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Properties of the energy Laplacian on the Sierpinski Gasket

Konstantinos Tsougkas

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Department of Mathematics
Uppsala University

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Abstract

In this paper we study the recent topic of analysis on fractals, taking as our primary focus the Sierpinski Gasket. We examine how a Laplacian is created on it in regards to different measures and we study its properties. Then we extend some results of the standard Laplacian to the Kusuoka one.

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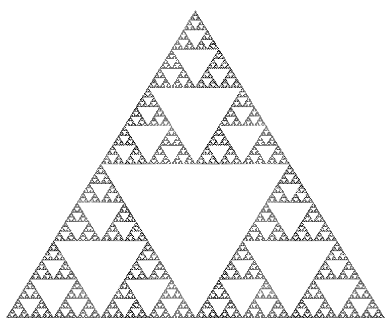
1 Introduction

Let us define the Sierpinski gasket, SG. Take an equilateral triangle in $\mathbb{R}^2$ with vertices $\{q_i\}$ and define maps $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ with $F_i = \frac{1}{2}(x - q_i) + q_i$ for $i = 0, 1, 2$. Then, the Sieprinski Gasket is the unique compact set satisfying

$$SG = \bigcup_{i=0}^{2} F_i SG.$$



Also, as a convention we may refer to the Sierpinski Gasket as SG or K. If $w = (w_1, \ldots, w_m)$ is a finite word, we can also define the mapping

$$F_w = F_{w_1} \circ \cdots \circ F_{w_m}.$$

We call $F_w K$ a cell of level m. The Sierpinski Gasket may be viewed as an approximation of a sequence of graphs Γ_m with vertices V_m and adjacency relations $x \sim_m y$. That means, that for $x, y \in V_m$ we have that

$$x \sim_m y \iff x, y \in F_w(V_0)$$

for some word w of length m. We want the vertices to be nested $V_0 \subseteq V_1 \subseteq V_2 \ldots$ with the union $V_* = \bigcup_{m=0}^{\infty} V_m$, a dense set of the fractal. In the case of the Sierpinski Gasket, we take $V_0 = \{q_i\}_{i=0}^{2}$ as the vertices of an original equilateral triangle and define

$$V_m = \bigcup_{i=0}^{2} F_i(V_{m-1}).$$

We call V_0 the boundary of the Sierpinski Gasket and every point in V_* a junction point. We will be concerned with functions $u : K \to \mathbb{R}$.

Now, we create a measure on the Sierpinski Gasket. In this thesis we will focus on two measures, namely the standard measure and the Kusuoka measure. The standard measure is a special case of a self-similar measure created in the following way:
Assign probability weights μ_i with

$$\sum_{i=0}^{2} \mu_i = 1 \text{ ,with each } \mu_i > 0 \text{ and then set}$$

$$\mu(F_w K) = \prod_{i=0}^{2} \mu_{w_i} \text{ for } |w| = m.$$

Then, for the standard measure we just set all $\mu_i = 1/3$. On the Sierpinski Gasket the standard invariant measure μ satisfies

$$\mu(F_w F_i SG) = \frac{1}{3}\mu(F_w SG), \quad i = 0, 1, 2, \text{ for any word } w.$$

We also have the self-similar identity

$$\mu(A) = \sum_i \mu_i \mu(F_i^{-1} A).$$

Now, after having defined a measure we can define integrals and create and enrich our theory. This is standard as in usual calculus, and since we have uniform continuity due to the compact set K it suffices to define it as

$$\int_K f \mathrm{d}\mu = \lim_{m \to \infty} \sum_{|w|=m} f(x_w)\mu(F_w K).$$

A key concept in the theory of analysis on fractals plays the concept of energy. On each graph G we construct an energy $E(u, v)$ for two functions u and v with:

$$E_G(u, v) = \sum_{x \sim y} (u(x) - u(y)) (v(x) - v(y))$$

where the sum extends over all the edges of the graph. For $u = v$ we simply denote $E_G(u)$. This is a bilinear form. If we have a graph G and G' with vertices V and V' respectively, and $V \subset V'$ and we also have a function u

defined on V, then we can extend it on V' in many possible ways. However, there is at least one way to do it so that it minimizes $E_{G'}(u)$. Such an energy-minimizing extension will be called a harmonic extension and write it $\tilde{u}$. In the Sierpinski Gasket, it turns out that $E_{G'}(\tilde{u}) = \frac{3}{5}E_G(u)$. This leads us to define the so called renormalized graph energies, $\mathcal{E}_m(u) = (r^{-m}E_m(u))$. In our case, it turns out $r = \frac{3}{5}$. There exists a simple extension algorithm in order to obtain a harmonic extension of a function defined on V_m to V_{m+1}. This algorithm is called the "$\frac{1}{5} - \frac{2}{5}$ rule" and it goes as follows:

If on a given cell V_m we have that the function takes boundary values a, b, c and the junctions points of the next level sub-cell, directly opposite of these points respectively take values x, y, z then we have that

$$x = \frac{2}{5}(b + c) + \frac{1}{5}a$$

$$y = \frac{2}{5}(a + c) + \frac{1}{5}b$$

$$z = \frac{2}{5}(a + b) + \frac{1}{5}c$$

This equations are obtained easily by using calculus. By computing the value of the energy, and taking the derivatives of x, y, z equal to zero to minimize it, and then solving accordingly. If $\tilde{u}$ is the harmonic extension of u then it holds that $\mathcal{E}_{m+1}(\tilde{u}) = \mathcal{E}_m(u)$ and in general we have that $\mathcal{E}_m(u) \leq \mathcal{E}_{m+1}(u)$. We define a harmonic function h to be one that minimizes $\mathcal{E}_m(h)$ at all levels for the given boundary values on V_0. This can be done by following the "$\frac{1}{5} - \frac{2}{5}$" rule and thus by this rule, it is obvious that a harmonic function is defined completely by its boundary values. Thus, we have a space of harmonic functions called $\mathcal{H}_0$ which is three-dimensional with a basis $\{h_0, h_1, h_2\}$ with $h_i(q_j) = \delta_{i,j}$. Thus we have that for harmonic extension functions at each point $x \in V_{m+1} \setminus V_m, \quad \tilde{u}(x)$ is the average of the values at the four neighboring points in V_{m+1}. In a similar fashion as standard analysis, we have a maximum principle for harmonic functions too, and thus they take their maximum value at the boundary. We also have the following key lemma.

Lemma 1.1. *Let u, v be defined on V_m, let $\tilde{u}$ be the harmonic extension of u and let v' be any extension of v to V_{m+1}. Then*

$$\mathcal{E}_{m+1}(\tilde{u}, v') = \mathcal{E}_m(u, v).$$

We define then the energy of u as

$$\mathcal{E}(u) = \lim_{m\to\infty} \mathcal{E}_m(u) \text{ or similarly}$$

$$\mathcal{E}(u,v) = \lim_{m\to\infty} \mathcal{E}_m(u,v)$$

This energy plays a central role in our theory and we call functions u such that $\mathcal{E}(u) < \infty$ as functions with finite energy and we denote $u \in dom\mathcal{E}$. A very important property of functions of finite energy is that they are continuous. In fact, they are Hölder continuous. Moreover, $dom\mathcal{E}$ is dense in $C(K,\mathbb{R})$ and also $dom\mathcal{E}$ forms an algebra. This energy $\mathcal{E}(u,v)$ is also a bilinear form and forms an inner product on the space of $dom\mathcal{E}$ modulo constants. In fact, $dom\mathcal{E}/constants$ forms a Hilbert space with that inner product.

We are now ready to define the main object of our study, the Laplacian.

Definition 1.2. Let $u \in dom\mathcal{E}$. Then, $u \in dom\Delta\mu$ and $\Delta_\mu u = f$ if

$$\mathcal{E}(u,v) = -\int_K fv\mathrm{d}\mu \text{ for all } v \in dom_0\mathcal{E}$$

where $dom_0\mathcal{E}$ denotes the functions of finite energy that vanish on the boundary.

