

UPPSALA UNIVERSITET
MATEMATISKA INSTITUTIONEN

MASTER'S THESIS

**KNOCK OUT POWER OPTIONS IN FOREIGN EXCHANGE
MARKETS**

COURSE: DEGREE PROJECT E IN MATHEMATICS

Student's name: Sicaio, Tomé Eduardo

Supervisor: Prof. Johan Tysk

tomeeduardosicaio@uem.mz

Civic registration number: 790603-P112

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Abstract

In recent years, the pressure of governments in maintaining currency parity has led to the break down of quite a few exchange rate mechanisms and has, thus, strengthened the need for companies, in particular, to make foreign exchange hedging decisions in order to avoid erosion of profit margins. This thesis deals with the pricing of Foreign exchange options. The Knock out options and the Power options will be treated in the sense that their payoffs are computed in closed form. We also combine the previous options in order to yield a new financial product. We find an explicit pricing formula for such financial product. Additionally, a new product of First generation exotic options, called Knock out power options, is described and its payoff is computed and compared with the payoff of the Knock out options.

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Introduction

Foreign exchange options are of great importance because nowadays are basic ingredients to international trading, they are many and each with its strategy for hedging.

Wytup [16] in his work describes the Power options whose payoff is just a vanilla option (call or put) raised to the power of n , that is, $[(X_T - K)^+]^n$, $[(K - X_T)^+]^n$ in the symmetric case; and each term of vanilla option (call or put) is raised to the power of n , that is, $(X_T^n - K^n)^+$, $(K^n - X_T^n)^+$ in the asymmetric case. Therefore, we encounter two kinds of Power options, the asymmetric and symmetric.

Furthermore, Wytup [16] also realizes that these options (Power options) are always equipped with high payoff compared to vanilla options. This, limits the risk of a short position even as the option premium for the holder.

This is one of the reasons that motivates speculators to invest in Power options once they request a high option premium. Yet, Wytup [16] and Tompkins [15] discuss hedging possibilities for Power options and they mention that these options could be hedged using a combination of vanilla options with different strike prices.

Knock out options are Barrier options whose payoff is a Vanilla option if up to maturity time the underlying foreign exchange rate never hits the barrier agreed previously.

The holder of the Knock out option provides protection against the rising of the foreign exchange rate and according to this protection the holder has to pay a premium.

Lipton [7] discusses Barrier options for Foreign exchange options and give different forms of calculation of their payoff under risk neutral valuation.

Wytup [16] describes some of advantages and disadvantages of Knock out power options. One of the advantages are: Knock out options are cheaper than Vanilla options, Knock out options provide a conditional protection against, for example, stronger USD/weaker SEK, Knock out options give a complete participation, for example, in a weaker USD/stronger SEK. Some of the disadvantages of Knock out options are: The exchange rate may hit the barrier before maturity time and another one is having to pay a premium.

One of the main aims in this thesis is to combine both options, Knock out options and Power options, and forming a new financial product in foreign exchange option which we will call it as Knock out power option. And we will compute its pricing function under risk neutral valuation using different methods.

The thesis consists of five chapters, the first one describe foreign exchange options, and herein,

we give an overview about foreign exchange options, the dynamics of foreign exchange rate are derived and is also described the pricing partial differential equation (known as Black-Scholes partial differential equation) for foreign exchange options.

In chapter two we describe the joint density of Brownian motion and its maximum and minimum. For this purpose we discuss the Reflection Principle in which we derive the main probability result that will be used in description of the joint density of Brownian motion and its maximum and minimum.

In the third chapter we describe the types of Foreign exchange options and special attention is given to Knock out options and Power options, and we compute the payoff under risk neutral valuation using different methods.

In the fourth chapter we describe our new financial product, Knock out power option, and also evaluate the payoff under risk neutral of the Down and out asymmetric power call option and Up and out symmetric power call option, using different methods. We also describe the sensitivity's analysis and we discuss the static hedging for Up and out power call option.

In the last chapter, the fifth, we describe the conclusions about our new financial product, Knock out power option.

Chapter 1

Foreign Exchange Options

1.1 Foreign Exchange Options Overview

Foreign Exchange Options are of great importance in the business world, allowing international trading over all the world. These options give the holder the right but not the obligation to exchange one currency into another currency at a determined exchange rate on a maturity date previously agreed. Nowadays, trade is just a small part of the domain of foreign exchange market whose leading participants are Commercial Banks, Brokers, The International Monetary Market, Investors, Money Managers, Funds and Central Banks.

The market of foreign exchange options is one of the oldest, by Shamah [12], just because between 9000 to 6000 BCE (Before Common/Current/Christian Era) was seen cattle as well as camels being used as the first and the form of money in exchange options.

The principal characteristic of any foreign exchange option for its holder are those of limited risk and, at the same time, profit potential is unlimited.

Foreign exchange options have been developed to protect companies from some randomness of the exchange rate displacement. They are used by companies as contingent cover because their exposure can lead to unnecessary losses agreement on currency transactions.

1.2 Exchange Rate Dynamics

Before we go deeper in derivation of the Exchange Rate Dynamics let us have a look in some technical concepts which will be need along the section. These definitions can also be found at least on Shreve [13], Neftci [11], Khoshnevisan [5] and Shamah [12].

Definition 1.2.1. An *Exchange rate* between two currency is the value at which one currency worth to be swapped for another.

Definition 1.2.2. A *probability space* consists of a sample space Ω , a set of events \mathcal{F} and a probability measure $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$. The sample Ω is a nonempty set, the collection \mathcal{F} satisfies the

following properties

1. $\emptyset \in \mathcal{F}$;
2. Whenever $A \in \mathcal{F}$, its complement $A^c \in \mathcal{F}$;
3. whenever $A_1, A_2, \dots \in \mathcal{F}$, then their union $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

In addition the **probability measure** \mathbb{P} is a function that assigns to each element $\omega \in \mathcal{F}$ a number in $[0, 1]$ so that

1. $\mathbb{P}(\Omega) = 1$
2. Whenever A_1, A_2, \dots is a sequence of disjoint events in \mathcal{F} , then $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

Definition 1.2.3. Let $(\Omega, \mathbb{P}, \mathcal{F})$ be a probability space. A real-valued function X defined on Ω with the property that for every Borel subset B of \mathbb{R} , the subset of Ω given by $\{X \in B\} = \{\omega \in \Omega; X(\omega) \in B\}$ is in \mathcal{F} , will be called **random variable**.

Definition 1.2.4. Let $X(t)$ be a collection of random variables, where t is a parameter that runs over an index set T . The collection of random variables in an indexed time set is called **stochastic process**.

Definition 1.2.5. A **Wiener process** W is a stochastic process which satisfies the following properties

1. $W(0) = 0$.
2. The process W has independent increments, i.e. if $r < s \leq t < u$ then $W(u) - W(t)$ and $W(t) - W(r)$ are independent stochastic variable.
3. For $s < t$ the stochastic variable $W(t) - W(s)$ has Gaussian distribution with mean zero and standard deviation $\sqrt{t - s}$
4. W has continuous trajectories.

Let $X(t)$ be the Spot Exchange Rate at time t quoted as the quotient between the number of units of the domestic currency and the the number of units of the foreign currency. We will use the notation in Wystup [16] for a quote of foreign exchange rate, as follows, FOR-DOM, which mean that one unit of foreign currency worth FOR-DOM units of domestic currency. For example, in the case of USD-SEK with a spot of 6.3800, this means that one unit of USD worth 6.3800SEK. We regard that B_f and B_d are bank accounts of the foreign currency and domestic currency,

respectively. We suppose that under objective probability measure, the Spot Exchange Rate $X(t)$ has a dynamics of a Geometric Brownian Motion [1]

$$dX = \alpha_X X dt + \sigma_X X d\bar{W}, \quad (1.1)$$

where $\alpha_X, \sigma_X, \bar{W}$ are constant drift, constant volatility and scalar Wiener Process, respectively. We consider that the foreign bank account and the domestic bank account have the following dynamics

$$dB_f = r_f B_f dt, \quad (1.2)$$

$$dB_d = r_d B_d dt, \quad (1.3)$$

where r_f, r_d are interest rates of the foreign account and domestic account, respectively. According to our exposure B_f units of foreign currency worth $X \cdot B_f$ units in the domestic currency. Thus, we regard that

$$\bar{B}_f = X \cdot B_f, \quad (1.4)$$

the dynamics of \bar{B}_f will be obtained applying Ito's product rule [13] to (1.4), and using the equation (1.1) thus

$$\begin{aligned} d\bar{B}_f &= X dB_f + B_f dX + (dX)(dB_f) \\ &= X r_f B_f dt + X B_f \alpha_X dt + X B_f \sigma_X d\bar{W} + 0 \\ &= \bar{B}_f (r_f + \alpha_X) dt + \bar{B}_f \sigma_X d\bar{W}. \end{aligned}$$

We know that the interest rate in the domestic account is r_d , so we can write the above equation under martingale measure Q as follows

$$d\bar{B}_f = r_d \bar{B}_f dt + \bar{B}_f \sigma_X dW. \quad (1.5)$$

Now, we seek for the dynamics of the spot exchange rate X under martingale measure Q . Using the relation (1.4) we can write X as follows

$$X = \frac{\bar{B}_f}{B_f}. \quad (1.6)$$

Thus, the Q -dynamics of spot exchange rate X will be obtained applying Ito's product rule to (1.6) and using relations (1.2) and (1.5) yields

$$\begin{aligned} dX &= \bar{B}_f dB_f^{-1} + B_f^{-1} d\bar{B}_f + (d\bar{B}_f)(dB_f^{-1}) \\ &= -\bar{B}_f B_f^{-2} dB_f + B_f^{-1} \bar{B}_f r_d dt + B_f^{-1} \bar{B}_f \sigma_X dW - 0 \\ &= -\bar{B}_f B_f^{-2} B_f r_f dt + B_f^{-1} \bar{B}_f r_d dt + B_f^{-1} \bar{B}_f \sigma_X dW \\ &= -r_f X dt + r_d X dt + \sigma_X X dW \\ &= X(r_d - r_f) dt + \sigma_X X dW. \end{aligned}$$

Therefore, the dynamics of spot exchange rate under martingale measure is given by

$$dX = X(r_d - r_f)dt + \sigma_X X dW. \quad (1.7)$$

Now, we compute the solution to the dynamics of spot exchange rate. Let $Z = \ln X$, applying Ito's formula we get

$$\begin{aligned} dZ &= \frac{1}{X}dX - \frac{1}{2} \frac{1}{X^2}(dX)^2 \\ &= (r_d - r_f)dt + \sigma_X dW - \frac{1}{2} \sigma_X^2 dt \\ &= \left(r_d - r_f - \frac{1}{2} \sigma_X^2 \right) dt + \sigma_X dW, \end{aligned}$$

integrating from t to T yields

$$Z(T) - Z(t) = \left(r_d - r_f - \frac{1}{2} \sigma_X^2 \right) (T - t) + \sigma_X (W(T) - W(t)),$$

we know that $Z = \ln X$, so

$$\ln X(T) - \ln X(t) = \left(r_d - r_f - \frac{1}{2} \sigma_X^2 \right) (T - t) + \sigma_X (W(T) - W(t)).$$

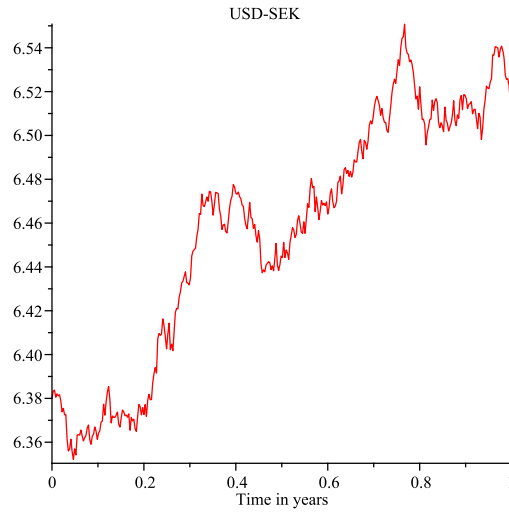


Figure 1.1: Spot Exchange Rate

Therefore, solution to the dynamics of spot exchange rate is

$$X(T) = X(t) \cdot e^{\left(r_d - r_f - \frac{1}{2} \sigma_X^2 \right) (T - t) + \sigma_X (W(T) - W(t))}, \quad (1.8)$$

where $(W(T) - W(t))$ has a normal distribution with zero mean and standard deviation $\sqrt{T - t}$.

The graph in the Figure 1.1 shows a simulated path, using Monte Carlo method [2], of exchange rate quoted in (USD-SEK) with spot 6.3800, $r_{\text{SEK}} = 12\%$, $r_{\text{USD}} = 8\%$, $\sigma_X = 0.43$, Maturity $T = 1$ year.

1.3 Pricing Foreign Exchange Options

Our aim herein, is to derive the pricing equation for foreign exchange options. Similar results can be found on Björk [1], Jiang[4] and Kwok [6]. Now, regarding the models given in the previous section defined by (1.1), (1.2) and (1.3) and a simple contingent claim of the form $\Phi(X(T))$, where Φ is a contract function.

Let us denote the relative portfolio invested in foreign exchange and the derivative by $u_{\bar{B}_f}$ and u_F , respectively. The dynamics for the value V of the portfolio is as follows

$$dV = V \left(u_{\bar{B}_f} \frac{d\bar{B}_f}{\bar{B}_f} + u_F \frac{dF}{F} \right), \quad (1.9)$$

where $d\bar{B}_f = \bar{B}_f(r_f + \alpha_X)dt + \bar{B}_f\sigma_X d\bar{W}$ and $F = F(t, X(t))$ is the pricing function whose dynamics is, applying Ito's formula and using (1.1), as follows

$$\begin{aligned} dF &= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial X} dX + \frac{1}{2} \frac{\partial^2 F}{\partial X^2} (dX)^2 \\ &= \frac{\partial F}{\partial t} dt + (X\alpha_X dt + X\sigma_X d\bar{W}) \frac{\partial F}{\partial X} + \frac{1}{2} \sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} dt \\ &= \left(\frac{\partial F}{\partial t} + X\alpha_X \frac{\partial F}{\partial X} + \frac{1}{2} \sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} \right) dt + \frac{\partial F}{\partial X} X\sigma_X d\bar{W} \\ &= F\alpha_F dt + F\sigma_F d\bar{W}, \end{aligned}$$

where

$$\alpha_F = \frac{\frac{\partial F}{\partial t} + X\alpha_X \frac{\partial F}{\partial X} + \frac{1}{2} \sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2}}{F} \quad \text{and} \quad \sigma_F = \frac{\frac{\partial F}{\partial X} X\sigma_X}{F}. \quad (1.10)$$

Taking the dynamics of \bar{B}_f and F to (1.9) and, collecting dt term and $d\bar{W}$ term yields

$$dV = V(u_{\bar{B}_f}(r_f + \alpha_X) + u_F\alpha_F)dt + V(u_{\bar{B}_f}\sigma_X + u_F\sigma_F)d\bar{W}.$$

We have to find the relative portfolio in a such way that

$$\begin{cases} u_{\bar{B}_f}\sigma_X + u_F\sigma_F = 0, \\ u_{\bar{B}_f} + u_F = 1. \end{cases}$$

Solving the above system of linear equations we get the following solution to the system

$$\begin{cases} u_F = -\frac{\sigma_X}{\sigma_F - \sigma_X}, \\ u_{\bar{B}_f} = \frac{\sigma_F}{\sigma_F - \sigma_X}. \end{cases} \quad (1.11)$$

So the $d\bar{W}$ term of the portfolio value will vanish and we will remain with

$$dV = V(u_{\bar{B}_f}(r_f + \alpha_X) + u_F\alpha_F)dt.$$

The theory of arbitrage free pricing implies that we must have

$$u_{\bar{B}_f}(r_f + \alpha_X) + u_F\alpha_F = r_d. \quad (1.12)$$

Substituting (1.11) into (1.12) and multiplying the entire equation by $\sigma_F - \sigma_X$ we get

$$(r_f + \alpha_X)\sigma_F - \sigma_X\alpha_F = r_d(\sigma_F - \sigma_X).$$

Taking (1.10) into the above equation we yield the following partial differential equation,

$$\frac{\partial F}{\partial t} + (r_d - r_f)X \frac{\partial F}{\partial X} + \frac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} - r_d F = 0,$$

the so called Black-Scholes partial differential equation or in a short way the Black-Scholes equation.

Proposition 1.3.1. *The pricing function $F(t, x)$ of the claim $\Phi(X_T)$ solves the boundary value problem*

$$\begin{cases} \frac{\partial F}{\partial t} + (r_d - r_f)x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 F}{\partial x^2} - r_d F = 0, \\ F(T, x) = \Phi(x). \end{cases}$$

Now, we can apply the Feynman-Kac representation theorem, which may be found on Björk [1], in the above proposition to give us a risk neutral valuation formula.

Proposition 1.3.2. *The pricing function has the representation*

$$F(t, x) = e^{-r_d(T-t)} \mathbb{E}_{t,x}^Q[\Phi(X_T)],$$

where the Q -dynamics of X are given by

$$dX(t) = (r_d - r_f)X(t)dt + \sigma_X X(t)dW(t).$$

Proof. First we have to apply Ito's formula to the pricing function $F(t, X(t))$ as follows

$$dF(t, X) = \left(\frac{\partial F}{\partial t} + (r_d - r_f)X \frac{\partial F}{\partial X} + \frac{1}{2} \sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} \right) dt + X \sigma_X \frac{\partial F}{\partial X} dW.$$

Using Proposition 1.3.1 we can write the above equation in the following way

$$dF(t, X) = r_d F dt + X \sigma_X \frac{\partial F}{\partial X} dW.$$

Integrating the above equation in $t \leq s \leq T$ yields

$$F(s, X(s)) - F(t, X(t)) = r_d \int_t^s F(\tau, X(\tau)) d\tau + \sigma_X \int_t^s X(\tau) \frac{\partial F(\tau, X(\tau))}{\partial X} dW(\tau).$$

Furthermore, the process $X(\tau) \frac{\partial F(\tau, X(\tau))}{\partial X}$ is integrable and we take the expected value, the stochastic integral will be zero and remain the following equation

$$E_{t,x}^Q[F(s, X(s))] - E_{t,x}^Q[F(t, X(t))] = r_d \int_t^s E_{t,x}^Q[F(\tau, X(\tau))] d\tau.$$

Differentiating the above result with respect to s and integrating from t to T and regarding the initial condition, we get

$$F(t, x) = e^{-r_d(T-t)} E_{t,x}^Q[\Phi(X(T))].$$

□

Chapter 2

The Joint Distribution of Brownian Motion and its Maximum and Minimum

In this section we are going to find the joint density of Brownian motion and its maximum or minimum. Similar results, for example, for maximum was discussed in Shreve [13] who has shown the joint density of Brownian motion and its maximum. We also will derive the basic result about Reflection principle which was discussed in Kwok [6] and Shreve [13]. Using reflection principle we are going to derive the basic probability result which will help us to derive the joint density of Brownian motion and its maximum or minimum.

