one defines a stochastic process $V(t, S_t)$ as the value of the contingent claim χ at time $t < T$ and adds the mathematical assumption that $V \in C^{1,2}$.

Now if one uses Itô's formula and introduces dynamics for portfolios and then uses certain hedging positions (see [1]) and the above assumptions one can derive that

$$\begin{cases} \frac{\partial V}{\partial t} + rX_t\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 V}{\partial x^2} - rV = 0 \\ V(T, X_T) = \Phi(X_T) \end{cases} \tag{3}$$

which according to Feynman-Kač has the solution

$$V(t, X_t) = e^{-r(T-t)}E[\Phi(X_T)].$$

If one replaces the stochastic variables with deterministic ones (3) becomes

$$\begin{cases} \frac{\partial V}{\partial t}(t,x) + rx\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t,x) - rV(t,x) = 0 \\ V(T, x) = \Phi(x) \end{cases}$$

which is called the Black-Scholes equation.

8. Optimal Stopping Theory

This section contains a simplified examination of optimal stopping theory. For a more rigorous analysis of optimal stopping theory see [3].

Definition 8.1. (Stopping time)
A random variable $\tau : \Omega \to [0, \infty]$ *is called a Markov time if*
$\{\tau \leq t\} \in \mathcal{F}_t \ \forall t \geq 0$. *A Markov time is called a stopping time if* $\tau < \infty$.

Note: $\{\tau \leq t\} \in \mathcal{F}_t$ means that we can determine, standing at time t if the "stopping event" has occurred or not.

8.1. **Example.** Let X_t be a process in discrete time which is adapted to $\mathcal{F}_t$. Let $\tau := \inf\{n \leq 0; X_n \in A\}$ i.e τ measures the time when the process takes values in A for the first time.

Here τ is a stopping time because: $\{\tau \leq n\} = \{w \in \Omega; \tau(w) \leq n\}$ for some probability space Ω and

$$\begin{aligned}\{w \in \Omega; \tau(w) \leq n\} &= \{w \in \Omega; X_t(w) \in A, \text{ for some } t \leq n\} \\ &= \bigcup_{t=1}^{n} \{X_t \in A\}.\end{aligned}$$

Since X_t is adapted $\Rightarrow \{X_t \in A\} \in \mathcal{F}_t \subseteq \mathcal{F}_n \Rightarrow \{\tau \leq n\} \in \mathcal{F}_n$

8.2. **Optimal stopping strategy in continuous time.** Assume one is holding a contingent claim at time t that expires at a future time T. The value of the contingent claim is assumed to depend on a stock, X_t, with the risk-neutral dynamics

$$dX_t = rX_t dt + \sigma X_t dW_t$$

Within the time interval $[t, T]$ one has the option to either exercise or continue to hold the contingent claim. Standing at a fixed point in time, s, the value of exercising the contingent claim at that time is $\Phi(X_s)$ were Φ is a contract function. Here, depending on the price levels of X_s and the nature of Φ, s might not be the optimal time to "stop" i.e exercising the contingent claim.

To analyze this problem further one defines

$$V(t, x) = \sup_{t \leq \tau \leq T} E_{t,x}[e^{-r(\tau - t)}\Phi(\tau, X_\tau)].$$

So standing at (t, x), were $x = X_t$, $V(t, x)$ is by definition the optimal value one can achieve from exercising the contingent claim at some time in the future between t and T.
Let $\hat{\tau}_t \in [t, T]$ be the stopping time that satisfies

$$V(t, x) = E_{t,x}[e^{-r(\hat{\tau}_t - t)}\Phi(X_\tau)].$$

Then $\hat{\tau}_t$ is the optimal stopping time when standing at (t, x).
Now the question is to find certain conditions that $V(t, x)$ must satisfy.