If we mean the standard Laplacian, that is the Laplacian with the standard measure, then we can simply write Δu instead of $\Delta_\mu u$ without any confusion. We have that $dom\Delta_\mu$ is a real vector space. However just by the definition it is not so clear that there are any nontrivial functions in $dom\Delta_\mu$. But a bit later on we will see a theorem which shows that it is a very rich space because for every continuous function f there exists $u \in dom\Delta_\mu$ such that $\Delta_\mu u = f$. Initially, a very important fact is that the space contains harmonic functions and they have Laplacian zero, which also is an equivalent way of defining harmonic functions. This fact holds true for all measures μ.

Theorem 1.3. *If $h \in \mathcal{H}_0$, we have that $h \in dom\Delta_\mu$ and $\Delta_\mu h = 0$. Conversely, if $u \in dom\Delta_\mu$ and $\Delta_\mu u = 0$ then u is harmonic.*

Proof. We know that harmonic functions h have the property that $\mathcal{E}_m(h,v)$ is independent of m, so $\mathcal{E}(h,v) = \mathcal{E}_0(h,v) = 0$ since v vanishes on the boundary. Thus $\Delta_\mu h = 0$.

For the opposite direction we will use some very specific functions v. Let m and a point $x \in V_m \notin V_0$. Define $\psi_x^{(m)}$ as the piecewise harmonic function at level m that satisfies $\psi_x^{(m)}(y) = \delta_{xy}$ for all $x, y \in V_m$. Then we have that $\psi_x^m \in dom_0\mathcal{E}$ since $x \notin V_0$. Then, by using the fact that $\Delta_\mu u = 0$ we get that $\mathcal{E}(u, \psi_x^{(m)}) = 0$. But then by reversing the roles of u and v we have that $\mathcal{E}(u, \psi_x^{(m)}) = \mathcal{E}_m(u, \psi_x^{(m)})$. But then, this condition that $\mathcal{E}_m(u, \psi_x^{(m)}) = 0$ means that

$$\sum_{y \sim_m x} (u(x) - u(y)) = 0$$

and that $u|_{V_m}$ is harmonic. But this is true for all m, and thus u is harmonic.

$\square$

A very big drawback of $dom\Delta_\mu$ is that it is not closed under multiplication. If $u \in dom\Delta_\mu$ then $u^2 \notin dom\Delta_\mu$. This will be explored later, and this fact is completely dependent on the measure μ. This major disadvantage however can be lifted if we create a different Laplacian with a different measure. Namely, Kusuoka created a measure called the Kusuoka measure and with that measure we have that the domain of its Laplacian is closed under multiplication. The major scope of this thesis is investigating the properties and the differences between the two Laplacians, with the Kusuoka and the standard measure.

The definition we used above for the Laplacian is called a weak definition and there is an equivalent pointwise formula. First, we define a graph Laplacian

$$\Delta_m u(x) = \sum_{y \sim_m x} (u(y) - u(x)) \text{ for all } x \in V_m \diagdown V_0.$$

Then our pointwise formula would be the following:

$$\Delta_\mu u(x) = \lim_{m \to \infty} r^{-m} \left(\int_K \psi_x^{(m)} \mathrm{d}\mu \right)^{-1} \Delta_m(x).$$

In the case of the standard Laplacian, we simply get

$$\Delta u(x) = \frac{3}{2} \lim_{m \to \infty} 5^m \Delta_m u(x)$$

The reason for this, is that we already have that $r = \frac{3}{5}$ and thus we only need to estimate

$$\int_K \psi_x^{(m)} \mathrm{d}\mu.$$

To do this, let $x \in K$. Then the function $\psi_x^{(m)}$ has support in the two m-cells meeting at x. If $F_w K$ is one of these cells with vertices x, y, z then we get that

$$\psi_x^{(m)} + \psi_y^{(m)} + \psi_z^{(m)} = 1$$

and this holds as an identity. Thus

$$\int_{F_w K} (\psi_x^{(m)} + \psi_y^{(m)} + \psi_z^{(m)}) \mathrm{d}\mu = \mu(F_w K) = \left(\frac{1}{3}\right)^m .$$

But due to symmetry all the summands have the same integral, so

$$\int_{F_w K} \psi_x^{(m)} \mathrm{d}\mu = \frac{1}{3^{m+1}}.$$

Along with the other m-cell, we get finally that

$$\int_{F_w K} \psi_x^{(m)} \mathrm{d}\mu = \frac{2}{3^{m+1}}.$$

We can also create normal derivatives at the boundary points $\{q_i\}$.

Definition 1.4. Let x a boundary point and u a continuous function on K. We say $\partial_n u(x)$ exists if the right hand side limit exists and

$$\partial_n u(x) = \lim_{m \to \infty} r^{-m} \sum_{y \sim_m x} (u(x) - u(y)).$$

This is called the normal derivative at the point x.

For the case of the Sierpinski Gasket this can be viewed as

$$\partial_n u(x) = \lim_{m \to \infty} \left(\frac{5}{3}\right)^m (2u(q_i) - u(F_i^m q_{i+1}) - u(F_i^m q_{i-1})) .$$

We can also localize the definition of the normal derivative. If we have a cell $F_w K$ and $x = F_w q_i$ be a boundary point of that cell, then we can say that $\partial_n u(x)$ with respect to the cell $F_w K$ is the same formula with the y in the formula lie inside $F_w K$. We can also view this as a scaling property for the normal derivative, namely

$$\partial_n u(F_w q_i) = r^{-|w|} \partial_n (u \circ F_w)(q_i).$$

However at this point it is important to note that a junction point has two addresses $F_w q_i$ and $F_{w'} q_{i'}$ and thus there is another local derivative on that point x with respect to the other cell. Thus a normal derivative is not only with respect to any junction point but also viewed according to the cell of that junction point.

However we have an important connection to these two derivatives, and that is that they sum to zero. This is called the matching condition for normal derivatives at x.

Proposition 1.5. *Suppose $u \in dom\Delta_\mu$. Then at each junction point $x = F_w q_i = F_{w'} q_{i'}$, the local derivatives exist and*

$$\partial_n u(F_w q_i) + \partial_n u(F_{w'} q_{i'}) = 0$$

Proof. The existence follows from the scaling property. Now, we have that

$$\partial_n u(F_w q_i) + \partial_n u(F_{w'} q_{i'}) = \lim_{m\to\infty} r^{-m} \sum_{y \sim_m x} (u(x) - u(y))$$

since the neighbors y of x lie in either $F_w K$ or $F_{w'} K$. However since we have that $u \in dom\Delta_\mu$ we know that

$$\lim_{m\to\infty} r^{-m} \left(\int_K \psi_x^{(m)} \mathrm{d}\mu \right)^{-1} \Delta_m u(x)$$

exists and since $\int_K \psi_x^{(m)} \mathrm{d}\mu \to 0$ we see that the above right hand side limit is also zero.

□

In a similar fashion, we also define the tangential derivatives:

Definition 1.6. Let q_i be a boundary point. The tangential derivative at the point q_i is the limit

$$\partial_T u(q_i) = \lim_{m\to\infty} 5^m (u(F_i^m(q_{i+1}) - F_i^m(q_{i-1}))$$

if the limit exists.

Note, that the tangential derivative may not always exist. However, later on, we will have a theorem confirming its existence under suitable conditions.

One of the most important equations which is a very powerful tool for our theory is that of the Gauss-Green formula.

Theorem 1.7. *Suppose u and v are in $dom\Delta_\mu$. Then*

$$\int_K (\Delta_\mu u) v \mathrm{d}\mu - \int_K (\Delta_\mu v) u \mathrm{d}\mu = \sum_{V_0} (u \partial_n v - v \partial_n u).$$

If we choose $v = 1$ in the above formula, then we get that

$$\int_K (\Delta_\mu u) \mathrm{d}\mu = \sum_{V_0} \partial_n u.$$

Another version of the formula is also

$$\mathcal{E}(u, v) = -\int_K (\Delta_\mu u) v \mathrm{d}\mu + \sum_{V_0} v \partial_n u.$$

We also note a consequence of the formula. If $u \in dom\Delta_\mu$ and $\partial_n u(x) = 0$ for every $x \in V_\star$ then u is constant. To see this, by choosing $v = 1$ in the Gauss-Green formula we have that $\int_{F_w K} \Delta_\mu u \mathrm{d}_\mu = \sum_{v_0} \partial_n u = 0$ for all cells $F_w K$ which imples $\Delta_\mu = 0$ and thus u is harmonic. But we know how to compute the normal derivatives of harmonic functions, and we don't get all zeros unless the function is constant.