2.1 Reflection Principle

Consider a standard Brownian motion $W(t)$, let $\bar{M}(T)$ be the maximum of Brownian motion $W(t)$, suppose that the maximum $\bar{M}(T)$ is greater than a positive m , fix a time T and regard that the first passage time τ_m , in which the Brownian motion path reaches the level m for the first time, $\tau_m < T$, see Figure 2.1.

Let us define $\bar{W}(t)$ as the mirror reflection of $W(t)$ at level m between the time interval $[\tau_m, T]$ and consider $\bar{W}(t)$ being as follows

$$\bar{W}(t) = \begin{cases} W(t), & 0 \leq t < \tau_m, \\ 2m - W(t), & \tau_m \leq t < T. \end{cases}$$

The events $\{W(T) < w\}$ and $\{\bar{W}(T) > 2m - w\}$ are equivalent, obviously from τ_m onward. For $\tau_m \leq t \leq T$, hold the following equality

$$\bar{W}(\tau_m + \delta) - \bar{W}(\tau_m) = -(W(\tau_m + \delta) - W(\tau_m)), \quad \delta > 0. \quad (2.1)$$

The first passage time τ_m , will not affect the Brownian motion at onward time since it depends only on trajectory's history of Brownian motion, $\{W(t) : 0 \leq t \leq \tau_m\}$.

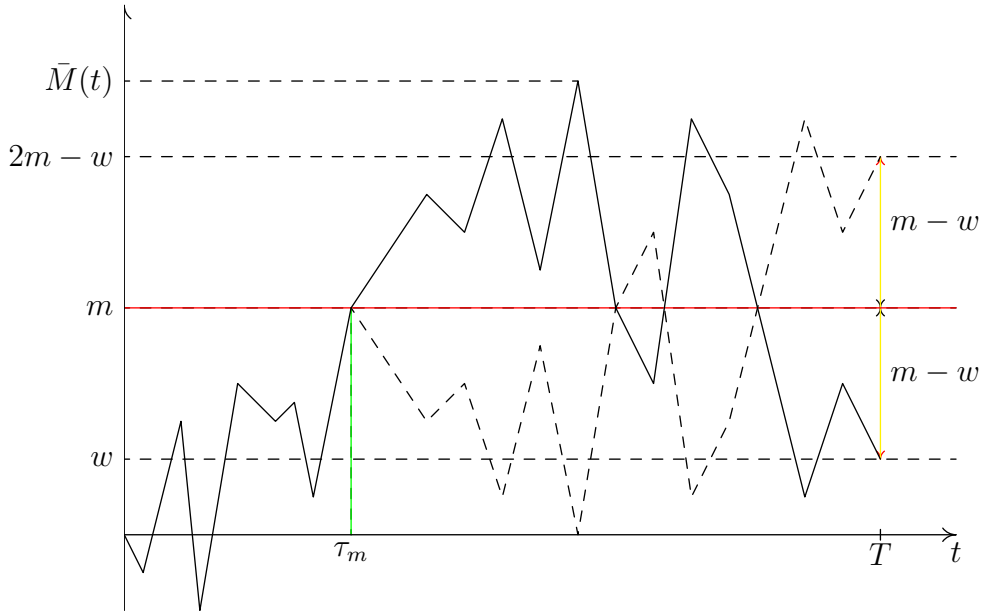


Figure 2.1: Brownian path with reflection after first passage time

The Brownian motion increments in (2.1) have the same distribution, it follows from the strong Markov property of Brownian motion.

The maximum of Brownian motion, $\bar{M}(T)$ is greater than m if and only if the first passage time is lesser than T . Now, suppose that $W(T) \leq w$, then $\bar{W}(T) \geq 2m - w$ and together with (2.1) we obtain

$$\begin{aligned}
 P(\bar{M}(T) \geq m, W(T) \leq w) &= P(\tau_m \leq T, W(T) \leq w) \\
 &= P(\bar{W}(T) \geq 2m - w) \\
 &= P(W(T) \geq 2m - w),
 \end{aligned}$$

for all $w \leq m$ and $m > 0$.

Proposition 2.1.1. *The probability of standard Brownian motion $W(t)$ being lesser than a fixed level w and the Maximum $\bar{M}(t)$ being greater than a fixed $(2m - w)$ at time T , as described in Figure 2.1, is given by the formula*

$$P(\bar{M}(T) \geq m \wedge W(T) \leq w) = P(W(T) \geq 2m - w), \quad w \leq m, \quad m > 0. \quad (2.2)$$

2.2 Joint density of Brownian motion and its maximum

In this section we will follow the theory described in Shreve [13] and Kwok [6]. Let $W(t)$ be a Brownian motion and let us denote by $\bar{M}(T) = \max_{0 \leq t \leq T} W(t)$ the maximum of Brownian

motion. We also denote by τ_m the first passage time at level $m > 0$. Furthermore, $\bar{M}(T) \geq m$ if and only if the first passage time τ_m is less or equal to time t .

Thus, in order to find the joint density of $(\bar{M}(T), W(T))$ we are going to apply the result from Proposition 2.1.1 derived in previous section about reflection principle, so

$$P(\bar{M}(T) \geq m \wedge W(T) \leq w) = P(W(T) \geq 2m - w), \quad w \leq m, \quad m > 0. \quad (2.3)$$

Regarding the reflection principle equality (2.3) and supposing that $f_{\bar{M},W}(m, w)$ is the joint density of $(\bar{M}(T), W(T))$ for $T > 0$ we have

$$\int_m^\infty \int_{-\infty}^w f_{\bar{M},W}(x, y) dy dx = \int_{2m-w}^\infty \frac{e^{-\frac{1}{2T}z^2}}{\sqrt{2\pi T}} dz. \quad (2.4)$$

Now, we may differentiate (2.4) in order to m to get

$$-\int_{-\infty}^w f_{\bar{M},W}(m, y) dy = \frac{2}{\sqrt{2\pi T}} e^{-\frac{(2m-w)^2}{2T}}. \quad (2.5)$$

Herein, we differentiate (2.5) with respect to w to obtain

$$f_{\bar{M},W}(m, w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{-\frac{(2m-w)^2}{2T}}.$$

The following two theorem below were discussed on Shreve [13].

Theorem 2.2.1. *Let $\bar{M}(T)$ be the maximum of Brownian motion $W(t)$. The joint density of $(\bar{M}(T), W(T))$ under measure Q , for $T > 0$ is*

$$f_{\bar{M},W}(m, w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{-\frac{(2m-w)^2}{2T}}, \quad \text{on } D = \{(m, w) : w \leq m, \quad m \geq 0\}$$

and zero on complement of D .

Theorem 2.2.2. *Let $W(t)$ be a Brownian motion with drift α and let $\bar{M}(T)$ be the maximum of $W(t)$. The joint density of $(\bar{M}(T), W(T))$ under measure \bar{Q} , for $T > 0$ is*

$$f_{\bar{M},W}(m, w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{(2m-w)^2}{2T}}, \quad \text{on } D = \{(m, w) : w \leq m, \quad m \geq 0\}$$

and zero on complement of D .

2.3 Joint density of Brownian motion and its minimum

Consider $W(t)$ as a Brownian motion and denote by $\underline{M}(T) = \min_{0 \leq t \leq T} W(t)$ the minimum of Brownian motion $W(t)$. We also denote by τ_m the first passage time at level $m < 0$, in this case, $\underline{M}(T) \leq m$ if and only if the first passage time is less or equal to time T . Therefore, in order to find the joint density of $(\underline{M}(T), W(T))$ we are going to apply the reflection principle, thus

$$P(\underline{M}(t) \leq m \wedge W(t) \geq w) = P(W(t) \leq 2m - w), \quad w \geq m, \quad m < 0. \quad (2.6)$$

According to the reflection principle (2.6) and supposing that $f_{\underline{M},W}(m, w)$ is the joint density of $(\underline{M}(T), W(T))$ for $T > 0$ we have

$$\int_{-\infty}^m \int_w^{\infty} f_{\underline{M},W}(x, y) dx dy = \int_{-\infty}^{2m-w} \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T}z^2} dz. \quad (2.7)$$

Differentiating (2.7) with respect to m yields

$$\int_w^{\infty} f_{\underline{M},W}(m, y) dy = \frac{1}{\sqrt{2\pi T}} e^{-\frac{(2m-w)^2}{2T}}. \quad (2.8)$$

Now, differentiating (2.8) with respect to w yields

$$-f_{\underline{M},W}(m, w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{-\frac{(2m-w)^2}{2T}}.$$

Theorem 2.3.1. *Let $\underline{M}(T)$ be the minimum of Brownian motion $W(T)$. The joint density of $(\underline{M}(T), W(T))$ under measure Q , for $T > 0$ is given by the formula*

$$f_{\underline{M},W}(m, w) = \frac{2(w-2m)}{T\sqrt{2\pi T}} e^{-\frac{(2m-w)^2}{2T}}, \quad \text{on } D = \{(m, w) : w \geq m, \quad m \leq 0\}$$

and zero on complement of D .

Theorem 2.3.2. *Let $W(T)$ be a Brownian motion with drift α and let $\underline{M}(T)$ be the minimum of $W(T)$. The joint density of $(\underline{M}(t), W(t))$ under measure \bar{Q} , for $T > 0$ is*

$$f_{\underline{M},W}(m, w) = \frac{2(w-2m)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{(2m-w)^2}{2T}}, \quad \text{on } D = \{(m, w) : w \geq m, \quad m \leq 0\}$$

and zero on complement of D .

Chapter 3

Types of Foreign Exchange Options

There exist many types of Foreign exchange options, Shamah [12] and Wystup [16] subdivide Foreign exchange options in two types, the First generation exotic options and the Second generation exotic options.

In the First generation exotic options we can find at least options such as Barrier options, Digital options, Touch options, Rebates options, Asian options, Lookback options, Forward Start options, Ratchet options, Cliquet options, Power options and Quanto options.

In the Second generation exotic options we can find at least options such as Corridors options, Faders options, Exotic Barrier options, Step up and Step down options, Forward on the harmonic average options, Variance options and Volatility swaps.

Of the above options we have just mentioned we will have a deep look into options, the Knock out options which belong to the class of Barrier options and Power options which is First generation exotic option.

The payoff's sum of a Down and Out option(call/put) and Down and In option (call/put) is always equal to the payoff of regular vanilla option (call/put) whereas the payoff's sum of an Up and Out option(call/put) and Up and In option(call/put) is also equal to the payoff of regular vanilla option(call/put). Therefore, it is obvious that Barrier options are cheaper than Vanilla options and that is why most dealers traded these kind of options in many cases.

3.1 Power Options

Power options are those whose payoff function is entirely or by parts raised to the power of order n . There are two types of Power Options, the Asymmetric Power Options and Symmetric Power Options [16]. The difference between these options, Asymmetric and Symmetric reside in the way as are their payoffs computed for the payoff of Asymmetric, this is entirely raised to the power of n while the payoff of Symmetric each parcel is raised to the power of n .

At maturity time T , Wystup [16] suggests, for Symmetric power call options and Symmetric

power put options, the following payoff functions

$$V(T) = (\max(X_T - K, 0))^n, \quad (3.1)$$

$$V(T) = (\max(K - X_T, 0))^n, \quad (3.2)$$

respectively. Here, X_T is the underlying at maturity time T and K the strike price. The payoff functions for Asymmetric power call options and Asymmetric power put options are given by

$$V(T) = \max(X_T^n - K^n, 0), \quad (3.3)$$

$$V(T) = \max(K^n - X_T^n, 0), \quad (3.4)$$

respectively.

All kind of Power options in foreign exchange options satisfy the same Black-Scholes partial differential equation

$$\begin{cases} \frac{\partial F}{\partial t} + (r_d - r_f)x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 F}{\partial x^2} - r_d F = 0, \\ F(T, x) = \Phi(x). \end{cases}$$

Wystup [16], in his work, only computes the discounted payoff for an Asymmetric power call option. In this project we will present discounted payoff for all Power options.

3.1.1 Discounted Payoff for Power Options

We assume that the dynamics model of the underlying foreign exchange rate is a geometric Brownian motion

$$dX(t) = X(t)(r_d - r_f)dt + X(t)\sigma_X dW(t)$$

under risk neutral measure Q . The spot of the underlying at maturity time T is given by

$$X_T = X_0 e^{(r_d - r_f - \frac{1}{2}\sigma_X^2)T + \sigma_X \sqrt{T}Z}$$

- Risk neutral valuation for Asymmetric power call options

$$\begin{aligned} C_{APO} &= e^{-r_d T} \mathbb{E}_{0,x}^Q [\max(X_T^n - K^n, 0)] = e^{-r_d T} \mathbb{E}_{0,x}^Q [(X_T^n - K^n) \mathbf{1}_{\{X_T > K\}}] \\ &= e^{-r_d T} \mathbb{E}_{0,x}^Q [X_T^n \mathbf{1}_{\{X_T > K\}}] - e^{-r_d T} K^n \mathbb{E}_{0,x}^Q [\mathbf{1}_{\{X_T > K\}}] \\ &= e^{-r_d T} \int_{-\infty}^{\infty} X_T^n \mathbf{1}_{\{X_T > K\}} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz - e^{-r_d T} K^n \int_{-\infty}^{\infty} \mathbf{1}_{\{X_T > K\}} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz \\ &= e^{-r_d T} X_0^n \int_{-z_0}^{\infty} e^{n(r_d - r_f - \frac{1}{2}\sigma_X^2)T + n\sigma_X \sqrt{T}z} \cdot \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz - e^{-r_d T} K^n \int_{-z_0}^{\infty} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz \end{aligned}$$

$$\begin{aligned}
&= e^{-r_d T} X_0^n e^{n(r_d - r_f + \frac{n-1}{2}\sigma_X^2)T} \int_{-z_0}^{\infty} \frac{e^{-\frac{1}{2}(z - n\sigma_X\sqrt{T})^2}}{\sqrt{2\pi}} dz - e^{-r_d T} K^n N(z_0) \\
&= e^{-r_d T} X_0^n e^{n(r_d - r_f + \frac{n-1}{2}\sigma_X^2)T} N(z_0 + n\sigma_X\sqrt{T}) - e^{-r_d T} K^n N(z_0)
\end{aligned}$$

where

$$z_0 = \frac{\ln \frac{X_0}{K} + (r_d - r_f - \frac{1}{2}\sigma_X^2)T}{\sigma_X\sqrt{T}} \quad \text{and} \quad N(x) = \int_{-\infty}^x \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz.$$

Proposition 3.1.1. *The price of an Asymmetric power call option, with exercise price K and maturity time T is given by the formula*

$$C_{APO}(t, x, T, B) = e^{-r_d(T-t)} X_0^n e^{n(r_d - r_f + \frac{n-1}{2}\sigma_X^2)(T-t)} N(z_0 + n\sigma_X\sqrt{T-t}) - e^{-r_d(T-t)} K^n N(z_0) \quad (3.5)$$

where N is the cumulative distribution function for the $N[0, 1]$ and

$$z_0 = \frac{\ln \frac{X_0}{K} + (r_d - r_f - \frac{1}{2}\sigma_X^2)(T-t)}{\sigma_X\sqrt{T-t}}.$$

This result from Proposition 3.1.1 can also be found on Wystup[16] and Tompkins[15].

- Risk neutral valuation for Asymmetric power put options

$$\begin{aligned}
P_{APO} &= e^{-r_d T} \mathbb{E}_{0,x}^Q[\max(K^n - X_T^n, 0)] = e^{-r_d T} \mathbb{E}_{0,x}^Q[(K^n - X_T^n)\mathbf{1}_{\{X_T < K\}}] \\
&= e^{-r_d T} K^n \mathbb{E}_{0,x}^Q[\mathbf{1}_{\{X_T < K\}}] - e^{-r_d T} \mathbb{E}_{0,x}^Q[X_T^n \mathbf{1}_{\{X_T < K\}}] \\
&= e^{-r_d T} K^n \int_{-\infty}^{\infty} \mathbf{1}_{\{X_T < K\}} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz - e^{-r_d T} \int_{-\infty}^{\infty} X_T^n \mathbf{1}_{\{X_T < K\}} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz \\
&= e^{-r_d T} K^n \int_{-\infty}^{z_0} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz - e^{-r_d T} X_0^n \int_{-\infty}^{z_0} e^{n(r_d - r_f - \frac{1}{2}\sigma_X^2)T + n\sigma_X\sqrt{T}z} \cdot \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz \\
&= e^{-r_d T} K^n N(z_0) - e^{-r_d T} X_0^n e^{n(r_d - r_f + \frac{n-1}{2}\sigma_X^2)T} \int_{-\infty}^{z_0} \frac{e^{-\frac{1}{2}(z - n\sigma_X\sqrt{T})^2}}{\sqrt{2\pi}} dz \\
&= e^{-r_d T} K^n N(z_0) - e^{-r_d T} X_0^n e^{n(r_d - r_f + \frac{n-1}{2}\sigma_X^2)T} N(z_0 - n\sigma_X\sqrt{T})
\end{aligned}$$

in this case

$$z_0 = -\frac{\ln \frac{X_0}{K} + (r_d - r_f - \frac{1}{2}\sigma_X^2)T}{\sigma_X\sqrt{T}}.$$

Proposition 3.1.2. *The price of an Asymmetric power put option, with exercise price K and maturity time T is given by the formula*

$$P_{APO}(t, x, T, K) = e^{-r_d(T-t)} K^n N(z_0) - e^{-r_d(T-t)} X_0^n e^{n(r_d - r_f + \frac{n-1}{2}\sigma_X^2)(T-t)} N(z_0 - n\sigma_X \sqrt{(T-t)}), \quad (3.6)$$

where N is the cumulative distribution function for the $N[0, 1]$ and

$$z_0 = -\frac{\ln \frac{X_0}{K} + (r_d - r_f - \frac{1}{2}\sigma_X^2)(T-t)}{\sigma_X \sqrt{T-t}}.$$

Remark : This result from Proposition 3.1.2 is not in the literature as far as we know.