Let s be the smallest element in $[t, T]$ s.t $V(s, X_s) = \Phi(X_s)$. If $V(s, X_s) = \Phi(X_s)$ it means that the optimal value one can receive is when exercising

the contingent claim at time s which is to say that $\hat{\tau}_t = s$.
This leads to a better definition of $\hat{\tau}_t$ namely

$$\hat{\tau}_t = \inf\{t \leq s \leq T; V(s, X_s) = \Phi(X_s)\}.$$

So the optimal strategy is to hold the contingent claim until some time $\hat{\tau}_t$ when $V(\hat{\tau}_t, X_{\hat{\tau}_t}) = \Phi(X_{\hat{\tau}_t})$. This leads to that $V(s, X_s) > \Phi(X_s)\ \forall s < \hat{\tau}_t$ which leads to the inequality

$$V(\tau, X_\tau) \geq \Phi(X_\tau), \forall \tau \in [t, T].$$

To reach new conclusions on the behavior of $V(t, x)$ we use Itô's formula on the process $fe^{-rt}V(t, X_t)$

$$d(e^{-rt}V(t, X_t)) = e^{-rt}(-r + \mathcal{L})V(t, X_t)dt + e^{-rt}\sigma\frac{\partial V}{\partial x}dW_t$$

where $\mathcal{L} = \frac{\partial}{\partial t} + rX_t\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 X_t^2\frac{\partial^2}{\partial x^2}$.

The solution to this SDE is

$$e^{-r(t+h)}V(t+h, X_{t+h}) = e^{-rt}V(t, x) + \int_t^{t+h} e^{-rs}(-r+\mathcal{L})V(s, X_s)\,ds + \int_t^{t+h} e^{-sr}\sigma\frac{\partial V}{\partial x}\,dW_s$$

where $\mathcal{L} = \frac{\partial}{\partial t} + rX_s\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 X_s^2\frac{\partial^2}{\partial x^2}$.

Multiplying both sides with e^{rt} and taking expectations yields

$$\text{(4)}\quad e^{-rh}E_{x,t}[V(t+h, X_{t+h})] = V(t, x) + E_{t,x}[\int_t^{t+h} e^{-r(s-t)}(-r + \mathcal{L})V ds].$$

Now standing at (t, x), what happens to V if it is not optimal to stop i.e $V(t, x) > \Phi(t, x)$. This means that the optimal value is in the future. So instead of exercising the contingent claim at t one holds it till $t + h$. Here we also let h tend to zero, to minimize loss of information of V between t and $t+h$. This means that the value of V must be the discounted expected value of this future V.

$$V(t, x) = e^{-rh}E_{t,x}[V(t + h, X_{t+h})]$$

Inserting this into (4) gives

$$E_{t,x}[\int_t^{t+h} e^{-r(s-t)}(-r + \mathcal{L})V(s, X_s)ds] = 0.$$

If one divides this expression with h and continue to let h tend to zero it will result in the integrand evaluated at t. This means that

$$\mathcal{L}V(t, x) = rV(t, x).$$

Alternatively if we had $V(t, x) = \Phi(t, x)$ it would not be optimal to continue to hold the contingent claim till $t+h$. This leads to the strict inequality

$$V(t, x) > e^{-rh}E_{t,x}[V(t + h, X_{t+h})].$$

Inserting this into (4) yields

$$\mathcal{L}V(t,x) < rV(t,x).$$

Proposition 8.2. *Assume that V is enough differentiable and* $V(t,x) = \sup_{t\leq\tau\leq T} E_{t,x}[e^{-r(\tau-t)}\Phi(\tau, X_\tau)]$.

Define the region $C := \{(t,x); V(t,x) > \Phi(t,x)\}$ then the following holds:

$$\begin{cases} V(T,x) = \Phi(T,x) \\ \mathcal{L}V(t,x) = rV(t,x) & \forall (t,x) \in C \\ \mathcal{L}V(t,x) < rV(t,x) & \forall (t,x) \notin C \end{cases}$$

where $\mathcal{L} = \frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}$.

The optimal stopping time standing at (t,x) is

$$\hat{\tau}_t = \inf\{s \geq t; V(s, X_s) = \Phi(s, X_s)\}$$

9. The American Put Option

When the opportunity of exercising an option at any time before maturity exists things get complicated. Here the contract function, in contrast to the European option, can not depend solely on the price of the asset at maturity. When maturity is finite there does not exist an analytic formula for the pricing of an American put option. However, in the case of the call option, it does exist.