A big scope of the theory is to provide solutions to the "differential equation" $\Delta_\mu u = f$ for a continuous f. To derive a solution we create a special function called Green's function.

Definition 1.8. Green's function is defined by

$$G(x, y) = \lim_{M \to \infty} G_M(x, y)$$

with $G_M(x, y)$ defined by

$$G_M(x, y) = \sum_{m=0}^{M} \sum_{z' \in V_{m+1} \setminus V_m} g(z, z') \psi_z^{(m+1)}(x) \psi_{z'}^{(m+1)}(y)$$

where $g(z, z') = 0$ when z and z' are not in the same cell of level $m + 1$,

$$g(z, z) = \frac{9}{50} \left(\frac{3}{5}\right)^m \quad \text{for} \quad z \in V_{m+1} \setminus V_m$$

and

$$g(z, z') = \frac{3}{50} \left(\frac{3}{5}\right)^m$$

for $z \neq z'$, z and z' in the same level $m + 1$ cell.

Then, now that we have defined Green's function, we arrive at this very important result that gives us existence of a huge class of functions in $dom\Delta_\mu$. Using the Green's function is also the main tool we use for solving differential equations on fractals.

Theorem 1.9. *On the Sierpinski Gasket, the Dirichlet problem*

$$-\Delta_\mu u = f \ , \ u_{|V_0} = 0$$

has a unique solution in $dom\Delta_\mu$ for any continuous f, given by

$$u(x) = \int_K G(x,y)f(y)\mathrm{d}\mu(\mathrm{y})$$

where $G(x,y)$ is the Green's function. If we don't have Dirichlet boundary conditions then the solution is given by

$$u(x) = \int_K G(x,y)f(y)\mathrm{d}\mu(\mathrm{y}) + h(x)$$

where $h(x)$ is a harmonic function with the same boundary values as u.

We have already seen that a function h such that $\Delta_\mu h = 0$ is called harmonic and the space of those functions is $\mathcal{H}_0$. We can also define spaces of multiharmonic functions in the same way by defining

$$\mathcal{H}_k = \{u \mid \Delta_\mu^{k+1} u = 0\} \quad \text{for} \quad k = 0, 1, 2, ...$$

Then this space has dimension $3k+3$ and we are interested in creating a basis. There have been many different construction of basis but one way of doing it that is particularly interesting is creating "monomials" that are the analogue of the functions $\{\frac{x^j}{j!}\}$ on the real line. These monomials are also centred around a specific boundary point with the ulterior motive of creating Taylor series.

Definition 1.10. We define the monomials $\{P_{ki} \in \mathcal{H}_k\}$ to have the k-jet consisting of all 0's except for one 1:

$$\Delta^j P_{ki}(q_0) = \delta_{jk}\delta_{i1}$$
$$\partial_n \Delta^j P_{ki}(q_0) = \delta_{jk}\delta_{i2}$$
$$\partial_T \Delta^j P_{ki}(q_0) = \delta_{jk}\delta_{i3}$$
$$\text{for} \quad j \leqslant k.$$

From that we can observe that this means $\Delta P_{ki} = P_{(k-1)i}$ and thus we can recursively find P_{ki} by

$$P_{ki}(x) = -\int_K G(x,y)P_{(k-1)i}(y)\mathrm{d}\mu(\mathrm{y}) + h(x)$$

for a harmonic function $h(x)$ defined by the $j=0$ case.

Definition 1.11. Define as follows

$$\alpha_j = P_{j1}(q_1),\ \beta_j = P_{j2}(q_1),\ \gamma_j = P_{j3}(q_1)n_j = \partial_n P_{j1}(q_1),\ t_j = \partial_T P_{j2}(q_1).$$

Note that by symmetry we have $P_{j1}(q_2) = \alpha_j$, $P_{j2}(q_2) = \beta_j$ and $P_{j3}(q_2) = -\gamma_j$, so that all values of monomials at boundary points are expressible in terms of α's, β's and γ's.

Then, in order to obtain some decay rates for the monomials, we use the following relations

Lemma 1.12. *The following recursion relations hold:*

$$\alpha_j = \frac{4}{5^j - 5}\sum_{\ell=1}^{j-1}\alpha_{j-\ell}\alpha_\ell \quad \textit{for} \quad j \geq 2$$

$$\gamma_j = \frac{4}{5^{j+1} - 5}\sum_{\ell=0}^{j-1}\alpha_{j-\ell}\gamma_\ell \quad \textit{for} \quad j \geq 1$$

$$\beta_j = \frac{1}{5^j - 1}\sum_{\ell=0}^{j-1}\Big(\frac{2}{5}5^{j-\ell}\alpha_{j-\ell}\beta_\ell - \frac{2}{3}\alpha_{j-\ell}5^\ell\beta_\ell + \frac{4}{5}\alpha_{j-\ell}\beta_\ell\Big) \quad \textit{for} \quad j \geq 1,$$

with initial data $\alpha_0 = 1$, $\alpha_1 = 1/6$, $\beta_0 = -1/2$, $\gamma_0 = 1/2$. *In particular,*

$$\gamma_j = 3\alpha_{j+1}.$$

Lemma 1.13. *There exists a constant* c *such that*

$$0 < a_j < c(j!)^{-log5/log2}$$

for all j*.*

Then by using the above two lemmas we get the following main theorem.

Theorem 1.14. *(i) For any $r < \infty$ there exists c_r such that*

$$\|P_{j1}\|_\infty \leqslant c_r r^{-j}$$

or more precisely

$$\lim_{j\to\infty} \frac{1}{j}\|P_{j1}\|_\infty = -\infty.$$

(ii) There exists c such that

$$\|P_{j2}\|_\infty \leqslant c{\lambda_2}^{-j}$$

and

$$\lim_{j\to\infty} -{\lambda_2}^j P_{j2} = \phi$$

where ϕ is a λ_2-Neumann eigenfunction of Δ which is R_0-symmetric and vanishes ok F_0K, the limit existing uniformly and in energy.

The proofs of the above lemmas and theorem are quite lengthy and detailed and can be found at [3]. Having found these rates for the monomials, a theory of Taylor series can be created. It should also be noted, that the polynomials, i.e the sum of the monomials P_{ij}, or equivalently solutions of $\Delta_\mu^n P = 0$, behave differently from the standard polynomials we are used to in analysis, and many key results that are true in standard analysis, do not hold here. A key example is the Stone-Weierstrass theorem. Polynomials cannot uniformly approximate all continuous functions.

To see this, first we must note a very interesting and curious fact about the Laplacian on SG. As usual in analysis we define eigenfunctions and eigenfunction in the natural way as solutions of

$$\Delta u = \lambda u$$

and the Dirichlet boundary conditions if the boundary vanishes, while the Neumann boundary conditions if the normal derivatives vanish at the boundary. A striking difference however between standard analysis and analysis on fractals is that in standard analysis we cannot have joint Dirichlet-Neumann eigenfunctions. However, surprisingly, in this theory of Laplacian on fractals we can have certain eigenfunctions also seen as "localized eigenfunctions" that satisfy both Dirichlet and Neumann conditions.

Then, let u be a joint Dirichlet-Neumann eigenfunction. Thus if $P(x)$ is a polynomial we have that

$$\int_K P(x)u(x)\mathrm{d}\mu = 0$$

since by the definition of the monomials we have that $\Delta P_{ki} = P_{(k-1)i}$ and thus we can use repeatedly the Gauss-Green formula to reduce the order of the polynomial since due to the joint $D - N$ conditions we have that all the terms $u(q_i)$=0 and $\partial_n u(q_i) = 0$.