Proposition 3.1.3. *Consider an Asymmetric European power call and an Asymmetric European power put both with strike price K and exercise time T . Denoting the corresponding pricing functions by $C_{APO}(t, x, T, K)$ and $P_{APO}(t, x, T, K)$ we have the following relation,*

$$C_{APO}(t, x, T, K) - P_{APO}(t, x, T, K) = e^{-r_d(T-t)} x^n e^{n(r_d - r_f + \frac{n-1}{2}\sigma_X^2)(T-t)} - e^{-r_d(T-t)} K^n. \quad (3.7)$$

- Risk neutral valuation for Symmetric power call options

$$\begin{aligned} C_{SPO} &= e^{-r_d T} \mathbb{E}_{0,x}^Q [(\max(X_T - K, 0))^n] = e^{-r_d T} \mathbb{E}_{0,x}^Q [((X_T - K)\mathbf{1}_{\{X_T > K\}})^n] \\ &= e^{-r_d T} \mathbb{E}_{0,x}^Q \left[\sum_{j=0}^n \binom{n}{j} (-K)^j X_T^{n-j} \mathbf{1}_{\{X_T > K\}} \right] = e^{-r_d T} \sum_{j=0}^n \binom{n}{j} (-K)^j \mathbb{E} [X_T^{n-j} \mathbf{1}_{\{X_T > K\}}] \\ &= e^{-r_d T} \sum_{j=0}^n \binom{n}{j} (-K)^j \int_{-\infty}^{\infty} X_T^{n-j} \mathbf{1}_{\{X_T > K\}} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz \\ &= e^{-r_d T} \sum_{j=0}^n \binom{n}{j} (-K)^j X_0^{n-j} \int_{-z_0}^{\infty} e^{(n-j)(r_d - r_f - \frac{1}{2}\sigma_X^2)T + (n-j)\sigma_X \sqrt{T}z} \cdot \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz \\ &= e^{-r_d T} \sum_{j=0}^n \binom{n}{j} (-K)^j X_0^{n-j} e^{(n-j)(r_d - r_f + \frac{n-j-1}{2}\sigma_X^2)T} \int_{-z_0}^{\infty} \frac{e^{-\frac{1}{2}(z - (n-j)\sigma_X \sqrt{T})^2}}{\sqrt{2\pi}} dz \\ &= e^{-r_d T} \sum_{j=0}^n \binom{n}{j} (-K)^j X_0^{n-j} e^{(n-j)(r_d - r_f + \frac{n-j-1}{2}\sigma_X^2)T} N(z_0 + (n-j)\sigma_X \sqrt{T}) \end{aligned}$$

where

$$z_0 = \frac{\ln \frac{X_0}{K} + (r_d - r_f - \frac{1}{2}\sigma_X^2)T}{\sigma_X \sqrt{T}}.$$

Proposition 3.1.4. *The price of an Symmetric power call option, with exercise price K and maturity time T is given by the formula*

$$C_{SPO}(t, x, T, K) = e^{-r_d(T-t)} \sum_{j=0}^n \binom{n}{j} (-K)^j X_0^{n-j} e^{(n-j)(r_d - r_f + \frac{n-j-1}{2}\sigma_X^2)(T-t)} N(z_0 + (n-j)\sigma_X\sqrt{T-t}), \quad (3.8)$$

where N is the cumulative distribution function for the $N[0, 1]$ and

$$z_0 = \frac{\ln \frac{X_0}{K} + (r_d - r_f - \frac{1}{2}\sigma_X^2)(T-t)}{\sigma_X\sqrt{T-t}}.$$

Remark : This result from Proposition 3.1.4 is valid for a positive integer n and Tompkins [15] discusses similar result.

- Risk neutral valuation for Symmetric power put options

$$\begin{aligned} P_{SPO} &= e^{-r_d T} \mathbb{E}_{0,x}^Q[(\max(K - X_T, 0))^n] = e^{-r_d T} \mathbb{E}_{0,x}^Q [((K - X_T)\mathbf{1}_{\{X_T < K\}})^n] \\ &= e^{-r_d T} \mathbb{E}_{0,x}^Q \left[\sum_{j=0}^n \binom{n}{j} K^j (-1)^{n-j} X_T^{n-j} \mathbf{1}_{\{X_T < K\}} \right] = e^{-r_d T} \sum_{j=0}^n \binom{n}{j} K^j (-1)^{n-j} \mathbb{E} [X_T^{n-j} \mathbf{1}_{\{X_T < K\}}] \\ &= e^{-r_d T} \sum_{j=0}^n \binom{n}{j} K^j (-1)^{n-j} \int_{-\infty}^{\infty} X_T^{n-j} \mathbf{1}_{\{X_T < K\}} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz \\ &= e^{-r_d T} \sum_{j=0}^n \binom{n}{j} K^j (-1)^{n-j} X_0^{n-j} \int_{-\infty}^{z_0} e^{(n-j)(r_d - r_f - \frac{1}{2}\sigma_X^2)T + (n-j)\sigma_X\sqrt{T}z} \cdot \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz \\ &= e^{-r_d T} \sum_{j=0}^n \binom{n}{j} K^j (-1)^{n-j} X_0^{n-j} e^{(n-j)(r_d - r_f + \frac{n-j-1}{2}\sigma_X^2)T} \int_{-\infty}^{z_0} \frac{e^{-\frac{1}{2}(z - (n-j)\sigma_X\sqrt{T})^2}}{\sqrt{2\pi}} dz \\ &= e^{-r_d T} \sum_{j=0}^n \binom{n}{j} K^j (-1)^{n-j} X_0^{n-j} e^{(n-j)(r_d - r_f + \frac{n-j-1}{2}\sigma_X^2)T} N(z_0 - (n-j)\sigma_X\sqrt{T}) \end{aligned}$$

in this case

$$z_0 = -\frac{\ln \frac{X_0}{K} + (r_d - r_f - \frac{1}{2}\sigma_X^2)T}{\sigma_X\sqrt{T}}.$$

Proposition 3.1.5. *The price of an Symmetric power put option, with exercise price K and maturity time T is given by the formula*

$$P_{SPO}(t, x, T, K) = e^{-r_d(T-t)} \sum_{j=0}^n \binom{n}{j} K^j (-1)^{n-j} X_0^{n-j} e^{(n-j)(r_d - r_f + \frac{n-j-1}{2}\sigma_X^2)(T-t)} N(z_0 - (n-j)\sigma_X\sqrt{T-t}), \quad (3.9)$$

where N is the cumulative distribution function for the $N[0, 1]$ and

$$z_0 = -\frac{\ln \frac{X_0}{K} + (r_d - r_f - \frac{1}{2}\sigma_X^2)(T-t)}{\sigma_X \sqrt{T-t}}.$$

Remark : This result from Proposition 3.1.5 is not in the literature as far as we know and is valid for a positive integer n .

European power call and European power put being symmetric or asymmetric their payoff functions are homogeneous of degree n with respect to spot and strike price. For example, let us consider $a > 0$ and a pricing function of a symmetric power call then the following relation holds

$$C_{SPO}(t, ax, T, aK) = a^n C_{SPO}(t, x, T, K).$$

3.1.2 Call and put bets

In this section we will follow Lipton [7]. Call (Put) bets are financial variables whose give a unit of domestic currency if up to maturity time T the foreign exchange rate X is above (below) the strike price K . The payoff functions, at maturity time, for call and put bets are as follows

$$\begin{aligned} V(T) &= H(X_T - K), \\ V(T) &= H(K - X_T), \end{aligned}$$

respectively. Here

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Now, suppose that X is a foreign exchange rate and has dynamics of a geometric Brownian motion. Consider that at maturity time, X can be written as follows $X_T = x \exp \left\{ (r_d - r_f - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z \right\}$. However, the discounted payoff of a call bet in domestic terms is

$$V^{CB} = e^{-r_d T} \mathbb{E}_{0,x}^Q [H(X_T - K)] = e^{-r_d T} \int_{-\infty}^{\infty} [H(X_T - K)] \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz = e^{-r_d T} \int_{-z_0}^{\infty} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} dz = e^{-r_d T} N(z_0),$$

$$\text{where } z_0 = \frac{\ln \frac{X_0}{K} + (r_d - r_f - \frac{1}{2}\sigma^2)(T)}{\sigma_X \sqrt{T}}.$$

Proposition 3.1.6. *The price, in domestic terms, of call and put bets with exercise price K and maturity time T is given by the formula,*

$$\begin{aligned} V^{CB}(t, x, T, K) &= e^{-r_d(T-t)} N(z_0), \\ V^{PB}(t, x, T, K) &= e^{-r_d(T-t)} N(-z_0), \end{aligned}$$

respectively. Here, $z_0 = \frac{\ln \frac{X_0}{K} + (r_d - r_f - \frac{1}{2}\sigma^2)(T-t)}{\sigma_X \sqrt{T-t}}$.

3.2 Knock Out Options

Knock Out Options are Barrier Options that are exercised if up to maturity time never reach the barrier otherwise expire worthless.

There are several options of type Knock out but we are going to concentrate ourselves in these two types the Up and Out Options and the Down and Out Options nevertheless we are going to restrict our study in barriers of European Style, that is, if the spot at expiration does not hit the barrier, the owner will exercise the option.

Shreve [13] deals with Knock out option of type Up and Out call he also computed the its discounted payoff similar results we can also find on Kwok [6], Hull [3] and Björk [1] who have gone beyond and additionally priced Barrier options of type Down and out , Up and in, down and in.

Consider the dynamics' model of foreign exchange rate as having the dynamics of a geometric Brownian motion.

Regard that the contractual parameters, X_T is the underlying foreign exchange rate, K exercise price, T expiration time and B the barrier. The payoff functions are as shown below according to Jiang [4]

$$\begin{aligned} \text{Down and Out} & \begin{cases} (X_T - K)^+ \mathbf{1}_{\{X_t > B, t \in [0, T]\}} & \text{CALL} \\ (K - X_T)^+ \mathbf{1}_{\{X_t > B, t \in [0, T]\}} & \text{PUT} \end{cases} \\ \text{Up and Out} & \begin{cases} (X_T - K)^+ \mathbf{1}_{\{X_t < B, t \in [0, T]\}} & \text{CALL} \\ (K - X_T)^+ \mathbf{1}_{\{X_t < B, t \in [0, T]\}} & \text{PUT} \end{cases} \end{aligned}$$

The payoff diagrams are presented on Figure 3.1, and it is easy to see that for an Up and out

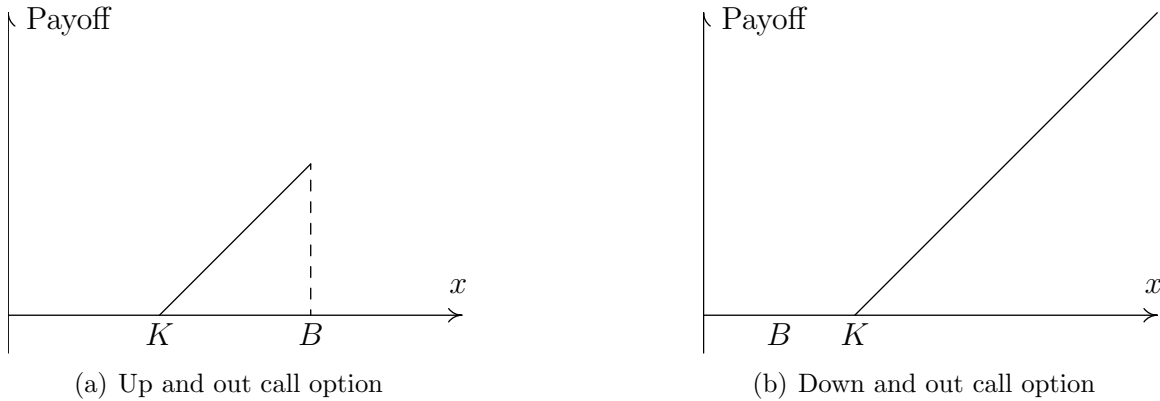


Figure 3.1: Knock out call option

call option, on Figure 3.1(a), it pays off if up to maturity time the underlying foreign exchange rate is between the exercise price K and the upper barrier B otherwise is zero, we can also say that the option is out of the money if the spot lies below the strike price K , at the money if the

spot lies at exercise price and out of the money if the spot lies above the upper barrier B . On the other hand, for a Down and out option, on Figure 3.1(b), it pays off if up to exercise date the underlying foreign exchange rate is above the lower barrier B , otherwise is zero, herein we can also say the option is in the money if the spot always lies above the lower barrier B and out of the money if the spot lies below the lower barrier B . Similar diagrams can also be found on Wystup [16] Lipton [7] and Shamah [12].

Regard a geometric Brownian motion as mentioned before

$$dX(t) = (r_d - r_f)X(t)dt + \sigma X(t)d\hat{W}(t) \quad (3.10)$$

which represents the dynamics of the Exchange rate, the underlying risky asset, where $\hat{W}(t)$, $0 \leq t \leq T$, is a Brownian motion under risky neutral measure, r_d the domestic interest rate and r_f the foreign exchange rate.

3.2.1 Up and Out Options

Up and Out Options of European style, are Barrier options, whose payoff is a vanilla option if up to maturity time the underlying foreign exchange rate never hit an upper barrier agreed previously.

- Up and out call options

Let us consider an European call, with maturity time T , exercise price K and an upper barrier B . We also have to assume that $K < B$ in order to keep the option in the money. We have already solved the stochastic differential equation (3.10) and its solution is $X(t) = X(0) \exp\{(r_d - r_f - \frac{1}{2}\sigma^2)t + \sigma\hat{W}(t)\}$ which can be written in a short form as follows

$$X(t) = xe^{\sigma W(t)}, \quad (3.11)$$

where $W(t) = \alpha t + \hat{W}$, $\alpha = \frac{1}{\sigma}(r_d - r_f - \frac{1}{2}\sigma^2)$, $X(0) = x$.

Let M be the maximum of Brownian motion $W(t)$. According to the definition of an up and out option $\max_{0 \leq t \leq T} X(t) = x \exp(\sigma M) \leq B$, therefore the option payoff is

$$\begin{aligned} (X(T) - K)^+ \mathbf{1}_{\{X_t < B, t \in [0, T]\}} &= (X(T) - K)^+ \mathbf{1}_{\{x \exp(\sigma M) \leq B\}} \\ &= (x \exp(\sigma W) - K) \mathbf{1}_{\{x \exp(\sigma W) \geq K \wedge x \exp(\sigma M) \leq B\}} \\ &= (x \exp(\sigma W) - K) \mathbf{1}_{\{W \geq k \wedge M \leq b\}} \end{aligned}$$

where $k = \frac{1}{\sigma} \ln \frac{K}{x}$ and $b = \frac{1}{\sigma} \ln \frac{B}{x}$.

The discounted payoff, under risk neutral, of an Up and out call option is

$$\begin{aligned} F(0, x) &= e^{-r_d T} E^Q_{t,x} \left[(X(T) - K)^+ \mathbf{1}_{\{\max_{0 \leq t \leq T} X_t < B\}} \right] \\ &= e^{-r_d T} E^Q_{t,x} \left[(x \exp(\sigma W) - K) \mathbf{1}_{\{W \geq k \wedge M \leq b\}} \right] \end{aligned}$$

$$\begin{aligned}
&= e^{-r_d T} \int_k^b \int_{w^+}^b (xe^{\sigma w} - K) \frac{2(2m-w)}{T\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2} \alpha^2 T - \frac{1}{2T} (2m-w)^2 \right\} dm dw \\
&= -e^{-r_d T - \frac{1}{2} \alpha^2 T} \int_k^b (xe^{\sigma w} - K) \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T} (2m-w)^2 \right\} \Big|_{w^+}^b dw \\
&= -e^{-r_d T - \frac{1}{2} \alpha^2 T} \int_k^b xe^{\sigma w} \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T} (2b-w)^2 \right\} dw \\
&\quad + e^{-r_d T - \frac{1}{2} \alpha^2 T} \int_k^b xe^{\sigma w} \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T} w^2 \right\} dw \\
&\quad + Ke^{-r_d T - \frac{1}{2} \alpha^2 T} \int_k^b \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T} (2b-w)^2 \right\} dw \\
&\quad - Ke^{-r_d T - \frac{1}{2} \alpha^2 T} \int_k^b \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T} w^2 \right\} dw \\
&= -I_1 + I_2 + I_3 - I_4.
\end{aligned}$$

Now, we will compute each of these integrals and we get,

$$\begin{aligned}
I_1 &= e^{-r_f T} \left(\frac{B}{x} \right)^{2\frac{r_d - r_f}{\sigma^2} + 1} x \left[N \left(\frac{\ln \frac{x}{B} - (r_d - r_f + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right) - N \left(\frac{\ln \frac{xK}{B^2} - (r_d - r_f + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right) \right], \\
I_2 &= e^{-r_f T} x \left[N \left(\frac{\ln \frac{B}{x} - (r_d - r_f + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right) - N \left(\frac{\ln \frac{K}{x} - (r_d - r_f + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right) \right], \\
I_3 &= e^{-r_d T} \left(\frac{B}{x} \right)^{2\frac{r_d - r_f}{\sigma^2} - 1} K \left[N \left(\frac{\ln \frac{x}{B} - (r_d - r_f - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right) - N \left(\frac{\ln \frac{xK}{B^2} - (r_d - r_f - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right) \right], \\
I_4 &= e^{-r_d T} K \left[N \left(\frac{\ln \frac{B}{x} - (r_d - r_f - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right) - N \left(\frac{\ln \frac{K}{x} - (r_d - r_f - \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right) \right].
\end{aligned}$$

A similar and detailed computation of these integrals can be found in the next chapter and also in Shreve [13].

Putting all the above integrals together we obtain

$$\begin{aligned}
F(0, x) = & -e^{-r_f T} \left(\frac{B}{x}\right)^{2\frac{r_d - r_f}{\sigma^2} + 1} x \left[N\left(\frac{\ln \frac{x}{B} - (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln \frac{xK}{B^2} - (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \right] \\
& + e^{-r_f T} x \left[N\left(\frac{\ln \frac{B}{x} - (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln \frac{K}{x} - (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \right] \\
& + e^{-r_d T} \left(\frac{B}{x}\right)^{2\frac{r_d - r_f}{\sigma^2} - 1} K \left[N\left(\frac{\ln \frac{x}{B} - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln \frac{xK}{B^2} - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \right] \\
& - e^{-r_d T} K \left[N\left(\frac{\ln \frac{B}{x} - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln \frac{K}{x} - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \right].
\end{aligned}$$

The payoff function of Up and out call option can be written as a proposition using financial variables as follows

Proposition 3.2.1. *The price of an Up and Out Call Option with an upper barrier B , exercise price K and maturity time T , regarding that $K < B$, is given by the formula*

$$\begin{aligned}
C_{UAO}(t, x, T, K, B) = & C(t, x, T, K) - C(t, x, T, B) \\
& - \left(\frac{B}{x}\right)^{2\frac{r_d - r_f}{\sigma^2} - 1} \left[C\left(t, \frac{B^2}{x}, T, K\right) - C\left(t, \frac{B^2}{x}, T, x\right) \right] \\
& - (B - K) \left[V^{CB}(t, x, T, B) - \left(\frac{B}{x}\right)^{2\frac{r_d - r_f}{\sigma^2} - 1} V^{CB}(t, B, T, x) \right],
\end{aligned}$$

where $C(t, x, T, K)$ is a call option in foreign exchange options with spot x , strike price K , Maturity time T , and $V^{CB}(t, x, T, B)$ is a call bit in foreign exchange options with strike price B .