Proposition 9.1. *The price of an American call option with finite maturity coincides with its European counterpart.*

Proof. This is proved by showing that the optimal stopping time for the option equals maturity. The optimal stopping problem is

$$\sup_{t\leq\tau\leq T} E_{t,x}[e^{-r(\tau-t)}\max(X_\tau - K, 0)].$$

Define a new process

$$Z_s = e^{-r(s-t)}\max(X_s - K, 0) = e^{rt}\max(e^{-rs}X_s - e^{-rs}K, 0)$$

and prove that Z_s is a submartingale.

This is done in two steps.

Step 1: $Y_t := e^{-rt}X_t - e^{-rt}K$. When s increases, so does $-e^{-rs}K$ hence, even though it is a deterministic function, $-e^{-rs}K$ is a submartingale. $e^{-rt}X_t$ were X_t has the dynamics under the risk neutral valuation i.e

$$dX_t = rX_t dt + \sigma X_t dW_t$$

is, following from theorem 6.2, a martingale.

So it follows that Y_t is a submartingale i.e $E[Y_t|\mathcal{F}_s] \geq E[Y_s], \forall s \leq t$.

Step 2: $\gamma(y) = \max\{y, 0\}$ is a convex function of y, hence by Jensens's inequality

$$E[\gamma(Y_t)|\mathcal{F}_s] \geq \gamma(Y_s), \forall s \leq t$$

and thus Z_t is a submartingale .

Hence

$$E_{t,x}[Z_T] \geq E_{t,x}[Z_\tau] \text{ , } \forall \tau \leq T$$

□

So, following the results of proposition 9.1, one focuses on the American put option. Thus, in this context one assumes that the contract function, Φ, has the form $\Phi(x) = max\{K - x, 0\} = (K - x)^+$. With an American put option one hopes for a decrease in the value of the underlying asset so that it is below the strike price K. Let X_t be the process that resembles the value of the underlying asset with the risk-neutral dynamics: $dX_t = rX_t dt + \sigma X_t dW_t$.

One assumes that there is a price level, call it $b(t)$, s.t when X goes below this level it is optimal to stop. For $b(t)$ to be optimal it must be smaller than K i.e be "in the money". Also by theorem 5.1 the solution to the SDE of the underlying asset looks like $X_t = x\exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t)$ so it is reasonable to demand that $b(t) > 0$.

Theorem 9.2. *Assume that* $V(t, x) = \sup_{t \leq \tau \leq T} E_{t,x}[e^{-r(\tau - t)}(K - X_\tau)^+]$. *If there exist a* $b(t)$ *and* $u(t, x)$ *that satisfies:*

$$\begin{cases} u(T,x) = (K - X_T)^+ & & 1. \\ u(t,x) > (K - x)^+ & x > b & 2. \\ u(t,x) = (K - x)^+ & x \leq b & 3. \\ \mathcal{L}u(t,x) = ru(t,x) & x > b & 4. \\ \mathcal{L}u(t,x) < ru(t,x) & x \leq b & 5. \\ \frac{\partial}{\partial x}u(t,x) = -1 & x = b & 6. \end{cases}$$

where $\mathcal{L} = \frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}$, *then it follows that* $u(t, x) = V(t, x)$

Proof. Use Itô's formula on $e^{-rt}u(t, X_t)$. Let $\tau \in [t, T]$ be a stopping time.

$$e^{-r\tau}u(\tau, X_\tau) = e^{-rt}u(t,x) + \int_t^\tau e^{-rs}(-r + \mathcal{L})u(s, X_s)\,ds + \int_t^\tau e^{-rs}\frac{\partial u}{\partial x}(s, X_s)\,dW_s.$$

2. and 3. says that

$$e^{-r\tau}u(\tau, X_\tau) \geq e^{-r\tau}(K - X_\tau)^+$$

$$\Rightarrow e^{-r\tau}(K - X_\tau)^+ \leq e^{-rt}u(t,x) + \int_t^\tau e^{-rs}(-r + \mathcal{L})u(s, X_s)\,ds + \int_t^\tau e^{-rs}\frac{\partial u}{\partial x}(s, X_s)\,dW_s.$$