2 Energy measures and the Kusuoka measure

While with the standard measure we know many results concerning the behavior of functions on the Sierpinski gasket, we see that the standard measure has also many drawbacks. Namely, the domain of its Laplacian is not an algebra. To overcome this we create a different measure called the Kusuoka measure. Many results that are known about the standard measure still remain open if we use the Kusuoka measure instead. To define the Kusuoka measure we need to define first the energy measures. Define a measure ν_u by

$$\nu_u(F_w K) = r^{-|w|}\mathcal{E}(u \circ F_w).$$

This is called the energy measure ν_u. For energy measures and $u, v \in dom\mathcal{E}$ we have the important carré du champs formula

$$\int_K f \mathrm{d}\nu_{u,v} = \frac{1}{2}\mathcal{E}(fu, v) + \frac{1}{2}\mathcal{E}(u, fv) - \frac{1}{2}\mathcal{E}(f, uv).$$

Definition 2.1. Let $\{h_1, h_2\}$ be an orthonormal basis for the space of harmonic functions modulo constants with respect to the energy inner product. Then the Kusuoka measure is defined as

$$\nu = \nu_{h_1} + \nu_{h_2}$$

Proposition 2.2. *The Kusuoka measure is independent on the choice of the orthonormal basis.*

Proof. Let $\{h, h'\}$ be an orthonormal basis for the Kusuoka measure and let $\{h_1, h_2\}$ be a different one such that $\nu' = \nu_{h_1} + \nu h_2$. Then there exist a, b, c, d such that $h_1 = ah + bh'$ and $h_2 = ch + dh'$. Because of the change of basis we have that the matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a rotation matrix and thus we have the properties

$$a^2 + c^2 = 1,\ b^2 + d^2 = 1,\ ab + cd = 0.$$

Then, we see that

$$
\begin{aligned}
\nu'(F_w K) &= r^{-|w|}(\mathcal{E}(h_1 \circ F_w) + \mathcal{E}(h_2 \circ F_w) \\
&= r^{-|w|}(a^2\mathcal{E}(h \circ F_w) + b^2\mathcal{E}(h' \circ F_w) \\
&\quad + c^2\mathcal{E}(h \circ F_w) + d^2\mathcal{E}(h' \circ F_w) \\
&\quad + ab\mathcal{E}(h \circ F_w, h' \circ F_w) + cd\mathcal{E}(h \circ F_w, h' \circ F_w)) \\
&= \nu_h(F_w K) + \nu_{h'}(F_w K)
\end{aligned}
$$

□

We can take as basis $h_1 = \frac{1}{\sqrt{2}}(0, 1, 1)$ and $h_2 = \frac{1}{\sqrt{6}}(0, 1, -1)$ to be the orthogonal harmonic functions defining the Kusuoka measure.

An equivalent definition for the Kusuoka measure would be to define it as

$$\nu' = \nu_{h_0} + \nu_{h_1} + \nu_{h_2}$$

but in this case we get that $\nu' = 3\nu$ with respect to the previous definition. Now, with this new measure, by exactly the same definition as before we create a different Laplacian, the Kusuoka Laplacian Δ_ν which has different properties. A very important result is that every energy measure is absolutely continuous with respect to the Kusuoka measure. In fact the Kusuoka measure is also singular with respect to the standard measure.

Now, in [7], a pointwise formula is obtained for the Kusuoka Laplacian.

Proposition 2.3. *Let $u \in dom\Delta_\nu$. Then for all $x \in V_* \diagdown V_0$ the following pointwise formula holds with uniform limit across $V_* \diagdown V_0$*

$$\Delta_\nu u(x) = 2 \lim_{m\to\infty} \frac{\Delta_m u(x)}{\Delta_m({h_1}^2 + {h_2}^2)(x)}$$

Proof. First of all, we have already mentioned before that the pointwise formula for any measure is

$$\Delta_\mu u(x) = \lim_{m\to\infty} r^{-m} \left(\int_K \psi_x^{(m)} \mathrm{d}\mu \right)^{-1} \Delta_\mu u(x).$$

To compute the $\left(\int_K \psi_x^{(m)} \mathrm{d}\mu \right)^{-1}$ where μ is now the Kusuoka measure, we use the carré du champs formula and thus we have

$$\int_K \psi_x^{(m)} \mathrm{d}\nu = \mathcal{E}(\psi_x^{(m)} h_1, h_1) + \mathcal{E}(\psi_x^{(m)} h_2, h_2) - \frac{1}{2}\mathcal{E}(\psi_x^{(m)}, {h_1}^2) - \frac{1}{2}\mathcal{E}(\psi_x^{(m)}, {h_2}^2).$$

But we have that

$$\mathcal{E}(\psi_x^{(m)} h_1, h_1) = \mathcal{E}_0(\psi_x^{(m)} h_1, h_1) = 0$$

$$\mathcal{E}(\psi_x^{(m)} h_2, h_2) = \mathcal{E}_0(\psi_x^{(m)} h_2, h_2) = 0.$$

And thus we have that

$$\int_K \psi_x^{(m)} \mathrm{d}\nu = -\frac{1}{2}\mathcal{E}(\psi_x^{(m)}, {h_1}^2 + {h_2}^2) = r^{-m}\Delta_m({h_1}^2 + {h_2}^2).$$

Concluding, we get that

$$\Delta_\nu u(x) = \lim_{m\to\infty} r^{-m}\left(\int_K \psi_x^{(m)} \mathrm{d}\mu\right)^{-1} \Delta_m u(x) = 2\lim_{m\to\infty} \frac{\Delta_m u(x)}{\Delta_m({h_1}^2 + {h_2}^2)(x)}$$

□

It would be interesting to try to see how similar $dom\Delta$ and $dom\Delta_\nu$ are. The following theorem shows us that they are quite different and in fact they only coincide on the space of harmonic functions.

Theorem 2.4. $dom\Delta \cap dom\Delta_\nu = \mathcal{H}_0$

Similarly as before with the standard measure, we can create polynomials P_{ij} that are the basis for multiharmonic functions for the Kusuoka measure. However, while the decay rates $\|P_{ij}\|_\infty$ are known for the polynomials of the standard measure, it remains an open problem to find estimates for the Kusuoka polynomials. In fact, numerical estimation shows that the decay rates are different. As for a self-similar identity, the Kusuoka measure is not self-similar in the way the standard measure is. However, we have the following result.

Proposition 2.5. *The Kusuoka measure satisfies the variable self-similar identity*

$$\sum_{i=0}^{2}\left(\left(\frac{1}{15} + \frac{12}{15}R_i\right)\nu\right) \circ {F_i}^{-1}$$

where $R_i = \frac{d\nu_i}{d\nu}$ *the Radon-Nikodym derivative of* ν_i.

Using this relation a scaling identity for the Kusuoka Laplacian can be derived. We will introduce the scaling identity in section 4, where we will also make extensive use of it. Perhaps the single most important advantage of the Kusuoka Laplacian over the standard one is that the domain of the Kusuoka Laplacian forms an algebra. This is a key fact that is not true for the standard Laplacian. A proof will be given for that in the next section by evaluating the properties of the decay rates of functions. Thus, for example if $u \in dom\Delta$ then $u^2 \notin dom\Delta$. However if $u \in dom\Delta_\nu$ then $u^2 \in dom\Delta_\nu$ and

$$\Delta_\nu u^2 = 2u\Delta_\nu u + 2\frac{d\nu_u}{d\nu}.$$

This advantage makes it clear that the Kusuoka Laplacian is one that is worth studying and perhaps despite its apparent disadvantages due to the lack of self-similarity, in some sense is better behaved than the standard one. However, we have also have the following theorem which shows us that the Radon-Nikodym derivative is not continuous.

Theorem 2.6. *Let h be harmonic function with $\nu_h = a\nu_0 + b\nu_1 + c\nu_2$ and h' be a harmonic function orthonormal to h under the energy inner product. Then, if C is a cell on K:*

$$i) \inf_{x \in C} \frac{d\nu_h}{d\nu} = 0$$

$$ii) \sup_{x \in C} \frac{d\nu_h}{d\nu} = \frac{2}{3}(a+b+c)$$

However we have some form of limited continuity, if we restrict the derivative to the set of vertices $V_\star$ then it is continuous on the edges of every triangle.