Remark : if $n = 1$ in Power option then $C(t, x, T, K) = C_{SPO}(t, x, T, K) = C_{APO}(t, x, T, K)$.

- Up and out put options The pricing boundary value problem for Down and out put option is given by

$$\begin{cases} \frac{\partial F}{\partial t} + (r_d - r_f)x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} - r_d F = 0, & 0 \leq x \leq B, 0 \leq t \leq T, \\ F(T, x) = (K - x)^+, & 0 < x < B, \\ F(t, B) = 0, F(t, x \rightarrow 0) \rightarrow e^{-r_f(T-t)}K, & 0 \leq x \leq B, 0 \leq t \leq T. \end{cases}$$

To solve this problem, we transform it using the relations $y = \ln \frac{x}{B}$, $F = Bu$. Though, the above problem becomes with constant coefficients. Next, we introduce a new function $u(t, y) = e^{\alpha y + \beta(T-t)}W(t, y)$ where $\alpha = -(r_d - r_f - \frac{1}{2}\sigma^2)/\sigma^2$ and $\beta = -r_d - (r_d - r_f - \frac{1}{2}\sigma^2)^2/(2\sigma^2)$. However, the problem reduces to the following backward heat equation

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial y^2} = 0, & -\infty < y < 0, 0 \leq t \leq T, \\ W(T, y) = e^{-\alpha y}(K_B - e^y)^+, & -\infty < x < 0, \\ W(t, 0) = 0, & 0 \leq t \leq T. \end{cases}$$

Indeed, we extend the above problem to hole real line using the image method and we write the solution using Green's function. Therefore the solution to the pricing boundary value problem for Down and out call is as follows,

$$\begin{aligned} F(t, x) = & e^{-r_d(T-t)}KN \left(-\frac{\ln \frac{x}{K} + (r_d - r_f - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \\ & - e^{-r_f(T-t)}xN \left(-\frac{\ln \frac{x}{K} + (r_d - r_f + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \\ & - e^{-r_d(T-t)}K \left(\frac{x}{B} \right)^{2\alpha} N \left(-\frac{\ln \frac{B^2}{xK} + (r_d - r_f - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \\ & + e^{-r_f(T-t)}B \left(\frac{x}{B} \right)^{2\alpha-1} N \left(-\frac{\ln \frac{B^2}{xK} + (r_d - r_f + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right). \end{aligned}$$

This result can be synthesized in the following proposition using financial variable as follows.

Proposition 3.2.2. *The price of an Up and Out put Option with an upper barrier B , exercise price K and maturity time T , regarding that $K < B$, is given by the formula*

$$P_{UAO}(t, x, T, K, B) = P(t, x, T, K) - \left(\frac{x}{B} \right)^{2\alpha} P\left(t, \frac{B^2}{x}, T, K\right)$$

where $P(t, x, T, K)$ is a put option in foreign exchange options with spot x , strike price K , Maturity time T .

Remark This result can also be found in Björk [1] and for $n = 1$ in Power option $P(t, x, T, K) = P_{SPO}(t, x, T, K) = P_{APO}(t, x, T, K)$.

3.2.2 Down and Out Options

Down and out Options of European style, are Barrier options, whose payoff is a vanilla option if up to maturity time the underlying foreign exchange rate never hit a lower Barrier agreed previously. The pricing boundary value problem for Down and out call is given by

$$\begin{cases} \frac{\partial F}{\partial t} + (r_d - r_f)x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} - r_d F = 0, & B \leq x < \infty, 0 \leq t \leq T, \\ F(T, x) = (x - K)^+, & B < x < \infty, \\ F(t, B) = 0, F(t, x \rightarrow \infty) \rightarrow e^{-r_f(T-t)}x, & B \leq x < \infty, 0 \leq t \leq T. \end{cases}$$

To solve this problem, we transform it using the relations $y = \ln \frac{x}{B}$, $F = Bu$. Though, the above problem becomes with constant coefficients. Next, we introduce a new function $u(t, y) = e^{\alpha y + \beta(T-t)}W(t, y)$ where $\alpha = -(r_d - r_f - \frac{1}{2}\sigma^2)/\sigma^2$ and $\beta = -r_d - (r_d - r_f - \frac{1}{2}\sigma^2)^2/(2\sigma^2)$. However, the problem reduces to the following backward heat equation

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial y^2} = 0, & 0 \leq x < \infty, 0 \leq t \leq T, \\ W(T, y) = e^{-\alpha y}(e^y - K_B)^+, & 0 < x < \infty, \\ W(t, 0) = 0, & 0 \leq t \leq T. \end{cases}$$

Indeed, we extend the above problem to hole real line using the image method and we write the solution using Green's function. Therefore the solution to the pricing boundary value problem for Down and out call is as follows,

$$\begin{aligned} F(t, x) = & e^{-r_f(T-t)}xN\left(\frac{\ln \frac{x}{K} + (r_d - r_f + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - e^{-r_d(T-t)}KN\left(\frac{\ln \frac{x}{K} + (r_d - r_f - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & - e^{-r_f(T-t)}B\left(\frac{x}{B}\right)^{-2(r_d - r_f)/\sigma^2}N\left(\frac{\ln \frac{B^2}{xK} + (r_d - r_f + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & + e^{-r_d(T-t)}K\left(\frac{x}{B}\right)^{-2(r_d - r_f)/\sigma^2 + 1}N\left(\frac{\ln \frac{B^2}{xK} + (r_d - r_f - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

This result can be synthesized in the following proposition using financial variable as follows.

Proposition 3.2.3. *The price of a Down and out call option with a barrier B , exercise price K and maturity time T , regarding that $K > B$ is given by the formula*

$$C_{DAO}(t, x, T, K, B) = C(t, x, T, K) - \left(\frac{x}{B}\right)^{-2(r_d - r_f)/\sigma^2 + 1}C\left(t, \frac{B^2}{x}, t, K\right),$$

where $C(t, x, T, K) = C_{APO}(t, x, T, K) = C_{SPO}(t, x, T, K)$ with $n = 1$.

Similar result can be found in Lipton [7].

Chapter 4

Knock Out Power Options

Knock out options are Barrier options which pay a Power options payoff if up to maturity the underlying asset never reach the barriers.

There are several types of Knock out power options but, we are only going to consider two kind of them, the Up and out power options and the Down and out power options.

Suppose that the contractual parameters X_T is the underlying exchange rate, K the strike price, B the barrier and T the maturity time, however, the payoff functions are as shown below

$$\begin{array}{l}
 \text{Down and Out} \left\{ \begin{array}{l}
 \text{Asymmetric} \left\{ \begin{array}{l}
 (X_T^n - K^n)^+ \mathbf{1}_{\{X_t > B, t \in [0, T]\}} \quad \text{CALL} \\
 (K^n - X_T^n)^+ \mathbf{1}_{\{X_t > B, t \in [0, T]\}} \quad \text{PUT}
 \end{array} \right. \\
 \text{Symmetric} \left\{ \begin{array}{l}
 [(X_T - K)^+]^n \mathbf{1}_{\{X_t > B, t \in [0, T]\}} \quad \text{CALL} \\
 [(K - X_T)^+]^n \mathbf{1}_{\{X_t > B, t \in [0, T]\}} \quad \text{PUT}
 \end{array} \right.
 \end{array} \right. \\
 \\
 \text{Up and Out} \left\{ \begin{array}{l}
 \text{Asymmetric} \left\{ \begin{array}{l}
 (X_T^n - K^n)^+ \mathbf{1}_{\{X_t < B, t \in [0, T]\}} \quad \text{CALL} \\
 (K^n - X_T^n)^+ \mathbf{1}_{\{X_t < B, t \in [0, T]\}} \quad \text{PUT}
 \end{array} \right. \\
 \text{Symmetric} \left\{ \begin{array}{l}
 [(X_T - K)^+]^n \mathbf{1}_{\{X_t < B, t \in [0, T]\}} \quad \text{CALL} \\
 [(K - X_T)^+]^n \mathbf{1}_{\{X_t < B, t \in [0, T]\}} \quad \text{PUT}
 \end{array} \right.
 \end{array} \right.
 \end{array}$$

The symbol S^+ denotes the positive parte of S , that is, $S^+ \equiv \max(S, 0)$.

The payoff diagrams are presented on Figure 4.1, and it is easy to see that for an Up and out Asymmetric power call option, on Figure 4.1(a), it pays off if up to maturity time the underlying foreign exchange rate is between the exercise price K and the upper barrier B otherwise is zero, we can also say that the option is out of the money if the spot lies below the strike price K , at the money if the spot lies at exercise price and out of the money if the spot lies above the upper

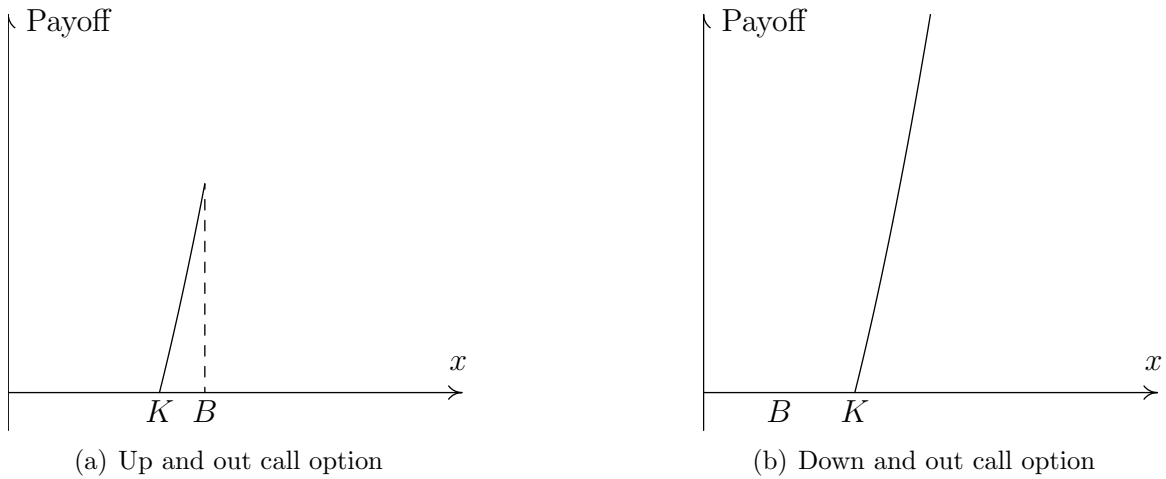


Figure 4.1: Asymmetric power call option

barrier B . On the other hand, for a Down and out Asymmetric power call option, on Figure 4.1(b), it pays off if up to exercise date the underlying foreign exchange rate is above the lower barrier B , otherwise is zero, herein we can also say the option is in the money if the spot always lies above the lower barrier B and out of the money if the spot lies below the lower barrier B .

The payoff diagrams are presented on Figure 4.2, and it is easy to see that for an Up and out

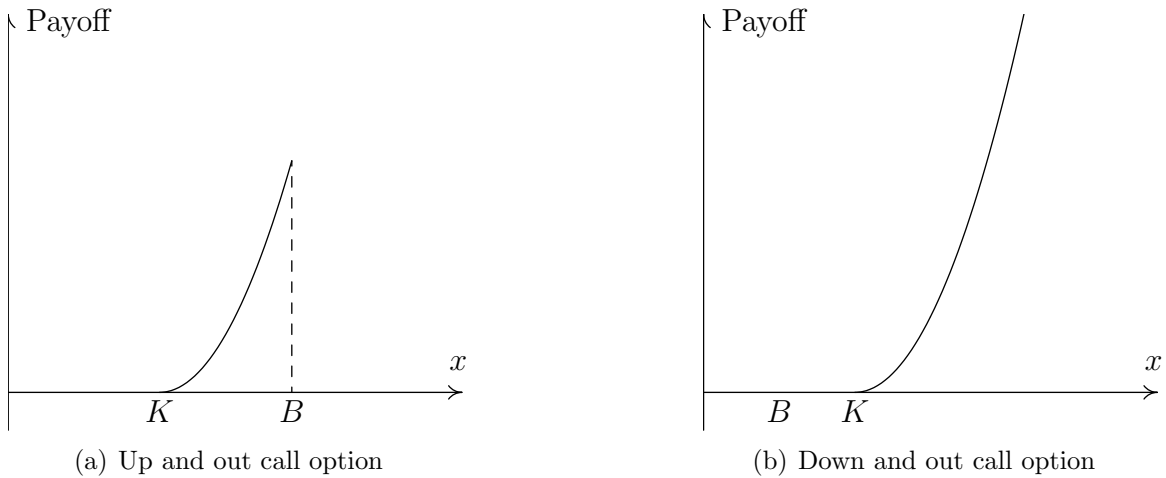


Figure 4.2: Symmetric power call option

symmetric power call option, on Figure 4.2(a), it pays off if up to maturity time the underlying foreign exchange rate is between the exercise price K and the upper barrier B otherwise is zero, we can also say that the option is out of the money if the spot lies below the strike price K , at the money if the spot lies at exercise price and out of the money if the spot lies above the upper barrier B . On the other hand, for a Down and out symmetric power call option, on Figure 4.2(b),

it pays off if up to exercise date the underlying foreign exchange rate is above the lower barrier B , otherwise is zero, herein we can also say the option is in the money if the spot always lies above the lower barrier B and out of the money if the spot lies below the lower barrier B .

All kind of Knock out Power options in foreign exchange options satisfy the same Black-Scholes partial differential equation

$$\begin{cases} \frac{\partial F}{\partial t} + (r_d - r_f)x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 F}{\partial x^2} - r_d F = 0, \\ F(T, x) = \Phi(x). \end{cases}$$

4.1 Discounted payoff valuation

4.1.1 Up and out Asymmetric Power call option

Consider an European Up and out asymmetric power call option with exercise time T , strike price K and an up and out barrier B . We assume that the strike price K is lesser than the Up and out barrier B , because otherwise, the option must knock out in order to be in the money and hence could only pay off zero.

Consider the spot foreign exchange rate

$$X_T = x \cdot e^{(r_d - r_f - \frac{1}{2}\sigma^2)T + \sigma \bar{W}},$$

where \bar{W} is a Brownian motion with mean zero and variance one. Let us write the above foreign exchange rate as follows

$$X_T = x \cdot e^{\sigma W}, \quad (4.1)$$

where $W = \alpha T + \bar{W}$, $\alpha = \frac{r_d - r_f - \frac{1}{2}\sigma^2}{\sigma}$ that is, W is a Brownian motion with mean αT and variance zero. Let

$$\bar{M}(T) = \max_{0 \leq t \leq T} W(t) \quad \text{and} \quad \underline{M}(T) = \min_{0 \leq t \leq T} W(t)$$

be the maximum and minimum of W , respectively. Regarding the following identity

$$\begin{aligned} X_T > K \wedge \max_{0 \leq t \leq T} X_t < B &\equiv x e^{\sigma W(T)} > K \wedge \max_{0 \leq t \leq T} x e^{\sigma W(t)} < B \\ &\equiv W(T) > \frac{1}{\sigma} \ln \frac{K}{x} \wedge x e^{\sigma \max_{0 \leq t \leq T} W(t)} < B \\ &\equiv W(T) > \frac{1}{\sigma} \ln \frac{K}{x} \wedge \max_{0 \leq t \leq T} W(t) < \frac{1}{\sigma} \ln \frac{B}{x} \\ &\equiv W(T) > k \wedge M(T) < b, \end{aligned}$$

where $k = \frac{1}{\sigma} \ln \frac{K}{x}$ and $b = \frac{1}{\sigma} \ln \frac{B}{x}$, the arbitrage free price for up and out power call option is given by

$$\begin{aligned}
F(0, x) &= e^{-r_d T} E^{\bar{Q}}_{t,x} \left[(X_T^n - K^n)^+ \mathbf{1}_{\{\max_{0 \leq t \leq T} X_t < B\}} \right] \\
&= e^{-r_d T} E^{\bar{Q}}_{t,x} \left[(X_T^n - K^n) \mathbf{1}_{\{X_T > K \wedge \max_{0 \leq t \leq T} X_t < B\}} \right] \\
&= e^{-r_d T} E^{\bar{Q}}_{t,x} \left[(X_T^n - K^n) \mathbf{1}_{\{W(T) > k \wedge M(t) < b\}} \right] \\
&= e^{-r_d T} \int_k^b \int_{w^+}^b (x^n e^{n\sigma w} - K^n) \frac{2(2m-w)}{T\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2} \alpha^2 T - \frac{1}{2T} (2m-w)^2 \right\} dm dw \\
&= e^{-r_d T - \frac{1}{2} \alpha^2 T} \int_k^b (x^n e^{n\sigma w} - K^n) \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T} (2m-w)^2 \right\} \Big|_{w^+}^b dw \\
&= e^{-r_d T - \frac{1}{2} \alpha^2 T} \int_k^b (x^n e^{n\sigma w} - K^n) \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T} (2b-w)^2 \right\} dw \\
&\quad - e^{-r_d T - \frac{1}{2} \alpha^2 T} \int_k^b (x^n e^{n\sigma w} - K^n) \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T} w^2 \right\} dw \\
&= I_1 - I_2.
\end{aligned}$$