4. and 5. gives

$$e^{-r\tau}(K - X_\tau)^+ \leq e^{-rt}u(t,x) + \int_t^\tau e^{-rs}\frac{\partial u}{\partial x}(s, X_s)\,dW_s.$$

Now taking expected values on both sides yields:

$$E_{t,x}[e^{-r\tau}(K - X_\tau)^+] \leq e^{-rt}u(t,x) \Leftrightarrow E[e^{-r(\tau - t)}(K - X_\tau)^+] \leq u(t,x)$$

$$\Rightarrow \sup_{t \leq \tau \leq T} E_{t,x}[e^{-r(\tau - t)}(\tau, X_\tau)^+] \leq u(t,x)$$

To prove the reverse inequality use the same arguments but change τ to $\tau^* := T \wedge \tau_b$, where $\tau_b := \inf\{t \le s \le T; X_s \le b\}$. Then it follows that

$$e^{-r\tau^*}u(\tau^*, X_{\tau^*}) = e^{-rt}u(t,x) + \int_t^{\tau^*} e^{-rs}(-r+\mathcal{L})u(s,X_s)\,ds + \int_t^{\tau^*} e^{-rs}\frac{\partial u}{\partial x}(s,X_s)\,dW_s$$

Between t and τ^* X_s will always be above b so from 4. it follows that

$$(-r+\mathcal{L})u(s,X_s) = 0 \Rightarrow e^{-r\tau^*}u(\tau^*, X_{\tau^*}) = e^{-rt}u(t,x) + \int_t^{\tau^*} e^{-rs}\frac{\partial u}{\partial x}(s,X_s)\,dW_s$$

1. and 3. $\Rightarrow u(\tau^*, X_{\tau^*}) = (K - X_{\tau^*})^+$ which leads to

$$e^{-r\tau^*}(K - X_{\tau^*})^+ = e^{-rt}u(t,x) + \int_t^{\tau^*} e^{-rs}\frac{\partial u}{\partial x}(s,X_s)\,dW_s$$

and multiplying both sides with e^{rt} and taking expected values yields:

$$u(t,x) = E_{t,x}[e^{-r(\tau^*-t)}(K - X_{\tau^*})^+] \le \sup_{t \le \tau \le T} E_{t,x}[e^{-r(\tau-t)}(\tau, X_\tau)^+]$$

$$\Rightarrow u(t,x) = V(t,x)$$

□

10. The Perpetual American Put Option

When dealing with an American put option that has a finite maturity, unlike its European counter part, no analytic formula for its value is known. If however one restricts only to the case when there is no maturity or equivalently when it is equal to infinity, an analytic formula can be derived. With $T = \infty$ the option is called a perpetual option.

When the option has no expiring date the value of the option does not depend on time but only on the price levels of the underlying asset i.e its derivative with respect to time is zero. The optimal price frontier, b, becomes a constant.

Now one is looking for b and $V(x)$ that satisfy the conditions of theorem 9.2, and as a consequence of the theorem, the solution to the perpetual American put option has been obtained.

It follows from condition 4. in theorem 9.2 that

$$\frac{1}{2}\sigma^2 x^2 V''(x) + rxV'(x) - rV(x) = 0$$

for $x > b$.

This ODE can be rewritten as an Cauchy Euler equation:

$$x^2V''(x) + \alpha x V'(x) + \beta V(x) = 0, \alpha = \frac{2r}{\sigma^2}, \beta = -\alpha. \tag{5}$$

This is solved by seeking a solution of the form $V(x) = x^m$ which by inserting into (5) gives $V(x) = C_1 x + C_2 x^{-\alpha}$.