3 Local behavior of functions in domΔ_μ

Now, in this section, we would like to turn our attention to some results concerning the local behavior of functions and namely their rate of convergence to junction points. Let q_i any boundary point and define

$$\varepsilon_m^i(u) = \sup_{F_i^m K} |u(x) - u(q_i)|.$$

We would like to get some results about how fast it is decaying to zero.

We will split our analysis into two parts. First with the standard measure and next with the Kusuoka measure.

For the standard measure, we have the following:

Lemma 3.1. *Let $u \in dom\Delta$ and consider any (m-1)-cell with boundary vertices y_0, y_1, y_2 and let x_0, x_1, $x_2 \in V_m \diagdown V_{m-1}$ be the vertex in that cell with x_j opposite y_j. Then*

$$u(x_2) = \frac{2}{5}(u(y_0)+u(y_1))+\frac{1}{5}u(y_2)+\frac{2}{3}\frac{1}{5^m}(\frac{6}{5}\Delta u(x_2)+\frac{2}{5}\Delta u(x_1)+\frac{2}{5}\Delta u(x_0))+R_m$$

and so on with $R_m = o(5^{-m})$

Theorem 3.2. *Let $u \in dom\Delta$. If $\partial_n u(q_0) \neq 0$ then*

$$c_1 \left(\frac{3}{5}\right)^m \leqslant \varepsilon_m \leqslant c_2 \left(\frac{3}{5}\right)^m$$

while if $\partial u(q_0) = 0$ then

$$\varepsilon_m \leqslant cm5^{-m}$$

Proof. Without loss of generality assume that $u(q_0) = 0$. First, we will prove estimates for $u(F_0^m q_1)$ and $u(F_0^m q_2)$ and then show that these estimates transfer to the entire cell. Formula (2.4.9) in [4] gives us that

$$2u(q_0) - u(F_0^m q_1) - u(F_0^m q_2) = \left(\frac{3}{5}\right)^m \partial_n u(q_0) + O\left(\frac{1}{5}\right)^m.$$

Now, in our case this gives

$$u(F_0^m q_1) + u(F_0^m q_2) = -\left(\frac{3}{5}\right)^m \partial_n u(q_0) + O\left(\frac{1}{5}\right)^m.$$

Now, by using the lemma above, and subtracting at the points $F_0^m(q_1)$ and $F_0^m(q_2)$ we obtain that

$$u(F_0^m q_1) - u(F_0^m q_2) = \frac{1}{5}(u(F_0^{m-1} q_1) - u(F_0^{m-1} q_2)) + O(5^{-m}).$$

This is a recursion relation for the difference, and it is easy to see that this implies

$$|u(F_0^m q_1) - u(F_0^m q_2)| \leqslant cm5^{-m}.$$

Then, if $\partial_n u(q_0) \neq 0$ we get that

$$c_1 \left(\frac{3}{5}\right)^m \leqslant |u(F_0^m q_j)| \leqslant c_2 \left(\frac{3}{5}\right)^m$$

while if $\partial_n u(q_0) = 0$ we get that $|u(F_0^m q_j)| \leqslant cm5^{-m}$

Now, to see that these estimates transfer to the entire cell. We will use a generic argument here. We recall first the scaling identity for the Laplacian

$$\Delta(u \circ F_w) = 5^{-m} f \circ F_w.$$

Then, if $u \in dom\Delta$ and $\Delta u = f$ and $F_w K$ is an m-cell, $|w| = m$, and we also have that

$$|u(F_w q_i)| \leqslant a \quad \text{for} \quad i = 0, 1, 2$$

then we can write $u \circ F_w = h + g$ where h is the harmonic function taking the same boundary values as $u \circ F_w$ and

$$g(x) = -5^{-m} \int G(x,y) f(F_w y) \mathrm{d}\mu(\mathrm{y}).$$

Then by the maximum principle for harmonic functions we have that $|h| \leqslant a$ and we also have that

$$|g(x)| \leqslant c_0 \|f\|_\infty 5^{-m}$$

where $c_0 = \sup\limits_{y \in K} \int G(x,y) \mathrm{d}\mu(\mathrm{y})$ So we obtain

$$|u| \leqslant a + c_0 \|f\|_\infty 5^{-m} \quad \text{on} \quad F_w K.$$

$\square$

We know that the normal derivatives always exist. However while it is not true in general for the tangential ones, we have the following sufficient condition.

Theorem 3.3. *Assume $u \in dom\Delta$ and Δu satisfies a Hölder condition of some order. Then $\partial_T u(q_0)$ exists.*

Using the above decay rates, we can also prove the very important result which is the main weakness of the standard Laplacian, and that is the domain of the Laplacian does not form an algebra.

Corollary 3.4. *Let u be any nonconstant function in $dom\Delta$. Then u^2 is not in $dom\Delta$.*

Proof. Let x_0 be a junction point such that $\partial_n u(x_0) \neq 0$. We can always find such a point since we assumed u is non constant. Then we can write $u(x_0) = y$. Thus $u = (u - y) + y$ and thus $u^2 = (u - y)^2 + 2y(u - y) + y^2$. Obviously $2y(u - y) + y^2 \in dom\Delta$ and thus we must show that $(u - y)^2 \notin dom\Delta$. By localizing the decay rates we have that $\partial_n(u - y)(x_0) \neq 0$ and thus by squaring the decay rates we have

$$(c_1)^2 \left(\frac{3}{5}\right)^{2m} \leqslant \tilde{\varepsilon}_m \leqslant (c_2)^2 \left(\frac{3}{5}\right)^{2m}$$

where

$$\tilde{\varepsilon}_m = \sup |u - y| \quad \text{in the m-cells containing } x_0.$$

Then by assuming that $(u - y)^2$ is in $dom\Delta$ we have a contradiction since these decay rates are impossible to be also compatible with the decay rates of $\Theta\left(\frac{3}{5}\right)^m$. □

Now, we would like to prove similar results for the Kusuoka measure. We would like to have an estimate of how the Kusuoka measure changes as we are zooming in on individual cells. A simple first result is to study how it changes by zooming in one direction.

Lemma 3.5. *For the Kusuoka measure we have that:*

$$\nu(F_0^m K) = \left(\frac{3}{5}\right)^m + \left(\frac{1}{15}\right)^m$$

Proof. It is easy to see by the definition of the energy measures that

$$\nu_{h_1}(F_0^m K) = \left(\frac{3}{5}\right)^m$$

and that

$$\nu_{h_2}(F_0^m K) = \left(\frac{1}{15}\right)^m.$$

Then, since $\nu = \nu_{h_1} + \nu_{h_2}$ we get the result.

$\square$

However this result has been significantly strengthened in [2] to have that for an arbitrary junction point, and thus we have the following lemma.

Lemma 3.6. *For the sequence $\{F_w F_i^m K\}$ which converges to the point $F_w(q_i)$ we have that*

$$\nu(F_w F_i^m K) = \Theta\left(\frac{3}{5}\right)^m$$

To obtain some results about the decay rates, we will need some results about the Green's function. The following theorem gives us the necessary tools. The proof of the theorem can be found in [5]

Theorem 3.7. *If $\phi(x) = \int_K |G(x,y) - G(Rx,y)| \mathrm{d}\nu(\mathrm{y})$ where R is the reflection about q_0. Then,*

$$\phi(F_0^m q_1) = \Theta\left(\frac{3}{5}\right)^{2m}.$$

If $\xi(x) = \sup_{y \in K} |G(x,y)|$, then

$$\xi(F_0^m q_1) = \Theta\left(\frac{3}{5}\right)^m$$

Now we are ready to prove important results for the decay rates of functions.