Herein, we split up the integrals I_1 and I_2 as follows

$$\begin{aligned}
I_1 &= e^{-r_d T - \frac{1}{2} \alpha^2 T} \int_k^b (x^n e^{n\sigma w} - K^n) \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T} (2b-w)^2 \right\} dw \\
&= e^{-r_d T - \frac{1}{2} \alpha^2 T} x^n \int_k^b e^{n\sigma w} \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T} (2b-w)^2 \right\} dw \\
&\quad - e^{-r_d T - \frac{1}{2} \alpha^2 T} K^n \int_k^b \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T} (2b-w)^2 \right\} dw \\
&= I_{11} - I_{12}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I_2 &= e^{-r_d T - \frac{1}{2}\alpha^2 T} \int_k^b (x^n e^{n\sigma w} - K^n) \frac{1}{\sqrt{2\pi T}} \exp\left\{\alpha w - \frac{1}{2T}w^2\right\} dw \\
&= e^{-r_d T - \frac{1}{2}\alpha^2 T} x^n \int_k^b e^{n\sigma w} \frac{1}{\sqrt{2\pi T}} \exp\left\{\alpha w - \frac{1}{2T}w^2\right\} dw \\
&\quad - e^{-r_d T - \frac{1}{2}\alpha^2 T} K^n \int_k^b \frac{1}{\sqrt{2\pi T}} \exp\left\{\alpha w - \frac{1}{2T}w^2\right\} dw \\
&= I_{21} - I_{22}.
\end{aligned}$$

Now, let us compute the closed form of each integrals I_{11} , I_{12} , I_{21} , I_{22} as follows

$$\begin{aligned}
I_{11} &= e^{-r_d T - \frac{1}{2}\alpha^2 T} x^n \int_k^b e^{n\sigma w} \frac{1}{\sqrt{2\pi T}} \exp\left\{\alpha w - \frac{1}{2T}(2b-w)^2\right\} dw \\
&= e^{-r_d T - \frac{1}{2}\alpha^2 T} x^n \int_k^b \frac{1}{\sqrt{2\pi T}} \exp\left\{(n\sigma + \alpha)w - \frac{1}{2T}(2b-w)^2\right\} dw \\
&= e^{-r_d T + 2b(n\sigma + \alpha) + n\sigma\alpha T + \frac{1}{2}n^2\sigma^2 T} x^n \int_k^b \frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{1}{2T}(w - (2b + (n\sigma + \alpha)T))^2\right\} dw \\
&= e^{-r_d T + 2b(n\sigma + \alpha) + n\sigma\alpha T + \frac{1}{2}n^2\sigma^2 T} x^n \int_{\frac{k - (2b + (n\sigma + \alpha)T)}{\sqrt{T}}}^{\frac{b - (2b + (n\sigma + \alpha)T)}{\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= e^{-r_d T + 2b(n\sigma + \alpha) + n\sigma\alpha T + \frac{1}{2}n^2\sigma^2 T} x^n \left[N\left(\frac{b - (2b + (n\sigma + \alpha)T)}{\sqrt{T}}\right) - N\left(\frac{k - (2b + (n\sigma + \alpha)T)}{\sqrt{T}}\right) \right] \\
&= \exp\left(-r_d T + n\left(r_d - r_f + \frac{n-1}{2}\sigma^2\right)T\right) \left(\frac{B}{x}\right)^{2\frac{r_d - r_f + (n-\frac{1}{2})\sigma^2}{\sigma^2}} x^n \left[N\left(\frac{\ln\frac{x}{B} - (r_d - r_f + (n-\frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}\right) \right. \\
&= \left. -N\left(\frac{\ln\frac{xK}{B^2} - (r_d - r_f + (n-\frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}\right) \right].
\end{aligned}$$

$$\begin{aligned}
I_{12} &= e^{-r_d T - \frac{1}{2}\alpha^2 T} K^n \int_k^b \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T}(2b - w)^2 \right\} dw \\
&= e^{-r_d T - \frac{1}{2}\alpha^2 T + 2b\alpha + \frac{1}{2}\alpha^2 T} K^n \int_k^b \frac{1}{\sqrt{2\pi T}} \exp \left\{ -\frac{1}{2T}(w - (2b + \alpha T))^2 \right\} dw \\
&= e^{-r_d T + 2b\alpha} K^n \int_{\frac{k - (2b + \alpha T)}{\sqrt{T}}}^{\frac{b - (2b + \alpha T)}{\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= e^{-r_d T + 2b\alpha} K^n \left(N \left(\frac{b - (2b + \alpha T)}{\sqrt{T}} \right) - N \left(\frac{k - (2b + \alpha T)}{\sqrt{T}} \right) \right) \\
&= e^{-r_d T} \left(\frac{B}{x} \right)^{\frac{2(r_d - r_f - \frac{1}{2}\sigma^2)}{\sigma^2}} K^n \left(N \left(\frac{\ln \frac{x}{B} - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) - N \left(\frac{\ln \frac{xK}{B^2} + (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right).
\end{aligned}$$

$$\begin{aligned}
I_{21} &= e^{-r_d T - \frac{1}{2}\alpha^2 T} x^n \int_k^b e^{n\sigma w} \frac{1}{\sqrt{2\pi T}} \exp \left\{ \alpha w - \frac{1}{2T}w^2 \right\} dw \\
&= e^{-r_d T - \frac{1}{2}\alpha^2 T} x^n \int_k^b \frac{1}{\sqrt{2\pi T}} \exp \left\{ (n\sigma + \alpha)w - \frac{1}{2T}w^2 \right\} dw \\
&= e^{-r_d T - \frac{1}{2}\alpha^2 T + \frac{1}{2}(n\sigma + \alpha)^2 T} x^n \int_k^b \frac{1}{\sqrt{2\pi T}} \exp \left\{ -\frac{1}{2T}[w - (n\sigma + \alpha)T]^2 \right\} dw \\
&= e^{-r_d T + n\sigma\alpha T + \frac{1}{2}n^2\sigma^2 T} x^n \int_{\frac{k - (n\sigma + \alpha)T}{\sqrt{T}}}^{\frac{b - (n\sigma + \alpha)T}{\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= \exp \left\{ -r_d T + n \left(r_d - r_f - \frac{1}{2}\sigma^2 \right) T + \frac{1}{2}n^2\sigma^2 T \right\} x^n \left[N \left(\frac{b - (n\sigma + \alpha)T}{\sqrt{T}} \right) - N \left(\frac{k - (n\sigma + \alpha)T}{\sqrt{T}} \right) \right] \\
&= \exp \left\{ -r_d T + n \left(r_d - r_f + \frac{n-1}{2}\sigma^2 \right) T \right\} x^n \left[N \left(\frac{\ln \frac{B}{x} + (r_d - r_f + (n - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}} \right) + \right.
\end{aligned}$$

$$\begin{aligned}
& -N\left(\frac{\ln \frac{K}{x} + (r_d - r_f + (n - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}\right) \Big]. \\
I_{22} &= e^{-r_d T - \frac{1}{2}\alpha^2 T} K^n \int_k^b \frac{1}{\sqrt{2\pi T}} \exp\left\{\alpha w - \frac{1}{2T}w^2\right\} dw \\
&= e^{-r_d T - \frac{1}{2}\alpha^2 T + \frac{1}{2}\alpha^2 T} K^n \int_k^b \frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{1}{2T}(w - T\alpha)^2\right\} dw \\
&= e^{-r_d T} K^n \int_{\frac{k-T\alpha}{\sqrt{T}}}^{\frac{b-T\alpha}{\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= e^{-r_d T} K^n \left[N\left(\frac{b-T\alpha}{\sqrt{T}}\right) - N\left(\frac{k-T\alpha}{\sqrt{T}}\right) \right] \\
&= e^{-r_d T} K^n \left[N\left(\frac{\ln \frac{B}{x} - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln \frac{K}{x} - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \right].
\end{aligned}$$

Putting all together, that is, $F(0, x) = I_1 - I_2 = I_{11} - I_{12} - I_{21} + I_{22}$, we have

$$\begin{aligned}
F(0, x) &= e^{-r_d T} \left\{ -e^{n\left(r_d - r_f + \frac{n-1}{2}\sigma^2\right)T} \left(\frac{B}{x}\right)^{2\frac{r_d - r_f + (n-\frac{1}{2})\sigma^2}{\sigma^2}} x^n \left[N\left(\frac{\ln \frac{x}{B} - (r_d - r_f + (n - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}\right) \right. \right. \\
&\quad \left. \left. - N\left(\frac{\ln \frac{xK}{B^2} - (r_d - r_f + (n - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}\right) \right] + \left(\frac{B}{x}\right)^{\frac{2(r_d - r_f - \frac{1}{2}\sigma^2)}{\sigma^2}} K^n \left[N\left(\frac{\ln \frac{x}{B} - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \right. \right. \\
&\quad \left. \left. - N\left(\frac{\ln \frac{xK}{B^2} - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \right] + \exp\left\{n\left(r_d - r_f + \frac{n-1}{2}\sigma^2\right)T\right\} x^n \right. \\
&\quad \cdot \left[N\left(\frac{\ln \frac{B}{x} - (r_d - r_f + (n - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln \frac{K}{x} - (r_d - r_f + (n - \frac{1}{2})\sigma^2)T}{\sigma\sqrt{T}}\right) \right] \\
&\quad \left. - K^n \left[N\left(\frac{\ln \frac{B}{x} - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln \frac{K}{x} - (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \right] \right\}.
\end{aligned}$$

This result can be written using financial variables as follows

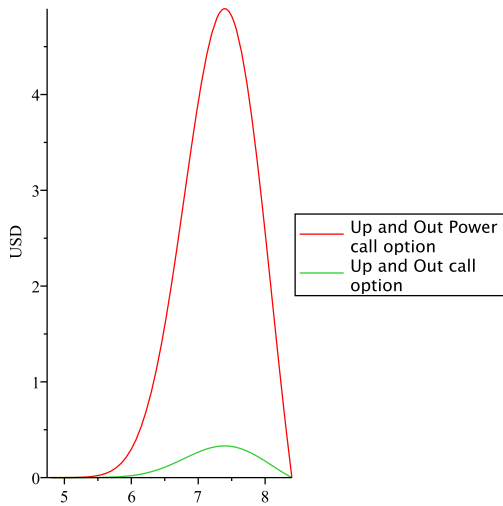
Proposition 4.1.1. *The price of an Up and out Asymmetric power call option with an upper barrier B , exercise price K and maturity time T , regarding that $K < B$, is given by the formula*

$$\begin{aligned}
C_{UAOAPO}(t, x, T, K, B) = & C_{APO}(t, x, T, K) - C_{APO}(t, x, T, B) \\
& - \left(\frac{B}{x}\right)^{2\frac{r_d - r_f}{\sigma^2} - 1} \left[C_{APO}\left(t, \frac{B^2}{x}, T, K\right) - C_{APO}\left(t, \frac{B^2}{x}, T, B\right) \right] \\
& - (B^n - K^n) \left[V^{CB}(t, x, T, B) - \left(\frac{B}{x}\right)^{2\frac{r_d - r_f}{\sigma^2} - 1} V^{CB}\left(t, \frac{B^2}{x}, T, B\right) \right],
\end{aligned}$$

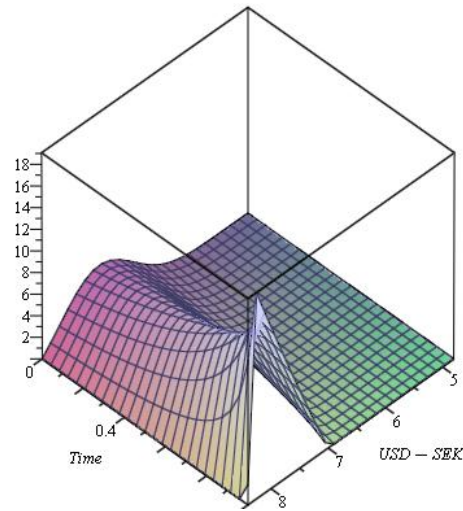
where $V^{CB}(t, x, T, K)$ is the price of a call bet which pays one unit of domestic currency if the foreign exchange rate at maturity is above K , and otherwise nothing.

Remark : This result from Proposition 4.1.1 is not in the literature as far as we know but, for $n = 1$ this result coincides with the Up and out call option.

It is clearly seen from Figure 4.3(a) that the payoff of Up and out power call option remain higher



(a) Payoff of Up and Out Asymmetric Power Call option and Up and Out Call Option



(b) Payoff: Up and out call option

Figure 4.3: Up and out Asymmetric power call options using (Volatility) $\sigma = 0.0853$, (Interest rates) $r_{USD} = 0.08$, $r_{SEK} = 0.12$, (Maturity time) $T = 1$ year, (Strike price) $K = 7$, (Upper barrier) $B = 8.4$, $n = 2$.

than the payoff of the Up and out call option. This fact can obviously motivate the Speculators to invest in a product with higher payoff. The Figure 4.3(b), it just shows the payoff surface of the Up and out asymmetric power options as function of time and foreign exchange rate.

4.1.2 Up and out Symmetric Power call option

Consider an European symmetric power call option with exercise time T , strike price K and an up and out barrier B . We assume that the strike price K is lesser than the up and out barrier B , because otherwise, the option must knock out in order to be in the money and hence could only pay off zero.

Following a similar procedure as used in the previous payoff valuation and applying Newton's binomial theorem we have the discounted payoff for an Up and out symmetric power call option stated by the following proposition.

Proposition 4.1.2. *The price of an Up and out symmetric power call option with an upper barrier B , exercise price K and maturity time T , regarding that $K < B$, is given by the formula*

$$F(0, x) = e^{-r_d T} \sum_{j=0}^n \binom{n}{j} (-K)^j x^{n-j} \exp \left(\left[r_d - r_f + \left(\frac{n-j-1}{2} \right) \sigma^2 \right] (n-j) T \right) \left\{ \left(\frac{B}{x} \right)^{2 \frac{[r_d - r_f + (n-j-\frac{1}{2})\sigma^2]}{\sigma^2}} \right. \\ \cdot \left[N \left(\frac{\ln \frac{x}{B} - [r_d - r_f + (n-j-\frac{1}{2})\sigma^2] T}{\sigma \sqrt{T}} \right) - N \left(\frac{\ln \frac{xK}{B^2} - [r_d - r_f + (n-j-\frac{1}{2})\sigma^2] T}{\sigma \sqrt{T}} \right) \right] \\ \left. - \left[N \left(\frac{\ln \frac{B}{x} - [r_d - r_f + (n-j-\frac{1}{2})\sigma^2] T}{\sigma \sqrt{T}} \right) - N \left(\frac{\ln \frac{K}{x} - [r_d - r_f + (n-j-\frac{1}{2})\sigma^2] T}{\sigma \sqrt{T}} \right) \right] \right\}.$$

Remark : This result from Proposition 4.1.2 is not in the literature as far as we know but, for $n = 1$ this result coincides with the Up and out call option.

4.1.3 Down and out Symmetric Power call option

Consider an European symmetric power call option with exercise time T , strike price K and a down and out barrier B . We assume that the strike price K is greater than the down and out barrier B , because otherwise, the option must knock out in order to be in the money and hence could only pay off zero.

The mathematical model for the European down and out symmetric power call is

$$\left\{ \begin{array}{ll} \frac{\partial F}{\partial t} + (r_d - r_f)x \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} - r_d F = 0, & B \leq x < \infty, 0 \leq t \leq T, \\ F(T, x) = [(x - K)^+]^n, & B < x < \infty, \\ F(t, B) = 0, F(t, x \rightarrow \infty) \rightarrow e^{-r_f(T-t)} x^n, & B \leq x < \infty, 0 \leq t \leq T. \end{array} \right.$$

Under the transformation

$$y = \ln \frac{x}{B},$$

$$F = Bu \quad (4.2)$$

the above problem becomes with constant coefficient as follows

$$\begin{cases} \frac{\partial u}{\partial t} + \left(r_d - r_f - \frac{1}{2}\sigma^2\right) \frac{\partial u}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial y^2} - r_d u = 0, & 0 \leq y < \infty, 0 \leq t \leq T, \\ u(T, y) = B^{n-1} [(e^y - K_B)^+]^n, & 0 < y < \infty, \\ u(t, 0) = 0, & 0 \leq t \leq T, \end{cases}$$

where $K_B = \frac{K}{B}$.

Introducing a new function W as follows

$$u = e^{\alpha y + \beta(T-t)} W, \quad (4.3)$$

where $\alpha = -\frac{1}{\sigma^2} \left(r_d - r_f - \frac{1}{2}\sigma^2\right)$ and $\beta = -r_d - \frac{1}{2\sigma^2} \left(r_d - r_f - \frac{1}{2}\sigma^2\right)^2$, we get the following final boundary value problem

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial y^2} = 0, & 0 \leq y < \infty, 0 \leq t \leq T, \\ W(T, y) = e^{-\alpha y} B^{n-1} [(e^y - K_B)^+]^n, & 0 < y < \infty, \\ W(t, 0) = 0, & 0 \leq t \leq T, \end{cases} \quad (4.4)$$

Applying the image method [14], define

$$\varphi(y) = \begin{cases} e^{-\alpha y} B^{n-1} [(e^y - K_B)^+]^n, & y > 0 \\ -e^{\alpha y} B^{n-1} [(e^{-y} - K_B)^+]^n, & y < 0. \end{cases}$$

It is clear that $\varphi(y) = -\varphi(-y)$ which means that $\varphi(y)$ is an odd function. Now, we consider the Cauchy problem on $\{-\infty \leq y < \infty, 0 \leq t \leq T\}$ as follows

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial y^2} = 0, & -\infty < y < \infty, 0 \leq t \leq T, \\ W(T, y) = \varphi(y), & -\infty < y < \infty. \end{cases} \quad (4.5)$$

Since its solution must be an odd function, thus in $\{0 \leq y < \infty, 0 \leq t \leq T\}$, $W(t, y)$ satisfies the final boundary value problem (4.4).