Since $V(x)$ is bounded by K it follows that $C_1 = 0$. Condition 3. and 6. in theorem 9.2 says respectively that $V(b) = (K - b)^+$ and $V^{'}(b) = -1$.

$$V(b) = C_2 b^{-\alpha} = (K - b)^+ = K - b$$
$$\Rightarrow C_2 = b^\alpha (K - b) \text{ and } V(x) = (\frac{b}{x})^\alpha (K - b).$$
$$V^{'}(x) = -\alpha b^\alpha x^{-(1+\alpha)}(K - b) \Rightarrow V^{'}(b) = -\alpha b^\alpha b^{-(1+\alpha)}(K - b) = -1$$

which is equivalent to

$$\begin{aligned} b &= \frac{\alpha K}{1+\alpha} \\ &= \frac{2rK}{\sigma^2}\frac{\sigma^2}{2r+\sigma^2} \\ &= \frac{2rK}{2r+\sigma^2}. \end{aligned}$$

So one arrives at the following formula for an American perpetual put option

$$V(x) = \begin{cases} \frac{b}{x}^\alpha (K - b) & x > b \\ K - x & x \le b \end{cases}$$

where

$$b = \frac{2rK}{2r+\sigma^2} \text{ and } \alpha = \frac{2r}{\sigma^2}.$$

11. JUMP TO DEFAULT MODELS

Under the BS-model stock prices, which have a positive initial value, can never drop to zero (this follows from theorem 5.1). History suggest that the probability of a so called "default" is non-zero. In this context default means that the financial entity in question fails to meet its financial obligations, such as interest payments on their loans. If this happens the value of this entity will drop to zero with no chance of a recovery. Here we do not assume that countries can default.

To incorporate default one can use poisson processes.
Let $N(t) =$ *the number of defaults between 0 and* t.

$$N(s + t) - N(s) \sim Po(\lambda \cdot t)$$

where λ is the default intensity.

Empirical evidence shows a positive correlation between corporate bond yields and credit default swap (CDS) spreads (see [4]). A CDS spread is the fee that the buyer of the CDS pays to the issuer. If the chances of default increases so does the fee, for having the insurance.

So in this context one expands the BS-model by saying that there exist a stock and two bonds. A corporate bond and a sovereign bond.

In the normal BS-model $\lambda = 0$, that is to say both bonds must have the same rate of return namely r. If however $\lambda > 0$ then it follows that if investors hold a corporate bond, they need to be compensated for taking a bigger risk in comparison if they held a government bond. This leads to assume that the rate of return of a corporate bond is $r + \lambda$. When using a risk neutral valuation the dynamics of a stock price becomes

$$dX_t = (r + \lambda)X_t + \sigma X_t dW_t - X_t dN_t$$

were $dN_t \sim po(\lambda \cdot dt)$

It is also assumed that λ is a decreasing function of the underlying asset, X_t. Using the above assumptions one can arrive at an ODE for a perpetual American put option with a non zero default intensity, λ, namely

$$\frac{1}{2}\sigma^2 x^2 V'' + (r + \lambda(x))xV' - (r + \lambda(x))V = -\lambda(x)K \tag{6}$$

Theorem 11.1. *Assume that V_1 solves the homogeneous differential equation*

$$V''(x) + p(x) \cdot V'(x) + q(x) \cdot V(x) = 0.$$

Then another linear independent solution, V_2 has the form:

$$V_2 = V_1 \cdot u(x), u(x) = \int^x \frac{1}{V_1^2} \exp(-\int^y p(z)\, dz)\, dy.$$

Theorem 11.2. *Assume that V_1 and V_2 are two linearly independent solutions to the homogeneous differential equation*

$$V''(x) + p(x) \cdot V'(x) + q(x) \cdot V(x) = 0.$$

The particular solution to the differential equation

$$V''(x) + p(x) \cdot V'(x) + q(x) \cdot V(x) = \beta(x)$$

has the form

$$V_p = u_1(x) \cdot V_1(x) + u_2(x) \cdot V_2(x)$$

where

$$\begin{cases} u_1' \cdot V_1 + u_2' \cdot V_2 = 0 \\ u_1' \cdot V_1' + u_2' \cdot V_2' = \beta(x) \end{cases}$$

This system has a solution if

$$det(\begin{bmatrix} V_1 & V_2 \\ V_1' & V_2' \end{bmatrix}) \neq 0$$

Theorem 11.3. (Cramer's Rule)

$$AX = B, A = \begin{bmatrix} a_{11} & ... & a_{1n} \\ ... & & ... \\ a_{n1} & ... & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix}$$

$$\Rightarrow x_i = \frac{det(A_i)}{det(A)}, A_i = \begin{bmatrix} a_{11} & ... & a_{1i-1} & b_1 & a_{1i+1} & ... & a_{1n} \\ ... & & & ... & & & \\ a_{n1} & ... & a_{ni-1} & b_n & a_{ni+1} & ... & a_{nn} \end{bmatrix}$$

i.e the i:th column of A is replaced by B.