Theorem 3.8. *If u is skew-symmetric, $u \in dom\Delta_\nu$, then*

$$\varepsilon_m(u) = O\left(\frac{3}{5}\right)^{2m}$$

Proof. Write

$$u(x) = \int_K G(x,y)f(y)\mathrm{d}\nu(\mathrm{y}) + h_1(x) = \tilde{u}(x) + h_1(x)$$

where $f = \Delta_\nu(u)$ is a continuous function and h_1 is a harmonic function taking the same boundary values as u. Note that u is skew-symmetric implies h_1 is skew-symmetric and hence

$$h_1 = \Theta\left(\frac{1}{5}\right)^m.$$

On the other hand,

$$\tilde{u}(x) = |\int_K G(x,y)f(y)\mathrm{d}\nu(\mathrm{y})|$$

$$= \frac{1}{2}|\int_K (G(x,y) - G(Rx,y)f(y)\mathrm{d}\nu(\mathrm{y})|$$

$$\leqslant \frac{1}{2}\|f\|_\infty \int_K |(G(x,y) - G(Rx,y)|\mathrm{d}\nu(\mathrm{y}).$$

By the theorem above we have $\sup_{\partial(F_0^m K)} \tilde{u} = O\left(\frac{3}{5}\right)^{2m}$. To extend the estimate to the entire cell, we fix $F_0^m K$ and write $\tilde{u} \circ F_0^m = \tilde{g} + h_2$, where h_2 is the harmonic function taking the same boundary values as $\tilde{u} \circ F_0^m$. Then, we have

$$\tilde{g}(x) = -\left(\frac{3}{5}\right)^m \left[\left(\frac{3}{5}\right)^m \int_K G(x,y)f(F_0^m y)\mathrm{d}\nu_{\mathrm{h}}(\mathrm{y}) + \left(\frac{1}{15}\right)^m \int_K G(x,y)f(F_0^m y)\mathrm{d}\nu_{\mathrm{h}'}(\mathrm{y})\right]$$

where h and h' are the harmonic functions used in the definition of the Kusuoka measure. Since f is continuous, we can regard the integrals as $O(1)$ so,

$$\tilde{g}(x) = O\left(\frac{3}{5}\right)^{2m}.$$

Together with the fact that $h_2 = \Theta\left(\frac{1}{5}\right)^m = O\left(\frac{3}{5}\right)^{2m}$ we are done.

$\square$

Theorem 3.9. *Let $u \in dom\Delta_\nu, u(q_0) = 0$, then*

$$\varepsilon_m(u) = O\left(\frac{3}{5}\right)^m$$

Proof. Similar as before, we can extend the results to the entire cell, so we will estimate only on the boundary. We have that

$$u(x) = \int_K G(x,y)\Delta_\nu u(y)\mathrm{d}\nu(\mathrm{y}) + h_1(x) = \tilde{u}(x) + h_1(x)$$

and h_1 is harmonic taking the same boundary values as u. Then we have that since $h_1(q_0) = 0$ it must be a linear combination of the orthonormal basis of harmonic functions for the Kusuoka measure, so $h_1 = O\left(\frac{3}{5}\right)^m$. Also, by the theorem above we have that $|\tilde{u}| = O\left(\frac{3}{5}\right)^m$ and thus we are done.

$\square$

In the same way that we have a sufficient condition for the existence of the tangential derivative for the standard Laplacian, we have a slightly different one also for the Kusuoka one.

Proposition 3.10. *If u is skew-symmetric and $u \in dom{\Delta_\nu}^2$ then $\varepsilon_m(u) = O\left(\frac{1}{5}\right)^m$ and $\partial_T u(q_0)$ exists.*

Using the following lemma, we can also obtain results about more general functions.

Lemma 3.11. *If u is symmetric, $u \in dom\Delta_\nu$ and $u(q_0) = 0$ then*

$$2u(q_0) = -\left(\frac{3}{5}\right)^m \partial_n u(q_0) + \left(\frac{3}{5}\right)^m \int_{F_0K} \psi_{q_0}^{(m)} \Delta_\nu u \mathrm{d}\nu$$

Theorem 3.12. *Let u be symmetric and $u(q_0) = 0$ and $u \in dom{\Delta_\nu}^{k+1}$ for $k = 1, 2, 3...$ with $\partial_n {\Delta_\nu}^j u(q_0) = 0$ and ${\Delta_\nu}^{j+1} u(q_0) = 0$ for $j < k$. Then*

$$\varepsilon_m(u) = O\left(\frac{3}{5}\right)^{(2k+1)m}$$

Proof. The estimate will be done similarly as before only on the boundary. The proof is one of induction. For $k = 1$ we have that $\partial_n u(q_0) = 0$ and $\Delta_\nu u \in dom|\Delta_\nu$ and that $\Delta_\nu u(q_0) = 0$. Then by the above lemma we have that

$$|u(F_0^m q_1)| \leqslant \frac{1}{2}\varepsilon_m(\Delta_\nu u)\left(\frac{3}{5}\right)^m \nu(F_0^m K).$$

By using then the decay rates above to estimate $\varepsilon_m(\Delta_\nu u)$ we are done. The induction is based on the above lemma.

□

Now, we will generalize the results in [5] to any random junction point. We will see that the convergence bounds obtained for the boundary points are exactly the same for all junction points of the Sierpinski Gasket.

Lemma 3.13. *Let y be a junction point such that $y = F_w(q_i)$. For any u in the domain of the Laplacian, we have that*

$$\sup_{F_w F_i^m K} |u(x) - u(y)| = \sup_{F_i^m K} |u \circ F_w(x) - u \circ F_w(q_i)|$$

for any word w and $m \in \mathbb{N}$.

Proof. Let w be a word and u be in the domain of the Laplacian. Then we know that u is continuous. We have that F_i are continuous functions, then F_w are also continuous, and since K is a compact set, then $u \circ F_w$ has a maximum and minimum value on $u \circ F_w(K)$ which is also compact. Then, we have that there exists a $y_1 \in F_w F_i^m K$ such that:

$$\sup_{F_w F_i^m K} |u(x) - u(y)| = |u(y_1) - u(y)|.$$

But then, since $y_1 \in F_w F_i^m K$ it must be that there exists a $y_2 \in K$ such that $y_1 = F_w F_i^m(y_2)$ So,

$$|u(y_1) - u(y)| = |u(F_w F_i^m(y_2)) - u(y)| = |(u \circ F_w)(F_i^m(y_2)) - u(y)|$$

$$\leqslant \sup_{F_i^m K} |(u \circ F_w)(K) - u(y)|.$$

Using the exact same argument, we obtain the other inequality as well.

□

Lemma 3.14. *Define* $\varepsilon_m^i(u) = \sup_{F_i^m K} |u(x) - u(q_i)|$, *for i=0,1,2. Then,* $\varepsilon_m^i(u) \leqslant O\left(\frac{3}{5}\right)^m$

Proof. Let R be the clockwise rotation around the center point of the Sierpinski gasket by 90 degrees. Namely, if (x_0, y_0) is the center of the Sierpinski gasket, then $R(x, y) = (x_0 + (y - y_0), y_0 - (x - x_0))$. Then, obviously R is a continuous function, and $(R \circ F_1)(K) = F_0 K$. Now, let $u \in dom\Delta$. We have that

$$\sup_{F_1^m K} |u(x)| = \sup_{F_1^m K} |u \circ R^{-1} \circ R(x)| = \sup_{F_1^m K} |(u \circ R^{-1}) \circ R(x)|.$$

But if we call $g = u \circ R^{-1}$ then we have that g is continuous, $F_1^m K$ is compact, and thus we have

$$\sup_{F_1^m K} |g(R(x))|$$

is actually attained by say, $y_1 \in F_1^m K$ and thus

$$\sup_{F_1^m K} |g(R(x))| = |g(R(y_1))|.$$

But then, $R(y_1) \in F_0^m K$ so

$$\sup_{F_1^m K} |g(R(x))| = |g(R(y_1))| \leqslant \sup_{F_0^m K} |g(x)|.$$

But since $g \in dom\Delta$ we have that $\sup_{F_0^m K} |g(x)| \leqslant O\left(\frac{3}{5}\right)^m$. So,

$$\sup_{F_1^m K} |u(x)| \leqslant O\left(\frac{3}{5}\right)^m.$$

In an exact similar way, but instead with rotation anticlockwise, we obtain that

$$\sup_{F_2^m K} |u(x)| \leqslant O\left(\frac{3}{5}\right)^m$$

□

Proposition 3.15. *The rate of convergence of a function in the domain of the Laplacian to an arbitrary junction point, is bounded by* $O\left(\frac{3}{5}\right)^m$