The solution of the Cauchy problem (4.5) can be written in the form of the Poisson formula as follows

$$\begin{aligned}
W(t, y) &= \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} \varphi(\xi) d\xi \\
&= \frac{B^{n-1}}{\sigma\sqrt{2\pi(T-t)}} \left[\int_0^{\infty} \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{-\alpha\xi} [(e^\xi - K_B)^+]^n d\xi \right. \\
&\quad \left. - \int_{-\infty}^0 \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{\alpha\xi} [(e^{-\xi} - K_B)^+]^n d\xi \right] \\
&= \frac{B^{n-1}}{\sigma\sqrt{2\pi(T-t)}} \left[\int_0^{\infty} \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{-\alpha\xi} [(e^\xi - K_B)^+]^n d\xi \right. \\
&\quad \left. - \int_0^{\infty} \exp\left\{-\frac{(y+\xi)^2}{2\sigma^2(T-t)}\right\} e^{-\alpha\xi} [(e^\xi - K_B)^+]^n d\xi \right] \\
&= \frac{B^{n-1}}{\sigma\sqrt{2\pi(T-t)}} \sum_{j=0}^n \binom{n}{j} (-K_B)^{n-j} \left[\int_{\ln K_B}^{\infty} \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{(j-\alpha)\xi} d\xi \right. \\
&\quad \left. - \int_{\ln K_B}^{\infty} \exp\left\{-\frac{(y+\xi)^2}{2\sigma^2(T-t)}\right\} e^{(j-\alpha)\xi} d\xi \right] \\
&= B^{n-1} \sum_{j=0}^n \binom{n}{j} (-K_B)^{n-j} [I_1 - I_2].
\end{aligned}$$

Now, let us compute the integrals I_1 and I_2

$$\begin{aligned}
I_1 &= \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{\ln K_B}^{\infty} \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{(j-\alpha)\xi} d\xi \\
&= \frac{\exp\left\{\frac{(j-\alpha)(2y+\sigma^2(T-t)(j-\alpha))}{2}\right\}}{\sigma\sqrt{2\pi(T-t)}} \int_{\ln K_B}^{\infty} \exp\left\{-\frac{(\xi - (y + \sigma^2(T-t)(j-\alpha)))^2}{2\sigma^2(T-t)}\right\} d\xi \\
&= \exp\left\{\frac{(j-\alpha)(2y + \sigma^2(T-t)(j-\alpha))}{2}\right\} N\left(\frac{-\ln K_B + y + \sigma^2(T-t)(j-\alpha)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{\ln K_B}^{\infty} \exp\left\{-\frac{(y+\xi)^2}{2\sigma^2(T-t)}\right\} e^{(j-\alpha)\xi} d\xi \\
&= \frac{\exp\left\{-\frac{(j-\alpha)(2y-\sigma^2(T-t)(j-\alpha))}{2}\right\}}{\sigma\sqrt{2\pi(T-t)}} \int_{\ln K_B}^{\infty} \exp\left\{-\frac{(\xi+y-\sigma^2(T-t)(j-\alpha))^2}{2\sigma^2(T-t)}\right\} d\xi \\
&= \exp\left\{-\frac{(j-\alpha)(2y-\sigma^2(T-t)(j-\alpha))}{2}\right\} N\left(\frac{-\ln K_B - y + \sigma^2(T-t)(j-\alpha)}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

Thus, put the value of the integrals I_1 and I_2 into $W(t, y)$ and becomes

$$\begin{aligned}
W(t, y) &= B^{n-1} \sum_{j=0}^n \binom{n}{j} (-K_B)^{n-j} \left[e^{\frac{(j-\alpha)(2y+\sigma^2(T-t)(j-\alpha))}{2}} \cdot N\left(\frac{-\ln K_B + y + \sigma^2(T-t)(j-\alpha)}{\sigma\sqrt{T-t}}\right) \right. \\
&\quad \left. - e^{-\frac{(j-\alpha)(2y-\sigma^2(T-t)(j-\alpha))}{2}} \cdot N\left(\frac{-\ln K_B - y + \sigma^2(T-t)(j-\alpha)}{\sigma\sqrt{T-t}}\right) \right]
\end{aligned}$$

Back to the function $u(t, y)$ by (4.3), we get

$$\begin{aligned}
u(t, y) &= B^{n-1} e^{\alpha y + \beta(T-t)} \sum_{j=0}^n \binom{n}{j} (-K_B)^{n-j} \left[e^{\frac{(j-\alpha)(2y+\sigma^2(T-t)(j-\alpha))}{2}} N\left(\frac{-\ln K_B + y + \sigma^2(T-t)(j-\alpha)}{\sigma\sqrt{T-t}}\right) \right. \\
&\quad \left. - e^{-\frac{(j-\alpha)(2y-\sigma^2(T-t)(j-\alpha))}{2}} N\left(\frac{-\ln K_B - y + \sigma^2(T-t)(j-\alpha)}{\sigma\sqrt{T-t}}\right) \right].
\end{aligned}$$

Back to the original function $F(t, x)$ by (4.2), we get

$$\begin{aligned}
F(t, x) &= B^n e^{\beta(T-t)} \sum_{j=0}^n \binom{n}{j} \left(-\frac{K}{B}\right)^{n-j} e^{\frac{\sigma^2(T-t)(j-\alpha)^2}{2}} \left[\left(\frac{x}{B}\right)^j N\left(\frac{\ln \frac{x}{K} + \sigma^2(T-t)(j-\alpha)}{\sigma\sqrt{T-t}}\right) \right. \\
&\quad \left. - \left(\frac{x}{B}\right)^{2\alpha-j} N\left(\frac{\ln \frac{B^2}{xK} + \sigma^2(T-t)(j-\alpha)}{\sigma\sqrt{T-t}}\right) \right].
\end{aligned}$$

We finish this section with the following result which can be written using financial variables as follows

Proposition 4.1.3. *The price of a Down and out symmetric power call option with a lower barrier B , exercise price K and maturity time T , regarding that $K > B$, is given by the formula*

$$C_{DOSPO}(t, x, T, K, B) = C_{SPO}(t, x, T, K) - \left(\frac{x}{B}\right)^{2\alpha} C_{SPO}\left(t, \frac{B^2}{x}, T, K\right), \quad (4.6)$$

where $C_{DOSPO}(t, x, T, K, B)$ is the price of a Down and out symmetric power call option and $C_{SPO}(t, x, T, K)$ is the price of Symmetric power call option under risk neutral valuation.

Remark : This result from Proposition 4.1.3 is not in the literature as far as we know but, for $n = 1$ this result coincides with the Down and out call option. This price corresponds to a dynamic hedging strategy consisting of buying $\Delta = \frac{\partial C_{DOSPO}(t, x, T, K, B)}{\partial x}$ units of foreign currency.

4.1.4 Down and out Asymmetric Power call option

Consider an European asymmetric power call option with exercise time T , strike price K and a down and out barrier B . We assume that the strike price K is greater than the down and out barrier B , because otherwise, the option must knock out in order to be in the money and hence could only pay off zero.

The mathematical model for the European down and out symmetric power call is

$$\begin{cases} \frac{\partial F}{\partial t} + (r_d - r_f)x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} - r_d F = 0, & B \leq x < \infty, 0 \leq t \leq T, \\ F(T, x) = (x^n - K^n)^+, & B < x < \infty, \\ F(t, B) = 0, F(t, x \rightarrow \infty) \rightarrow e^{-r_f(T-t)}x^n, & B \leq x < \infty, 0 \leq t \leq T. \end{cases}$$

Under the transformation

$$\begin{aligned} y &= \ln \frac{x}{B}, \\ F &= Bu, \end{aligned} \tag{4.7}$$

the above problem becomes with constant coefficients as follows

$$\begin{cases} \frac{\partial u}{\partial t} + \left(r_d - r_f - \frac{1}{2}\sigma^2\right) \frac{\partial u}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial y^2} - r_d u = 0, & 0 \leq y < \infty, 0 \leq t \leq T, \\ u(T, y) = B^{n-1}(e^{ny} - K_B)^+, & 0 < y < \infty, \\ u(t, 0) = 0, & 0 \leq t \leq T, \end{cases}$$

where $K_B = \frac{K^n}{B^n}$.

Introduce a new function W , as follows

$$u = e^{\alpha y + \beta(T-t)}W, \tag{4.8}$$

where $\alpha = -\frac{1}{\sigma^2}\left(r_d - r_f - \frac{1}{2}\sigma^2\right)$ and $\beta = -r_d - \frac{1}{2\sigma^2}\left(r_d - r_f - \frac{1}{2}\sigma^2\right)^2$.

By the transformation (4.8), we get a final boundary value for W in $\{(t, y) : 0 \leq y < \infty, 0 \leq t \leq T\}$ as follows

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial y^2} = 0, & 0 \leq y < \infty, 0 \leq t \leq T, \\ W(T, y) = e^{-\alpha y} B^{n-1} (e^{ny} - K_B)^+, & 0 < y < \infty, \\ W(t, 0) = 0, & 0 \leq t \leq T, \end{cases} \quad (4.9)$$

Applying the image method [14], define

$$\varphi(y) = \begin{cases} e^{-\alpha y} B^{n-1} (e^{ny} - K_B)^+, & y > 0; \\ -e^{\alpha y} B^{n-1} (e^{-ny} - K_B)^+, & y < 0. \end{cases}$$

It is clear that $\varphi(y) = -\varphi(-y)$ which means that $\varphi(y)$ is an odd function. Now, we consider the Cauchy problem on $\{(t, y) : -\infty \leq y < \infty, 0 \leq t \leq T\}$ as follows

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial y^2} = 0, & -\infty < y < \infty, 0 \leq t \leq T, \\ W(T, y) = \varphi(y), & -\infty < y < \infty. \end{cases} \quad (4.10)$$

Since its solution must be an odd function, thus in $\{(t, y) : 0 \leq y < \infty, 0 \leq t \leq T\}$, $W(t, y)$ satisfies the final boundary value problem (4.9).

The solution of the Cauchy problem (4.10) can be written in the form of the Poisson formula as follows

$$\begin{aligned} W(t, y) &= \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} \varphi(\xi) d\xi \\ &= \frac{B^{n-1}}{\sigma\sqrt{2\pi(T-t)}} \left[\int_0^{\infty} \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{-\alpha\xi} (e^{n\xi} - K_B)^+ d\xi \right. \\ &\quad \left. - \int_{-\infty}^0 \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{\alpha\xi} (e^{-n\xi} - K_B)^+ d\xi \right] \\ &= \frac{B^{n-1}}{\sigma\sqrt{2\pi(T-t)}} \left[\int_0^{\infty} \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{-\alpha\xi} (e^{n\xi} - K_B)^+ d\xi \right. \\ &\quad \left. - \int_0^{\infty} \exp\left\{-\frac{(y+\xi)^2}{2\sigma^2(T-t)}\right\} e^{-\alpha\xi} (e^{n\xi} - K_B)^+ d\xi \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{B^{n-1}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \left[\exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} - \exp\left\{-\frac{(y+\xi)^2}{2\sigma^2(T-t)}\right\} \right] e^{-\alpha\xi}(e^{n\xi} - K_B)^+ d\xi \\
&= \frac{B^{n-1}}{\sigma\sqrt{2\pi(T-t)}} \int_{\frac{1}{n}\ln K_B}^\infty \left[\exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} - \exp\left\{-\frac{(y+\xi)^2}{2\sigma^2(T-t)}\right\} \right] e^{-\alpha\xi}(e^{n\xi} - K_B) d\xi.
\end{aligned}$$

Back to the function $u(t, y)$ by (4.8), we get

$$\begin{aligned}
u(t, y) &= \frac{B^{n-1}e^{\alpha y + \beta(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\frac{1}{n}\ln K_B}^\infty \left[\exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} - \exp\left\{-\frac{(y+\xi)^2}{2\sigma^2(T-t)}\right\} \right] e^{-\alpha\xi}(e^{n\xi} - K_B) d\xi \\
&= \frac{B^{n-1}e^{\alpha y + \beta(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \left[\int_{\frac{1}{n}\ln K_B}^\infty \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{(n-\alpha)\xi} d\xi - \int_{\frac{1}{n}\ln K_B}^\infty \exp\left\{-\frac{(y+\xi)^2}{2\sigma^2(T-t)}\right\} e^{(n-\alpha)\xi} d\xi \right. \\
&\quad \left. - K_B \int_{\frac{1}{n}\ln K_B}^\infty \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{-\alpha\xi} d\xi + K_B \int_{\frac{1}{n}\ln K_B}^\infty \exp\left\{-\frac{(y+\xi)^2}{2\sigma^2(T-t)}\right\} e^{-\alpha\xi} d\xi \right] \\
&= B^{n-1}e^{\alpha y + \beta(T-t)} [I_1 - I_2 - I_3 + I_4].
\end{aligned}$$

Now, let us compute the integrals I_1 , I_2 , I_3 and I_4 .

$$\begin{aligned}
I_1 &= \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{\frac{1}{n}\ln K_B}^\infty \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{(n-\alpha)\xi} d\xi \\
&= \frac{\exp\{(n-\alpha)(2y + \sigma^2(T-t)(n-\alpha))/2\}}{\sigma\sqrt{2\pi(T-t)}} \int_{\frac{1}{n}\ln K_B}^\infty \exp\left\{-\frac{(\xi - (y + \sigma^2(T-t)(n-\alpha)))^2}{2\sigma^2(T-t)}\right\} d\xi \\
&= e^{(n-\alpha)(2y + \sigma^2(T-t)(n-\alpha))/2} N\left(\frac{-\frac{1}{n}\ln K_B + y + \sigma^2(T-t)(n-\alpha)}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{\frac{1}{n}\ln K_B}^\infty \exp\left\{-\frac{(y+\xi)^2}{2\sigma^2(T-t)}\right\} e^{(n-\alpha)\xi} d\xi \\
&= \frac{\exp\{-(n-\alpha)(2y - \sigma^2(T-t)(n-\alpha))/2\}}{\sigma\sqrt{2\pi(T-t)}} \int_{\frac{1}{n}\ln K_B}^\infty \exp\left\{-\frac{(\xi + y - \sigma^2(T-t)(n-\alpha))^2}{2\sigma^2(T-t)}\right\} d\xi
\end{aligned}$$

$$= e^{-(n-\alpha)(2y-\sigma^2(T-t)(n-\alpha))/2} N\left(\frac{-\frac{1}{n} \ln K_B - y + \sigma^2(T-t)(n-\alpha)}{\sigma\sqrt{T-t}}\right).$$

$$\begin{aligned} I_3 &= \frac{K_B}{\sigma\sqrt{2\pi(T-t)}} \int_{\frac{1}{n} \ln K_B}^{\infty} \exp\left\{-\frac{(y-\xi)^2}{2\sigma^2(T-t)}\right\} e^{-\alpha\xi} d\xi \\ &= \frac{K_B \exp\{-\alpha(2y-\sigma^2(T-t)\alpha)/2\}}{\sigma\sqrt{2\pi(T-t)}} \int_{\frac{1}{n} \ln K_B}^{\infty} \exp\left\{-\frac{(\xi-(y-\sigma^2(T-t)\alpha))^2}{2\sigma^2(T-t)}\right\} d\xi \\ &= K_B e^{-\alpha(2y-\sigma^2(T-t)\alpha)/2} N\left(\frac{-\frac{1}{n} \ln K_B + y - \sigma^2(T-t)\alpha}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

$$\begin{aligned} I_4 &= \frac{K_B}{\sigma\sqrt{2\pi(T-t)}} \int_{\frac{1}{n} \ln K_B}^{\infty} \exp\left\{-\frac{(y+\xi)^2}{2\sigma^2(T-t)}\right\} e^{-\alpha\xi} d\xi \\ &= \frac{K_B \exp\{\alpha(2y+\sigma^2(T-t)\alpha)/2\}}{\sigma\sqrt{2\pi(T-t)}} \int_{\frac{1}{n} \ln K_B}^{\infty} \exp\left\{-\frac{(\xi+y+\sigma^2(T-t)\alpha)^2}{2\sigma^2(T-t)}\right\} d\xi \\ &= K_B e^{\alpha(2y+\sigma^2(T-t)\alpha)/2} N\left(\frac{-\frac{1}{n} \ln K_B - y - \sigma^2(T-t)\alpha}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

Thus, put the value of the integrals I_1 , I_2 , I_3 and I_4 into $u(t, y)$ and becomes

$$\begin{aligned} u(t, y) &= B^{n-1} e^{\alpha y + \beta(T-t)} \left[e^{(n-\alpha)(2y+\sigma^2(T-t)(n-\alpha))/2} N\left(\frac{-\frac{1}{n} \ln K_B + y + \sigma^2(T-t)(n-\alpha)}{\sigma\sqrt{T-t}}\right) \right. \\ &\quad - e^{-(n-\alpha)(2y-\sigma^2(T-t)(n-\alpha))/2} N\left(\frac{-\frac{1}{n} \ln K_B - y + \sigma^2(T-t)(n-\alpha)}{\sigma\sqrt{T-t}}\right) \\ &\quad - K_B e^{-\alpha(2y-\sigma^2(T-t)\alpha)/2} N\left(\frac{-\frac{1}{n} \ln K_B + y - \sigma^2(T-t)\alpha}{\sigma\sqrt{T-t}}\right) \\ &\quad \left. + K_B e^{\alpha(2y+\sigma^2(T-t)\alpha)/2} N\left(\frac{-\frac{1}{n} \ln K_B - y - \sigma^2(T-t)\alpha}{\sigma\sqrt{T-t}}\right) \right]. \end{aligned}$$

Back to the original function $F(t, x)$ by (4.7), we get

$$\begin{aligned}
F(t, x) = & e^{\beta(T-t)} \left[x^n e^{\sigma^2(n-\alpha)^2(T-t)/2} N\left(\frac{\ln \frac{x}{K} + \sigma^2(T-t)(n-\alpha)}{\sigma\sqrt{T-t}}\right) \right. \\
& - B^n \left(\frac{x}{B}\right)^{2\alpha-n} e^{\sigma^2(n-\alpha)^2(T-t)/2} N\left(\frac{\ln \frac{B^2}{xK} + \sigma^2(T-t)(n-\alpha)}{\sigma\sqrt{T-t}}\right) \\
& - K^n e^{\sigma^2\alpha^2(T-t)/2} N\left(\frac{\ln \frac{x}{K} - \sigma^2(T-t)\alpha}{\sigma\sqrt{T-t}}\right) \\
& \left. + K^n \left(\frac{x}{B}\right)^{2\alpha} e^{\sigma^2\alpha^2(T-t)/2} N\left(\frac{\ln \frac{B^2}{xK} - \sigma^2(T-t)\alpha}{\sigma\sqrt{T-t}}\right) \right].
\end{aligned}$$

We end this section with the following result which can be written using financial variables as follows

Proposition 4.1.4. *The price of a Down and out asymmetric power call option with a lower barrier B , exercise price K and maturity time T , regarding that $K > B$, is given by the formula*

$$C_{DAOAPO}(t, x, T, K, B) = C_{APO}(t, x, T, K) - \left(\frac{x}{B}\right)^{2\alpha} C_{APO}\left(t, \frac{B^2}{x}, T, K\right), \quad (4.11)$$

where $C_{DAOAPO}(t, x, T, K, B)$ is the price of a Down and out asymmetric power call option and $C_{APO}(t, x, T, K)$ is the price of Asymmetric Power Call Option under risk neutral valuation.

Remark : This result from Proposition 4.1.4 is not in the literature as far as we know but, for $n = 1$ this result coincide with the Up and out call option.

This price corresponds to a dynamic hedging strategy consisting of buying $\Delta = \frac{\partial C_{DAOAPO}(t, x, T, K, B)}{\partial x}$ units of foreign currency.