For proofs of all three of the above theorems see [5].

To make calculations easier we write (6) as

$$V^{''} + \alpha\frac{1}{x}V^{'} - \alpha\frac{1}{x^2}V = \beta$$
$$\alpha = \frac{2(r+\lambda(x))}{\sigma^2}, \beta = \frac{-2K\lambda(x)}{\sigma^2 x^2}.$$

One solution to the homogenous differential equation,

$$V^{''} + \alpha\frac{1}{x}V^{'} - \alpha\frac{1}{x^2}V = 0$$

is $V_1 = x$.
Theorem 11.1 gives the second linearly independent solution, namely $V_2 = V_1 \cdot u(x)$, where $u(x) = \int^x \frac{1}{V_1^2} \exp(-\int^y \frac{\alpha}{z}\,dz)\,dy$.

To find the particular solution, V_p, to (6) one uses theorem 11.2.

$$V_p = V_1 \cdot u_1(x) + V_2 \cdot u_2(x)$$

$$\begin{cases} u_1{\prime}V_1 + u_2{\prime}V_2 = 0 \\ u_1{\prime}V_1{\prime} + u_2{\prime}V_2{\prime} = \beta \end{cases}$$

which can also be written in matrix form:

$$\begin{bmatrix} V_1 & V_2 \\ V_1{\prime} & V_2{\prime} \end{bmatrix}\begin{bmatrix} u_1{\prime} \\ u_2{\prime} \end{bmatrix} = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

$$V_1^{'} = 1$$

$$\begin{aligned} V_2^{'} &= (V_1 u)^{'} = V_1^{'}u + V_1 u^{'} \\ &= u + V_1\frac{d}{dx}(\int^x \frac{1}{V_1^2}\exp(-\int^y \frac{\alpha}{z}\,dz)\,dy) \\ &= u + V_1(\frac{1}{V_1^2}\exp(-\int^x \frac{\alpha}{z}\,dz)) \\ &= u + \frac{1}{V_1}\exp(-\int^x \frac{\alpha}{z}\,dz). \end{aligned}$$

We now apply theorem 11.3.

$$A = \begin{bmatrix} V_1 & V_2 \\ V_1\prime & V_2\prime \end{bmatrix}, A_1 = \begin{bmatrix} 0 & V_2 \\ \beta & V_2\prime \end{bmatrix}, A_2 = \begin{bmatrix} V_1 & 0 \\ V_1\prime & \beta \end{bmatrix}$$

$$det(A_1) = -\beta V_2, det(A_2) = \beta V_1,$$

$$\begin{aligned} det(A) &= V_1V_2^{'} - V_1^{'}V_2 \\ &= V_1(u + \frac{1}{V_1}\exp(-\int^x \frac{\alpha}{z}\,dz)) - V_2 \\ &= V_2 + \exp(-\int^x \frac{\alpha}{z}\,dz) - V_2 \\ &= \exp(-\int^x \frac{\alpha}{z}\,dz). \end{aligned}$$

Theorem 11.3 gives

$$u_1^{'} = -\beta V_2 \exp(\int^x \frac{\alpha}{z}\,dz), u_2^{'} = \beta V_1 \exp(\int^x \frac{\alpha}{z}\,dz)$$

$$\Rightarrow u_1 = -\int^x \beta V_2 \exp(\int^y \frac{\alpha}{z}\,dz)\,dy, u_2 = \int^x \beta V_1 \exp(\int^y \frac{\alpha}{z}\,dz)\,dy.$$

$$\Rightarrow V_p = -V_1 \int^x \beta V_2 \exp(\int^y \frac{\alpha}{z}\,dz)\,dy + V_2 \int^x \beta V_1 \exp(\int^y \frac{\alpha}{z}\,dz)\,dy$$