Proof. Define

$$\varepsilon_m^i(u) = \sup_{F_i^m K} |u(x) - u(q_i)|.$$

Now, let y be any junction point that is not in V_0. Then, it is known that y has two addresses, namely $y = F_w(q_i)$ and $y = F_{w'}(q_i')$ with $w \neq w'$ and $i \neq i'$. Then, let $\{A_m\}_m$ be a sequence of cells with $\{A_m\}_m \to y$. Define, $\tilde{\varepsilon}_m(u) = \sup_{A_m} |u(x) - u(y)|$. Then, it is clear that

$$\tilde{\varepsilon}_m(u) \leqslant \sup_{F_w F_i^m K} |u(x) - u(y)| + \sup_{F_{w'} F_{i'}^m K} |u(x) - u(y)|$$

(Since, it is true that $\{F_w F_i^m K\} \to y$ and $\{F_{w'} F_{i'}^m K\} \to y$). Then, using the Lemma above, we have that

$$\tilde{\varepsilon}_m(u) \leqslant \sup_{F_i^m K} |(u \circ F_w(x)) - u \circ F_w(q_i)| + \sup_{F_{i'}^m K} |(u \circ F_{w'}(x)) - u \circ F_{w'}(q_{i'})|.$$

But, if we call now $u \circ F_w(x) = g(x)$ and $u \circ F_{w'}(x) = h(x)$ then it is clear that both g and h belong in the domain of the Laplacian. Thus,

$$\tilde{\varepsilon}_m(u) \leqslant \varepsilon_m^i(g) + \varepsilon_m^{i'}(h) \leqslant O\left(\frac{3}{5}\right)^m.$$

□

4 Local Solvability of Differential equations for the energy Laplacian

In this section we are interested in studying the solvability of differential equations of the form

$$\Delta_\mu u = f.$$

We have already seen that due to the existence of the Green's function we have solutions. However what is of particular interest is that we have solvability for other certain subsets of K.

Theorem 4.1. *Let Ω be an open subset of K not containing any points of V_0. Then the equation*

$$-\Delta_\mu u = f \quad \text{on } \Omega$$

has a solution for any continuous f on Ω.

This result is independent of the measure μ and the proof can be found in [8]. Of course the solution is not unique since we can always add an harmonic function on Ω.

Now, we are interested in the analogue of Picard existence and uniqueness theorem for the local solvability of

$$-\Delta_\nu u(x) = F(x, u(x)).$$

We already have results for the standard Laplacian in [8]. We are interested in extending them for the Kusuoka Laplacian as well. We will follow the proof of [8] in section 2 "Local Solvability" making the appropriate changes that are required by using Δ_ν instead of Δ.

Let $F(x, u)$ denote a continuous function from $K \times \mathbb{R}$ to $\mathbb{R}$ which satisfies also a Local Lipschitz condition in the u-variables:

$$\text{for every } T > 0, \text{ there exists } M_T < \infty \text{ such that}$$

$$|F(x,u) - F(x,u')| \leqslant M_T |u - u'|, \text{ provided } |u|, |u'| \leqslant T.$$

Theorem 4.2. *Given F satisfying the Lipschitz condition above, for every A there exists m such that for all choices of $\{a_j\}$ with $|a_j| \leqslant A$, the equation $-\Delta_\nu u(x) = F(x, u(x))$ on $F_w K$ for any $|w| = m$ and boundary conditions $u(F_w q_j) = a_j$ with $V_0 = \{q_1, q_2, q_3\}$ has a unique solution.*

First of all, before we begin the proof, we note that there is a typo in [2]. At Theorem 2.3 the expression

$$Q_j = \frac{1}{15} + \frac{12}{25}\frac{d\nu_j}{d\nu}$$

should be replaced with

$$Q_j = \frac{1}{25} + \frac{12}{25}\frac{d\nu_j}{d\nu}.$$

Then, we have the following scaling property for the energy Laplacian.

$$\Delta_\nu(u \circ F_j) = \frac{3}{5}Q_j(\Delta_\nu u) \circ F_j$$

for $Q_j = \frac{1}{15} + \frac{12}{15}\frac{d\nu_j}{d\nu}$. And using this, on page 8 of [2] we get that

$$\Delta_\nu(u \circ F_w) = \left(\frac{3}{5}\right)^m Q_w(\Delta_\nu u) \circ F_w$$

for $w = (w_1, ..., w_m)$ a finite word of length m and $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$ and

$$Q_w = Q_{w_m} \cdot (Q_{w_{m-1}} \circ F_{w_m}) \cdot (Q_{w-2} \circ F_{w-1} \circ F_{w_m}) \cdots (Q_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}).$$

In the following, we will use the estimate from [2]

$$\nu(F_i^m K) = \Theta\left(\frac{3}{5}\right)^m$$

from which it is obvious that

$$\nu(F_i^m K) = O\left(\frac{3}{5}\right)^m.$$

Now we are ready to prove the theorem.

Proof. First, we study the case in which all $a_j = 0$. By changing the variable $x \to F_w x$ where w is a word of length m, the original equation becomes

$$-\Delta_\nu(u \circ F_w)(x) = \left(\frac{3}{5}\right)^m Q_w(x) F(F_w x, u \circ F_w(x))$$

for $x \in K$. Let $v = u \circ F_w$. Then, the equation along with the boundary conditions above becomes

$$-\Delta_\nu v(x) = \left(\frac{3}{5}\right)^m Q_w(x) F(F_w x, v(x)) \text{ on K}$$
$$v_{|V_0} = 0$$

which we can write in an equivalent way to

$$v(x) = \left(\frac{3}{5}\right)^m \int_K G(x,y) Q_w(y) F(F_w y, v(y)) \mathrm{d}\nu(\mathrm{y}).$$

Let $\mathcal{G}v(x)$ to be

$$\mathcal{G}v(x) = \left(\frac{3}{5}\right)^m \int_K G(x,y) Q_w(y) F(F_w y, v(y)) \mathrm{d}\nu(\mathrm{y}).$$

The space of continuous functions v is a Banach space and thus to obtain our result it suffices to show that $\mathcal{G}$ satisfies the hypotheses of the contractive mapping principle on a suitable ball ($\|v\|_\infty \leqslant T$) . Since $G(x,y)$ is continuous and bounded (let G_0 be an upper bound), we have that

$$|\mathcal{G}v(x)| \leqslant \left(\frac{3}{5}\right)^m \|Q_w(y)\|_\infty G_0 F_T$$

where F_T is an upper bound for $|F(x,u)|$ for $x \in K$ and $|u| \leqslant T$ First, to bound $\|Q_w(y)\|_\infty$ we have that

$$\|Q_w(y)\|_\infty = \|Q_{w_m}(y) \cdot (Q_{w_{m-1}} \circ F_{w_m} y) \cdot (Q_{w-2} \circ F_{w-1} \circ F_{w_m} y) \cdots (Q_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m} y)\|_\infty$$
$$\leqslant \|Q_{w_m} \cdot Q_{w_{m-1}} \cdots Q_{w_1}\|_\infty \leqslant \|Q_{w_m}\|_\infty \cdot \|Q_{w_{m-1}}\|_\infty \cdots \|Q_{w_1}\|_\infty.$$

However, we have that for any $j = 0, 1, 2$ that

$$\|Q_j\|_\infty = \|\frac{1}{15} + \frac{12}{15}\frac{d\nu_j}{d\nu}\|_\infty \leqslant \frac{1}{15} + \frac{12}{15}\|\frac{d\nu_j}{d\nu}\|_\infty.$$

Here, we will use the key result from [2], Theorem 3.5 page 10. This theorem proves not only the discontinuity of the Radon-Nikodym derivative but also gives us bounds on its values with respect to the energy measure taken. In our case we see that the Radon-Nikodym derivative is bounded by $\frac{2}{3}$. Then

$$\|Q_j\|_\infty \leqslant \frac{1}{15} + \frac{12}{15} \cdot \frac{2}{3} = \frac{3}{5}.$$

And thus, we have that $\|Q_w\|_\infty \leqslant (\frac{3}{5})^m$. Then, $|\mathcal{G}(x)| \leqslant \left(\frac{3}{5}\right)^{2m} G_0 F_T$ so we just have to take m large enough such that $\left(\frac{3}{5}\right)^{2m} G_0 F_T \leqslant T$ to conclude that $\mathcal{G}$ maps the ball to itself. For v and v' in this ball, we have that

$$|\mathcal{G}v(x) - \mathcal{G}v'(x)| \leqslant \left(\frac{3}{5}\right)^{2m} G_0 M_T \|v - v'\|_\infty$$

so we get the contractive mapping estimate as long as $\left(\frac{3}{5}\right)^{2m} G_0 M_T < 1$.