In both cases, asymmetric and symmetric down and out power call options, if we set $n = 1$ and consider $r_d = r_f$ we can write, using put and call symmetry [16], both equalities (4.6) and (4.11) in the following form

$$\begin{aligned}
C_{DAOSPO}(t, x, T, K, B) &= C_{DAOAPO}(t, x, T, K, B) \\
&= C_{SPO}(t, x, T, K) - \frac{K}{B} P_{SPO}\left(t, x, T, \frac{B^2}{K}\right) \\
&= C_{APO}(t, x, T, K) - \frac{K}{B} P_{APO}\left(t, x, T, \frac{B^2}{K}\right).
\end{aligned}$$

The above relations follow from the fact that for $n = 1$ the symmetric and asymmetric call options are equivalents to vanilla call options and for vanilla put options the equivalence holds with the symmetric and asymmetric put options.

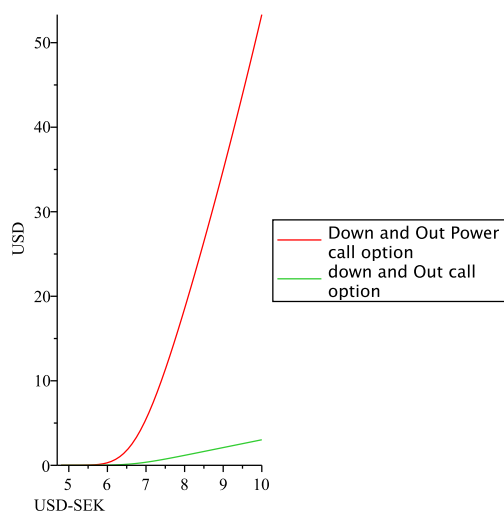


Figure 4.4: Discounted payoff of Down and out asymmetric power call and Down and out call (Strike price) $K = 7$, (Lower barrier) $B = 4.8$, (Interest rates) $r_{USD} = 0.08$, $r_{SEK} = 0.12$, (Volatility) $\sigma = 0.0853$, (Maturity time) $T = 1\text{year}$, $n = 2$

The most important thing in the above relations they show the static hedging which consist of buying and selling vanilla options in order to get a replicated one of down and out call.

In Figure 4.4, it is clearly seen that the Down and out asymmetric power call has a higher discounted payoff function than the simple Down and out call.

4.2 The Greeks for Knock Out Power Options

Greeks are sensitivity analysis and they measure the dimension of risk involved in taking a position in an option. There are many kind of parameters for sensitivity's analysis such as, Delta, Theta, Gamma, Vega and Rho. The foreign exchange rate sensitivity, delta, denoted by Δ , is the rate of change between the discounted payoff and the underlying price of the foreign exchange rate. The time sensitivity, Θ , also called theta is the rate of change between the portfolio and time. The second order exchange rate sensitivity, gamma, denoted by Γ , is the rate of change between delta and underlying price of the foreign exchange rate. The volatility sensitivity, vega, denoted by \mathcal{V} , is the rate of change between between the discounted payoff and the volatility of the dynamics of foreign exchange rate. The interest rate sensitivity, Rho, denoted by ρ , is the rate of change between the discounted payoff and the domestic interest rate (or foreign interest rate) of the foreign exchange rate dynamics.

A portfolio which does not vary with respect to small increments of the parameters is called neutral. Thus, delta neutral portfolio is the one which is insensitive to small changes of the foreign exchange rate, that is, $\Delta = 0$.

Let V be the discounted payoff of the Down and out asymmetric power call option. The Greeks are mathematically defined as follows

$$\Delta = \frac{\partial V}{\partial x} \quad (4.12)$$

$$\Gamma = \frac{\partial^2 V}{\partial x^2} \quad (4.13)$$

$$\rho = \frac{\partial V}{\partial r_d} \quad (4.14)$$

$$\Theta = \frac{\partial V}{\partial t} \quad (4.15)$$

$$\mathcal{V} = \frac{\partial V}{\partial \sigma} \quad (4.16)$$

Herein, we will compute some of Greeks for a Down and Out asymmetric Power Option and in these some of them will be presented graphically. It is slightly comfortable compute Δ of Down and out asymmetric power option using the closed formula (4.11). However, we get

$$\begin{aligned} \Delta = & n e^{-r_d(T-t)} x^{n-1} e^{n(r_d-r_f+\frac{n-1}{2})(T-t)} N(z_1) - \frac{2\alpha}{B} \left(\frac{x}{B}\right)^{2\alpha-1} C_{APO}\left(t, \frac{B^2}{x}, T, K\right) \\ & + \frac{n}{x} \left(\frac{x}{B}\right)^{2\alpha} e^{-r_d(T-t)} \left(\frac{B^2}{x}\right)^n e^{n(r_d-r_f+\frac{n-1}{2})(T-t)} N(z_1^*), \end{aligned}$$

where

$$z_1 = \frac{\ln \frac{x}{K} + (r_d - r_f + (n - \frac{1}{2})\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \quad z_1^* = \frac{\ln \frac{B^2}{xK} + (r_d - r_f + (n - \frac{1}{2})\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$

The Figure 4.5 presents surfaces of Delta and Gamma as function of time and foreign exchange rate. It is important to note that Γ is very large in the neighborhood of expiration time and exercise price.

Looking in Figure 4.6(a) it is clearly seen that the delta of Vanilla call and the delta of Down and out option, under the parameters considered, have almost the same shape which mean that they can be hedged with closer foreign exchange rates.

The deltas of Up and out and Up and out asymmetric power have the same behavior, they differ in their absolute values because the absolute value of delta of Up and out asymmetric power is grater or equal than the absolute value of delta of Up and out for any spot value.

Similar behavior is observed for delta of Asymmetric power and delta of Down and out asymmetric power where this is always bellow of the delta of Asymmetric power.

We point out that the delta of the Asymmetric power is always above the deltas of all option for any value of spot. The Up and out symmetric power is the option which attain the lowest delta value for some value of spot.

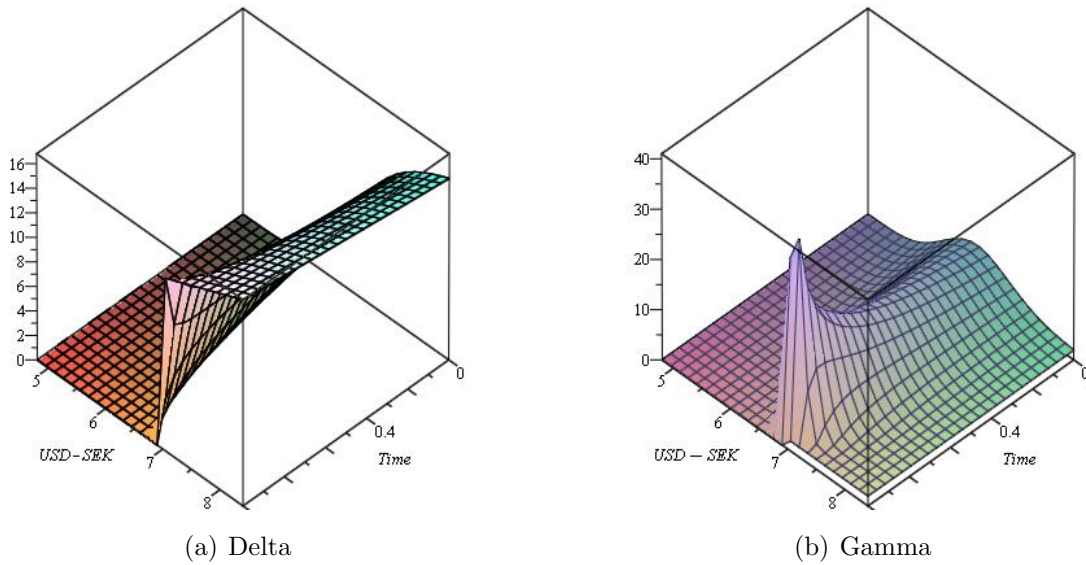


Figure 4.5: Down and out asymmetric power call options with (Volatility) $\sigma = 0.0853$, (Interest rates) $r_{USD} = 0.08$, $r_{SEK} = 0.12$, (Maturity time) $T = 1$ year, (Strike price) $K = 7$, (Lower barrier) $B = 4.8$, $n = 2$.

Since Delta quantifies the sensitivity of theoretical value of an option to change in price foreign exchange rate and shows that how much the value of an option should vary when the price of the underlying exchange rate change in one unity. Thus, for those options whose delta is huge their option value will have a huge variation when the price of the underlying foreign exchange rate vary by one unity.

In Figure 4.6(b), we present the comparison of Gammas of Vanilla, Asymmetric power, Down and out, Down and out asymmetric, Up and out and finally Up and out asymmetric power.

We point out that the Gamma of Vanilla and the Gamma of Down and out have the same behavior and are non-negative.

The gamma of Up and out and the gamma of Up and out asymmetric power have the same course and the absolute value of gamma of the Up and out asymmetric power is greater or equal to the absolute value of gamma of the Up and out for any value of spot.

Considering Vanilla and Asymmetric power we observe that their gammas are always positive and the gamma of Vanilla is ever bellow or equal to the gamma of Asymmetric power.

Looking at the Greeks in Figure 4.6 of asymmetric power options compared to Vanilla and Knock out options the power of the elements of Power options is well reflected in the exposures.

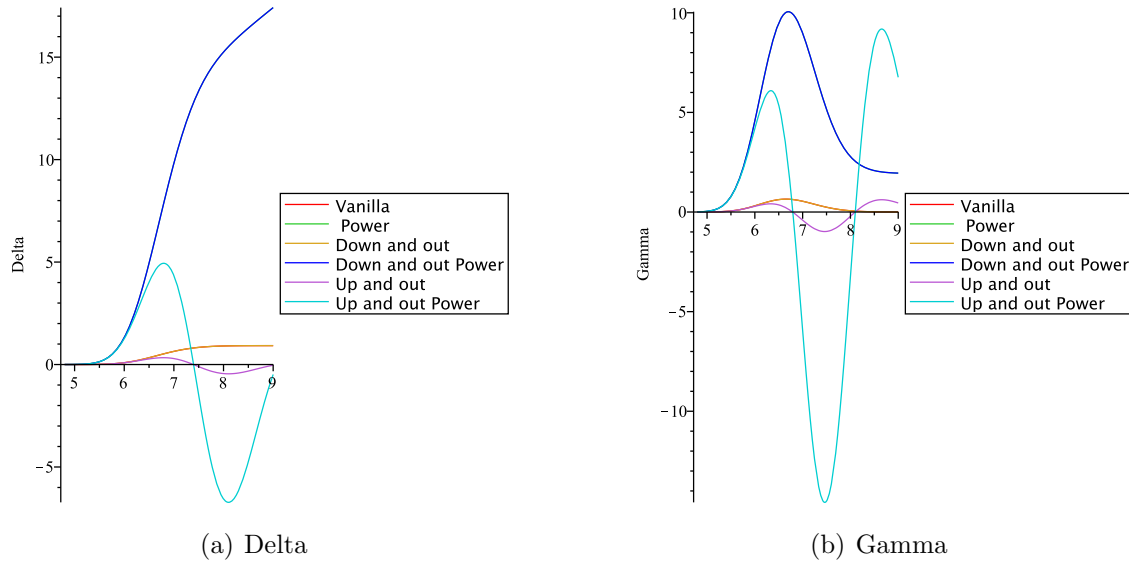


Figure 4.6: Delta's comparison and Gamma's comparison of call options (Volatility) $\sigma = 0.0853$, (Interest rates) $r_{USD} = 0.08$, $r_{SEK} = 0.12$, Maturity Time $T = 1$ year, (Strike price) $K = 7$, (Lower barrier) $B = 4.8$, (Upper barrier) $B = 8.4$, $n = 2$.

4.3 Static Hedging

In this section we will discuss the static hedging of the Up and out asymmetric power call option by creating it synthetically from power call options and power put options. To do so, we will follow the sophisticated case discussed in Lipton [7] for up and out call option. Maruhn [9] has also described the static hedging for up and out call option.

A static hedge is a strategy which is fixed and is created onetime to hedge a prevailing position or option. It is not adjusted at all, once created, in contrast with a dynamic hedge. An overview of static hedge is to provide an exact and necessary protection so far and in addition including the option maturity date.

One of the reasons of hedging is to be immune to unpleasant surprises, faced by the portfolio, such as sharp increment, for example, in price of foreign exchange rate.

Up and out power call option and Down and out power put option have discontinuity point in the boundary, this makes their portfolio difficulty for hedging.

The boundary value problem for an Up and out asymmetric power call options is given by

$$\begin{cases} \frac{\partial F}{\partial t} + (r_d - r_f)x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} - r_d F = 0, & 0 \leq x \leq B, 0 \leq t \leq T, \\ F(T, x) = (x^n - K^n)^+, & 0 \leq x < B, \\ F(0, x) = F(t, B) = 0, & 0 \leq x \leq B, 0 \leq t \leq T. \end{cases}$$

The method we will use to price and hedge consist of creating a fictitious barrier which is shifted a bit outward. This takes into account short selling restrictions. One way for hedging Up and out asymmetric power call option consist of, as described in Lipton[7], replacing the Dirichlet boundary condition, in the above problem, at the barrier B by

$$B \frac{\partial F(t, B)}{\partial x} + \mathcal{A}F(t, B) = 0, \quad (4.17)$$

where \mathcal{A} is the leverage coefficient. Let us denote the condition (4.17) by

$$V(t, x) = x \frac{\partial F(t, x)}{\partial x} + \mathcal{A}F(t, x) \quad (4.18)$$

This relation, at time T , can be written as follows

$$\begin{aligned} V(T, x) &= nx^n H(x^n - K^n) + \mathcal{A}(x^n - K^n)^+ + nK^n H(x^n - K^n) - nK^n H(x^n - K^n) \\ &= n(x^n - K^n)H(x^n - K^n) + \mathcal{A}(x^n - K^n)^+ + nK^n H(x^n - K^n) \\ &= n(x^n - K^n)^+ + \mathcal{A}(x^n - K^n)^+ + nK^n H(x^n - K^n) \\ &= (n + \mathcal{A})(x^n - K^n)^+ + nK^n H(x^n - K^n). \end{aligned}$$

Now, we are going to show that function $V(t, x)$, in this case relation (4.18), satisfies the Black-Scholes partial differential equation

$$\begin{aligned} \frac{\partial V}{\partial t} + (r_d - r_f)x \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - r_d V &= x \frac{\partial^2 F}{\partial x \partial t} + \mathcal{A} \frac{\partial F}{\partial x} + (r_d - r_f)x^2 \frac{\partial^2 F}{\partial x^2} + (r_d - r_f)x \mathcal{A} \frac{\partial F}{\partial x} \\ &\quad + \frac{1}{2}\sigma^2 x^3 \frac{\partial^3 F}{\partial x^3} + \mathcal{A} \frac{\partial^2 F}{\partial x^2} - r_d x \frac{\partial F(t, x)}{\partial x} - r_d \mathcal{A}F(t, x) \\ &= x \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial t} + (r_d - r_f)x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} - r_d F \right) + \mathcal{A} \left(\frac{\partial F}{\partial t} \right. \\ &\quad \left. + (r_d - r_f)x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} - r_d F \right) = 0. \end{aligned}$$

Thus, collecting all together, we have the following pricing problem

$$\begin{cases} \frac{\partial V}{\partial t} + (r_d - r_f)x \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - r_d V = 0, & 0 \leq x \leq B, 0 \leq t \leq T, \\ V(T, x) = (n + \mathcal{A})(x^n - K^n)^+ + nK^n H(x^n - K^n), & 0 \leq x < B, \\ V(0, x) = V(t, B) = 0, & 0 \leq x \leq B, 0 \leq t \leq T. \end{cases}$$

In this case, V is equal to the price of the portfolio consisting of $(n + \mathcal{A})$ Up and out symmetric power call and nK^n Up and out symmetric call bets. Therefore, the solution to the above problem can be written as follows

$$V(t, x) = (n + \mathcal{A})C_{UAOSPCO}(t, x, T, K, B) + nK^n V^{CB}(t, x, T, K). \quad (4.19)$$

Solving the first order partial differential equation (4.18) with trivial condition at $x = 0$ we get the replicated portfolio given as follows

$$F(t, x) = x^{-\mathcal{A}} \int_0^x \chi^{\mathcal{A}-1} V(t, \chi) d\chi, \quad (4.20)$$

where $V(t, \chi)$ is defined by (4.19).

The Figure 4.7 present both graphs, the one of replicated portfolio with fictitious barrier and the

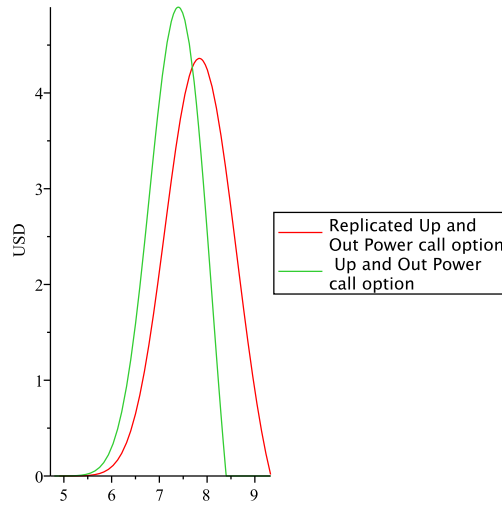


Figure 4.7: Replicated payoff function of Up and out asymmetric power call with (Strike price) $K = 7$, (Leverage coefficient) $\mathcal{A} = 10$, (Interest rates) $r_{USD} = 0.08$, $r_{SEK} = 0.12$, (Volatility) $\sigma = 0.0853$, (Upper barrier) $B = 8.4$, (Maturity time) $T = 1\text{year}$, $n = 2$.

one of original portfolio. We point out that replicated portfolio coincide with the original one when the leverage coefficient tends to infinity. If the fictitious barrier is getting closer to the original one then both options value, of the replicated and the original, are getting closer. Whenever the fictitious barrier is moving up the maximum value of the replicated portfolio is decreasing and vice versa.