If λ is assumed to be constant the values of V_1 and V_2 are as follows.

$$V_1 = x,$$

$$\begin{aligned} V_2 &= x\int^x \frac{1}{y^2}\exp(-\alpha\int^y \frac{1}{z}\,dz)\,dy \\ &= x\int^x \frac{1}{y^2}\exp(-\alpha log(y))\,dy \\ &= x\int^x \frac{1}{y^{\alpha+2}}\,dy \\ &= \frac{-1}{(\alpha+1)x^{\alpha}} \end{aligned}$$

Set $\beta = \frac{\gamma}{x^2}, \gamma = \frac{-2K\lambda}{\sigma^2}$ and insert V_1 and V_2 into V_p yeilds

$$\begin{aligned} V_p &= \frac{x\gamma}{\alpha+1}\int^x \frac{1}{y^2}\,dy - \frac{\gamma}{(\alpha+1)x^\alpha}\int^x y^{\alpha-1}\,dy \\ &= \frac{-\gamma}{\alpha+1} - \frac{\gamma}{\alpha(\alpha+1)} \\ &= \frac{-\gamma}{\alpha} \\ &= \frac{2K\lambda}{\sigma^2}\frac{\sigma^2}{2(r+\lambda)} \\ &= \frac{K\lambda}{(r+\lambda)} \end{aligned}$$

So the solution to (6) where λ is constant is

$$V = C_1 x + C_2 x^{-\alpha} + \frac{K\lambda}{(r+\lambda)}.$$

Now we would like V to satisfy the conditions in theorem 9.2.

$$\begin{cases} V(x) \le K & (i) \\ V(b) = K - b & (ii) \\ \frac{\partial V}{\partial x}(b) = -1 & (iii) \end{cases}$$

$$(i) \Rightarrow C_1 = 0, (ii) \Rightarrow V(b) = C_2 b^{-\alpha} + \frac{K\lambda}{(r+\lambda)} = K - b.$$

$$\text{This leads to that } C_2 = b^\alpha\Big(K - b - \frac{K\lambda}{(r+\lambda)}\Big).$$

$$\Rightarrow V(x) = \Big(\frac{b}{x}\Big)^\alpha\Big(K - b - \frac{K\lambda}{(r+\lambda)}\Big) + \frac{K\lambda}{(r+\lambda)}$$

$$(iii) \Rightarrow \frac{\partial V}{\partial x}\Big|_{x=b} = -\alpha b^\alpha x^{-(\alpha+1)}\Big(K - b - \frac{K\lambda}{(r+\lambda)}\Big)\Big|_{x=b} = -1$$

which is equivalent to $\frac{-\alpha}{b}\Big(K - b - \frac{K\lambda}{(r+\lambda)}\Big) = -1 \Leftrightarrow b = \frac{\alpha}{\alpha+1}\Big(K - \frac{K\lambda}{(r+\lambda)}\Big)$

$$\begin{aligned} \alpha = \frac{2(r+\lambda)}{\sigma^2} \Rightarrow \frac{\alpha}{\alpha+1} &= \frac{2(r+\lambda)}{\sigma^2}\frac{\sigma^2}{2(r+\lambda)+\sigma^2} \\ &= \frac{2(r+\lambda)}{2(r+\lambda)+\sigma^2} \end{aligned}$$

$$K - \frac{K\lambda}{(r+\lambda)} = \frac{rK}{r+\lambda}$$

$$\Rightarrow b = \frac{2(r+\lambda)}{2(r+\lambda)+\sigma^2} \cdot \frac{rK}{r+\lambda} = \frac{2rK}{2(r+\lambda)+\sigma^2}.$$

Now one has arrived at the following formula for an perpetual American put option on a stock with default intensity λ namely

$$V(x) = \begin{cases} (\frac{b}{x})^{\alpha}(K - b - \frac{K\lambda}{(r+\lambda)}) + \frac{K\lambda}{(r+\lambda)} & \text{if } x > b \\ K - x, & \text{if } x \leq b \end{cases}$$

where $b = \frac{2rK}{2(r+\lambda)+\sigma^2}$ and $\alpha = \frac{2(r+\lambda)}{\sigma^2}$.

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