Now, to modify the proof for the case of general $\{a_j\}$. Let $h(x)$ denote the harmonic function which satisfies the same boundary conditions as u. Then $(w = u - h)_{|F_w V_0} = 0$ and solves the equation

$$-\Delta_\nu w = F(x, h(x) + w(x))$$

so it is the same as in the initial special case with F being changed to

$$F'(x, u) = F(x, h(x) + u).$$

Note that $|h(x)| \leqslant A$, so by taking $T = 2A$, we have

$$|F'(x, u)| \leqslant F_T \text{ if } \|u\|_\infty \leqslant A,$$

and we can apply the same argument as before.

□

5 Approximation by functions vanishing in a neighborhood of the boundary

Now, we will generalize some results found in [6] in section 7 "Spline cut-offs" from the standard Laplacian to the Kusuoka Laplacian. It is proven in [10] that we have a "weak=strong" property for the Laplacian, as long as certain conditions are satisfied for the test functions, namely that $v(q_i) = 0$ and $\partial_n v(q_i) = 0$. These conditions have been weakened in [6] to the smaller class of functions v such that they vanish on a neighborhood of the boundary. Our goal in this section is to obtain a similar result for the Kusuoka Laplacian as well.

Lemma 5.1. *For any $u \in \mathcal{H}_1$ we have that*

$$\|\Delta_\nu u_{|F_wK}\|_\infty \leqslant c\left(\left(\frac{5}{3}\right)^{2m} \sup_{\partial F_wK} |u| + \left(\frac{5}{3}\right)^{m} \sup_{\partial F_wK} |\partial_n u|\right).$$

Proof. Since $\Delta_\nu u$ is harmonic, we simply have to bound its values on $\partial F_wK = F_w\partial K$. Now, for w equal to the empty word, the above estimate is an immediate consequence of the basis for $\mathcal{H}_1$ defined in [6]. The general case then follows from the scaling property for the Kusuoka Laplacian, where we substitute $\|Q_w\|_\infty = \left(\frac{3}{5}\right)^m$ and the scaling property for the normal derivatives as well.

□

Lemma 5.2. *Let Δ_ν be the Kusuoka Laplacian and $f \in dom\Delta_\nu$ such that it vanishes along with its normal derivatives on the boundary. Then*

$$f_{|\partial F_i^m K} = O\left(\frac{3}{5}\right)^{2m}$$

and

$$\partial_n f_{|\partial F_i^m K} = O\left(\frac{3}{5}\right)^{m}.$$

Proof. For the first part:

First, let $i = 0, 1, 2$. Then,

$$u(x) = \frac{1}{2}(u(x) + u(R_ix)) + \frac{1}{2}(u(x) - u(R_ix))$$

where R_i is the reflection around the point q_i. For shorter notation let's call the symmetric part u_+ and the skew symmetric part u_-. Then, $\partial_n u_-(q_i) = 0$ since always for a skew-symmetric function the normal derivatives are zero. But then, this implies that also $\partial u_+(q_i) = 0$ since we assumed that u has its normal derivatives zero on the boundary. Then we have that

$$\sup_{\partial F_i^m K} |u| \leqslant \sup_{\partial F_i^m K} |u_-| + \sup_{\partial F_i^m K} |u_+| \leqslant O\left(\frac{3}{5}\right)^{2m}$$

by using theorem 3.1 and 3.6 from [5].

For the second part:

We use the Gauss-Green formula localized to $F_i^m K$ with the functions $u = f$ and $v = h$ where h is the harmonic function taking the values 1,-1,0 on the boundary points of $F_i^m K$ (the value 1 at the point q_i) Then, this gives us

$$-\int_{F_i^m K} h\Delta f \mathrm{d}\nu = \sum_{x\in V_0} f(F_i^m x)\partial_n h(F_i^m x) - h(F_i^m x)\partial_n f(F_i^m x).$$

By assumption, $\partial_n f(F_i^m q_i) = 0$ so the only term in the right hand side of the form $-h(F_i^m x)\partial_n f(F_i^m x)$ that occurs is the single value at the vertex where h assumes the value -1. The integral on the left side is $O\left(\frac{3}{5}\right)^m$ since h and f are uniformly bounded and the measure is $O\left(\frac{3}{5}\right)^m$ and the terms of the form $f(F_i^m x)\partial_n h(F_i^m x)$ are $O\left(\frac{3}{5}\right)^m$ since $\partial_n h(F_i^m x) = O\left(\frac{5}{3}\right)^m$ and

$$f(F_i^m x) = O\left(\frac{3}{5}\right)^{2m}$$

by the first part of this lemma.

$\square$

Using these two lemmas, we are ready to prove the following theorem.

Theorem 5.3. *For the Kusuoka Laplacian on SG, suppose that $f \in dom\Delta_\nu$ and f vanishes together with its normal derivatives on the boundary. Then there exists a sequence of functions $\{f_m\}$ with each $f_m \in dom\mathcal{E}$ vanishing in a neighborhood of the boundary with $f_m \to f$ uniformly, $\mathcal{E}(f_m - f) \to 0$ and $\Delta_\nu f_m \to \Delta_\nu f$ in $L^p(d\nu)$ for any $p < \infty$.*

Proof. As in the proof of theorem 7.1 on [6], we choose f_m so that $f_m = f$ on Ω_m with support in Ω_{m+1}. On each of the sets $F_i^m K$ we take f_m to be the spline locally in $\mathcal{S}(\mathcal{H}_1, V_{m+1})$ so that $f_m = f$ and $\partial_n f_m = \partial_n f$ at the two boundary points of $F_i^m K$ not equal in q_i and $f_m = 0$ and $\partial_n f_m = 0$ in the other 4 vertices in $V_{m+1} \cap F_i^m K$. Because we have matched the values of the functions and the normal derivatives, the functions f_m will be in $dom\Delta_\nu$. We will show that

$$\int_{F_i^m K} |\Delta_\nu f_m|^p \mathrm{d}\nu \to 0 \text{ as } m \to \infty \text{ for every } p < \infty.$$

After this, the proof is identical to the standard Laplacian case since no other differences between the standard and the energy Laplacian arise. We will use the previous lemma. We have then that:

$$\|\Delta_\nu f_{m|F_i^m K}\|_\infty \leqslant c[\left(\frac{5}{3}\right)^{2m} \sup_{\partial F_i^m K} |f| + \left(\frac{5}{3}\right)^{m} \sup_{\partial F_i^m K} |\partial_n f|]$$

and by using the lemmas above, we have that $\|\Delta_\nu f_{m|F_i^m K}\|_\infty \leqslant C$ and thus since $\nu(F_i^m K) = O\left(\frac{3}{5}\right)^m$ we get that

$$\int_{F_i^m K} |\Delta_\nu f_m|^p \mathrm{d}\nu \to 0$$

□

Then, the proof for the following interesting Corollary on [6] is exactly identical to the standard Laplacian case by using the above theorem and can be found at [6].

Corollary 5.4. *Let Δ be the Kusuoka Laplacian on SG. If $u \in L^2(d\nu)$ and $f \in L^2(d\nu)$ (respectively, f is continuous), and*

$$\int_K u \Delta v \mathrm{d}\nu = \int_K f v \mathrm{d}\nu$$

for all $v \in dom_C(\Delta_\nu)$ vanishing on a neighborhood of the boundary, then $u \in dom_{L^2}(\Delta_\nu)$ (respectively $dom_C(\Delta_\nu)$ and $\Delta_\nu = f$.

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