Now, set $n = 2$ and let us consider the payoff of an Up and out asymmetric call option,

$$(X_T^2 - K^2)^+ \mathbf{1}_{\{X_t < B, t \in [0, T]\}}.$$

This payoff can be written as follows,

$$(X_T^2 - K^2) \mathbf{1}_{\{X_T > K, X_t < B, t \in [0, T]\}}.$$

Furthermore

$$X_T^2 - K^2 - (X_T - K)^2 = 2K(X_T - K).$$

Though, we can multiply the above relation by $\mathbf{1}_{\{X_T > K, X_t < B, t \in [0, T]\}}$ and we get

$$(X_T^2 - K^2)^+ \mathbf{1}_{\{X_t < B, t \in [0, T]\}} - [(X_T - K)^+]^2 \mathbf{1}_{\{X_t < B, t \in [0, T]\}} = 2K(X_T - K)^+ \mathbf{1}_{\{X_t < B, t \in [0, T]\}}.$$

This equality shows that an Up and out asymmetric power call option can be hedged by purchasing $2K$ units of Down and out call option and buying a Symmetric up and out power option.

4.4 Speculation with Knock Out Power Options

In this section we are going to deal with speculation using Knock out power options. About speculation we will slightly follow the discussion in Moosa [10]. We can view speculation as an opportunistic financial transaction that leverages oscillations of the exchange rates in the market for getting high profits.

Instead of getting high profits, there is another view of speculation, called insurance motive. The insurance motive views speculation as an alternative for lacking insurance markets, in that, the incomes from commerce are connected to differences in the willing of traders to care risk on a first plan.

Moosa [10], describes some reasons for speculation from insurance view point. The liquidity motive, in which the traders may have to sell assets in a period of financial necessity or just buy in order to get a variety of assets. Next reason is the divergence of priors, in which assets are transacted because agents are in disagreement on their coming value. Here the speculative behavior is induced by differences in beliefs among agents. Another motive puts speculators as intermediate and their participation depend on how they are compensate for their services.

Madura and Fox [8] describe the percentage change in a foreign currency as the product of inverse of spot X_{t-1} at earlier date and the difference between the spot X_t at current time and the spot X_{t-1} at earlier date, that is,

$$\frac{X_t - X_{t-1}}{X_{t-1}} \cdot 100.$$

We say that a currency is appreciating if the percentage change of the exchange rate is positive and we say that is depreciating if the the percentage change of the exchange rate is negative.

A call option on a given currency can be bought by speculators, those expecting currency appreciation. A put option on a given currency can be bought by speculators those expecting currency depreciation.

Let us consider the case in Table 4.1 which simulate the USD appreciation. Suppose that the options are in the money and let the current spot value be equal to 7.3. Considering call options, if a speculator expects that the percentage change of the spot value will be positive, say 5%, at maturity time. Then the speculator can buy a call and if his forecast of the percentage change in spot value is correct, he can exercise the option getting a percentage change in the option value

Nr.	Call options	Option Value	Change in Spot value				
			0.001%	0.01%	0.1%	1%	5%
1	Vanilla	0.5798	0.0098	0.0979	0.9805	9.9463	52.2320
2	Asymmetric Power	8.7821	0.0101	0.1013	1.0143	10.3197	54.9724
3	Down and Out	0.5798	0.0098	0.0979	0.9805	9.9463	52.2320
4	Up and Out	0.3268	0.0020	0.0195	0.1882	1.2052	-9.6079
5	Down and out asymmetric Power	8.7821	0.0101	0.1013	1.0143	10.3197	54.9724
6	Up and out asymmetric Power	4.8301	0.0020	0.0203	0.1963	1.2838	-9.3093
Spot value is equal to 7.3			Percentage change in discounted payoff				

Table 4.1: USD appreciation, (Strike price) $K = 7$, (Lower barrier) $B = 4.8$, (interest rates) $r_{USD} = 0.08$, $r_{SEK} = 0.12$, (Volatility) $\sigma = 0.0853$, (Upper barrier) $B = 8.4$, (Maturity time) $T = 1\text{year}$, $n = 2$.

of about 52.23%, for example, in exercising Vanilla or Down and out Options at the forecasted foreign exchange rate.

A speculator can also get a huge percentage change in the option value of about 54.97%, for example, in exercising the Symmetric power option or the Down and out symmetric power option at the forecasted foreign exchange rate.

The feature of making huge profits is not observed, for example, in exercising Up and out call or Up and out symmetric power call where the percentage change in the option value is about -9.61% and -9.31% , respectively.

Yet, in Table 4.1, we point out that buying a call a speculator is making money with the appreciation of USD when he exercises Vanilla, Asymmetric power, Down and out or Down and out asymmetric power while for Up and out or Up and out asymmetric power the range of speculative action is limited.

From Table 4.2, about the depreciation of USD, is seen that buying a call there is no speculative action because the speculator will be losing money since all percentage changes in option values are negative. One way to avoid the non-speculative situation is to buy a put option, thus the speculator gets a speculative strategy to make money.

Now, let us consider the case when a call option is out of the money. As we can see from Table 4.3 even the call options are out of the money, if the forecast of a speculator indicate an appreciation of the underlying foreign exchange rate, the percentage change in discounted payoff remain positive and it grows according to the growth of the percentage change in spot value.

Still, power options (Asymmetric, Asymmetric Down and out, Asymmetric Up and out) remain with higher option value and with higher percentage change in discounted payoff function than the regular options, say Vanilla, Down and out, Up and out.

Nr.	Call options	Option Value	Change in spot value				
			-0.001%	-0.01%	-0.1%	-1%	-5%
1	Vanilla	0.5798	-0.0098	-0.0979	-0.9772	-9.6177	-43.9625
2	Asymmetric Power	8.7821	-0.0101	-0.1012	-1.0103	-9.9147	-44.8199
3	Down and Out	0.5792	-0.0098	-0.0979	-0.9772	-9.6177	-43.9625
4	Up and Out	0.3268	-0.0020	-0.0196	-0.2029	-2.6652	-24.5545
5	Down and out asymmetric Power	8.7821	-0.0101	-0.1012	-1.0103	-9.9147	-44.8199
6	Up and out asymmetric Power	4.8301	-0.0020	-0.0204	-0.2109	-2.7461	-24.9039
Spot value is equal to 7.3			Percentage change in discounted payoff				

Table 4.2: USD depreciation, (Strike price) $K = 7$, (Lower barrier) $B = 4.8$, (Interest rates) $r_{USD} = 0.08$, $r_{SEK} = 0.12$, (Volatility) $\sigma = 0.0853$, (Upper barrier) $B = 8.4$, (Maturity time) $T = 1\text{year}$, $n = 2$.

Nr.	Call options	Option Value	Change in Spot value				
			0.001%	0.01%	0.1%	1%	5%
1	Vanilla	0.1199	0.0180	0.1804	0.1804	19.1403	118.2870
2	Asymmetric Power	1.7627	0.0182	0.1824	1.8340	19.3778	120.6399
3	Down and Out	0.1199	0.0180	0.1804	1.8142	19.1403	118.2870
4	Up and Out	0.1086	0.0164	0.1639	1.6450	17.0476	94.1317
5	Down and out asymmetric Power	1.7627	0.0182	0.1824	1.8340	19.3758	120.6399
6	Up and out asymmetric Power	1.5867	0.0165	0.1650	1.6561	17.1748	95.1512
Spot value is equal to 6.5			Percentage change in discounted payoff				

Table 4.3: USD appreciation, (Strike price) $K = 7$, (Lower barrier) $B = 4.8$, (Interest rates) $r_{USD} = 0.08$, $r_{SEK} = 0.12$, (Volatility) $\sigma = 0.0853$, (Upper barrier) $B = 8.4$, (Maturity time) $T = 1\text{year}$, $n = 2$.

In Table 4.4, also about the USD appreciation, we can see that for one month contract functions of Up and out asymmetric call and Up and out call, these options are non-speculative in the vicinity of the current exchange rate, that is, for any percentage change in spot value the speculator expects to lose money once the percentage in payoff is negative.

It is also observable that Down and out asymmetric power call as well as Down and out call are speculative for any maturity time.

However, the option value, in the vicinity of the lower barrier, is pretty small when the contract has a short maturity time, whereas the option value is high when the contract has a long exercise time.

In Figure 4.8 about the Up and out call and the Up and out asymmetric power call we can see that for different maturities the discounted payoff function attains its maximum value with the

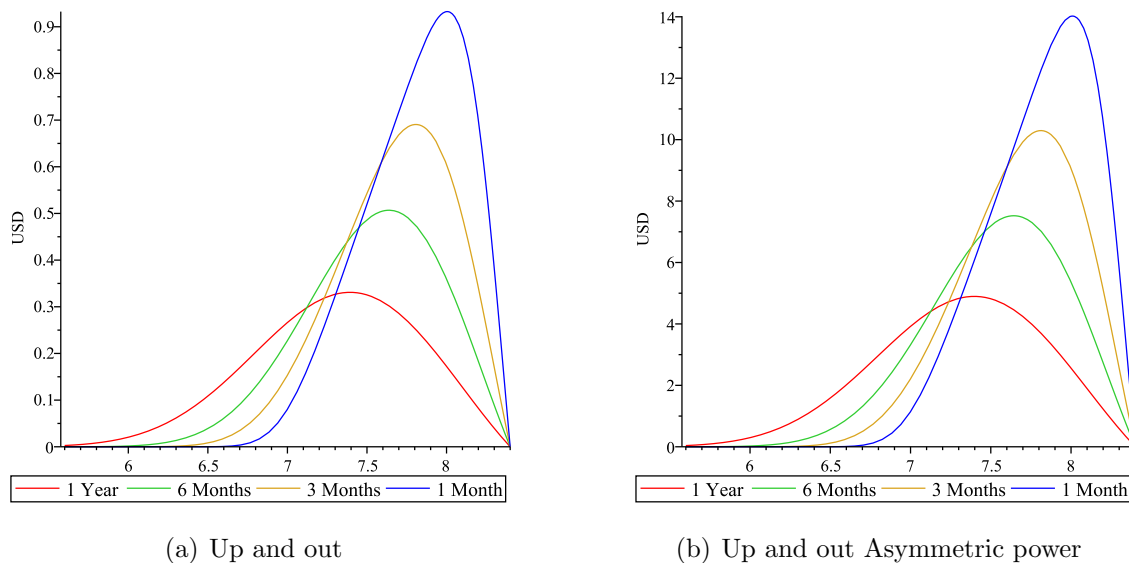


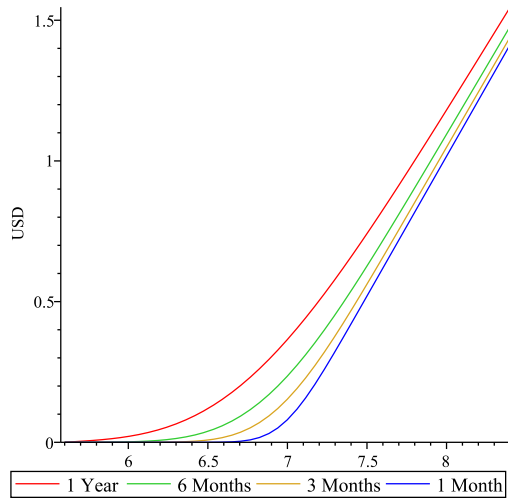
Figure 4.8: Call options with different Maturities (Volatility) $\sigma = 0.0853$, (Interest rates) $r_{USD} = 0.08$, $r_{SEK} = 0.12$, Strike price $K = 7$, (Upper barrier) $B = 8.4$, $n = 2$.

shortest maturity time. But, these contracts with short maturity time have small range of spot in which the payoff is high. Furthermore, for any maturity time the Up and out asymmetric power call options is more speculative than its regular Up and out call option.

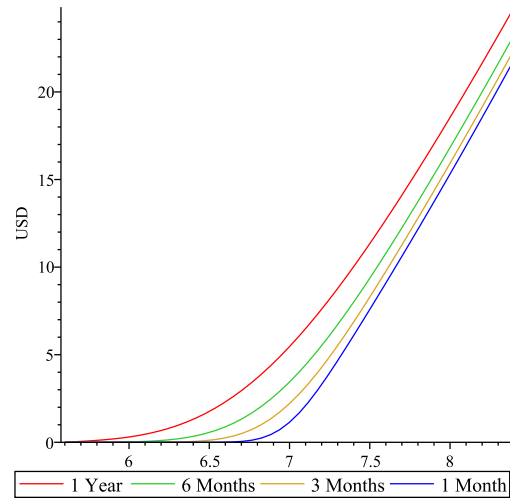
On the other hand, the contracts with the highest maturity time, their discounted payoff function has the lowest maximum value, but with a huge spot range of high payoff.

We see in Figure 4.9, about Down and out call and Down and out asymmetric power call as well as Figure 4.10 about Vanilla call and Asymmetric power call, all these options have a contrary behavior with respect to the Up and out asymmetric power call and Up and out call, that is, the payoff function is high when the exercise time is long and small when the maturity time is short.

Vanilla call and Asymmetric power call, see Figure 4.10, for different representative maturity time, have the same behavior as Down and out call and Down and out asymmetric call, that is, their payoff functions are high with a long maturity time and narrow with a short maturity time. According to the Figure 4.9 and Figure 4.10, it is clearly seen that, Vanilla call, Asymmetric power call, Down and out call and Down and out asymmetric power call are more speculative with long exercise time, once their payoff functions are high with long maturity time.

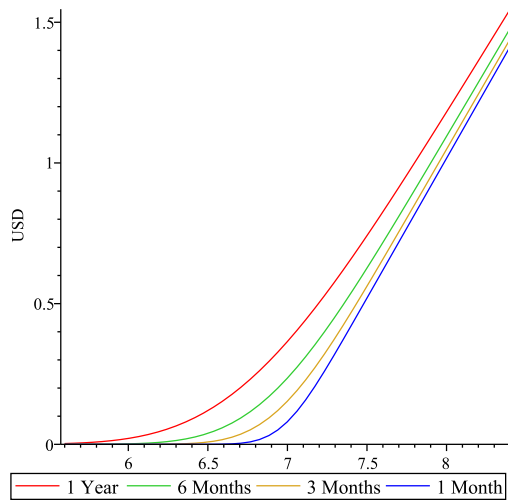


(a) Down and out

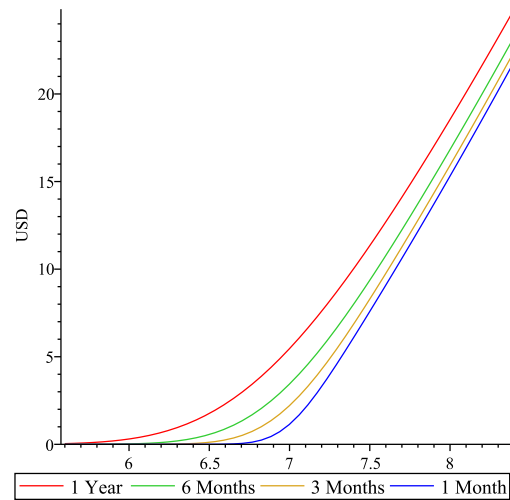


(b) Down and out Asymmetric power

Figure 4.9: Call options with different Maturities (Volatility) $\sigma = 0.0853$, (Interest rates) $r_{USD} = 0.08$, $r_{SEK} = 0.12$, (Strike price) $K = 7$, (Lower barrier) $B = 4.8$, $n = 2$.



(a) Vanilla



(b) Asymmetric power

Figure 4.10: Call options with different Maturities (Volatility) $\sigma = 0.0853$, (Interest rates) $r_{USD} = 0.08$, $r_{SEK} = 0.12$, (Strike price) $K = 7$, $n = 2$.

Nr.	Call options	Option Value	Change in Spot value					Mat
			0.01%	0.1%	1%	5%	10%	
1	Vanilla	0.6703	0.0354	0.3542	3.5714	18.493	38.539	1Y
2	Asymmetric Power	13.222	0.0380	0.3806	3.8442	20.062	42.231	
3	Down and Out	0.0073	5.6096	56.094	560.77	2801.9	5606.5	
4	Up and Out	0.0111	-0.0042	-0.0428	-0.4709	-3.2771	-8.7465	
5	DAO asymmetric Power	0.1569	5.6096	56.094	560.80	2805.0	5629.0	
6	UAO asymmetric Power	0.1629	-0.0042	-0.0428	-0.4706	-3.2757	-8.7439	
1	Vanilla	0.3589	0.0513	0.5142	5.2131	27.658	59.288	6M
2	Asymmetric Power	6.2035	0.0535	0.5357	5.4393	29.047	62.805	
3	Down and Out	0.0059	5.6096	56.095	560.83	2807.2	5644.6	
4	Up and Out	0.0231	0.0113	0.1122	1.0590	3.8559	3.9714	
5	DAO asymmetric Power	0.1075	5.6096	56.095	560.88	2810.9	5671.4	
6	UAO asymmetric Power	0.3398	0.0113	0.1123	1.0603	3.8622	3.9830	
1	Vanilla	0.1627	0.0777	0.7789	7.9762	44.161	99.459	3M
2	Asymmetric Power	2.5948	0.0795	0.7966	8.1663	45.457	103.14	
3	Down and Out	0.0042	5.6097	56.096	561.04	2824.8	5770.3	
4	Up and Out	0.0350	0.0411	0.4111	4.0691	19.094	33.640	
5	DAO asymmetric Power	0.6855	5.6097	56.097	561.10	2829.3	5803.4	
6	UAO asymmetric Power	0.5133	0.0412	0.4115	4.0743	19.121	33.697	
1	Vanilla	0.0248	0.1646	1.6555	17.562	113.51	306.56	1M
2	Asymmetric Power	0.3677	0.1658	1.6677	17.705	114.87	312.04	
3	Down and Out	0.0014	5.6106	56.111	562.91	2980.1	6882.6	
4	Up and Out	0.0188	0.1471	1.4781	15.481	93.460	225.20	
5	DAO asymmetric Power	0.0206	5.6106	56.112	562.99	2986.6	6933.5	
6	UAO asymmetric Power	0.2745	0.1474	1.4815	15.520	93.769	226.18	
Spot value is equal to 5.61			Percentage change in discounted payoff					

Table 4.4: USD appreciations, (Strike price) $K = 7$, (Lower barrier) $B = 5.6$, (Interest rates) $r_{USD} = 0.02$, $r_{SEK} = 0.03$, (Volatility) $\sigma = 0.20$, (Upper barrier) $B = 8.4$, (Maturity time) $T = 1\text{year}$, $n = 2$, DAO - Down and out, UAO - Up and out.

Conclusion

We have established explicit formulas for Symmetric and Asymmetric down and out power calls as well as for Symmetric and Asymmetric up and out power calls. To do this we used the Reflection Principle for Brownian motion.

The main difference between these instruments and standard Barrier options is that these instruments have a potentially higher discounted payoff.

Therefore, these instruments can be used for speculation with power call options for an investor believing in an increasing exchange rate, and the put options for someone with the opposite view.

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